James Ruse Agricultural High School

2022 **Year 12 Trial HSC Examination**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/ or calculations

Total marks: 100

Section I – 10 marks (pages 2–4)
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6–15)
- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1 Which expression is equal to $\int x^2 \sin x \, dx$?

A. $-x^2 \cos x - \int 2x \cos x \, dx$

B. $-2x \cos x + \int x^2 \cos x \, dx$

C. $-x^2 \cos x + \int 2x \cos x \, dx$

D. $-2x \cos x - \int x^2 \cos x \, dx$

2 Given that $(1+i)^n = ai$, where a is a non-zero real constant, then $(1+i)^{2n+2}$ simplifies to

A. a^4

B. $2a^2 i$

C. $1 + a^2 i$

D. $-2a^2 i$

3 A line in 3D space has equation given by $\underset{\sim}{r} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$, where λ is a real constant.

Which of the following statements is true?

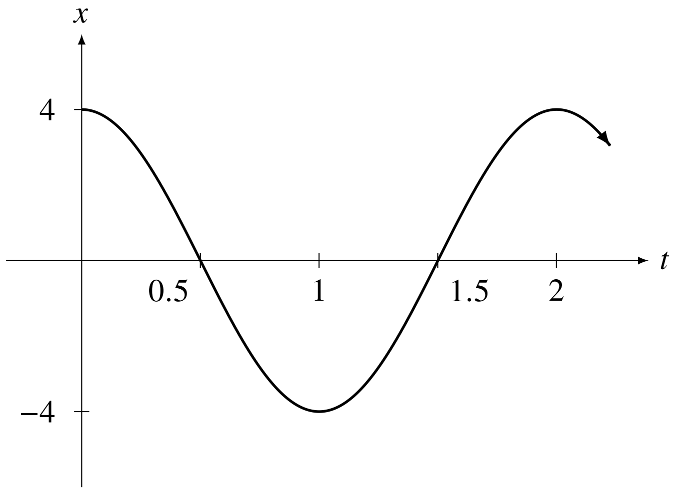
A. The line passes through the origin.

B. The point $(-2, 5, 1)$ lies on the line.

C. The point $(2, 2, 2)$ lies on the line.

D. The vector $4\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$ points in the direction of the line.

4 A particle moves in simple harmonic motion represented by the displacement-time graph below.



Which of the following represents the velocity of the particle as a function of time?

A. $v(t) = 4\cos \pi t$

B. $v(t) = \pi \cos \pi t$

C. $v(t) = -4\pi \sin \pi t$

D. $v(t) = -4 \sin \pi t$

5 Which of the following options is the contrapositive of the statement:

"You win the game if you know the rules but are not overconfident."

A. If you lose the game, then you don't know the rules or you are overconfident.

B. If you don't know the rules or are overconfident, you lose the game.

C. If you know the rules of are overconfident, then you win the game.

D. A necessary condition that you know the rules or you are not overconfident is that you win the game.

6 If $1 + ki$ a root of the quadratic $z^2 + kz + 5$, where k is a real number, what is the value of k?

A. $k = 2$ only

B. $k = -2$ only

C. $k = 2$ and $k = -2$

D. No real value of k exists.

7 Let $z = \sqrt{3} + i$. If $z^n + (\overline{z})^n$ is rational, which of the following is NOT a possible value of n?

A. 2

B. 3

C. 5

D. 6

8 Consider the statements below.

I. $f(x) \le g(x) \iff f'(x) \le g'(x)$

II. $f(x) \le g(x)$ for all $x \in [a, b] \iff \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$

Which of the following options are true?

A. I and II are both false.

B. I is false but II is true.

C. I is true but II is false.

D. I and II are both true.

9 A particle is moving along a straight line so that initially its displacement is $x = 1$, its velocity is $v = 2$, and its acceleration is $a = 4$. Which is a possible equation describing the motion of the particle?

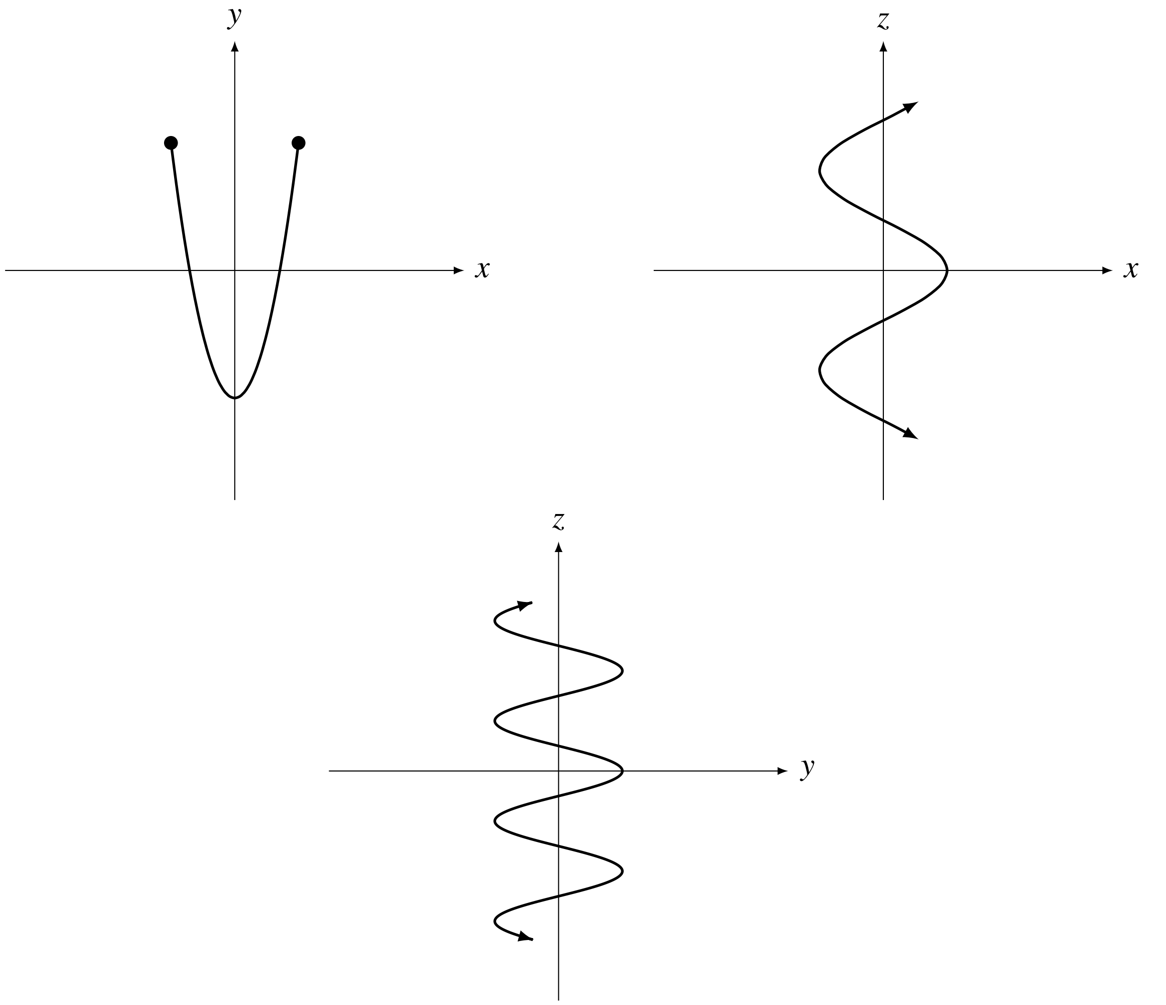
A. $v = 2\sin(x - 1) + 2$

B. $v = 2 + 4\ln x$

C. $v^2 = 4(x^2 - 2)$

D. $v = x^2 + 2x + 4$

10 Let $\underset{\sim}{r}(t)$ be a curve in 3D space. The following diagrams show projections of $\underset{\sim}{r}(t)$ onto the xy, xz and yz planes. The diagrams are NOT to scale.



Which of the following is the correct vector representation of $\underset{\sim}{r}(t)$?

A. $\underset{\sim}{r}(t) = (2\sin t)\underset{\sim}{i} + (4\cos 2t)\underset{\sim}{j} + (t)\underset{\sim}{k}$

B. $\underset{\sim}{r}(t) = (2\sin t)\underset{\sim}{i} + (4\sin 2t)\underset{\sim}{j} + (t)\underset{\sim}{k}$

C. $\underset{\sim}{r}(t) = (2\cos t)\underset{\sim}{i} + (4\cos 2t)\underset{\sim}{j} + (t)\underset{\sim}{k}$

D. $\underset{\sim}{r}(t) = (2\cos t)\underset{\sim}{i} + (4\sin 2t)\underset{\sim}{j} + (\sqrt{t})\underset{\sim}{k}$

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use the Question 11 Writing Booklet

(a) (i) Write the negation of the statement P below using logic symbols. **1**

$$P: \quad \forall x \in \mathbb{Z}^+, \exists y \in \mathbb{Z}^+, y = x - 1.$$

(ii) Prove that the original statement P is false by providing a counterexample. **1**

(b) (i) Prove that $x + y \geq 2\sqrt{xy}$ for positive real numbers x and y. **1**

(ii) Hence, or otherwise, find the minimum value of the function **2**

$$f(x) = \frac{9x^2 \sin^2 x + 4}{x \sin x}$$

in the domain $0 < x < \pi$.

Question 11 continues on page 7

Question 11 (continued)

(c) Compute the following integrals.

(i) $\displaystyle\int \frac{1}{\sqrt{3+2x-x^2}}\,dx$ **2**

(ii) $\displaystyle\int \frac{x+7}{1-x^2}\,dx$ **3**

(d) Determine whether the line through the points $(2, 0, 9)$ and $(-4, 1, 5)$ and the line given by $\underset{\sim}{r} = \begin{bmatrix} 5 \\ 2 \\ -8 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ -9 \\ -3 \end{bmatrix}$ are parallel, perpendicular or neither. You must show all your working. **2**

(e) Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be non-zero vectors and $\underset{\sim}{F}(t) = e^{2t}\underset{\sim}{a} + e^{-2t}\underset{\sim}{b}$. Prove that $\underset{\sim}{F}''(t)$ has the same direction as $\underset{\sim}{F}(t)$. **2**

(f) Show that a particle which moves according to the equation $v^2 = 36 - 6x - 2x^2$ is undergoing simple harmonic motion, where v is the velocity of the particle and x is the displacement of the particle. **2**

End of Question 11

Question 12 (14 marks) Use the Question 12 Writing Booklet

(a) Find the cartesian equation of the vector function $\underset{\sim}{r}(t) = \begin{bmatrix} \frac{1}{2}(\cos t - \sin t) \\ \cos^3 t - \sin^3 t \end{bmatrix}$. **4**

(b) A particle of unit mass is projected vertically upwards from the ground at a speed of V ms^{-1}. The particle is acted on by both gravity, and air resistance of magnitude $\dfrac{v^2}{40}$, where v is the velocity of the particle measured in ms^{-1}. After t seconds, the particle's height from the ground, is x metres.

(i) Draw a force diagram illustrating all the forces acting on the particle while the particle is moving upwards and derive the equation of motion $\ddot{x} = -\left(g + \dfrac{v^2}{40}\right)$. **2**

(ii) Show that the greatest height the particle reaches is $h = 20 \ln\left(\dfrac{40g + V^2}{40g}\right)$. **3**

(iii) Find the time taken to reach this greatest height. **3**

Having reached its maximum height, the particle falls back down towards its initial point of projection. Assume that only gravity and air resistance act on the particle.

(iv) Find the terminal velocity of the particle on its way down. **2**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Prove that for all complex numbers z, $e^z \neq 0$. **3**

(b) Use mathematical induction to prove that the number of diagonals of a convex polygon with n vertices is $\dfrac{1}{2}n(n-3)$ for $n \geq 4$. **3**

(c) A function $f(x)$ with domain $\mathcal{D}$ is called *injective* if

$$\forall x_1, x_2 \in \mathcal{D} \qquad x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

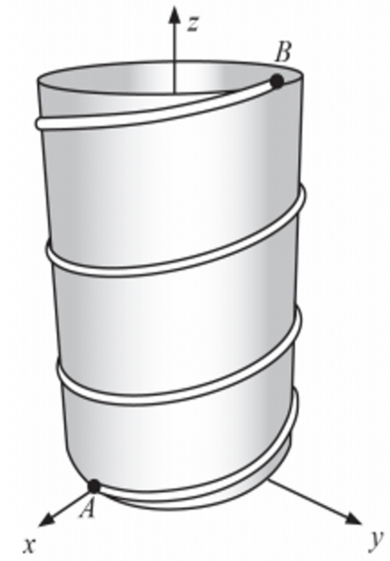
(i) Write down an equivalent statement for $f(x)$ to be *injective*. **1**

(ii) Hence, prove that $f(x) = 3x+4$ is an *injective* function across its natural domain. **2**

(d) Three points A, B and C have position vectors $-2\underset{\sim}{a} + 3\underset{\sim}{b} + 5\underset{\sim}{c}$, $\underset{\sim}{a} + 2\underset{\sim}{b} + 3\underset{\sim}{c}$ and $7\underset{\sim}{a} - \underset{\sim}{c}$ respectively. The point B lies between A and C. Show that A, B and C are collinear. **2**

Question 13 continues on page 10

(e) The stairs of a cylindrical shaped tower spiral upwards from the ground to an observation deck at point B, as shown below.



The path begins at point A on the ground along the x-axis and finishes at B. The time taken in seconds for Hannah to walk along the spiral path starting at A is presented by t. It takes Hannah 70π seconds to reach point B. Her position on this path can be represented by

$$\underset{\sim}{r}(t) = \begin{bmatrix} 15\cos(0.5t) \\ 15\sin(0.5t) \\ 0.3t \end{bmatrix} \text{ metres.}$$

(i) Determine the height of the observation deck above the ground correct to 2 decimal places. **1**

Hannah walks one loop around the tower and ends up directly above point A.

(ii) At what time and height does this occur? **1**

(iii) Find Hannah's speed at this point. **2**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) Let P be the point which represents the complex number $z = x + iy$ in the complex plane.

(i) Sketch the curve traced out by P in the complex plane if $|z-1-i| = \mathrm{Im}(z+1+i)$. **3**
You MUST draw the curve on the axes provided for you on the back of the Multiple Choice sheet.

(ii) Show that $-(\overline{iz}) = y + ix$. **1**

Suppose now that the point Q represents the complex number $-(\overline{iz})$.

(iii) Sketch the curve traced out by Q on the same diagram as part (i). Show all necessary features. **2**

(b) For $n = 0, 1, 2, \ldots$ define

$$I_n = \int_0^{\pi/4} \tan^n \theta \, d\theta.$$

(i) Show that $I_1 = \frac{1}{2}\ln 2$. **1**

(ii) Show that for $n \geq 2$, **3**

$$I_n + I_{n-2} = \frac{1}{n-1}.$$

(iii) For $n \geq 2$, explain why $I_n < I_{n-2}$, and deduce that **3**

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}.$$

(iv) By using the recurrence relation in part (ii), find I_5 and deduce that **2**

$$\frac{2}{3} < \ln 2 < \frac{3}{4}.$$

End of Question 14

Question 15 (14 marks) Use the Question 15 Writing Booklet

(a) Let $\mathcal{S}_1$ be a sphere with equation $\left| \underset{\sim}{r} - \begin{bmatrix} 0 \\ -1 \\ 6 \end{bmatrix} \right| = 15.$

(i) Show that the point $P(4.2, -1, 0.4)$ lies inside the sphere $\mathcal{S}_1$. **1**

(ii) Find the equation of the line ℓ passing through the centre of the sphere $\mathcal{S}_1$ and the point P. Express the direction vector of your line as a unit vector. **2**

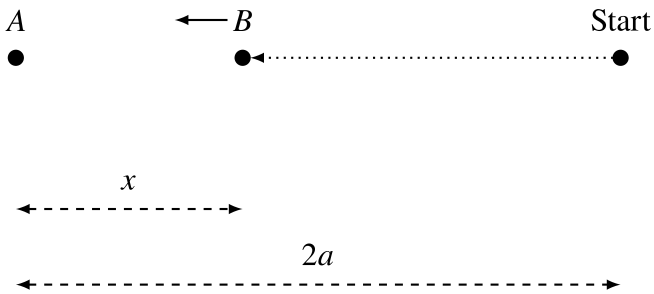
Another sphere $\mathcal{S}_2$ has radius 10 and centre at the point P. The intersection between $\mathcal{S}_1$ and $\mathcal{S}_2$ is a circle C. You are given that the line ℓ is perpendicular to the plane which the circle C lies in.

(iii) The radius of sphere $\mathcal{S}_2$ remains fixed at 10 but its centre is now free to move along the line ℓ. Find all possible vector equations of sphere $\mathcal{S}_2$ such that it is internally tangent to $\mathcal{S}_1$. **3**

Question 15 continues on page 13

(b) Two particles with opposite charges are attracted to each other with a force numerically equal to $\frac{k^2}{x^2}$, where x is their distance apart in metres.

Particle A is fixed and particle B, of mass m kg, is released at a distance $2a$ metres from A, as shown in the diagram below.



(i) Show that the speed v of particle B can be given by $v^2 = \frac{2k^2}{m}\left(\frac{1}{x} - \frac{1}{2a}\right)$. **3**

(ii) Find the time taken for particle B to reach halfway to A from the start. **5**

End of Question 15

Question 16 (16 marks) Use the Question 16 Writing Booklet

(a) Let w be the fifth root of unity with smallest positive argument.

(i) Show that $1 + w + w^2 + w^3 + w^4 = 0$. **1**

(ii) Hence, or otherwise, show that **2**

$$1 + 2w + 3w^2 + 4w^3 + 5w^4 = \frac{5}{w - 1}.$$

(iii) By expressing $z^5 - 1$ as a product of its factors, deduce that **2**

$$(1 - w)(1 - w^2)(1 - w^3)(1 - w^4) = 5.$$

(iv) If k is a positive integer, show that **3**

$$1 + w^k + w^{2k} + w^{3k} + w^{4k} = \begin{cases} 5, & \text{if } k \text{ is divisible by } 5 \\ 0, & \text{otherwise.} \end{cases}$$

(v) Let ℓ be the largest integer such that $5\ell \le n$. Use the binomial theorem to show that for $n \in \mathbb{Z}^+$ **3**

$$\frac{1}{5}\left[2^n + (1 + w)^n + (1 + w^2)^n + (1 + w^3)^n + (1 + w^4)^n\right]$$
$$= \binom{n}{0} + \binom{n}{5} + \binom{n}{10} + \cdots + \binom{n}{5\ell}.$$

Question 16 continues on page 15

(b) (i) Show that, for non-zero vectors **u** and **v**, and real scalars α and β, **1**

$$\mathrm{proj}_{\beta\mathbf{v}}\,(\alpha\mathbf{u}) = \alpha\ \mathrm{proj}_{\mathbf{v}}\,(\mathbf{u})\,.$$

(ii) Consider the sequences of vectors $\{\mathbf{a_n}\}$ and $\{\mathbf{b_n}\}$ defined by **4**

$$\mathbf{a_n} = \frac{1}{5^n}\begin{bmatrix}(-1)^{n+1}\times 2\\ 3\\ (-1)^n\end{bmatrix} \qquad \text{and} \qquad \mathbf{b_n} = \frac{1}{2^n}\begin{bmatrix}1\\ -8\\ 7\end{bmatrix}$$

for all integers $n \geq 0$.

A sequence of projection vectors $\{\mathbf{c_n}\}$ is defined by $\mathbf{c_n} = \mathrm{proj}_{\mathbf{a_n}}\,(\mathbf{b_n})$ for all integers $n \geq 0$.

Find $\displaystyle\sum_{n=0}^{\infty} \mathbf{c_n}$.

End of Examination.

correct solutions

Ext 2 M/C

Section I

10 Marks
Attempt Question 1 – 10.
Allow approximately 15 minutes for this section.

Use the multiple choice answer sheet below to record your answers to Question 1 – 10.

Select the alternative: A, B, C or D that best answers the question.

Colour in the response oval completely.

Sample:

$2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, draw a cross through the incorrect answer and colour in the new answer

ie A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word "*correct*" and draw an arrow as follows:

correct

A ● B ~~●~~ C ○ D ○

Trial HSC Examination 2022
Mathematics Extension 2

Multiple Choice Answer Sheet

Student ID number: | | | | |

Completely colour in the response oval representing the most correct answer.

1	A	○	B	○	C	●	D	○
2	A	○	B	○	C	○	D	●
3	A	○	B	○	C	●	D	○
4	A	○	B	○	C	●	D	○
5	A	●	B	○	C	○	D	○
6	A	○	B	●	C	○	D	○
7	A	○	B	○	C	●	D	○
8	A	●	B	○	C	○	D	○
9	A	●	B	○	C	○	D	○
10	A	○	B	○	C	●	D	○

Mark: /10

MC Brief Explanations:

(Q1) $\int x^2 \sin x \, dx$

$u = x^2 \qquad v' = \sin x$

$u' = 2x \qquad v = -\cos x$

$= -x^2 \cos x + \int 2x \cos x \, dx$

C

(Q2) $(1+i)^{2n+2} = (1+i)^2 \left[(1+i)^n\right]^2$

$= 2i \times (ai)^2$

$= -2a^2 i$

D

(Q3) $\lambda = 1 \rightarrow \underset{\sim}{x} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

C

(Q4) $x(t) = 4\cos(\pi t)$

$v(t) = -4\pi \sin(\pi t)$

C

(Q5) $(\underbrace{\text{Know Rules} \wedge \text{not overconfident}}_{A}) \underbrace{\Rightarrow}_{\Rightarrow} \underbrace{\text{win game}}_{B}$

$\underbrace{\text{Lose game}}_{\neg B} \underbrace{\Rightarrow}_{\Rightarrow} \underbrace{(\text{Don't know rules} \vee \text{overconfident})}_{\neg A}$

A

(Q6) $z^2 + hz + 5$ has roots $1 \pm ki$

$\therefore \quad \sum \alpha = \frac{-b}{a}$

$2 = -h$

B

(Q7) $z = \sqrt{3} + i$

$= 2e^{i\pi/6}$

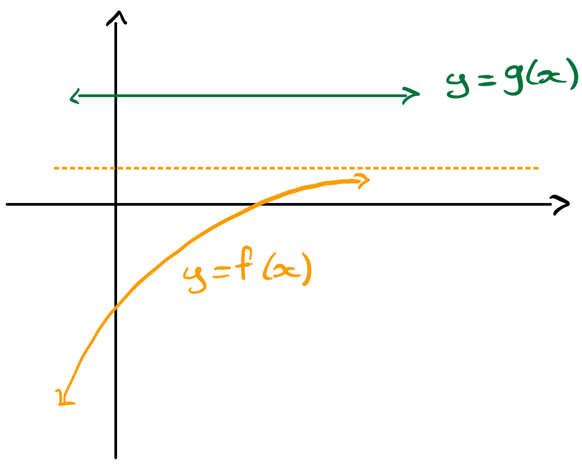
$z^n + \bar{z}^n = 2^n\left(e^{i\pi n/6} + e^{-i\pi n/6}\right)$

$= 2^n \times 2\cos\left(\frac{n\pi}{6}\right) \in \mathbb{Q} \iff \cos\left(\frac{n\pi}{6}\right) \in \mathbb{Q}$

$\therefore n = 5$

C

(Q8) Statement I:

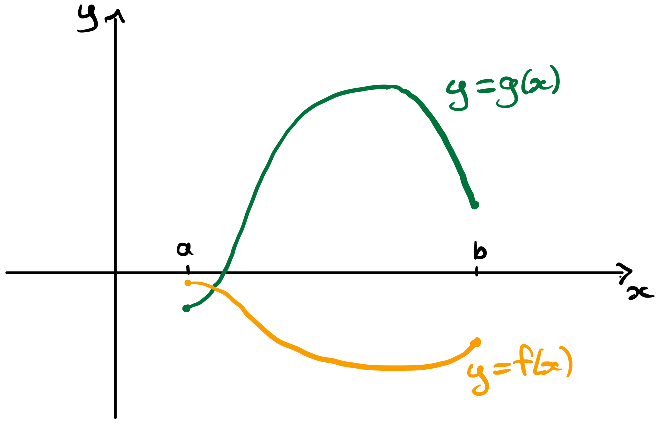


The diagram shows a clear counterexample, since

$f(x) \leq g(x)$ but

$f'(x) \geq 0 = g(x)$

Statement II :



The diagram shows a clear counter example since

$$\int_a^b f(x)\,dx \leq 0 \leq \int_a^b g(x)\,dx$$

but

$f(x) \nleq g(x)$ for all $x \in [a, b]$.

$\therefore$ Option A

(Q9) $t=0,\ x=1,\ v=2,\ a=4$

$x=1,\ v=2 \rightarrow$ eliminates C & D

$a = v\dfrac{dv}{dx}$

$= [2\sin(x-1)+2]\ 2\cos(x-1)$

$x=1 \rightarrow a = 2 \times 2$
$= 4$

A

$a = v\dfrac{dv}{dx}$

$= (2+4\ln x)\dfrac{4}{x}$

$x=1 \rightarrow a = 2 \times \dfrac{4}{1}$

$= 8$

(Q10) Clearly not D since $z = \sqrt{t} \geq 0$ but $z \in \mathbb{R}$.

Clearly not A or B since the x-z plane shows $2\cos(z) = x$, not $2\sin(z) = x$.

C

MATHEMATICS EXT2 : Question 11

Suggested Solutions	Marks	Marker's Comments
ai, $\exists x \in \mathbb{Z}^+,\ \forall y \in \mathbb{Z}^+,\ y \neq x-1$	1	
ii, $x = 1$ is a counterexample since		
$x - 1 = 1 - 1$		
$= 0 \notin \mathbb{Z}^+$	1	
bi, $(\sqrt{x} - \sqrt{y})^2 \geq 0$		
$x - 2\sqrt{x}\sqrt{y} + y \geq 0$		
$x + y \geq 2\sqrt{xy}$	1	
ii, $f(x) = \dfrac{9x^2\sin^2 x + 4}{x \sin x}$		
$= 9x\sin x + \dfrac{4}{x \sin x}$		
$\geq 2\sqrt{9\cancel{x\sin x} \times \dfrac{4}{\cancel{x \sin x}}}$	1	
(using (i) and noting both terms are positive)		
$= 12$		
$\therefore$ min value of $f(x)$ is 12.	1	
ci, $\displaystyle\int \frac{1}{\sqrt{3 + 2x - x^2}}\,dx$		
$= \displaystyle\int \frac{1}{\sqrt{4 - (x-1)^2}}\,dx$	1	

MATHEMATICS EXT2 : Question 11

Suggested Solutions	Marks	Marker's Comments
$= \sin^{-1}\left(\dfrac{x-1}{2}\right) + C$	1	
ii) $\dfrac{x+7}{1-x^2} = \dfrac{A}{1-x} + \dfrac{B}{1+x}$		
$x+7 = A(1+x) + B(1-x)$		
$x = 1 \rightarrow 8 = A(2)$		
$A = 4$	1	
$x = -1 \rightarrow 6 = 2B$		
$B = 3$	1	
$\therefore \int \dfrac{x+7}{1-x^2}\,dx = \int\left(\dfrac{4}{1-x} + \dfrac{3}{1+x}\right)dx$		
$= 3\ln\lvert 1+x\rvert - 4\ln\lvert 1-x\rvert + C$	1	
d) let $A = (2, 0, 9)$ $B = (-4, 1, 5)$		
$\overrightarrow{AB} = \begin{pmatrix} -4-2 \\ 1-0 \\ 5-9 \end{pmatrix}$		
$= \begin{pmatrix} -6 \\ 1 \\ -4 \end{pmatrix}$	1	
let $\underset{\sim}{v} = \begin{pmatrix} 0 \\ -9 \\ -3 \end{pmatrix}$		

MATHEMATICS EXT2 : Question 11

Suggested Solutions	Marks	Marker's Comments
Clearly $\underset{\sim}{v} \neq \lambda \overrightarrow{AB}$ for $\lambda \in \mathbb{R}$, so they are not parallel.		
$\underset{\sim}{v} \cdot \overrightarrow{AB} = -6 \times 0 + 1 \times -9 + -4 \times -3$		
$= -9 + 12$		
$= 3$		
$\neq 0$		
$\therefore$ not perpendicular		
$\therefore$ lines are neither parallel nor perpendicular	1	
e, $\underset{\sim}{F}(t) = e^{2t}\underset{\sim}{a} + e^{-2t}\underset{\sim}{b}$		
$\underset{\sim}{F}'(t) = 2e^{2t}\underset{\sim}{a} - 2e^{-2t}\underset{\sim}{b}$	1	
$\underset{\sim}{F}''(t) = 4e^{2t}\underset{\sim}{a} + 4e^{-2t}\underset{\sim}{b}$		
$= 4\left(e^{2t}\underset{\sim}{a} + e^{-2t}\underset{\sim}{b}\right)$		
$= 4\underset{\sim}{F}(t)$		
$\therefore \underset{\sim}{F}(t)$ has the same direction as $\underset{\sim}{F}''(t)$.	1	

MATHEMATICS EXT2 : Question 11

Suggested Solutions	Marks	Marker's Comments
f) $v^2 = 36 - 6x - 2x^2$		
$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$		
$= \frac{d}{dx}\left(18 - 3x - x^2\right)$		
$= -3 - 2x$	1	
$a = -2\left(x - \frac{3}{2}\right)$		
which is of the form $a = -n^2(x - x_0)$		
Hence, particle moves in SHM.	1	

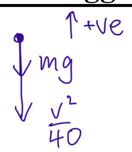
MATHEMATICS: Question 12
Extension 2

Suggested Solutions	Marks	Marker's Comments
a) $x = \frac{1}{2}(\cos t - \sin t)$, $2x = \cos t - \sin t$ $y = \cos^3 t - \sin^3 t$ $= (\cos t - \sin t)(\cos^2 t + \cos t \sin t + \sin^2 t)$ $= 2x(1 + \cos t \sin t)$	1	Simplifying to obtain an expression in $\cos t \sin t$.
OR Now, $4x^2 = \cos^2 t - 2\cos t \sin t + \sin^2 t$ $= 1 - 2\cos t \sin t$ $\therefore \cos t \sin t = \frac{1}{2}(1 - 4x^2)$		OR expressing x^3 as $x^3 = \frac{1}{8}(y - 3(\cos^3 t - \sin^3 t - (\cos t - \sin t))$
$y = 2x(1 + \frac{1}{2}(1 - 4x^2))$ $= 2x + x(1 - 4x^2)$ $= 3x - 4x^2$	1	Equation in x and y only
Using auxilliary angle: $x = \frac{\sqrt{2}}{2}\cos(t - \alpha)$ No need to evaluate α to obtain restriction $\therefore -\frac{\sqrt{2}}{2} \leq x \leq \frac{\sqrt{2}}{2}$	1	Considering a restriction
$\therefore y = 3x - 4x^2$, $-\frac{\sqrt{2}}{2} \leq x \leq \frac{\sqrt{2}}{2}$	1	Correct equation with correct restriction
Alternate method $y = 2x(1 + \frac{1}{2}\sin 2t)$ Using auxilliary angle: Let $R\sin(t - \alpha) = \cos t - \sin t$ $R\sin t\cos\alpha - R\cos t\sin\alpha = -\sin t + \cos t$ $R\cos\alpha = -1$ $-R\sin\alpha = 1$ $\frac{\sin\alpha}{\cos\alpha} = \tan\alpha = 1$ $R = \sqrt{2}$ $\alpha = \frac{\pi}{4}$ $\therefore x = \frac{\sqrt{2}}{2}\sin(t - \frac{\pi}{4})$ $\sqrt{2}x = \sin(t - \frac{\pi}{4})$ $t - \frac{\pi}{4} = \sin^{-1}(\sqrt{2}x)$ $2t = 2\sin^{-1}(\sqrt{2}x) + \frac{\pi}{2}$ $\sin 2t = \sin(2\sin^{-1}(\sqrt{2}x) + \frac{\pi}{2})$ $= \sin(2\sin^{-1}(\sqrt{2}x))\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\cos(2\sin^{-1}\sqrt{2}x)$ $= 0 + \cos(2\sin^{-1}\sqrt{2}x)$ $\therefore y = 2x(1 + \frac{1}{2}\cos(2\sin^{-1}\sqrt{2}x))$ OR $y = 2x(1 + \frac{1}{2}\sin(2\cos^{-1}\sqrt{2}x - \pi/2))$	4	Restriction is embedded in $\sin^{-1}\sqrt{2}x$.

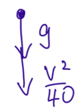
MATHEMATICS Extension ~~1~~2: Question 12

Suggested Solutions	Marks	Marker's Comments

b) (i)



OR "since $m=1$"



1 — Poorly done.

Maximum 1 mark for:

- resistance as $\frac{mv^2}{40}$
- force due to gravity as g without stating "$m=1$".

$m\ddot{x} = -mg - \frac{v^2}{40}$ by Newton's 2nd law

since $m=1$

$\ddot{x} = -g - \frac{v^2}{40}$

$= -\left(g + \frac{v^2}{40}\right)$

OR $\ddot{x} = -g - \frac{v^2}{40m}$

$m=1$

$\therefore \ddot{x} = -g - \frac{v^2}{40}$

$= -\left(g + \frac{v^2}{40}\right)$

1

(ii) Greatest height reached when $\dot{x} = 0$

Method 1:

$v\frac{dv}{dx} = -\left(g + \frac{v^2}{40}\right)$

$v\frac{dv}{dx} = -\left(\frac{40g + v^2}{40}\right)$

$\int_V^0 \frac{40v}{40g+v^2}\,dv = -\int_0^h dx$ — **1** — integral with correct bounds

$20\left[\ln|40g+v^2|\right]_V^0 = \left[-x\right]_0^h$ — **1** — correct integrals

$20\left(\ln(40g) - \ln(40g+V^2)\right) = -h + 0$, since $40g+v^2>0$

$h = 20\left(\ln(40g+V^2) - \ln(40g)\right)$

$= 20\ln\left(\frac{40g+V^2}{40g}\right)$ — **1** — result

- Students must show what the question is asking
  - h not H
  - $\left(\frac{40g+V^2}{40g}\right)$ not $\left|\frac{40g+V^2}{40g}\right|$
- Bounds cannot be the variable with respect to which you are integrating.

Metho .

$v\frac{dv}{dx} = -\left(g + \frac{v^2}{40}\right)$

$= -\left(\frac{40g + v^2}{40}\right)$

$\frac{dv}{dx} = -\left(\frac{40g+v^2}{40v}\right)$

$\frac{dx}{dv} = \frac{-40v}{40g+v^2}$

$x = -20\ln|40g+v^2| + C$ — **1** — integrating

when $x=0$, $v=V$

$x=h$, $v=0$

$0 = -20\ln(40g+V^2) + C$ since $40g+v^2>0$

$\therefore C = 20\ln(40g+V^2)$ — **1** — C

$\therefore x = -20\ln(40g+v^2) + 20\ln(40g+V^2)$

$h = -20\ln(40g+0) + 20\ln(40g+V^2)$

$= 20\ln\left(\frac{40g+V^2}{40g}\right)$ — **1** — h

MATHEMATICS Extension 2: Question 12

Suggested Solutions	Marks	Marker's Comments
(iii) $\frac{dv}{dt} = -\left(\frac{40g+v^2}{40}\right)$		
$\int_V^0 \frac{40}{40g+v^2}\,dv = -\int_0^T dt$ where T is time to reach greatest height	1	integral with correct bounds
$\left[\frac{40}{\sqrt{40g}}\tan^{-1}\frac{v}{\sqrt{40g}}\right]_V^0 = \left[-t\right]_0^T$	1	integrating
$\sqrt{\frac{40}{g}}\tan^{-1}\frac{0}{\sqrt{40g}} - \sqrt{\frac{40}{g}}\tan^{-1}\frac{V}{\sqrt{40g}} = -T+0$		
$T = \sqrt{\frac{40}{g}}\tan^{-1}\frac{V}{\sqrt{40g}}$	1	T
Alternative method:		
$\frac{dt}{dv} = \frac{-40}{40g+v^2}$		
$t = \frac{-40}{\sqrt{40g}}\tan^{-1}\frac{v}{\sqrt{40g}} + C$	1	integrating
when $t=0$, $v=V$		
$0 = -\sqrt{\frac{40}{g}}\tan^{-1}\frac{V}{\sqrt{40g}} + C$		
$\therefore t = -\sqrt{\frac{40}{g}}\left(\tan^{-1}\frac{v}{\sqrt{40g}} - \tan^{-1}\frac{V}{\sqrt{40g}}\right)$	1	finding C
max height when $v=0$		
$\therefore t = -\sqrt{\frac{40}{g}}\left(\tan^{-1}0 - \tan^{-1}\frac{V}{\sqrt{40g}}\right)$		
$= \sqrt{\frac{40}{g}}\tan^{-1}\frac{V}{\sqrt{40g}}$	1	t
(iv) $\uparrow \frac{v^2}{40}$, $\downarrow mg$; $\downarrow$ +ve; $m\ddot{x} = mg - \frac{v^2}{40}$		
$\ddot{x} = g - \frac{v^2}{40m}$, $m=1$		
$\therefore \ddot{x} = g - \frac{v^2}{40}$	1	
Terminal velocity when $\ddot{x}=0$		
$\therefore \frac{v^2}{40} = g$		
$v^2 = 40g$		
$v = \sqrt{40g}$ since downwards motion positive	1	
If motion defined as upwards positive then $v = -\sqrt{40g}$.		

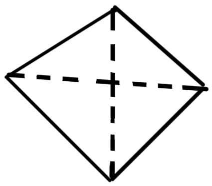
MATHEMATICS Extension 2: Question 13

Suggested Solutions	Marks	Marker's Comments
a) let $z = x + iy$, $x, y \in \mathbb{R}$ and suppose $e^z = 0$		
$\therefore e^{x+iy} = 0$		
$e^x e^{iy} = 0$		
$\therefore e^x = 0$ or $e^{iy} = 0$	1	
$e^x = 0$ as $x \in \mathbb{R}$, $e^x \neq 0$	1	
$e^{iy} = 0$		
$\lvert e^{iy} \rvert = 1$		
$\lvert 0 \rvert = 0$		
as $\lvert e^{iy} \rvert \neq \lvert 0 \rvert$		
$\therefore e^{iy} \neq 0$	1	
$\therefore$ neither e^x or e^{iy} can be 0		
$\therefore$ by contradiction, $e^z \neq 0$ for all complex numbers z.		
Incorrect methods:		
$e^z = 0$		
$\ln e^z = \ln 0$		max. 1 mark
$z = \ln 0$		
$\therefore$ no complex number z.		
NOTE: $\ln 0$ is only undefined for real numbers.		
$\log(z) = \ln\lvert z\rvert + i\arg(z)$ this is not part of our course		

MATHEMATICS Extension 2: Question 13

Suggested Solutions	Marks	Marker's Comments

b) base case: $n=4$



Any convex quadrilateral will have 2 diagonals.

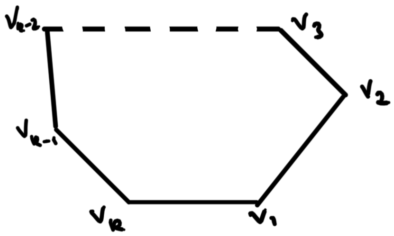
when $n=4$,

$$\tfrac{1}{2}(4)(4-3) = 2(1)$$
$$= 2$$

$\therefore$ true for $n=4$

Marks: 1 — base case

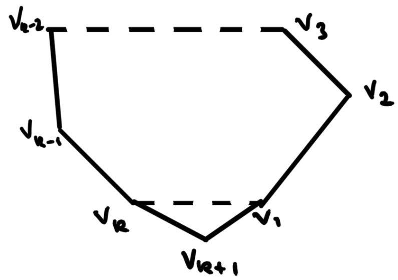
Assume true for $n=k$,



ie. any convex k-sided polygon will have $\tfrac{1}{2}k(k-3)$ diagonals.

Prove true for $n=k+1$,

ie. any convex $(k+1)$-sided polygon will have $\tfrac{1}{2}(k+1)(k-2)$ diagonals.



When one more side is added, a new vertex is formed. The $(k+1)$th vertex can form a line with the k other vertices except for V_k and V_1 as this becomes the sides.

ie. $k-2$ new diagonals.

The side V_k to V_1 will now become a diagonal as well.

$\therefore$ total new diagonals $= k-2+1$
$= k-1$

$\therefore$ total diagonals = diagonal in k-sided polygon $+ \; k-1$

Marks: 1 — correct explanation

most students said that the $(k+1)$th vertex makes diagonals will all other vertices except V_k and V_1
$\therefore k+1-2$
$= k-1$
this is incorrect

MATHEMATICS Extension 2: Question 13

Suggested Solutions	Marks	Marker's Comments
$= \frac{1}{2}k(k-3) + k-1$ (by assumption)		
$= \frac{1}{2}(k^2-3k+2k-2)$		
$= \frac{1}{2}(k^2-k-2)$		
$= \frac{1}{2}(k+1)(k-2)$	1	remaining working out and conclusion
$\therefore$ true for $n=k+1$		
$\therefore$ the statement is true by the principle of mathematical induction		
c) i) $\forall x_1, x_2 \in D$		
$f(x_1)=f(x_2) \Rightarrow x_1 = x_2$	1	one-to-one function was also accepted
ii) Suppose $f(x_1)=f(x_2)$ for all $x_1, x_2 \in \mathbb{R}$		
$3x_1+4 = 3x_2+4$	1	
$3x_1 = 3x_2$		
$x_1 = x_2$		
$\therefore f(x_1)=f(x_2) \Rightarrow x_1 = x_2$		
$\therefore f(x)=3x+4$ is injective across $\mathbb{R}$	1	need to show all working and have conclusion
If proving monotonically increasing, need to state $f'(x)>0$.		
If drawing a graph, need to show that it is a one-to-one function as it satisfies the horizontal and vertical line test.		

MATHEMATICS Extension 2: Question 13		
Suggested Solutions	**Marks**	**Marker's Comments**
d) $\vec{AB} = 3\underset{\sim}{a} - \underset{\sim}{b} - 2\underset{\sim}{c}$ $\vec{BC} = 6\underset{\sim}{a} - 2\underset{\sim}{b} - 4\underset{\sim}{c}$ $\vec{AC} = 9\underset{\sim}{a} - 3\underset{\sim}{b} - 6\underset{\sim}{c}$	1	having any two correct vectors
$\vec{BC} = 2\vec{AB}$ or $\vec{AC} = 3\vec{AB}$ $\therefore$ the points are collinear as $\vec{BC} \parallel \vec{AB}$ and B is a common point	1	showing one is a scalar multiple of the other and having the correct conclusion - parallel - common point.
e) i) height $= 0.3 \times 70\pi$ $= 21\pi$ $= 65.97344\ldots$ $= 65.97$m (2dp)	1	
ii) when $t=0$, $A = \begin{pmatrix} 15 \\ 0 \\ 0 \end{pmatrix}$ $\therefore$ for point above A: $15\cos(0.5t) = 15$ $\cos(0.5t) = 1$ $0.5t = 0, 2\pi, 4\pi, \ldots$ $\therefore t = 0, 4\pi, 8\pi, \ldots$ $\therefore t = 4\pi$ when $t = 4\pi$, height $= 0.3 \times 4\pi$ $= 1.2\pi$ $= 3.7699\ldots$ $= 3.77$m (2dp)	1	for time and height

MATHEMATICS Extension 2: Question 13		
Suggested Solutions	**Marks**	**Marker's Comments**
iii) $\underset{\sim}{r}'(t) = \begin{bmatrix} -7.5\sin(0.5t) \\ 7.5\cos(0.5t) \\ 0.3 \end{bmatrix}$	1	
at $t = 4\pi$		
Speed $= \sqrt{(-7.5\sin(0.5\times 4\pi))^2 + (7.5\cos(0.5\times 4\pi))^2 + 0.3^2}$		
$= 7.505997\ldots$		
$= 7.51$ m/s (2dp)	1	

MATHEMATICS Extension 2 : Question 1

Suggested Solutions	Marks Awarded	Marker's Comments
a i) $\lvert z-1-i\rvert = \text{Im}(z+1+i)$		well done
$\lvert(x-1)+(y-1)i\rvert = y+1$	1	some students failed to square the $(y+1)$, were awarded marks if they graphed their answer correctly.
$(x-1)^2+(y-1)^2=(y+1)^2$		
$(x-1)^2+y^2-2y+1=y^2+2y+1$		
$\underline{y=\frac{1}{4}(x-1)^2}$	1	
(ii) $-(\overline{iz}) = -(\overline{i(x+iy)})$		most students gained this mark
$= -(\overline{ix-y})$		
$= -(-ix-y)$		
$= y+ix$	1	
(iii) curves are inverse relations. Reflection of each other in line $y=x$	2	1 mark if just graphed $y=x$. 2nd mark for correct curve. • Some students misinterpreted question and only drew $y=x$.

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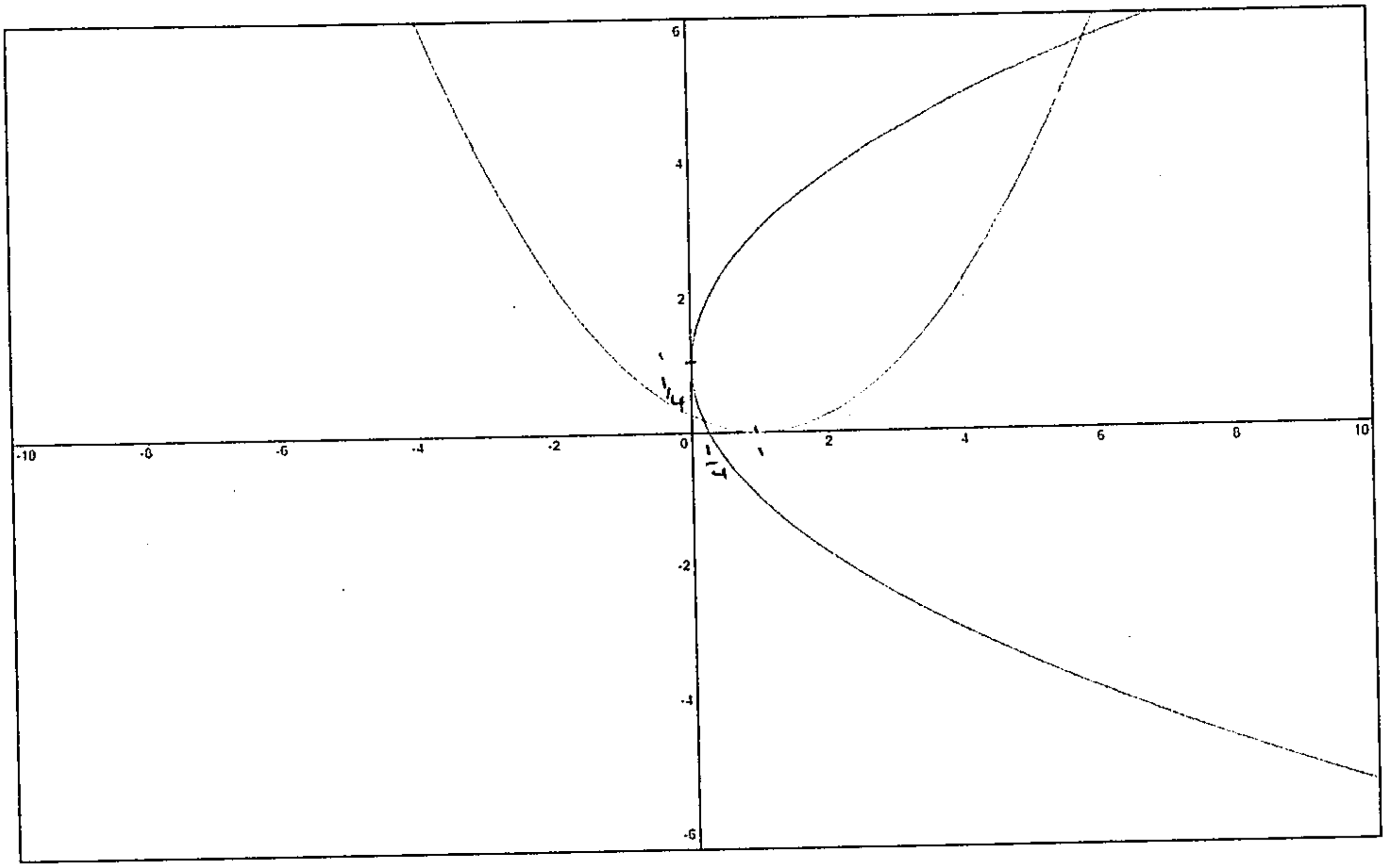
(2)



$y=\frac{1}{4}(x-1)^2$

$x=\frac{1}{4}(y-1)^2$

part i) 1 mark needed to have correct points and indicate intercepts on x and y-axis OR have graph in correct position.

- the use of different scales on each axis distorted some graphs
- students need to think about the scale they are using.

(iii) 2 marks for correct graph

- If part i) was incorrect if there graph was a reflection gained two marks.

MATHEMATICS Extension 2: Question 14

Suggested Solutions	Marks Awarded	Marker's Comments
ii) $I_1 = \int_0^{\pi/4} \tan x \, dx$ $= \Big[-\ln\lvert\cos x\rvert\Big]_0^{\pi/4}$ $= \ln\lvert 1\rvert - \ln\left\lvert\frac{1}{\sqrt{2}}\right\rvert$ $= \ln\lvert\sqrt{2}\rvert$ $= \underline{\frac{1}{2}\ln 2}$	1	• Some students said $\cos(0) = 0$ • Make sure last line is what you have been asked to prove Well done on the whole.
(ii) Either $I_n + I_{n-2}$ $= \int_0^{\pi/4} \tan^n x \, dx + \int_0^{\pi/4} \tan^{n-2} x \, dx$ $= \int_0^{\pi/4} \tan^{n-2} x(\tan^2 x + 1) \, dx$	1	1st mark was for some manipulation 2nd mark was for integral 3rd mark for answer
$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x \, dx$ $= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4}$ $= \frac{1}{n-1}$	1	• Students didn't recognise this and used integration by parts which was time consuming.

MATHEMATICS: Question 14

Suggested Solutions	Marks Awarded	Marker's Comments
OR $\int_0^{\pi/4} \tan^n x\,dx = \int_0^{\pi/4} \tan^{n-2} x(\tan^2 x)\,dx$	1	
$= \int_0^{\pi/4} \tan^{n-2} x(\sec^2 x - 1)\,dx$		
$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x\,dx - \int_0^{\pi/4} \tan^{n-2} x\,dx$	1	
$I_n = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$		
$I_n + I_{n-2} = \frac{1}{n-1}$	1	
(iii) Since $x \in [0, \frac{\pi}{4}]$ then $\tan x \in [0, 1]$ $\therefore \int_0^{\frac{\pi}{4}} \tan^n x\,dx < \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx$ Since if a fraction raised to a power is smaller than fraction raised to a smalle power $\left(\frac{1}{2}\right)^3 < \left(\frac{1}{2}\right)$	1 for explanation	• students failed to include the 0 for $\tan x$ • graph of $y = \tan^n x$ and $\tan^{n-2} x$ meet at 1 which was ignored by some students. • Students failed to give an explanation

MATHEMATICS Extension 1 : Question 14

Suggested Solutions	Marks Awarded	Marker's Comments
(iii) $I_n < I_{n-2}$		• some students didn't know where to start need to consider previous part.
$2I_n < I_n + I_{n-2}$		
$I_n + I_{n-2} = \frac{1}{n-1}$ part ii)		
$I_n < \frac{1}{2(n-1)}$	1	
$I_{n+2} < I_n$ (map n to n+2)		
$I_{n+2} + I_n < 2I_n$		
$\frac{1}{n+2-1} < 2I_n$		• Last statement should be what you are trying to prove
$\therefore \frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$	1	
Overall students need to improve setting out. Squashing working at bottom of page which won't be scanned.		

Year 12 **Task** Trials **MATHEMATICS** Ext2 **Question** 15

Suggested Solutions	Marks	Marker's Comments

15. a) i Distance from P to centre of S_1 is given by:

$$d = \sqrt{(4.2-0)^2 + (-1+1)^2 + (0.4-6)^2}$$

$$= \sqrt{(4.2)^2 + (-5.6)^2}$$

$$= \sqrt{49}$$

$$= 7$$

$$< 15 \qquad (1)$$

$\therefore$ P is closer to the centre than the surface of S_1

$\therefore$ P lies inside the sphere.

ii let $\underset{\sim}{v}$ be the direction vector of ℓ

$$\therefore \underset{\sim}{v} = \begin{pmatrix} 4.2 \\ -1 \\ 0.4 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 4.2 \\ 0 \\ -5.6 \end{pmatrix} \qquad (1)$$

$$\hat{\underset{\sim}{v}} = \begin{pmatrix} 4.2 \\ 0 \\ -5.6 \end{pmatrix} \times \frac{1}{7}$$

$$= \begin{pmatrix} 0.6 \\ 0 \\ -0.8 \end{pmatrix} \quad \underline{\underline{\text{or}}} \quad \begin{pmatrix} -0.6 \\ 0 \\ 0.8 \end{pmatrix}$$

$$\therefore \ell : \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 0.6 \\ 0 \\ -0.8 \end{pmatrix} \qquad (1)$$

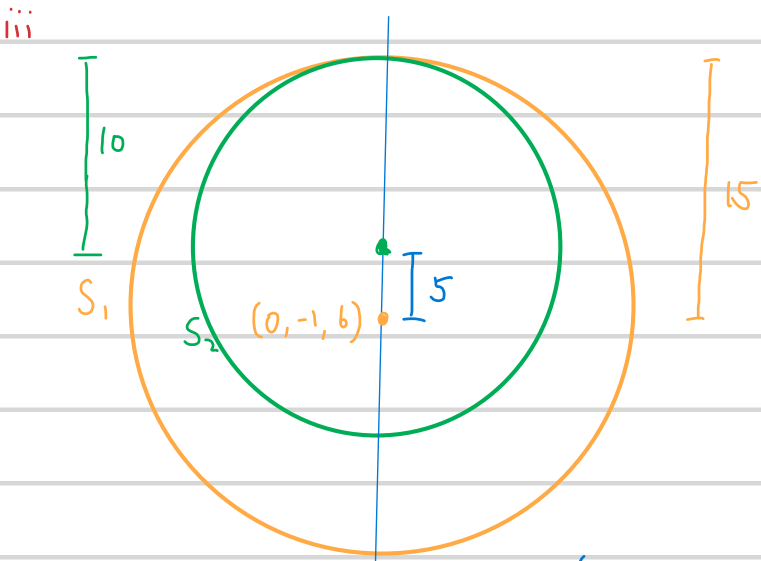
$\underline{\underline{\text{or}}}$ $\begin{pmatrix} -4.2 \\ -1 \\ 0.4 \end{pmatrix} + \lambda \begin{pmatrix} -0.6 \\ 0 \\ 0.8 \end{pmatrix}$

$\underline{\underline{\text{or}}}$ $\begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -0.6 \\ 0 \\ 0.8 \end{pmatrix}$ $\underline{\underline{\text{or}}}$ $\begin{pmatrix} 4.2 \\ -1 \\ 0.4 \end{pmatrix} + \lambda \begin{pmatrix} 0.6 \\ 0 \\ -0.8 \end{pmatrix}$

Year 12 **Task** Trials **MATHEMATICS** Ext2 **Question** 15

Suggested Solutions	Marks	Marker's Comments

iii



$$l: \underset{\sim}{r} = \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 0.6 \\ 0 \\ -0.8 \end{pmatrix}$$

For the spheres to be internally tangential, their centres must be 5 units apart. Let the centre of S_2 be (x, y, z)

$\therefore$ new centre will have positional vector

$$\begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} \pm 5 \begin{pmatrix} 0.6 \\ 0 \\ -0.8 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \text{ or } \begin{pmatrix} -3 \\ -1 \\ 10 \end{pmatrix}$$

$\therefore$ possible equations of S_2 are:

$$\left| \underset{\sim}{r} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \right| = 10 \quad \text{and} \quad \left| \underset{\sim}{r} - \begin{pmatrix} -3 \\ -1 \\ 10 \end{pmatrix} \right| = 10$$

1 Solving for λ correctly (λ takes different values depending on your answer in ii)

1 Equation of 1st sphere

1 Equation of 2nd sphere

Year 12 Task Trial MATHEMATICS Ext2 Question 15		
Suggested Solutions	Marks	Marker's Comments

b) i $m\ddot{x} = \dfrac{-k^2}{x^2}$ (Newton's 2nd law)

$\ddot{x} = \dfrac{-k^2}{mx^2}$ ①

$\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = \dfrac{-k^2}{mx^2}$

$\dfrac{1}{2}v^2 = \dfrac{-k^2}{m}\displaystyle\int \dfrac{1}{x^2}\,dx$

$v^2 = \dfrac{-2k^2}{m}\displaystyle\int \dfrac{1}{x^2}\,dx$

$= \dfrac{2k^2}{mx} + C$ ①

when $x = 2a$, $V = 0$

$\therefore 0 = \dfrac{2k^2}{m(2a)} + C$

$C = \dfrac{-k^2}{ma}$

$\therefore v^2 = \dfrac{2k^2}{mx} - \dfrac{k^2}{ma}$

$= \dfrac{2k^2}{m}\left(\dfrac{1}{x} - \dfrac{1}{2a}\right)$ ①

Year 12 Task Trials MATHEMATICS Ext 2 Question 15

Suggested Solutions	Marks	Marker's Comments

b) Alternate method, using the definite integral method

i $ma = \frac{-k^2}{x^2}$ (Newton's 2nd law of motion)

$a = \frac{-k^2}{mx^2}$

$v\frac{dv}{dx} = \frac{-k^2}{mx^2}$

$v\,dv = \frac{-k^2}{mx^2}dx$

$\int_0^v v\,dv = \int_{2a}^x \frac{-k^2}{mx^2}dx$

$\left[\frac{v^2}{2}\right]_0^v = \left[\frac{k^2}{mx}\right]_{2a}^x$

$\frac{v^2}{2} = \frac{k^2}{mx} - \frac{k^2}{2a}$

$= \frac{2k^2}{m}\left(\frac{1}{x} - \frac{1}{2a}\right)$

lines such as the following are strongly discouraged

$\int d\left(\frac{1}{2}v^2\right) = \int_{2a}^x \frac{k^2}{mx^2}dx$

- LHS is indefinite and RHS is definite

- There is no way to know whether the limits were fudged as we can't see how the two sides correlate

- Also in a show question, the limits are CRITICAL for the marker to see your logic in defining which one should be upper and which one is lower

Note: Students who missed the negative signs at the beginning of the question could only receive 2 out of 3 as a maximum if they were able to get to the following line without fudging.

$$v^2 = \frac{2k^2}{m}\left(\frac{1}{2a} - \frac{1}{x}\right)$$

Students who realised that something was wrong half way through the question and decided to "fudge" their way to the answer received only 1/3 as it is considered as 2 errors made.

There were a couple of exceptions where a CONVINCING argument was made involving redefinition of start and end points etc. These were rare.

Year 12 Task Trials MATHEMATICS Ext2 Question 15		
Suggested Solutions	Marks	Marker's Comments

ii $v^2 = \frac{2k^2}{m}\left(\frac{1}{x} - \frac{1}{2a}\right)$

$= \frac{2k^2}{m}\left(\frac{2a-x}{2ax}\right)$

$v = -\sqrt{\frac{2k^2}{m}} \times \sqrt{\frac{2a-x}{2ax}}$ (v is opposite to x) — 1

$\frac{dx}{dt} = -\sqrt{\frac{2k^2}{m}} \times \sqrt{\frac{2a-x}{2ax}}$

$= -\sqrt{\frac{k^2}{ma}} \times \sqrt{\frac{2a-x}{x}}$

$\int \frac{\sqrt{x}}{\sqrt{2a-x}}\,dx = -\sqrt{\frac{k^2}{ma}}\int dt$

$\int \frac{x}{\sqrt{2ax-x^2}}\,dx = \frac{-kt}{\sqrt{ma}} + C_1$ $(k>0)$ — 1

Consider LHS:

$\int \frac{x}{\sqrt{2ax-x^2}}\,dx = \int \frac{-2x+2a}{\sqrt{2ax-x^2}} \times \left(\frac{-1}{2}\right)dx + \int \frac{a}{\sqrt{2ax-x^2}}\,dx$

$= -\frac{1}{2}\int \frac{2a-2x}{\sqrt{2ax-x^2}}\,dx + \int \frac{a}{\sqrt{a^2-(x^2-2ax+a^2)}}\,dx$

$= -\frac{1}{2}\int \frac{2a-2x}{\sqrt{2ax-x^2}}\,dx + \int \frac{a}{\sqrt{a^2-(x-a)^2}}\,dx$

$= -\frac{1}{2} \times (2ax-x^2)^{\frac{1}{2}} \times 2 + a\sin^{-1}\left(\frac{x-a}{a}\right) + C_2$

$= a\sin^{-1}\left(\frac{x-a}{a}\right) - \sqrt{2ax-x^2} + C_2$

1 1

Year 12 Task Trials MATHEMATICS Ext 2 Question 15

Suggested Solutions	Marks	Marker's Comments
$\therefore a\sin^{-1}\left(\frac{x-a}{a}\right) - \sqrt{2ax - x^2} = \frac{-kt}{\sqrt{ma}} + C_3$		
when $t=0$, $x=2a$		
$\therefore \frac{a\pi}{2} = C_3$		
$\therefore a\sin^{-1}\left(\frac{x-a}{a}\right) - \sqrt{2ax-x^2} = \frac{-kt}{\sqrt{ma}} + \frac{a\pi}{2}$		
$\therefore$ when $x=a$,		
$-\sqrt{a^2} = \frac{-kt}{\sqrt{ma}} + \frac{a\pi}{2}$		
$\therefore \frac{kt}{\sqrt{ma}} = \frac{a\pi}{2} + a$ (since $a>0$)		
$t = \frac{\sqrt{ma}\,a}{k}\left(\frac{\pi}{2}+1\right)$	1	
or		

Year 12 **Task** Trials **MATHEMATICS** Ext 2 **Question** 15

Suggested Solutions	Marks	Marker's Comments

$$\int \frac{\sqrt{x}}{\sqrt{2a-x}}\,dx = \frac{-k}{\sqrt{ma}}\int dt$$

let $I = \int \frac{\sqrt{x}}{\sqrt{2a-x}}\,dx$

let $\sqrt{x} = \sqrt{2a}\sin\theta \quad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$

$x = 2a\sin^2\theta$

$\frac{dx}{d\theta} = 4a\sin\theta\cos\theta$

$dx = 4a\sin\theta\cos\theta\,d\theta$

$$\therefore I = \int \frac{\sqrt{2a}\sin\theta}{\sqrt{2a-2a\sin^2\theta}}\,4a\sin\theta\cos\theta\,d\theta$$

$$= \int \frac{\sqrt{2a}\sin\theta}{\sqrt{2a}\sqrt{1-\sin^2\theta}} \times 4a\sin\theta\cos\theta\,d\theta$$

$$= \int \frac{\sin\theta}{|\cos\theta|} \times 4a\sin\theta\cos\theta\,d\theta$$

$$= \int 4a\sin^2\theta\,d\theta \quad \left(\text{since } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

$$= 2a\int 1-\cos 2\theta\,d\theta$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

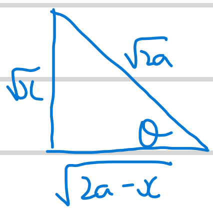
$= 1 - 2\sin^2\theta$

$\therefore \sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$= 2a\left[\theta - \frac{1}{2}\sin 2\theta\right] + C$$

$$= 2a\theta - a\sin 2\theta + C$$

$$= 2a\sin^{-1}\left(\frac{\sqrt{x}}{\sqrt{2a}}\right) - \sqrt{2ax-x^2} + C$$



$\sin 2\theta$

$= 2\sin\theta\cos\theta$

$= 2 \times \frac{\sqrt{x}}{\sqrt{2a}} \times \frac{\sqrt{2a-x}}{\sqrt{2a}}$

$= \frac{\sqrt{2ax-x^2}}{a}$

Year 12 Task Trials MATHEMATICS Ext2 Question 15

Suggested Solutions	Marks	Marker's Comments

$$\therefore 2a\sin^{-1}\left(\frac{\sqrt{x}}{\sqrt{2a}}\right) - \sqrt{2ax - x^2} = \frac{-kt}{\sqrt{ma}} + C$$

when $t = 0$, $x = 2a$

$$\therefore 2a \times \frac{\pi}{2} = C$$

$$\therefore C = a\pi$$

$$\therefore 2a\sin^{-1}\left(\frac{\sqrt{x}}{\sqrt{2a}}\right) - \sqrt{2ax - x^2} = \frac{-kt}{\sqrt{ma}} + a\pi$$

when $x = a$, $2a\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sqrt{a^2} = \frac{-kt}{\sqrt{ma}} + a\pi$

$$\frac{a\pi}{2} - a = \frac{-kt}{\sqrt{ma}} + a\pi \quad (a > 0)$$

$$\frac{kt}{\sqrt{ma}} = \frac{a\pi}{2} + a$$

$$t = \frac{\sqrt{m}\,a\sqrt{a}}{k}\left(\frac{\pi}{2} + 1\right)$$

<table>
<tr><th colspan="3">Year 12 Trial Q16</th></tr>
<tr><th>Suggested Solutions</th><th>Marks Awarded</th><th>Marker's Comments</th></tr>
<tr><td>
(a)
(i)
Let $z = 1 + w + w^2 + w^3 + w^4$ where $w = e^{2\pi i/5}$.
Then
$$wz = w + w^2 + w^3 + w^4 + w^5$$
$$= 1 + w + w^2 + w^3 + w^4 \qquad (w^5 = 1)$$
So
$$z - zw = 0$$
i.e.
$$z(1 - w) = 0$$
Since $w \neq 1$, it follows $z = 0$. Hence the desired result follows.

(ii)
Consider that
$$(1 + 2w + 3w^2 + 4w^3 + 5w^4)(w - 1)$$
$$= w + 2w^2 + 3w^3 + 4w^4 + 5w^5 - 1 - 2w - 3w^2 - 4w^3 - 5w^4$$
$$= -(1 + w + w^2 + w^3 + w^4) + 5w^5$$
$$= -(0) + 5 \cdot 1 \text{ by part (i) and since } w^5 = 1$$
$$= 5$$
Hence, as $w \neq 1$,
$$1 + 2w + 3w^2 + 4w^3 + 5w^4 = \frac{5}{w - 1}$$

(iii)
Given $P(z) := z^5 - 1$, we have for w^k where $k \in \mathbb{Z}$,
$$P(w^k) = (w^k)^5 - 1$$
$$= (w^5)^k - 1$$
$$= 1 - 1$$
$$= 0$$
so, by factor theorem, $P(z)$ may be factorised as
$$z^5 - 1 = (z - 1)(z - w)(z - w^2)(z - w^3)(z - w^4)$$
Also,
$$z^5 - 1 = (z - 1)(1 + z + z^2 + z^3 + z^4)$$
</td>
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1

1
Sufficient, logical progress.

1
Final conclusion.

1
Needed to demonstrate an understanding that w^k was a zero of the polynomial $z^5 - 1$ or something equivalent that meant one could
</td>
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Using the identity,
$$0 = w^5 - 1 = (w - 1)(1 + w + w^2 + w^3 + w^4)$$
along with $w \neq 1$ was accepted.

Many students took the geometric series
$$1 + w + w^2 + w^3 + w^4 + w^5 = \frac{w^6 - 1}{w - 1}$$
and differentiated, giving the desired result.
Differentiation of complex-valued functions is <u>NOT</u> part of the course and is therefore <u>NOT</u> something you are allowed to do.

The worst type of question you can hope for is one where the target is given to you. Why? Because you must explain why you're doing what you're doing when you're doing it. No matter how 'trivial' you think, it's not the point of the exercise. Ignore this advice at your peril.

Many students moved straight into producing a
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so, $(z-1)(1+z+z^2+z^3+z^4)$ $= (z-1)(z-w)(z-w^2)(z-w^3)(z-w^4) \quad \ldots (*)$ Hence for $z \neq 1$, $1+z+z^2+z^3+z^4 = (z-w)(z-w^2)(z-w^3)(z-w^4) \quad \ldots (\#)$ Now, we have then that $(\#)$ is an identity for all $z \neq 1$. But then LHS and RHS agree for at least five distinct values of z, hence the quartic polynomials are identical. Hence they also hold true for $z = 1$. Putting $z = 1$, in $(\#)$, we find $5 = (1-w)(1-w^2)(1-w^3)(1-w^4)$	produce the factorisation in w that was desired. 1 Second mark for correct conclusion.	factorisation in w as something to be accepted. No: you must justify. Marks were not deducted for not justifying why, after division by $z-1$, we are then allowed to make the substitution of $z=1$ since we are dealing with identities (benefit of doubt was given).
(iv) • One way of handling the problem: If $5 \mid k$, then $k = 5j$ for some $j \in \mathbb{Z}$, so $1+w^k+w^{2k}+w^{3k}+w^{4k} = 1+w^{5j}+w^{2(5j)}+w^{3(5j)}+w^{4(5j)}$ $= 1+w^{5j}+\left(w^{5j}\right)^2+\left(w^{5j}\right)^3+\left(w^{5j}\right)^4$ $= 1+1+1^2+1^3+1^4$ $= 5$	1 First mark for proving the case for $5 \mid k$	
And if $\neg 5 \mid k$, then we could note, since $w^{5k} = (w^5)^k = 1^k = 1$, that $0 = w^{5k}-1 = \left(w^k-1\right)\left(1+w^k+w^{2k}+w^{3k}+w^{4k}\right)$ Since 5 does not divide k, $k = 5j+r$ for $r = 1,2,3,4$, so $w^k - 1 = w^{5j+r}-1 = w^{5j}w^r - 1 = 1 \cdot w^r - 1 = w^r - 1 \neq 0$, so the only source of 0 in $\left(w^k-1\right)\left(1+w^k+w^{2k}+w^{3k}+w^{4k}\right) = 0$ is the factor $1+w^k+w^{2k}+w^{3k}+w^{4k}$ Hence $1+w^k+w^{2k}+w^{3k}+w^{4k} = 0$ If 5 does not divide k.	1 Second mark for sufficient progress dealing with remaining cases. 1 Final mark for complete proof.	Many students lost a mark for not completing their proof for the case where 5 does not divide k. 'Ellipsis-proofing' or using 'etc.' is not going to cut it when others are completing it fully. Until you perform the derivation, you're asserting a belief, not a fact. You are awarded for presentation of facts.
• The way most candidates handled the problem was through applying quotient-remainder theorem as follows. By quotient-remainder theorem, we may express all integers k in the form $k = 5j + r$ where $j \in \mathbb{Z}, r \in \{0,1,2,3,4\}$. So,		

$$\sum_{n=0}^{4} w^{nk} = \sum_{n=0}^{4} w^{n(5j+r)} = \sum_{n=0}^{4} (w^5)^{nj}\, w^{nr} = \sum_{n=0}^{4} 1 \cdot w^{nr}$$

$$= 1 + w^r + (w^r)^2 + (w^r)^3 + (w^r)^4$$

$$= \begin{cases} \dfrac{w^{5r}-1}{w^r-1} & \text{if } r \neq 0 \text{ (i.e. } w^r \neq 1) \\ 5 & \text{if } r = 0 \text{ (i.e. } w^r = 1) \end{cases}$$

$$= \begin{cases} \dfrac{1-1}{w^r-1} & \text{if } r \neq 0 \\ 5 & \text{if } r = 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } r \neq 0 \\ 5 & \text{if } r = 0 \end{cases}$$

So,

$$1 + w^k + w^{2k} + w^{3k} + w^{4k} = \begin{cases} 5 & \text{if } 5|k \\ 0 & \text{otherwise} \end{cases}$$

(v)

$$S := \frac{1}{5}\left[(1+1)^n + (1+w)^n + (1+w^2)^n + (1+w^3)^n + (1+w^4)^n\right]$$

$$= \frac{1}{5}\left[\sum_{j=0}^{n}\binom{n}{j} + \sum_{j=0}^{n}\binom{n}{j}w^j + \sum_{j=0}^{n}\binom{n}{j}w^{2j} + \sum_{j=0}^{n}\binom{n}{j}w^{3j} + \sum_{j=0}^{n}\binom{n}{j}w^{4j}\right]$$

$$= \frac{1}{5}\sum_{j=0}^{n}\binom{n}{j}\left(1 + w^j + w^{2j} + w^{3j} + w^{4j}\right)$$

Now, by (iv), $1 + w^j + w^{2j} + w^{3j} + w^{4j} = 0$ if 5 does not divide j, and the sum is 5 otherwise. Hence all terms carrying j not a multiple of 5 will make contributions of 0 to the sum. All terms with j a multiple of 5 will make a contribution of $\binom{n}{j}5$.

So, for $5\ell = \max\{5m \in \mathbb{N}_0 : 5m \leq n,\ n \geq 0 \text{ fixed}\}$,

$$S = \frac{1}{5}\sum_{\substack{j=0 \\ 5|j}}^{5\ell}\binom{n}{j}5$$

$$= \binom{n}{0} + \binom{n}{5} + \binom{n}{10} + \cdots + \binom{n}{5\ell}$$

(i.e. where ℓ is largest integer such that $5\ell \leq n$).

1
First mark for correctly expanding and manipulating the binomial sums.

1
Second mark for justifying why ~80% of the terms collapse to 0.

1
Final mark for correct conclusion.

Many students failed to provide sufficient justification as to why certain terms vanished. Again, **the worst problem to be given is one where you're given the answer!** You must justify every step as completely as possible. Don't take anything for granted.

(b) (i) $$\text{proj}_{\beta\mathbf{v}}(\alpha\mathbf{u}) = \frac{(\alpha\mathbf{u})\cdot(\beta\mathbf{v})}{(\beta\lvert\mathbf{v}\rvert)^2}\beta\mathbf{v}$$ $$= \frac{\alpha(\mathbf{u}\cdot\mathbf{v})}{\lvert\mathbf{v}\rvert^2}\mathbf{v}$$ $$= \alpha\ \text{proj}_{\mathbf{v}}\mathbf{u}$$	1	
(ii) Given $$\mathbf{a}_n = \frac{1}{5^n}\begin{bmatrix}(-1)^{n+1}\times 2\\ 3\\ (-1)^n\end{bmatrix} \qquad \mathbf{b}_n = \frac{1}{2^n}\begin{bmatrix}1\\ -8\\ 7\end{bmatrix}$$ then $$\mathbf{c}_n \coloneqq \text{proj}_{\,\mathbf{a}_n}\mathbf{b}_n$$ $$= \frac{1}{2^n}\text{proj}_{\begin{bmatrix}(-1)^{n+1}\times 2\\ 3\\ (-1)^n\end{bmatrix}}\left(\begin{bmatrix}1\\ -8\\ 7\end{bmatrix}\right)$$ $$= \frac{1}{2^n}\times\frac{((-1)^{n+1}\times 2\times 1 + 3\times(-8) + (-1)^n\times 7)}{((-1)^{n+1}\times 2)^2 + 3^2 + ((-1)^n)^2}\begin{bmatrix}(-1)^{n+1}\times 2\\ 3\\ (-1)^n\end{bmatrix}$$ $$= \frac{1}{2^n}\times\frac{5\times(-1)^n - 24}{14}\begin{bmatrix}(-1)^n\times(-2)\\ 3\\ (-1)^n\end{bmatrix}$$ Presence of $(-1)^n$ suggests alternation in c_n, where we note that for all integral $n \geq 0$, we get two forms from $\frac{5\times(-1)^n-24}{14}\begin{bmatrix}(-1)^n\times(-2)\\ 3\\ (-1)^n\end{bmatrix}$, with the only variation occurring because of $\frac{1}{2^n}$. Since we have alternation, we'll get one form when n is even, the other form when n is odd. We will partition the sum $$\sum_{n=0}^{\infty}\mathbf{c}_n$$ into a sum of even n and a sum of odd n.	1 First mark for correct calculation of $\mathbf{c}_n$ 1 Second mark for *sufficiently* identifying some pattern (i.e. odd and even expansion)	Some general remarks: • Some students projected vectors incorrectly (i.e. projecting $\mathbf{a}_n$ onto $\mathbf{b}_n$); • Mishandling of negatives; • Not expanding at all to observe a pattern (it should be noted that $(-1)^n$ where $n \in \mathbb{Z}$ is an alternator/switching function…when used alone, it usually breaks sums into two classes. Can also see $\cos(n\pi)$, $n \in \mathbb{Z}$ does the same); • Students dropping vectors, thereby summing only the

Let $n = 2k$. Then $$c_{2k} = \frac{1}{2^{2k}} \times \frac{5 \times (-1)^{2k} - 24}{14} \begin{bmatrix} (-1)^{2k} \times (-2) \\ 3 \\ (-1)^{2k} \end{bmatrix}$$ $$= \frac{1}{4^k} \times \frac{-19}{14} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$ $$= \frac{1}{4^k} \times \frac{19}{14} \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$ And for $n = 2k+1$, $$c_{2k+1} = \frac{1}{2^{2k+1}} \times \frac{5 \times (-1)^{2k+1} - 24}{14} \begin{bmatrix} (-1)^{2k+1} \times (-2) \\ 3 \\ (-1)^{2k+1} \end{bmatrix}$$ $$= \frac{1}{2} \times \frac{1}{4^k} \times \frac{-29}{14} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$ $$= \frac{1}{4^k} \times \frac{29}{28} \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$ Hence $$\sum_{n=0}^{\infty} \mathbf{c}_n = \sum_{k=0}^{\infty} \mathbf{c}_{2k} + \sum_{k=0}^{\infty} \mathbf{c}_{2k+1}$$ $$= \sum_{k=0}^{\infty} \frac{1}{4^k} \times \frac{19}{14} \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} + \sum_{k=0}^{\infty} \frac{1}{4^k} \times \frac{29}{28} \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$ $$= \left(\frac{19}{14} \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} + \frac{29}{28} \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \right) \sum_{k=0}^{\infty} \frac{1}{4^k}$$ $$= \frac{1}{28} \left(\begin{bmatrix} 18 \\ -201 \\ -9 \end{bmatrix} \right) \sum_{k=0}^{\infty} \frac{1}{4^k}$$ Now, if $$s_n := \sum_{k=0}^{n} \frac{1}{4^k} = 1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^n}$$ then $$\frac{1}{4} s_n = \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^n} + \frac{1}{4^{n+1}}$$	1 Third mark for correct calculation of **both** vectors for odd and even n.	constants and presenting that as the limiting sum (be careful with projection calculations, this often happens). Students have asked since the examination whether separating infinite sums like this is necessarily 'OK'. No, it's not necessarily OK, but in a contrived condition (i.e. HSC), any non-standard infinite sum will have passed sufficiency tests for convergence…the point is, you don't need to concern yourselves with it.

So, $$s_n - \frac{1}{4}s_n = \frac{3}{4}s_n = 1 - \frac{1}{4^{n+1}}$$ So $$s_n = \sum_{k=0}^{n} \frac{1}{4^k} = \frac{4}{3}\left(1 - \frac{1}{4^{n+1}}\right)$$ Taking the limit of partial sums, as $n \to \infty$, $$s_\infty = \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{4}{3}$$ Hence $$\sum_{n=0}^{\infty} \mathbf{c}_n = \frac{4}{3} \times \frac{1}{28}\left(\begin{bmatrix} 18 \\ -201 \\ -9 \end{bmatrix}\right) = \frac{1}{7}\begin{bmatrix} 6 \\ -67 \\ -3 \end{bmatrix}$$	1 Final mark for correct limiting sum of vectors in $\mathbf{c}_n$	