# Girraween High School

# 2020

TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

# Mathematics Extension 2

**Total Marks: 100**

**Section 1** (Pages 2 – 4) **10 Marks**

- Attempt Q1 - Q10
- Allow about 15 minutes for this section

**Section 2** (Pages 5-11) **90 marks**

- Attempt Q11 - Q16
- Allow about 2 hours and 45 minutes for this section

**General Instructions**

- Reading time: 5 minutes
- Working time: 3 Hours
- Write using a black or blue pen
- Board approved calculators may be used
- Laminated reference sheets are provided
- Answer multiple choice questions by completely colouring in the appropriate circle on your multiple choice answer sheet on the front page of your answer booklet.
- In questions 11-16 start all questions on a separate page in your answer booklet and show all relevant mathematical reasoning and/or calculations.

## Section 1 (10 marks)

**Attempt Questions 1-10**

**Allow about 15 minutes for this section**

**Question 1**

$$\int x^3 cosx.dx =$$

**(A)** $-x^3 sinx + 3\int x^2\, sinx.dx$ **(B)** $-x^3 sinx - 3\int x^2\, sinx.dx$

**(C)** $x^3 sinx - 3\int x^2\, sinx.dx$ **(D)** $x^3 sinx + 3\int x^2\, sinx.dx$

**Question 2**

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}.dx =$$

**(A)** $\frac{1}{2}\ln(1-e^{2x}) + C$ **(B)** $sin^{-1}(e^x) + C$ **(C)** $tan^{-1}(e^x) + C$

**(D)** $cos^{-1}(e^x) + C$

**Question 3**

$$2e^{\frac{5\pi i}{6}} =$$

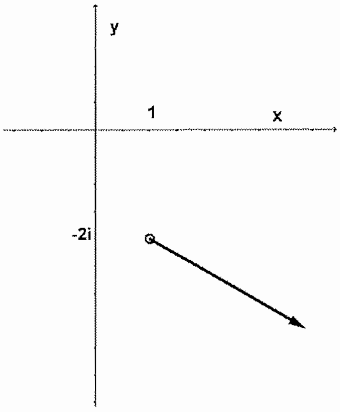
**(A)** $\sqrt{3} - i$ **(B)** $\sqrt{3} + i$ **(C)** $-\sqrt{3} - i$ **(D)** $-\sqrt{3} + i$

*Multiple choice continues on the following page*

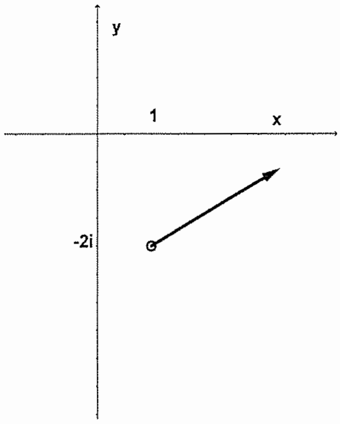
**Question 4**

Which of the following diagrams shows $Arg(z - 1 + 2i) = -\frac{\pi}{3}$?

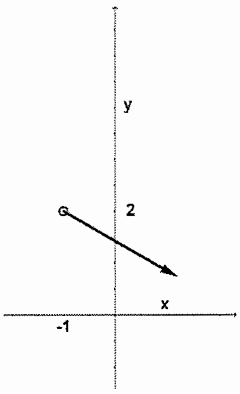
**(A)**



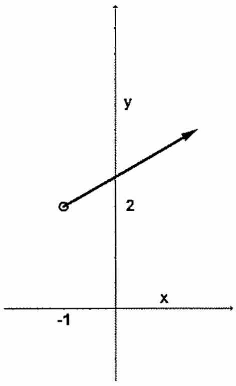

**(B)**




**(C)**




**(D)**




∀∃∈

**Question 5**

The contrapositive of "If it barks it's a dog" is

**(A)** "If it doesn't bark it isn't a dog"

**(B)** "If it's a dog it will bark"

**(C)** "If it doesn't bark it's a dog"

**(D)** "If it isn't a dog it doesn't bark"

**Question 6**

Which of the following is true?

**(A)** $\forall\, a \in Z^+ \;\exists\, b \in Z^+ : b = a^3$

**(B)** $\forall\, a \in Z^+ \;\exists\, b \in Z^+ : b = \sqrt[3]{a}$

**(C)** $\forall\, a \in Z^+ \;\exists\, b \in Z^+ : a = b^3$

**(D)** $\forall\, a \;\exists\, b \in Z^+ : b = a^3$

*Multiple choice continues on the following page*

**Question 7**

If $a$ and $b$ are *entirely imaginary* then which of the following is true

**(A)** $a^2 + b^2 \geq 2ab$ **(B)** $a^2 + b^2 \leq 2ab$ **(C)** $a^2 + b^2 = 2ab$

**(D)** Any of the above can happen.

**Question 8**

A particle moves in a straight line. At one point, $x = 6, v = 8$ and $a = 18$. An equation of motion for the particle could be

**(A)** $v^2 = \frac{x^3}{3} - 8$ **(B)** $v^2 = x^2 + 28$ **(C)** $v = x + 2$ **(D)** $v^2 = 3x^2$

**Question 9**

A particle moves with simple harmonic motion so that $v^2 = 27 - 18x - 9x^2$. The period and amplitude of the motion are

**(A)** Period = $\frac{\pi\sqrt{2}}{3}$ seconds, amplitude = $3m$ **(B)** Period = $\frac{2\pi}{3}$ seconds, amplitude = $2m$

**(C)** Period = $\frac{3\pi}{2}$ seconds, amplitude = $3m$ **(D)** Period = $\frac{\pi\sqrt{2}}{3}$ seconds, amplitude = $2m$

**Question 10**

The cartesian equation of the line $\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} + \boldsymbol{\lambda}(2\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k})$

is

**(A)** $\frac{x+1}{2} = \frac{y+2}{3} = \frac{z+1}{4}$ **(B)** $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+1}{4}$ **(C)** $x - 2 = \frac{y-2}{3} = z + 4$

**(D)** $x + 2 = \frac{y+2}{3} = z + 4$

*Examination continues on the following page*

**Section II (90 marks)**

**Attempt Questions 11-16**

**Allow about 2 hours and 45 minutes for this section**

Start the answers to each question on a separate page in your answer booklet.

In Questions 11-16 your responses should include all relevant mathematical reasoning and/ or calculations.

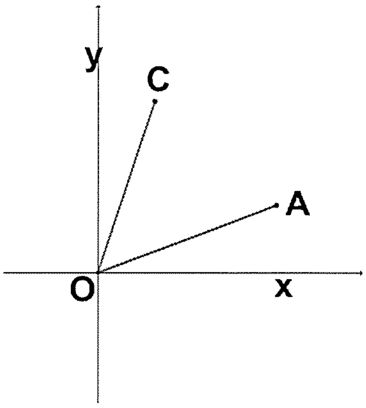
**Question 11** ***(15 marks)*** **Marks**

**(a)** If $z = 1 - i\sqrt{3}$

(i) Express $z$ in modulus/argument form **2**

(ii) Find $z^3$ and show that it is real. **2**

**(b)** If $O$ is the origin, $\overrightarrow{OA} = 2 + i$ and $\overrightarrow{OC} = 1 + 2i$ *(see diagram)*




(i) Find $B$ so that $OABC$ is a rhombus. **1**

(ii) By finding $\frac{\overrightarrow{OB}}{\overrightarrow{OA}}$, show that $\tan \angle AOB = \frac{1}{3}$ **2**

(iii) HENCE show that $tan^{-1}\left(\frac{1}{3}\right) + tan^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{4}$ **3**

***Question 11 continues on the following page***

**Question 11 (continued)** **Marks**

**(c)** (i) By letting $(x + iy)^2 = 3 - 4i$, find $\sqrt{3 - 4i}$ **3**

(ii) Hence solve the equation $z^2 + (4 - i)z + (3 - i) = 0$ **2**

**Question 12** ***(15 marks)*** **Marks**

**(a)** (i) Express $\frac{-13x-10}{(x+1)^2(x-2)}$ in the form $\frac{A}{(x+1)^2} + \frac{B}{(x+1)} + \frac{C}{(x-2)}$ **3**

(ii) Hence find $\int \frac{-13x-10}{(x+1)^2(x-2)} . dx$ **1**

**(b)** Find $\int \frac{1}{sinx-cosx-1} . dx$ **3**

**(c)** Find $\int e^x cosx. dx$ **2**

**(d)** (i) Show that $\frac{d}{dx} \cot x = -cosec^2\, x$ **1**

Let $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cot^n x. dx$

(ii) Show that $I_n = \frac{1}{n-1} - I_{n-2}$ **2**

(iii) Hence find $I_6$ **2**

(iv) Given that $I_n \to 0 \; as \; n \to \infty$, find $\lim_{n\to\infty} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$ **1**

***Examination continues on the following page***

**Question 13 (15 marks)** **Marks**

**(a)** Prove *by contraposition* that if $n^2 + 6n$ is even, then $n$ is even. **3**

**(b)** Prove *by contradiction* that $\log_3 11$ is irrational. **3**

**(c)** Prove by induction that $3^n \geq n^2$ for all positive integers $n \geq 1$ **3**

**(d)** Prove for all integers $x, y$ that if $10x + y$ is divisible by $17, 3y - 4x$ is also divisible by 17. **2**

**(e)** (i) Prove $a^2 + b^2 \geq 2ab$ for all $a, b \in R$ **1**

(ii) Hence or otherwise prove $a^4 + b^4 + c^4 + d^4 \geq 4abcd$ for all $a, b, c, d \in R$ **2**

(iii) Hence or otherwise prove $\frac{w+x+y+z}{4} \geq \sqrt[4]{wxyz}$ for all $w, x, y, z > 0$. **1**

**Question 14 (15 marks)**

**(a)** If $\underline{p} = \underline{i} + 2\underline{j} - \underline{k}$ and $\underline{q} = 2\underline{i} - \underline{j} + \underline{k}$

(i) Find $\underline{p}.\underline{q}$ **1**

(ii) Find the angle between $\underline{p}$ and $\underline{q}$. **1**

(iii) Find $Proj_{\underline{p}}\underline{q}$ **1**

*Question 14 continues on the following page*

**Question 14 (continued)** **Marks**

**(b)** (i) Show that the point $\begin{pmatrix} 12 \\ -6 \\ 6 \end{pmatrix}$ lies on the sphere **1**

$(x-10)^2 + (y+12)^2 + (z-14)^2 = 104.$

(ii) Show that the line $\begin{pmatrix} 12 \\ -6 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{-4} \end{pmatrix}$ forms a diameter of the sphere **2**

$(x-10)^2 + (y+12)^2 + (z-14)^2 = 104$ and find the other point at which the line intersects with the sphere.

(iii) Show that the line $\begin{pmatrix} -8 \\ 12 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{6} \\ \mathbf{-8} \\ \mathbf{5} \end{pmatrix}$ is skew to the line $\begin{pmatrix} 12 \\ -6 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{-4} \end{pmatrix}$ **4**

(iv) Find the points of intersection of $\begin{pmatrix} -8 \\ 12 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{6} \\ \mathbf{-8} \\ \mathbf{5} \end{pmatrix}$ and the sphere **3**

$(x-10)^2 + (y+12)^2 + (z-14)^2 = 104.$

(v) Show that the line $\begin{pmatrix} -8 \\ 12 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{6} \\ \mathbf{-8} \\ \mathbf{5} \end{pmatrix}$ passes directly above the centre of **2**

the sphere $(x-10)^2 + (y+12)^2 + (z-14)^2 = 104$ and find the point at which this happens.

***Examination continues on the following page***

**Question 15 (15 marks)** **Marks**

**(a)** The depth of the water at a wharf is regulated by the tide and can be modelled using simple harmonic motion. If at low tide at 7:00a.m. the depth is $5m$ and at high tide at 1:30p.m. the depth is $13m$

(i) Letting the time be measured in hours and $t = 0$ hours to be 7:00a.m. write a rule for the depth $(x)$ in terms of time $(t)$. **2**

(ii) A certain boat can only reach the wharf when the depth is *greater* than $8m$. What are the times this can happen between 7:00a.m. and the next low tide? **2**

**(b)** A $5kg$ projectile is launched at a speed of $600m/s$ at an angle of $30^o$ up from the horizontal. It experiences gravity of 50 Newtons and air resistance opposite to its direction of motion of $\frac{5}{6}v$ Newtons.

(i) Show that $\ddot{x} = -\frac{1}{6}\dot{x}$ and $\ddot{y} = -10 - \frac{1}{6}\dot{y}$ where $x$ is horizontal displacement and $y$ is vertical displacement. **1**

(ii) Show that the initial velocities in the horizontal and vertical directions are $300\sqrt{3}m/s$ and $300m/s$ respectively. **1**

(iii) Show that $\dot{x} = 300\sqrt{3}\, e^{-\frac{t}{6}}$ and $x = 1800\sqrt{3}(1 - e^{-\frac{t}{6}})$ **3**

(iv) Find the maximum possible horizontal range of the projectile the range it can never quite reach). **1**

(v) Show that $\dot{y} = 360e^{-\frac{t}{6}} - 60$ and $y = -2160e^{-\frac{t}{6}} - 60t + 2160$. **3**

(vi) Find the maximum height of the projectile. **2**

***Examination continues on the following page***

**Question 16 (15 marks)** **Marks**

**(a)** A projectile is launched vertically upwards from the ground at a speed of $Um/s$. It experiences acceleration due to gravity of $g\ m/s^2$ and acceleration due to air resistance of $kv^2 m/s^2$.

(i) If $x$ is the vertical height of the projectile above the ground, show that $x = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g+kv^2}\right)$. **3**

(ii) Show that the maximum height reached is $\frac{1}{2k}\ln\left(1+\frac{kU^2}{g}\right)$ metres. **1**

(iii) The projectile starts to fall from its maximum height. It continues to experience acceleration due to gravity of $g\ m/s^2$ and air resistance *against* its motion of $kv^2 m/s^2$. Letting down be positive, and the point where the projectile reaches its maximum height be $x = 0$, find the *terminal velocity* of the projectile in terms of $k$ and $g$ and show that $x = \frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right)$. **4**

(iv) Letting $T$ be the terminal velocity, $W$ be the impact velocity (the speed at which the projectile hits the ground) and keeping $U$ as the initial launch velocity, show that $\frac{1}{U^2}+\frac{1}{T^2}=\frac{1}{W^2}$. **2**

*Question 16 continues on the following page*

**Question 16 (continued)** **Marks**

**(b)** (i) Solve $z^5 - 1 = 0$. **1**

(ii) If $w$ is the root of $z^5 - 1 = 0$ with the smallest positive argument,

show that $w^2 + \frac{1}{w^2} + w + \frac{1}{w} = -1$. **2**

(iii) Hence show that $x = cos\frac{2\pi}{5}$ is a root of the equation $4x^2 + 2x - 1 = 0$. **1**

(iv) Hence find the exact value of $cos\frac{2\pi}{5}$. **1**

***END OF EXAMINATION!!!***

# Solutions: Y12 Trial Exam Ext 2 2020 p.1

New Syllabus.

Multiple Choice:

Q.(1) C (2) B (3) D (4) A (5) D (6) A (7) B (8) A (9) B (10) B

(1) $\int x^3 \cos x \, dx$  $u = x^3$  $v = \sin x$
$u' = 3x^2$  $v' = \cos x$

$= x^3 \sin x - 3\int x^2 \sin x \, dx$  (C)

(2) $\int \frac{e^x}{\sqrt{1-e^{2x}}} \, dx$  (B)

$= \sin^{-1}(e^x) + C$

[By $\int \frac{f'(x)}{\sqrt{1-[f(x)]^2}} \, dx = \sin^{-1}(f(x)) + C$].

(3) $2e^{\frac{5\pi i}{6}}$  (D)

$= 2 \times [\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}]$

(by Euler's theorem)

$= 2[-\frac{\sqrt{3}}{2} + \frac{i}{2}]$

$= -\sqrt{3} + i$

(4) (A)

(5) "If it isn't a dog it doesn't bark" (D)

(6) (A) "For all positive integers a there is a positive integer b such that $b = a^3$"
[Note: (B) & (C) said that the cube root of every positive integer was also a positive integer & (D) didn't say a was an integer.

(7) (B) If $a = xi$, $b = yi$
$x, y$ real then
$a^2 + b^2 = (xi)^2 + (yi)^2 = -(x^2+y^2)$
& $2ab = -2xy$.
As $x^2+y^2 \geq 2xy$, $x, y$ real
$-(x^2+y^2) \leq -2xy$.

(8) Using $a = \frac{d}{dx}\left[\frac{1}{2}v^2\right]$  (A)

in (A), $v^2 = \frac{x^3}{3} - 8$

$\frac{1}{2}v^2 = \frac{x^3}{6} - 4$

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{x^2}{2}$. In this case, $x = 6$, $a = \frac{6^2}{2} = 18$ &

$v^2 = \frac{6^3}{3} - 8 \Rightarrow v = 8$.

In (B) $a = x$ & would $= 6$, not 18

In (C) $a = x + 2$ & would $= 8$, not 18

In (D) $v^2 \neq 3x^2$ [$8^2 \neq 3 \times 6^2$].

(9) $v^2 = 27 - 18x - 9x^2$

Amplitude: $v^2 = 0$

$27 - 18x - 9x^2 = 0$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$  $x = -3$ or $1$.  (B)

Amplitude $= 2$.

Period: $v^2 = 27 - 18x - 9x^2$

$a = \frac{d}{dx}\left[\frac{1}{2}v^2\right] = -9(x-1)$.

By $\ddot{x} = -n^2 x$, Period $= \frac{2\pi}{3}$

(10) As $x = 2\lambda + 1$, $\lambda = \frac{x-1}{2}$

As $y = 3\lambda + 2$, $\lambda = \frac{y-2}{3}$

As $z = 4\lambda - 1$, $\lambda = \frac{z+1}{4}$

$\therefore \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+1}{4}$  (B)

Solutions: Y12 Trial: p-2

Q.(11.) (a) (i) $z = 1 - i\sqrt{3}$

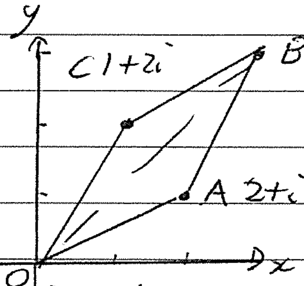
$= 2\,\text{cis}\left(-\frac{\pi}{3}\right)$

(ii) $z^3 = 2^3\,\text{cis}\left(\frac{-3\pi}{3}\right)$ (By De Moivre).

$= 8\,\text{cis}(-\pi)$

$= -8$, which is real.

(b)




(i) $B = 3+3i$.

(ii) $\frac{\overrightarrow{OB}}{\overrightarrow{OA}} = \frac{3+3i}{2+i} \times \frac{(2-i)}{(2-i)}$

$= \frac{9+3i}{5}$

$\text{Arg}\left(\frac{\overrightarrow{OB}}{\overrightarrow{OA}}\right) = \tan\angle AOB = \frac{\left(\frac{3}{5}\right)}{\left(\frac{9}{5}\right)}$

$= \frac{1}{3}$.

(iii) $\text{Arg}\,\overrightarrow{OA} = \tan\angle AOx = \frac{1}{2}$.

$\therefore$ & $\text{Arg}(\overrightarrow{OB}) = \tan\angle BOx = \frac{1}{1} = 1$.

$\therefore \angle AOx + \angle AOB = \angle BOx$

$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$.

(c) (i) $(x+iy)^2 = 3-4i$, $x, y$ real

$(x^2-y^2) + 2ixy = 3-4i$.

$\therefore x^2 - y^2 = 3$ equating reals (1)

$2xy = -4$ equating imaginaries.

$y = -\frac{2}{x}$ (2)

(11)(c)(i) (continued):

Sub. (2) in (1):

$x^2 - \left(\frac{-2}{x}\right)^2 = 3$.

$x^4 - 3x^2 - 4 = 0$

$(x^2-4)(x^2+1) = 0$

$x = \pm 2$, [$x \neq \pm i$ as $x$ is real].

As $y = -\frac{2}{x}$, $y = \mp 1$.

$\therefore \sqrt{3-4i} = \pm(2-i)$.

(ii) Solving $z^2 + (4-i)z + (3-i) = 0$

Noting $\Delta = b^2 - 4ac$

$= (4-i)^2 - 4 \times 1 \times (3-i)$

$= 3-4i$

$z = \frac{-(4-i) \pm \sqrt{\Delta}}{2 \times 1}$ [Quadratic formula].

As $\sqrt{\Delta} = \pm(2-i)$ [from (i)]

$z = \frac{-(4-i) \pm (2-i)}{2}$

$z = -1$ or $z = -3+i$

[Note that as the co-efficients in the original quadratic equation aren't real, complex solutions DON'T have to be conjugates of each other].

Q.(12)(a)(i) $\dfrac{-13x-10}{(x+1)^2(x-2)} = \dfrac{A}{(x+1)^2} + \dfrac{B}{(x+1)} + \dfrac{C}{(x-2)}$

$\therefore -13x-10 = A(x-2)+B(x+1)(x-2)+C(x+1)^2 \quad (1)$

Sub. $x=2$ in (1):

$-13\times 2-10 = C(2+1)^2 \Rightarrow C=-4.$

Sub. $x=-1$ in (1):

$-13\times -1-10 = A(-1-2) \Rightarrow A=-1.$

Sub. $x=0, A=-1, C=-4$ in (1):

$-10 = -2\times -1 - 2B - 4.$

$4 = B$

$\therefore A=-1, B=4$ & $C=-4$ & $\dfrac{-13x-10}{(x+1)^2(x-2)} = \dfrac{-1}{(x+1)^2} + \dfrac{4}{(x+1)} - \dfrac{4}{(x-2)}.$

(ii) $\displaystyle\int \frac{-13x-10}{(x+1)^2(x-2)}.dx$

$\displaystyle = \int -\frac{1}{(x+1)^2} + \frac{4}{(x+1)} - \frac{4}{(x-2)}.dx$

$\displaystyle = \frac{1}{x+1} + 4\ln(x+1) - 4\ln(x-2) + C.$

(b) $\displaystyle\int \frac{1}{\sin x - \cos x - 1}.dx$

Letting $t = \tan\left(\frac{x}{2}\right)$

$\dfrac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

$= \dfrac{1+t^2}{2}$

$\therefore dx = \dfrac{dx}{dt}.dt = \dfrac{2}{1+t^2}.dt$

$\displaystyle = \int \frac{1}{\frac{2t}{1+t^2} - \frac{(1-t^2)}{1+t^2} - 1} \cdot \frac{2}{1+t^2}.dt$

$\displaystyle = \int \frac{1}{2t-2}.dt$

$\displaystyle = \frac{1}{2}\int \frac{2}{2t-2}.dt$

$\displaystyle = \frac{1}{2}\int \frac{1}{t-1}.dt$

$= \frac{1}{2}\ln(t-1) + C$

Note also

$\frac{1}{2}\ln(2t-2)+C$ is also correct as it only differs from $\frac{1}{2}\ln(t-1)$ by $\frac{1}{2}\ln 2$ which is part of the constant $C$.

Q.(12)(c) $\int e^x \cos x . dx$ $\quad u = e^x \quad v = \sin x$

$u' = e^x \quad v' = \cos x$

Letting $I = \int e^x \cos x . dx$

$I = e^x \sin x - \int e^x \sin x . dx \quad (1)$

Taking $\int e^x \sin x . dx$ out of (1): $\quad u = e^x \quad v = -\cos x$

$u' = e^x \quad v' = \sin x$

$= -e^x \cos x + \int e^x \cos x . dx$

$\therefore \int e^x \sin x . dx = -e^x \cos x + I \quad (2)$

Sub. (2) in (1):

$I = e^x \sin x - \left[-e^x \cos x + I\right]$

$2I = e^x[\sin x + \cos x]$

$\therefore I = \int e^x \cos x . dx = \frac{1}{2} e^x [\sin x + \cos x] + C$

(d)(i) $\frac{d}{dx}(\cot x)$

$= \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right]$

$= \frac{-\sin x \times \sin x - \cos x \times \cos x}{\sin^2 x}$ (By quotient rule)

$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= -\frac{1}{\sin^2 x}$

$= -\operatorname{cosec}^2 x.$

PTO →

Y12 4U Trial p.5

Q.(12)(d)(ii) Let $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x \, dx$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x . \cot^2 x . dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x (\operatorname{cosec}^2 x - 1) . dx$$

$$= -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x . -\operatorname{cosec}^2 x \, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x . dx$$

$$I_n = -\left[\frac{\cot^{n-1} x}{n-1}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - I_{n-2}.$$

$$I_n = \left[\frac{0-1}{n-1}\right] - I_{n-2}$$

[as $\cot\frac{\pi}{2} = 0$, $\cot\frac{\pi}{4} = 1 \therefore \cot^{n-1}(\frac{\pi}{4}) = 1^{n-1} = 1$.

$$= \frac{1}{n-1} - I_{n-2}$$

(iii) Finding $I_6$.

$$I_0 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^0 x . dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 . dx$$

$$= \frac{\pi}{4}.$$

$$I_2 = 1 - I_0$$

$$= 1 - \frac{\pi}{4}.$$

$$I_4 = \frac{1}{3} - I_2$$

$$= \frac{1}{3} - \left(1 - \frac{\pi}{4}\right)$$

$$= \frac{1}{3} - 1 + \frac{\pi}{4}$$

$$= -\frac{2}{3} + \frac{\pi}{4}.$$

$$I_6 = \frac{1}{5} - I_4$$

$$= \frac{1}{5} + \frac{2}{3} - \frac{\pi}{4}$$

$$= \frac{13}{15} - \frac{\pi}{4}.$$

(iv) Using $I_4 = \frac{1}{3} - 1 + \frac{\pi}{4}$

$$I_6 = \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}.$$

$$I_8 = \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 + \frac{\pi}{4}$$

$$I_{10} = \frac{1}{9} - \frac{1}{7} + \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}.$$

$$I_{2n+2} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + .. + \frac{1}{2n+1} - \frac{\pi}{4}.$$

As $I_n \to 0$ as $n \to \infty$

$$\lim_{n\to\infty}\left(1 - \frac{1}{3} + \frac{1}{5} - .. + \frac{1}{2n+1} - \frac{\pi}{4}\right) = 0$$

$$\therefore \lim_{n\to\infty}\left(1 - \frac{1}{3} + \frac{1}{5} - .. + \frac{1}{2n+1}\right) = \frac{\pi}{4}.$$

Q.(13)(a) Contrapositive of "if $n^2+6n$ is even, $n$ is even" is "If $n$ is odd, $n^2+6n$ is odd"

Letting $n$ be odd i.e. $n=2k-1$, $k$ an integer.

$\therefore n^2+6n$

$= (2k-1)^2+6(2k-1)$

$= 4k^2+8k-5$

$= 2[2k^2+4k-2]-1$

which is odd as $2(2k^2+4k-2)$ is even.

(b) Let $\log_3 11$ be rational

i.e. $\log_3 11 = \frac{p}{q}$, $p, q$ integers.

$3^{\frac{p}{q}} = 11$

$3^p = 11^q$

which is not possible as both 3 & 11 are prime & any whole number can only have 1 set of prime factors [fundamental theorem of arithmetic)

$\therefore \log_3 11$ is irrational.

(c) Step 1: Show true for $n=1$.

LHS: $=3^1 = 3$

RHS: $=1^2 = 1$.

LHS > RHS

True for $n=1$.

Step 2: Assume true for $n=k$

i.e. $3^k > k^2$, $k$ an integer $\geq 1$.

Step 3: Prove true for $n=k+1$

i.e. $3^{k+1} \geq (k+1)^2$, $k$ an integer $\geq 1$.

LHS:

$3^{k+1}$

$= 3 \times 3^k$

$\geq 3k^2$ [by assumption)

$\geq (k+1)^2$ [by (2) below]

$\therefore$ If it is true for $n=k$ it will be true for $n=k+1$

$\therefore$ As it is true for $n=2$ it will be true for $n=2+1=3$ & so on for all positive integers $k$. It was also shown to be true for $n=1$ in Step 1

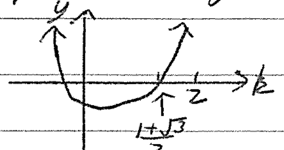
$\therefore 3^{\cdot} \geq n^2$ for all positive integers $n$.

Showing $3k^2 \geq (k+1)^2$

Solve $3k^2-(k+1)^2 \geq 0$

$2k^2-2k-1 \geq 0$

For $2k^2-2k-1=0$, $k=\frac{1+\sqrt{3}}{2}$ (or $\frac{1-\sqrt{3}}{2}$)




From diagram, $3k^2 \geq (k+1)^2$, $k \geq 2$.

10:56

Q. (13)(d) Let $10x + y = 17k$, $k$ an integer

$\therefore \quad y = 17k - 10x. \quad (1)$

$\therefore \quad 3y - 4x$

$= 3(17k - 10x) - 4x$

$= 51k - 30x - 4x$

$= 51k - 34x$

$= 17(3 - 2x)$

$= 17L$, $L = 3 - 2x$, an integer as $x$ is an integer.

$\therefore 3y - 4x$ is also divisible by 17.

(e)(i) If $a, b$ real,

$(a - b)^2 \geq 0$

$a^2 + b^2 - 2ab \geq 0$

$a^2 + b^2 \geq 2ab \quad (1)$

(ii) Using (i) above,

$a^4 + b^4 \geq 2a^2b^2$ & $c^4 + d^4 \geq 2c^2d^2$

$\therefore a^4 + b^4 + c^4 + d^4 \geq 2(a^2b^2 + c^2d^2)$

$\geq 2(2abcd)$ [as $a^2b^2 + c^2d^2 \geq 2abcd$ by (i)]

$= 4abcd.$

(iii) Letting $w = a^4$, $x = b^4$, $y = c^4$ & $z = d^4$,

$w + x + y + z \geq 4 \sqrt[4]{w} \times \sqrt[4]{x} \times \sqrt[4]{y} \times \sqrt[4]{z}$ $\div 4$

$$\frac{w + x + y + z}{4} \geq \sqrt[4]{wxyz}.$$

Q.(14)(a)(i) $\underset{\sim}{p} \cdot \underset{\sim}{q}$

$= (\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) \cdot (2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k})$

$= 2 - 2 - 1$

$= -1.$

(ii) $\cos \angle POQ = \dfrac{\underset{\sim}{p} \cdot \underset{\sim}{q}}{|\underset{\sim}{p}||\underset{\sim}{q}|}$

$= \dfrac{-1}{\sqrt{6} \times \sqrt{6}}$

$= \dfrac{-1}{6}.$

$\angle POQ = 99^\circ 36'$ [nearest minute].

(iii) $\text{Proj}_{\underset{\sim}{p}} \underset{\sim}{q} = \dfrac{\underset{\sim}{p} \cdot \underset{\sim}{q}}{|\underset{\sim}{p}|^2} \underset{\sim}{p}$

$= -\dfrac{1}{6}(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}).$

(b)(i) $(12-10)^2 + (-6+12)^2 + (6-14)^2 = 104.$

(ii) The line $\begin{pmatrix}12\\-6\\6\end{pmatrix} + \lambda\begin{pmatrix}1\\3\\-4\end{pmatrix}$ forms a diameter if it passes through the CENTRE of the sphere (the point $\begin{pmatrix}10\\-12\\14\end{pmatrix}$)

Showing this, $\begin{pmatrix}12\\-6\\6\end{pmatrix} + \lambda\begin{pmatrix}1\\3\\-4\end{pmatrix} = \begin{pmatrix}10\\-12\\14\end{pmatrix}$

$12 + \lambda = 10 \Rightarrow \lambda = -2$

$-6 + 3\lambda = -12 \Rightarrow \lambda = -2$

$6 - 4\lambda = 14 \Rightarrow \lambda = -2.$

Line passes through centre. By symmetry, other point of intersection will be $\begin{pmatrix}8\\-18\\22\end{pmatrix}$.

(iii) Checking to see if lines are skew:

$\begin{pmatrix}12\\-6\\6\end{pmatrix} + \lambda_1\begin{pmatrix}1\\3\\-4\end{pmatrix} = \begin{pmatrix}-8\\12\\2\end{pmatrix} + \lambda_2\begin{pmatrix}6\\-8\\5\end{pmatrix}$

$12 + \lambda_1 = -8 + 6\lambda_2 \Rightarrow \lambda_1 - 6\lambda_2 = -20 \quad (1)$

$-6 + 3\lambda_1 = 12 - 8\lambda_2 \Rightarrow 3\lambda_1 + 8\lambda_2 = 18 \quad (2)$

$6 - 4\lambda_1 = 2 + 5\lambda_2 \Rightarrow -4\lambda_1 - 5\lambda_2 = -4$

or $4\lambda_1 + 5\lambda_2 = 4 \quad (3)$

Using (1) & (2)

$(1)\times 3 = (3)$

$3\lambda_1 + 8\lambda_2 = 18 \quad (2)$

$3\lambda_1 - 18\lambda_2 = -60 \quad (3)$

$26\lambda_2 = 78$

$\lambda_2 = 3.$

Sub. $\lambda_2 = 3$ in (1)

$\lambda_1 - 18 = -20$

$\lambda_1 = -2.$

Sub. $\lambda_1 = -2, \lambda_2 = 3$ in (3) to see if it works:

$4\times -2 + 5\times 3 = 7 \neq 4.$

$\Rightarrow$ Lines don't intersect.

If you used (1) & (3)

$(1)\times 4 = (5)$

$4\lambda_1 + 5\lambda_2 = 4 \quad (3)$

$4\lambda_1 - 24\lambda_2 = -80 \quad (5)$

$29\lambda_2 = 84.$

$\lambda_2 = \frac{84}{29} = 2\frac{26}{29}.$

Sub. $\lambda_2 = 2\frac{26}{29}$ in (1):

$\lambda_1 - 6\times 2\frac{26}{29} = -20$

$\lambda_1 = -2\frac{18}{29}.$

Sub. $\lambda_1 = -2\frac{18}{29}, \lambda_2 = 2\frac{26}{29}$ in (2):

$3\times -2\frac{18}{29} + 8\times 2\frac{26}{29} = 15\frac{9}{29} \neq 18$

$\Rightarrow$ Lines don't intersect.

If you used (2) & (3)

$(2)\times 4 = (6) \quad (3)\times 3 = (7)$

$12\lambda_1 + 32\lambda_2 = 72 \quad (6)$

$12\lambda_1 + 15\lambda_2 = 12 \quad (7)$

$17\lambda_2 = 60$

$\lambda_2 = \frac{60}{17} = 3\frac{9}{17}.$

Sub. $\lambda_2 = 3\frac{9}{17}$ in (3)

$4\lambda_1 + 5\times 3\frac{9}{17} = 4$

$\lambda_1 = -\frac{58}{17} = -3\frac{7}{17}.$

Sub. $\lambda_1 = -3\frac{7}{17}, \lambda_2 = 3\frac{9}{17}$ in (1) to see if it works:

$-3\frac{7}{17} - 6\times 3\frac{9}{17}$

$= -24\frac{10}{17} \neq -20$

$\Rightarrow$ Lines don't intersect.

By observation, as $\begin{pmatrix}6\\-8\\5\end{pmatrix} \neq \lambda\begin{pmatrix}1\\3\\-4\end{pmatrix}$, direction vectors aren't //.

$\Rightarrow$ Lines don't intersect & aren't //. They are SKEW.

(iv) $(-8+6\lambda-10)^2 + (12-8\lambda+12)^2 + (2+5\lambda-14)^2 = 104$

$36(\lambda-3)^2 + 64(3-\lambda)^2 + (5\lambda-12)^2 = 104.$

$100(\lambda-3)^2 + (5\lambda-12)^2 = 104$

$125\lambda^2 - 720\lambda + 1044 = 104$

$25\lambda^2 - 144\lambda + 188 = 0$

$$\lambda = \frac{144 \pm \sqrt{(-144)^2 - 4\times 25\times 188}}{2\times 25}$$

$\lambda = \frac{94}{25}$ or $\lambda = 2.$

Points of intersection of line & sphere

$= \begin{pmatrix}-8\\12\\2\end{pmatrix} + \frac{94}{25}\begin{pmatrix}6\\-8\\5\end{pmatrix} = \begin{pmatrix}14\frac{14}{25}\\-18\frac{2}{25}\\20\frac{4}{5}\end{pmatrix}$ or $= \begin{pmatrix}-8\\12\\2\end{pmatrix} + 2\begin{pmatrix}6\\-8\\5\end{pmatrix} = \begin{pmatrix}4\\-4\\12\end{pmatrix}$

(v) $\begin{pmatrix}-8\\12\\2\end{pmatrix} + \lambda\begin{pmatrix}6\\-8\\5\end{pmatrix}$ passes through $\begin{pmatrix}10\\-12\\z\end{pmatrix}$ $\quad 6\lambda - 8 = 10 \Rightarrow \lambda = 3$

Check: $-8\lambda + 12 = -12 \Rightarrow \lambda = 3$

So if $\lambda = 3$, z co-ordinate $= 2 + 5\times 3 = 17.$

Point above centre of sphere line passes through is $\begin{pmatrix}10\\-12\\17\end{pmatrix}.$

Q.(15)(a) → Starts at BOTTOM so − cos

(i) → Period $= \frac{2\pi}{n} = 13$ hours.

$\times n$

$\therefore \cancel{n}\ 2\pi = 13n$

$n = \frac{2\pi}{13}$

→ Centre of "motion" $= 9$. Amplitude $= 4$.

∴ Depth is $x = -4\cos\left(\frac{2\pi t}{13}\right) + 9$.

(ii) Finding when $-4\cos\left(\frac{2\pi t}{13}\right) + 9 = 8$

$\cos\left(\frac{2\pi t}{13}\right) = \frac{1}{4}$.

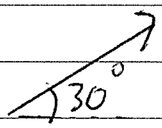
$\frac{2\pi t}{13} = 1{\cdot}3181\ldots$ or $4{\cdot}9650\ldots$

$t = $ 2h 43mins 37s. or 10h 16mins 22...

But to the nearest minute must be BETWEEN these times.

So $t = $ 9:44 a.m. to 5:16 p.m.

(b)



(i) Initial horizontal velocity:

Force $= 5\ddot{x} = -\frac{5}{6}v = ma$

$\xleftarrow{\quad} \frac{5}{6}v$

$\ddot{x} = -\frac{1}{6}v$.

$\ddot{y}$

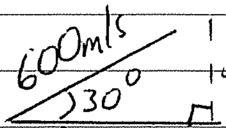
$\downarrow\ \downarrow$

$50 \quad \frac{5}{6}v$.

$F = ma = 5\ddot{y} = -50 - \frac{5}{6}v$

$\ddot{y} = -10 - \frac{1}{6}v$.

(ii) Initial $\dot{x}$ & $\dot{y}$:



$\dot{y} = 600\sin 30^\circ = 300$ m/s.

$\dot{x} = 600\cos 30^\circ = 300\sqrt{3}$ m/s.

Q.(15)(b)(iii) $\ddot{x} = \frac{dv}{dt}$ $(v = v_x) = -\frac{1}{6}v$

$$\therefore \frac{dt}{dv} = \frac{-6}{v}$$

$$t = \int \frac{-6}{v}\,dv$$

$$= -6\ln v + C$$

As $v = 300\sqrt{3}$ when $t = 0$,

$$0 = -6\ln 300\sqrt{3} + C$$

$$t = -6\ln v + 6\ln(300\sqrt{3})$$

$$= -6\ln\left(\frac{v}{300\sqrt{3}}\right)$$

$$\therefore e^{-\frac{t}{6}} = \frac{v}{300\sqrt{3}}$$

$$\underline{300\sqrt{3}e^{-\frac{t}{6}} = v_x}$$

$$x = \int 300\sqrt{3}\,e^{-\frac{t}{6}}\,dt$$

$$= -1800\sqrt{3}e^{-\frac{t}{6}} + C$$

As $x = 0$ when $t = 0$

$$0 = -1800\sqrt{3}e^{0} + C$$

$$\underline{x = 1800\sqrt{3}\left(1 - e^{-\frac{t}{6}}\right)}$$

(iv) As $t \to \infty$, $e^{-\frac{t}{6}} \to 0$ & $x \to 1800\sqrt{3}$.

Limiting range $= 1800\sqrt{3}$ m.

(v) $\ddot{y} = \frac{dv_y}{dt} = -10 - \frac{1}{6}v$

$$= \frac{-(60+v)}{6}.$$

$$\frac{dt}{dv} = \frac{-6}{60+v}.$$

$$t = \int \frac{-6}{60+v}\,dv$$

$$= -6\ln(60+v) + C$$

As $\dot{y} = 300$ when $t = 0$

$$0 = -6\ln(60+300) + C$$

$$6\ln 360 = C.$$

$$t = -6\ln(60+v) + 6\ln 360$$

$$t = -6\ln\left(\frac{60+v}{360}\right)$$

$$e^{-\frac{t}{6}} = \frac{60+v}{360}$$

$$\underline{360e^{-\frac{t}{6}} - 60 = \dot{y}}$$

Q.(15)(b)(v) (continued):

$$y = \int 360e^{-\frac{t}{6}} - 60 \, dt$$

$$= -2160e^{-\frac{t}{6}} - 60t + C$$

As $y = 0$ when $t = 0$

$$0 = -2160e^{0} - 60 \times 0 + C$$

$$2160 = C$$

$$y = -2160e^{-\frac{t}{6}} - 60t + 2160.$$

(vi) Maximum height: Is $y$ when $\dot{y} = 0$

$$360e^{-\frac{t}{6}} - 60 = 0$$

$$e^{-\frac{t}{6}} = \frac{1}{6}.$$

$$\& \ -\frac{t}{6} = \ln\left(\frac{1}{6}\right)$$

$$t = -6\ln\left(\frac{1}{6}\right)$$

$$= \underline{6\ln 6}.$$

$$\text{Max. height} = -2160e^{-\ln 6} - 60 \times 6\ln 6 + 2160$$

$$= \underline{1154.966... \text{ m.}}$$

So the maximum height the projectile reaches is
<u>approx. 1155m after 10.75.. seconds.</u>

Y12 Ext 2 p.13

Q.(16)(a)(i)

$\downarrow g \quad \downarrow kv^2$

$$\ddot{x} = v.\frac{dv}{dx} = -g-kv^2.$$

$$\frac{dv}{dx} = \frac{-g-kv^2}{v}$$

$$\frac{dx}{dv} = \frac{-v}{g+kv^2}.$$

$$x = \frac{-1}{2k}\int \frac{2kv}{g+kv^2}.dv$$

$$x = -\frac{1}{2k}\ln(g+kv^2)+C$$

As $x=0$ when $v=U$,

$$0 = -\frac{1}{2k}\ln(g+kU^2)+C$$

$$\frac{1}{2k}\ln(g+kU^2) = C$$

$$x = -\frac{1}{2k}\ln(g+kv^2)+\frac{1}{2k}\ln(g+kU^2)$$

$$x = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g+kv^2}\right).$$

(ii) Maximum height reached $v=0$

$$x = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g}\right)$$

$$x = \frac{1}{2k}\ln\left(1+\frac{kU^2}{g}\right).$$

(iii)

$\downarrow g$

$kv^2 \uparrow$

Going down: $\ddot{x} = g-kv^2.$

$$v.\frac{dv}{dx} = g-kv^2.$$

$$\frac{dv}{dx} = \frac{g-kv^2}{v}.$$

$$\frac{dx}{dv} = \frac{v}{g-kv^2}$$

$$x = -\frac{1}{2k}\int\frac{2kv}{g-kv^2}.dv$$

$$x = -\frac{1}{2k}\ln(g-kv^2)+C$$

As projectile is falling from rest, $0 = -\frac{1}{2k}\ln g + C \Rightarrow C = \frac{1}{2k}\ln g$

$$x = \frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right).$$

Q.(16)(a)(iv) T = terminal velocity [when $\ddot{x}=0$].

$$g-kv^2=0$$
$$v^2=\frac{g}{k}$$
$$v=\sqrt{\frac{g}{k}}$$
$$\therefore T=\sqrt{\frac{g}{k}}$$

W is impact velocity → velocity when projectile hits the ground, which is where $x=\frac{1}{2k}\ln\left(1+\frac{kU^2}{g}\right)$

So $\frac{1}{2k}\ln\left(\frac{g}{g-kW^2}\right)=\frac{1}{2k}\ln\left(1+\frac{kU^2}{g}\right)$

$$\frac{g}{(g-kW^2)}=1+\frac{kU^2}{g}.$$
$$\times g(g-kW^2)$$
$$g^2=g^2-gkW^2+kU^2g-k^2U^2W^2.$$
$$+gkW^2-g^2+k^2U^2W^2$$
$$gkW^2+k^2U^2W^2=kU^2g$$
$$\div \text{BS by } gkW^2U^2$$
$$\frac{1}{U^2}+\frac{k}{g}=\frac{1}{W^2}.$$

As T (Terminal Velocity) $=\sqrt{\frac{g}{k}}$, $\frac{1}{T^2}=\frac{k}{g}$.

$$\therefore \frac{1}{U^2}+\frac{1}{T^2}=\frac{1}{W^2} \quad \text{QED.}$$

(b)(i) $z^5-1=0$

$$z=\text{cis}\frac{2\pi}{5},\ \text{cis}\frac{4\pi}{5},\ \text{cis}\frac{6\pi}{5},\ \text{cis}\frac{8\pi}{5},\ 1.$$

(ii) $\omega$ is $\text{cis}\frac{2\pi}{5}$ [root with smallest positive argument].

Other roots $=\omega^2\left(\text{cis}\frac{4\pi}{5}\right)$, $\omega^3\left(\text{cis}\frac{6\pi}{5}\right)$, $\omega^4\left(\text{cis}\frac{8\pi}{5}\right)$ & 1.

$\therefore$ By sum of roots of $z^5-1=0$, $1+\omega+\omega^2+\omega^3+\omega^4=0$

$$\omega+\omega^2+\omega^3+\omega^4=-1.$$

As $\frac{\omega^4}{\omega^5}=\frac{\omega^4}{1}=\frac{1}{\omega}$ & $\frac{\omega^3}{\omega^5}=\frac{\omega^3}{1}=\omega$, $\omega+\omega^4+\omega^2+\omega^3=-1.$

$$\underline{\left(\omega+\frac{1}{\omega}\right)+\left(\omega^2+\frac{1}{\omega^2}\right)=-1.}$$

Q.(16)(b)(iii) As $w + \frac{1}{w} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\left(\frac{-2\pi}{5}\right) + i\sin\left(\frac{-2\pi}{5}\right)$

$= \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$

$= 2\cos\frac{2\pi}{5}$ (as cos even, sin odd).

& similarly $w^2 + \frac{1}{w^2} = 2\cos\frac{4\pi}{5}$.

$2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = -1.$

Given that $\cos\frac{4\pi}{5} = 2\cos^2\left(\frac{2\pi}{5}\right) - 1$ [by $\cos 2\theta = 2\cos^2\theta - 1$],

$2\cos\frac{2\pi}{5} + 4\cos^2\frac{2\pi}{5} - 2 = -1.$

$4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} - 1 = 0.$

Hence $\cos\frac{2\pi}{5}$ is a root of $4x^2 + 2x - 1 = 0$

(iv) Solving $4x^2 + 2x - 1 = 0$

$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 4 \times -1}}{2 \times 4}$

$= \frac{-2 \pm \sqrt{20}}{8}$

$= \frac{-2 \pm 2\sqrt{5}}{8}$

$= \frac{-1 \pm \sqrt{5}}{4}.$

As $\frac{2\pi}{5}$ is in $Q_1$, $\cos\frac{2\pi}{5}$ is positive.

$\therefore \cos\frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4}.$

END OF SOLUTIONS!!!