# Network Localization and Multi-Dimensional Scaling: Escaping Saddles and a Local Optimality Condition

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**Abstract**

In this paper, we focus on a class of problems characterized by solving a set of non-linear composite norm equations, where each equation represents a measurement that approximates a norm of a linear transformation. These problems manifest as non-linear least squares minimization problems, which are characterized by their non-convex and non-smooth nature, presenting challenges in finding (locally) optimal solutions. While existing optimization algorithms mostly concentrate on finding critical points of the associated least squares objective function, these functions often possess multiple non-global local minima and saddle points. These problems find wide applications in various domains, and within this category we focus our attention on two challenging problems: Wireless Sensor Network Localization and Multi-Dimensional Scaling. We establish that non-differentiable points correspond to maximum or saddle points, and we provide a constructive approach to determine descent directions at these points. Leveraging this, we propose a straightforward procedure to escape non-differentiable saddle points that is applicable in either centralized or distributed computational setting. Furthermore, we develop a necessary condition for differentiable points to be locally optimal, by exploiting the structure of the objective function of these problems.

**Keywords:** sensor network localization; multi-dimensional scaling; criticality; optimality condition; saddle point

## 1 Introduction

In this paper, we address the task of minimizing a non-linear, non-convex and non-smooth least squares function $\mathcal{H}\colon \mathbb{R}^q \to [0,\infty)$, which is expressed as

$$\underset{\mathbf{x}\in\mathbb{R}^q}{\text{minimize}} \quad \mathcal{H}(\mathbf{x}) \equiv \sum_{l=1}^{N} \left(\|\mathbf{A}_l\mathbf{x} + \mathbf{b}_l\| - d_l\right)^2, \tag{1}$$

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where $\|\cdot\|$ represents the Euclidean norm, and for given data matrices $\mathbf{A}_l \in \mathbb{R}^{q_l \times q}$, vectors $\mathbf{b}_l \in \mathbb{R}^{q_l}$ and scalars $d_l \in \mathbb{R}$, $l = 1, 2, \ldots, N$. Such optimization problems arise in several applied context, for example, in signal processing and unsupervised learning (see specific applications below). This minimization naturally manifests in several scenarios. For instance, consider a set of given scalar measurements $d_l \in \mathbb{R}$, $l = 1, 2, \ldots, N$, each representing a norm of a linear transformation of an unknown parameter $\mathbf{x} \in \mathbb{R}^q$ with some added noise. Mathematically, this involves finding $\mathbf{x} \in \mathbb{R}^q$ that solves the set of non-linear composite norm equations [26, 35]

$$d_l = \|\mathbf{A}_l \mathbf{x} + \mathbf{b}_l\| + \epsilon_l, \quad l = 1, 2, \ldots, N, \tag{2}$$

where $\epsilon_l \in \mathbb{R}$ is a noise term. By applying the maximum-likelihood estimation approach to approximate such $\mathbf{x} \in \mathbb{R}^q$, we arrive at Problem (1). As a result, any problem that involves estimating a set of equations of the form (2) can be addressed by solving the optimization problem in (1).

The optimization model in (1) is non-convex and non-smooth (i.e., non-differentiable), which presents challenges in finding (locally) optimal solutions. Recent advancements in non-convex optimization have mainly concentrated on finding critical points, which could technically be (locally) optimal solutions but also saddle points or even maximum points. However, in many interesting domains of applications it is possible to build descent optimization algorithms [2,3,8] that converge to critical points. The good news in this case is that the converging critical point can not be a (local) maximum point. Therefore, a grand challenge of optimization theory and practice is to avoid saddle points. This work aims at advancing the research on this challenging question by studying non-convex optimization problems as described in (1) and exploiting the more specific structure.

Our paper focuses on a specific sub-class within the class of problems in (1). We specifically investigate the scenario where the data matrices $\mathbf{A}_l$ are structured in the form $\mathbf{A}_l = \left[\cdots \mathbf{I}_n \cdots - \mathbf{I}_n \cdots\right] \in \mathbb{R}^n \times \mathbb{R}^q$, with $\mathbf{I}_n$ denoting the $n \times n$ identity matrix (all unspecified elements are 0), and the data vectors $\mathbf{b}_l$ are $\mathbf{0}_n$ for $l = 1, 2, \ldots, N$. In this case, Problem (1) is translated to the following optimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^q} \quad \mathcal{F}(\mathbf{x}) \equiv \sum_{l=1}^{N} \left(\|\mathbf{x}_{i_l} - \mathbf{x}_{j_l}\| - d_l\right)^2, \tag{3}$$

where $\mathbf{x}_{i_l}, \mathbf{x}_{j_l} \in \mathbb{R}^n$ are sub-vectors of the vector $\mathbf{x} \in \mathbb{R}^q$ corresponding to the position of $\mathbf{I}_n$ and $-\mathbf{I}_n$ in $\mathbf{A}_l$. This specific scenario captures, among others, the widely studied applications of Sensor Network Localization (SNL) and Multi-Dimensional Scaling (MDS), as we discuss below. Although this choice may initially seem restrictive, our theoretical framework can be readily extended to other problems. By thoroughly examining the structure in (3), we provide a clear and convenient framework for analysis and facilitate insightful discussions, enabling the extension and generalization of our findings to a broader range of related problems encompassed by the general class of problems given in (1).

The main goal of this paper it to deepen our understanding of optimization problems with structure as given in (3). We fully characterize its set of critical points and leverage this knowledge to also advance the algorithmic and practical fronts. Our main contributions are summarized now.

1. In Section 3, we prove that all non-differentiable points of $\mathcal{F}$ are either maximum or saddle points. Moreover, we show that each such point has an *explicit and easy-to-find* descent direction. Consequently, any minimum point of $\mathcal{F}$ must be a differentiable critical point.

2. In Section 4, we develop a procedure that escapes non-differentiable saddle points, thereby preventing minimization algorithms from getting trapped in a subset of the non-optimal critical points. This escape procedure can be implemented in both centralized or distributed computational settings.

3. As any minimum point of $\mathcal{F}$ must be a differentiable point, in Section 5, we formulate a necessary condition for a differentiable critical point to be a local minimum point. This condition can be easily verified in centralized and distributed computational settings.

# 2 Main Motivating Applications

In this section, we will discuss two prominent applications that provide the motivation for studying the problem of solving the non-linear composite norm equations with the structure as given in Problem (3).

The first application is Multi-Dimensional Scaling (MDS), which is a popular tool for dimensionality reduction and data visualization [40]. Formally, given a symmetric matrix $\mathbf{D} \in \mathbb{R}^{K \times K}$, where $\mathbf{D}_{ij} = \mathbf{D}_{ji}$ denotes the dissimilarity between two data points $\mathbf{o}_i, \mathbf{o}_j \in \mathbb{R}^p$ defined mathematically as $\mathbf{D}_{ij} = \|\mathbf{o}_i - \mathbf{o}_j\|$. MDS aims to find lower-dimensional representations $\mathbf{x}_i \in \mathbb{R}^n$, $n < p$, for each data point $\mathbf{o}_i \in \mathbb{R}^p$. These representations should satisfy the condition that the distances $\|\mathbf{x}_i - \mathbf{x}_j\|$ approximate the original dissimilarity $\mathbf{D}_{ij}$ within a certain tolerance $\epsilon_{ij} \in \mathbb{R}$. In essence, MDS seeks to find $K$ vectors $\mathbf{x}_i \in \mathbb{R}^n$ such that [14, 15]

$$\mathbf{D}_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\| + \epsilon_{ij}, \quad i, j \in \{1, 2, \ldots, K\}.$$

We easily see that MDS fits into the framework of equations in (2) by representing $\mathbf{x} \in \mathbb{R}^{nK}$ as the concatenation of all unknowns $\mathbf{x}_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, K$, into a single vector.

In the second application, Sensor Network Localization (SNL), the objective is to find the location of each sensor in a deployed sensor network, utilizing distance measurements between neighboring sensors [1]. Formally, we consider a set of $K$ sensors, each located at an unknown location $\mathbf{x}_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, K$. Given a set $\mathcal{E}$ comprising pairs of neighboring sensors $i$ and $j$, with positive noisy distance measurements $d_{ij} > 0$ between them, the SNL problem is typically formulated as finding $K$ vectors $\mathbf{x}_i \in \mathbb{R}^n$ such that [30, 33]

$$d_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\| + \epsilon_{ij}, \quad (i, j) \in \mathcal{E},$$

where $\epsilon_{ij}$ denotes distance measurement noise. Notice that we result with a similar set of equations as in MDS, and therefore it fits into the setting of equations given in (2).

A few other well-studied applications that fall under the category of the set of equations in (2) are the Source Localization problem [7, 18, 23] and the Phase Retrieval problem [16, 39]. All these well-known applications have be studied in several optimization settings. The most popular optimization model to handle these non-linear composite norm equations is the least-squares explicitly formulated in (3). Before presenting our study on this optimization model, we review existing works that are relevant for the task of understanding the critical points of the function $\mathcal{F}$.

To this end, we recall that a first-order *critical point* for a non-convex and non-smooth function $\mathcal{F}$ is a point $\mathbf{x} \in \mathbb{R}^q$ for which the zero vector belongs to its limiting sub-differential set [25], denoted by $\partial \mathcal{F}(\mathbf{x})$, i.e., $\mathbf{0}_q \in \partial \mathcal{F}(\mathbf{x})$. In the context of non-smooth analysis, a critical point can be either a differentiable or a non-differentiable point. When a point $\mathbf{x} \in \mathbb{R}^q$ is differentiable, then the condition $\mathbf{0}_q \in \partial \mathcal{F}(\mathbf{x})$ reduces to the equality $\mathbf{0}_q = \nabla \mathcal{F}(\mathbf{x})$, where $\nabla \mathcal{F}(\mathbf{x})$ is the gradient of $\mathcal{F}$ at $\mathbf{x} \in \mathbb{R}^q$.

We mention that, even though the problem is non-convex and non-smooth, criticality serves as a necessary condition for local optimality, implying that any global or local minimum point must also be a first-order critical point of the objective function. Therefore, in this paper, we first aim at characterizing the set of critical points of $\mathcal{F}$ and explore their relationship with (locally) optimal minimum points.

In order to better understand the critical points of the given function at hand, beyond the definition of the sub-differential set, it will be beneficial to exploit different structural properties of the function $\mathcal{F}$. One line of research that can be relevant for our case is from the domain of Difference-of-Convex (DC) programming. A general DC programming problem can be expressed in the following form

$$\min_{\mathbf{x}\in\mathbb{R}^q} \{\Psi(\mathbf{x}) \equiv \varphi(\mathbf{x}) - \psi(\mathbf{x})\}, \tag{DC}$$

where $\varphi, \psi \colon \mathbb{R}^q \to (-\infty, \infty]$ are convex and (possibly) non-smooth. Indeed, we easily see that Problem (3), and also the general Problem (1), can be formulated in the form by defining

$$\varphi(\mathbf{x}) = \sum_{l=1}^{N} \left( \|\mathbf{x}_{i_l} - \mathbf{x}_{j_l}\|^2 + d_l^2 \right) \quad \text{and} \quad \psi(\mathbf{x}) = 2\sum_{l=1}^{N} d_l \|\mathbf{x}_{i_l} - \mathbf{x}_{j_l}\|.$$

By framing Problem (3) as a DC programming problem, we can leverage existing insights from this domain to shed some light on the critical points of the function $\mathcal{F}$. For instance, in DC programming problems, a necessary condition for optimality is known as a *DC-critical point* [28], which is a point $\mathbf{x} \in \mathbb{R}^q$ satisfying $\partial\varphi(\mathbf{x}) \cap \partial\psi(\mathbf{x}) \neq \emptyset$, where here $\partial$ denotes the sub-differential set of a convex function[1]. The concept of criticality in DC programming has been studied in several papers, see for instance [37, 38]. See also [21, 29] for a concise introduction to this notion.

However, when it comes to characterizing the minimum points of Problem (3), it is important to note that the notion of DC-criticality does not provide a better understanding than the classical notion of criticality, as DC-criticality can not differentiate between minimum and maximum points. For example, consider the one-dimensional DC function $\Psi(x) = x^2 - |x|$. In this case, $x = 0$ is a DC-critical point that is also a maximum point.

A more restrictive concept than DC-criticality is a *directional-stationary point* [28], often referred to as a d-stationary point, which is a point $\mathbf{x} \in \mathbb{R}^q$ where no feasible descent directions exist. In the context of Problem (DC), the work [6] shows that if the function $\varphi$ of Problem (3) is smooth, then any d-stationary point is a differentiable point of the function $\Psi$. This result implies that any optimal solution of Problem (3) must be a point $\mathbf{x} \in \mathbb{R}^q$ where the gradient $\nabla\mathcal{F}(\mathbf{x})$ exists and $\nabla\mathcal{F}(\mathbf{x}) = \mathbf{0}_q$. It is important to note that the function $\mathcal{F}$ of Problem (3) is non-smooth, so its gradient is not defined for its non-differentiable points. Consequently, any critical point of Problem (3) that is not a d-stationary point has at least one descent direction and is therefore not a minimum point of the problem. However, determining such descent directions is a challenging task as currently there is no clear way to identify such directions for Problem (3).

To conclude, even though framing the function $\mathcal{F}$ as a DC function exploits a certain structure, the understanding of its critical points remains very limited. In the following sections, we show that the structure of $\mathcal{F}$ is generous enough to enable us the characterize its critical points, develop a simple procedure that escapes non-differentiable saddle points, and even devise a necessary condition for a differentiable critical point to qualify as a local minimum point.

# 3 Characterization of Extremum Points

In this section, we explore the extremum points of the Problem (5), which will provide an important and useful ground for the understanding of critical points in the following sections.

To achieve this goal, since the function $\mathcal{F}$ is non-differentiable, we first recall the notion of directional derivatives. Let $\phi \colon \mathbb{R}^q \to (-\infty, \infty]$ be a proper function and let $\mathbf{x} \in \operatorname{int}(\operatorname{dom}(\phi))$. The

[1] In the convex setting, the limiting sub-differential coincides with the "regular" sub-differential.

directional derivative in the direction of the vector $\mathbf{d} \in \mathbb{R}^q$ is defined by

$$\phi'(\mathbf{x};\mathbf{d}) \equiv \lim_{\epsilon \to 0^+} \frac{\phi(\mathbf{x}+\epsilon\mathbf{d}) - \phi(\mathbf{x})}{\epsilon}. \tag{4}$$

The function $\phi$ is said to be *differentiable* at $\mathbf{x} \in \mathbb{R}^q$ if the gradient vector $\nabla\phi(\mathbf{x}) \in \mathbb{R}^q$ exists. If $\phi$ is continuously differentiable over an open set $U \subseteq \mathbb{R}^q$ that contains the point $\mathbf{x} \in \mathbb{R}^q$, then $\phi'(\mathbf{x};\mathbf{d}) = \nabla\phi(\mathbf{x})^T \mathbf{d}$ for any $\mathbf{d} \in \mathbb{R}^q$ [4].

For the sake of simplicity in the subsequent analysis and for ease of index notation, we express Problem (3) using the terminology of the SNL problem. In other words, we rewrite Problem (3) equivalently as

$$\min_{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_K \in \mathbb{R}^n} \quad \mathcal{F}(\mathbf{x}) \equiv \sum_{(i,j)\in\mathcal{E}} \left(\|\mathbf{x}_i - \mathbf{x}_j\| - d_{ij}\right)^2, \tag{5}$$

and we use this formulation interchangeably with Problem (3).

Due to the non-differentiability of the norm function at $\mathbf{0}_n$, the non-differentiable points of $\mathcal{F}$ are precisely the points $\mathbf{x} \in \mathbb{R}^{nK}$ where $\mathbf{x}_i = \mathbf{x}_j \in \mathbb{R}^n$ for some pair $(i,j) \in \mathcal{E}$. For simplicity of developments, for any pair $(i,j) \in \mathcal{E}$, we define the function $\mathcal{F}_{ij} \colon \mathbb{R}^{nK} \to \mathbb{R}$ as

$$\mathcal{F}_{ij}(\mathbf{x}) \equiv \left(\|\mathbf{x}_i - \mathbf{x}_j\| - d_{ij}\right)^2, \tag{6}$$

and we notice that $\mathbf{x} \in \mathbb{R}^{nK}$ is a non-differentiable point of $\mathcal{F}_{ij}$ if and only if $\mathbf{x}_i = \mathbf{x}_j$. Following (6), it holds that

$$\mathcal{F}(\mathbf{x}) = \sum_{(i,j)\in\mathcal{E}} \mathcal{F}_{ij}(\mathbf{x}), \tag{7}$$

and notice that a point $\mathbf{x} \in \mathbb{R}^{nK}$ is a non-differentiable point of $\mathcal{F}$ if and only if it is a non-differentiable point of $\mathcal{F}_{ij}$, for some pair $(i,j) \in \mathcal{E}$.

We begin with a few simple properties of the directional derivative of $\mathcal{F}$. To this end, for any vector $\mathbf{d} \in \mathbb{R}^{nK}$ we denote by $\mathbf{d}_i$, $i = 1,2,\ldots,K$, the sub-vector obtained from $\mathbf{d}$ by taking its $n(i-1)+1$ to $n \cdot i$ coordinates.

**Lemma 1.** *Let $(i,j) \in \mathcal{E}$.*

*(i) Let $\mathbf{x} \in \mathbb{R}^{nK}$ be a non-differentiable point of $\mathcal{F}_{ij}$. Then, for any $\mathbf{d} \in \mathbb{R}^{nK}$, it holds that*

$$\mathcal{F}'_{ij}(\mathbf{x};\mathbf{d}) = -2d_{ij}\|\mathbf{d}_i - \mathbf{d}_j\|.$$

*In particular, $\mathcal{F}'_{ij}(\mathbf{x};\mathbf{d}) < 0$ if $\mathbf{d}_i \neq \mathbf{d}_j$ and $\mathcal{F}'_{ij}(\mathbf{x};\mathbf{d}) = 0$ otherwise.*

*(ii) Let $\mathbf{x} \in \mathbb{R}^{nK}$ be a point satisfying $\mathcal{F}_{ij}(\mathbf{x}) = 0$. Then, $\mathcal{F}_{ij}(\epsilon\mathbf{x}) > 0$ for any $\epsilon \neq 1$.*

*Proof.* (i). We recall that $\mathbf{x}_i = \mathbf{x}_j$ for any non-differentiable point of $\mathcal{F}_{ij}$. From (4) we have

$$\begin{aligned}\mathcal{F}'_{ij}(\mathbf{x};\mathbf{d}) &= \lim_{\epsilon \to 0^+} \frac{\mathcal{F}_{ij}(\mathbf{x}+\epsilon\mathbf{d}) - \mathcal{F}_{ij}(\mathbf{x})}{\epsilon} = \lim_{\epsilon \to 0^+} \left(\epsilon\|\mathbf{d}_i - \mathbf{d}_j\|^2 - 2d_{ij}\|\mathbf{d}_i - \mathbf{d}_j\|\right) \\ &= -2d_{ij}\|\mathbf{d}_i - \mathbf{d}_j\|,\end{aligned}$$

and the result immediately follows.

(ii). Notice that $\mathcal{F}_{ij}(\mathbf{x}) = 0$ if and only if $\|\mathbf{x}_i - \mathbf{x}_j\| = d_{ij}$. Now, the point $\epsilon\mathbf{x}$ for any $\epsilon \neq 1$ satisfies $\mathcal{F}_{ij}(\epsilon\mathbf{x}) = (\epsilon\|\mathbf{x}_i - \mathbf{x}_j\| - d_{ij})^2 = d_{ij}^2(\epsilon - 1)^2 > 0$, as required. $\square$

Recall that $\mathbf{x} \in \mathbb{R}^q$ is a *local minimum* point of a function $\phi\colon \mathbb{R}^q \to (-\infty, \infty]$, if $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$ for all $\mathbf{y} \in \operatorname{dom}(\phi)$ such that $\|\mathbf{x} - \mathbf{y}\| \leq \epsilon$ for some $\epsilon > 0$. In addition, a local minimum point $\mathbf{x} \in \mathbb{R}^q$ of $\phi$ is *global*, if it attains the minimal value of $\phi$ over its domain. Similarly, we define *local* and *global maximum* points. Moreover, an *extremum* point of $\phi$ is either a local minimum or a local maximum point. Of course, any global extremum point (if exists) is also a local extremum point.

Before we characterize the extremum points of $\mathcal{F}$, we recall the notion of descent and ascent directions. We say that $\mathbf{d} \in \mathbb{R}^q$ is a *descent direction* of $\phi\colon \mathbb{R}^q \to (-\infty, \infty]$ if $\phi'(\mathbf{x}; \mathbf{d})$ exists and is negative. Similarly, if it is positive then $\mathbf{d}$ is an *ascent direction*. This notion is important since it is well-known (see, for instance, [5, Lemma 8.2]) that if $\mathbf{d} \in \mathbb{R}^q$ is a descent direction of $\phi$ at $\mathbf{x} \in \mathbb{R}^q$, then there exists some $\bar{\epsilon} > 0$ such that $\phi(\mathbf{x} + \epsilon\mathbf{d}) < \phi(\mathbf{x})$ for all $\epsilon \in (0, \bar{\epsilon}]$. Hence, $\mathbf{x}$ is not a minimum point[2]. Similar results hold for ascent directions.

We recall that a point $\mathbf{x} \in \mathbb{R}^q$ is a *stationary point* of a function $\phi\colon \mathbb{R}^q \to (-\infty, \infty]$ if the gradient $\nabla\phi(\mathbf{x})$ exits and is the vector of all zeros.

Now, we are ready to provide a characterization of the extremum points of the functions $\mathcal{F}_{ij}$. To this end, for any point $\mathbf{x} \in \mathbb{R}^{nK}$, we denote by $\mathcal{B}_{ij}[\mathbf{x}] \subset \mathbb{R}^{nK}$ the closed ball centered at $\mathbf{x}$ with radius $d_{ij} > 0$.

**Lemma 2.** *(i) Let $(i, j) \in \mathcal{E}$. A point $\mathbf{x} \in \mathbb{R}^{nK}$ is a stationary point of $\mathcal{F}_{ij}$ if and only if it is a global minimum point of $\mathcal{F}_{ij}$.*

*(ii) Let $(i, j) \in \mathcal{E}$. Then, $\mathbf{x} \in \mathbb{R}^{nK}$ is a non-differentiable point of $\mathcal{F}_{ij}$ if and only if it is a local maximum point of $\mathcal{F}_{ij}$.*

*(iii) Any non-differentiable point of $\mathcal{F}$ is a local maximum point of $\mathcal{F}_{ij}$ for some $(i, j) \in \mathcal{E}$.*

*Proof.* (i). Recall that the differentiable points of $\mathcal{F}_{ij}$ are exactly the points $\mathbf{x} \in \mathbb{R}^{nK}$ for which $\mathbf{x}_i \neq \mathbf{x}_j$. In particular, the gradient $\nabla\mathcal{F}_{ij}(\mathbf{x})$ exists and simple calculation show that

$$\nabla_{\mathbf{x}_i}\mathcal{F}_{ij}(\mathbf{x}) = \frac{2\left(\|\mathbf{x}_i - \mathbf{x}_j\| - d_{ij}\right)}{\|\mathbf{x}_i - \mathbf{x}_j\|}\left(\mathbf{x}_i - \mathbf{x}_j\right) = -\nabla_{\mathbf{x}_j}\mathcal{F}_{ij}(\mathbf{x}),$$

where $\nabla_{\mathbf{x}_i}\mathcal{F}_{ij}$ is the gradient of the partial function $\mathbf{x}_i \mapsto \mathcal{F}_{ij}(\mathbf{x})$ (which can also be viewed as the sub-vector of $\nabla\mathcal{F}_{ij}$ corresponding to the coordinates of $\mathbf{x}_i \in \mathbb{R}^n$). Also, $\nabla_{\mathbf{x}_l}\mathcal{F}_{ij}(\mathbf{x}) = \mathbf{0}_n$ for any $l \in \{1, 2, \ldots, K\}$ such that $l \notin \{i, j\}$. Therefore, $\nabla\mathcal{F}_{ij}(\mathbf{x})$ is the vector of all zeros (i.e., $\mathbf{x}$ is a stationary point) if and only if $\|\mathbf{x}_i - \mathbf{x}_j\| = d_{ij}$ if and only if $\mathcal{F}_{ij}(\mathbf{x}) = 0$ if and only if $\mathbf{x}$ is a global minimum point of the non-negative function $\mathcal{F}_{ij}$.

(ii). Let $\mathbf{x} \in \mathbb{R}^{nK}$ be a non-differentiable point of $\mathcal{F}_{ij}$ and we will prove that it is a local maximum point of $\mathcal{F}_{ij}$. To this end, we prove that $\mathcal{F}_{ij}(\mathbf{y}) \leq \mathcal{F}_{ij}(\mathbf{x})$ for any $\mathbf{y} \in \mathcal{B}_{ij}[\mathbf{x}]$. Recall that for any non-differentiable point it holds that $\mathbf{x}_i = \mathbf{x}_j$, hence we get from the triangle inequality

$$\|\mathbf{y}_i - \mathbf{y}_j\| \leq \|\mathbf{y}_i - \mathbf{x}_i\| + \|\mathbf{y}_j - \mathbf{x}_j\| \leq 2d_{ij},$$

where the last inequality is due to the fact that $\mathbf{y} \in \mathcal{B}_{ij}[\mathbf{x}]$. Hence, $\|\mathbf{y}_i - \mathbf{y}_j\| - 2d_{ij} \leq 0$ and we get

$$\mathcal{F}_{ij}(\mathbf{y}) = \left(\|\mathbf{y}_i - \mathbf{y}_j\| - d_{ij}\right)^2 = \|\mathbf{y}_i - \mathbf{y}_j\|\left(\|\mathbf{y}_i - \mathbf{y}_j\| - 2d_{ij}\right) + d_{ij}^2 \leq d_{ij}^2 = \mathcal{F}_{ij}(\mathbf{x}), \tag{8}$$

where the last equality follows from the fact that $\mathbf{x}_i = \mathbf{x}_j$, and the required result follows.

[2] Conversely, if there exists some $\bar{\epsilon} > 0$ such that $\phi(\mathbf{x} + \epsilon\mathbf{d}) \leq \phi(\mathbf{x})$ for all $\epsilon \in (0, \bar{\epsilon}]$, then if the directional derivative exists, it follows that $\phi'(\mathbf{x}; \mathbf{d}) \leq 0$ and hence $\mathbf{d}$ is a non-ascent direction (that is, $\mathbf{d}$ is either a descent direction or that the directional derivative is 0). Similarly, if $\phi(\mathbf{x} + \epsilon\mathbf{d}) \geq \phi(\mathbf{x})$ we obtain that $\mathbf{d}$ is a non-descent direction.

For the converse direction, we will prove that if $\mathbf{x} \in \mathbb{R}^{nK}$ is a local maximum point of $\mathcal{F}_{ij}$, then it is also a non-differentiable point of $\mathcal{F}_{ij}$. More precisely, we will prove that $\mathbf{x}_i = \mathbf{x}_j$. Assume on the contrary that $\mathbf{x}$ is differentiable point. Meaning, the gradient $\nabla \mathcal{F}_{ij}(\mathbf{x})$ exists and $\mathbf{x}_i \neq \mathbf{x}_j$. From the continuity of $\mathcal{F}_{ij}$, there exists an open set $U \subseteq \mathbb{R}^{nK}$ such that $\mathbf{y}_i \neq \mathbf{y}_j$ for all $\mathbf{y} \in U$, and $\mathcal{F}_{ij}$ is continuously differentiable over $U$.

Now, either $\nabla \mathcal{F}_{ij}(\mathbf{x})$ is a non-zero vector, or it is the vector of all zeros. If $\nabla \mathcal{F}_{ij}(\mathbf{x}) \neq \mathbf{0}_{nK}$, then $\nabla \mathcal{F}_{ij}(\mathbf{x})$ is an ascent direction of $\mathcal{F}_{ij}$ at $\mathbf{x}$ (since $\nabla \mathcal{F}_{ij}(\mathbf{x})^T \nabla \mathcal{F}_{ij}(\mathbf{x}) = \|\nabla \mathcal{F}_{ij}(\mathbf{x})\|^2 > 0$), in contrary to the assumption that $\mathbf{x}$ is a local maximum point. If $\nabla \mathcal{F}_{ij}(\mathbf{x}) \equiv \mathbf{0}_{nK}$, then from item (i) it follows that $\mathcal{F}_{ij}(\mathbf{x}) = 0$. From Lemma 1(ii) it follows that any neighborhood containing $\mathbf{x}$ attains a function value that is strictly greater than 0, which again contradicts the assumption that $\mathbf{x}$ is a local maximum point.

(iii). Recall that if $\mathcal{F}$ is non-differentiable at $\mathbf{x} \in \mathbb{R}^{nK}$, then there exists at least one pair $(i,j) \in \mathcal{E}$ such that $\mathbf{x}_i = \mathbf{x}_j$, which means that $\mathbf{x}$ is also a non-differentiable point of $\mathcal{F}_{ij}$. The result now follows from item (ii). $\square$

Based on this result, we would like to provide a few more direct consequences regarding the extremum points of $\mathcal{F}$.

*Remark* 1. (i) All local minimum points of $\mathcal{F}_{ij}$, for any $(i,j) \in \mathcal{E}$, are necessarily global. Indeed, let $\mathbf{x} \in \mathbb{R}^{nK}$ be a local minimum point. From Lemma 2(ii) it follows that $\mathbf{x}$ must be a differentiable point of $\mathcal{F}_{ij}$. In particular, the gradient $\nabla \mathcal{F}_{ij}(\mathbf{x})$ exists at any local minimum of $\mathcal{F}_{ij}$. If $\nabla \mathcal{F}_{ij}(\mathbf{x}) \neq \mathbf{0}_{nK}$, then surely $-\nabla \mathcal{F}_{ij}(\mathbf{x})$ is a descent direction, which contradicts the fact that it is a local minimum. If $\nabla \mathcal{F}_{ij}(\mathbf{x}) \equiv \mathbf{0}_{nK}$, then it is a global minimum point (see Lemma 2(i)).

(ii) The function $\mathcal{F}_{ij}$ has no global maximum points since it is unbounded from above. Indeed, for any point $\tilde{\mathbf{x}} \in \mathbb{R}^{nK}$ such that $\tilde{\mathbf{x}}_i = \mathbf{1}_n$ (where $\mathbf{1}_n \in \mathbb{R}^n$ denotes the vector of all ones) and $\tilde{\mathbf{x}}_l = \mathbf{0}_n$ for all $l \neq i$ it holds that $\mathcal{F}_{ij}(\alpha\tilde{\mathbf{x}}) = (\alpha\sqrt{n} - d_{ij})^2 \to \infty$ as $\alpha \to \infty$. Therefore, all local maximum points are necessarily non-global.

(iii) The function $\mathcal{F}$ of Problem (3) (equivalently, Problem (5)) has no global maximum points since it is unbounded from above. Indeed, $\mathcal{F}$ is the sum of the non-negative functions $\mathcal{F}_{ij}$, for all $(i,j) \in \mathcal{E}$. Then, following the same arguments of item (ii) yields that all local maximum points of $\mathcal{F}$ are necessarily non-global.

Now, we are ready to state and prove the main result of this section. To this end, for any $\mathbf{x} \in \mathbb{R}^{nK}$ we denote by $\mathcal{E}_{\min}(\mathbf{x}) \subseteq \mathcal{E}$ the subset of all pairs $(i,j) \in \mathcal{E}$ for which $\mathbf{x}$ is a local minimum point of $\mathcal{F}_{ij}$. Meaning, if $(i,j) \in \mathcal{E}_{\min}(\mathbf{x})$ then $\mathbf{x}$ is a local minimum point of the function $\mathcal{F}_{ij}$. Similarly, we define the subset $\mathcal{E}_{\max}(\mathbf{x})$. In addition, for any $\mathbf{x} \in \mathbb{R}^{nK}$, we define the subset $\mathcal{E}_{\text{nm}}(\mathbf{x}) \subseteq \mathcal{E}$ as the subset of all pairs $(i,j) \in \mathcal{E}$ for which $\mathbf{x}$ is not a local minimum nor a local maximum point of $\mathcal{F}_{ij}$. Meaning, if $(i,j) \in \mathcal{E}_{\text{nm}}(\mathbf{x})$ then $\mathbf{x}$ is not an extremum point of the function $\mathcal{F}_{ij}$. Clearly, it holds that $\mathcal{E} = \mathcal{E}_{\min}(\mathbf{x}) \cup \mathcal{E}_{\max}(\mathbf{x}) \cup \mathcal{E}_{\text{nm}}(\mathbf{x})$.

This union is also disjoint. To see this, if $(i,j) \in \mathcal{E}_{\min}(\mathbf{x})$, then $\mathbf{x}$ is a local minimum point of $\mathcal{F}_{ij}$. From Remark 1(i) we know that $\mathbf{x}$ must be a global minimum point, and from Lemma 2(i) we get that $\mathbf{x}$ must be a differentiable point of $\mathcal{F}_{ij}$ (since all stationary points are differentiable by their definition). In addition, if $(i,j) \in \mathcal{E}_{\max}(\mathbf{x})$ then $\mathbf{x}$ is a non-differentiable point of $\mathcal{F}_{ij}$ (see Lemma 2(ii)). Therefore, the three subsets $\mathcal{E}_{\min}(\mathbf{x})$, $\mathcal{E}_{\max}(\mathbf{x})$ and $\mathcal{E}_{\text{nm}}(\mathbf{x})$ are disjoint.

**Theorem 1.** *Let $\mathbf{x} \in \mathbb{R}^{nK}$ be a non-differentiable point of $\mathcal{F}$. Let $\mathbf{d} \in \mathbb{R}^{nK}$ be such that $\mathbf{d}_i \neq \mathbf{d}_j$ for all $(i,j) \in \mathcal{E}_{\max}(\mathbf{x})$. Then, either $\mathbf{d}$ or $-\mathbf{d}$ is a descent direction of $\mathcal{F}$ at $\mathbf{x}$.*

*Proof.* Since $\mathbf{x}$ is a non-differentiable point of $\mathcal{F}$, it follows from Lemma 2(iii) that $\mathcal{E}_{\max}(\mathbf{x}) \neq \emptyset$. From Lemma 1(i) we know that for any $\mathbf{d} \in \mathbb{R}^{nK}$ such that $\mathbf{d}_i \neq \mathbf{d}_j$ for all $(i,j) \in \mathcal{E}_{\max}(\mathbf{x})$ it holds that $\mathcal{F}'_{ij}(\mathbf{x};\mathbf{d}) < 0$ (and such $\mathbf{d}$ surely exists since the set $\mathcal{E}_{\max}(\mathbf{x})$ is finite). Meaning, such $\mathbf{d}$ is a descent direction of $\mathcal{F}_{ij}$ at $\mathbf{x}$ for all $(i,j) \in \mathcal{E}_{\max}(\mathbf{x})$.

If $\mathbf{d}$ is also a descent direction of $\mathcal{F}$ at $\mathbf{x}$ then we are done. Therefore, let us assume that $\mathbf{d}$ is a non-descent direction of $\mathcal{F}$ at $\mathbf{x}$ (that is, $\mathcal{F}'(\mathbf{x};\mathbf{d}) \geq 0$), and we will prove that $-\mathbf{d}$ is indeed a descent direction of $\mathcal{F}$ at $\mathbf{x}$.

Since $\mathcal{F}'(\mathbf{x};\mathbf{d}) = \sum_{(i,j)\in\mathcal{E}} \mathcal{F}'_{ij}(\mathbf{x};\mathbf{d})$, it holds that

$$0 \leq \mathcal{F}'(\mathbf{x};\mathbf{d}) = \sum_{(i,j)\in\mathcal{E}_{\max}(\mathbf{x})} \mathcal{F}'_{ij}(\mathbf{x};\mathbf{d}) + \sum_{(i,j)\in\mathcal{E}_{\min}(\mathbf{x})} \mathcal{F}'_{ij}(\mathbf{x};\mathbf{d}) + \sum_{(i,j)\in\mathcal{E}_{\mathrm{nm}}(\mathbf{x})} \mathcal{F}'_{ij}(\mathbf{x};\mathbf{d}), \tag{9}$$

where we used the fact that the three sub-sets above are disjoint. From Lemma 2(i) we know that $\nabla\mathcal{F}_{ij}(\mathbf{x}) = \mathbf{0}_{nK}$ for any $(i,j) \in \mathcal{E}_{\min}(\mathbf{x})$, and therefore

$$0 = \pm\nabla\mathcal{F}_{ij}(\mathbf{x})^T \mathbf{d} = \mathcal{F}'_{ij}(\mathbf{x};\pm\mathbf{d}), \quad \forall (i,j) \in \mathcal{E}_{\min}(\mathbf{x}). \tag{10}$$

Plugging (10) into (9) yields

$$0 \leq \mathcal{F}'(\mathbf{x};\mathbf{d}) = \sum_{(i,j)\in\mathcal{E}_{\max}(\mathbf{x})} \mathcal{F}'_{ij}(\mathbf{x};\mathbf{d}) + \sum_{(i,j)\in\mathcal{E}_{\mathrm{nm}}(\mathbf{x})} \mathcal{F}'_{ij}(\mathbf{x};\mathbf{d}). \tag{11}$$

Now, since $\mathcal{F}'_{ij}(\mathbf{x};\mathbf{d}) < 0$ for any $(i,j) \in \mathcal{E}_{\max}(\mathbf{x})$ (see Lemma 1(i)), it follows from (11) that

$$0 < \sum_{(i,j)\in\mathcal{E}_{\mathrm{nm}}(\mathbf{x})} \mathcal{F}'_{ij}(\mathbf{x};\mathbf{d}). \tag{12}$$

Recall that $\mathbf{x}$ is not an extremum point of $\mathcal{F}_{ij}$, for all $(i,j) \in \mathcal{E}_{\mathrm{nm}}(\mathbf{x})$. In particular, $\mathbf{x}$ is not a local maximum point of $\mathcal{F}_{ij}$, and from Lemma 2(ii) we get that $\mathbf{x}$ must be a differentiable point of $\mathcal{F}_{ij}$. This means that, $\mathbf{x}_i \neq \mathbf{x}_j$ for all $(i,j) \in \mathcal{E}_{\mathrm{nm}}(\mathbf{x})$. From the continuity of each function $\mathcal{F}_{ij}$, there exists an open set $U \subseteq \mathbb{R}^{nK}$ that contains $\mathbf{x}$, such that $\mathbf{y}_i \neq \mathbf{y}_j$ for all $\mathbf{y} \in U$. This means that $\mathcal{F}_{ij}$ is continuously differentiable over $U$, and therefore $\mathcal{F}'_{ij}(\mathbf{x};\mathbf{d}) = \nabla\mathcal{F}_{ij}(\mathbf{x})^T \mathbf{d}$ for all $(i,j) \in \mathcal{E}_{\mathrm{nm}}(\mathbf{x})$. Thus,

$$\sum_{(i,j)\in\mathcal{E}_{\mathrm{nm}}(\mathbf{x})} \mathcal{F}'_{ij}(\mathbf{x};-\mathbf{d}) = -\sum_{(i,j)\in\mathcal{E}_{\mathrm{nm}}(\mathbf{x})} \nabla\mathcal{F}_{ij}(\mathbf{x})^T \mathbf{d} = -\sum_{(i,j)\in\mathcal{E}_{\mathrm{nm}}(\mathbf{x})} \mathcal{F}'_{ij}(\mathbf{x};\mathbf{d}) < 0, \tag{13}$$

where the last inequality follows from (12).

Last, recall that we picked $\mathbf{d}$ such that $-\mathbf{d}_i \neq -\mathbf{d}_j$ for any $(i,j) \in \mathcal{E}_{\max}(\mathbf{x})$. Then, from Lemma 1(i) we get

$$0 > \sum_{(i,j)\in\mathcal{E}_{\max}(\mathbf{x})} \mathcal{F}'_{ij}(\mathbf{x};-\mathbf{d}). \tag{14}$$

Summing (13) and (14), we derive from (10) that

$$0 > \sum_{(i,j)\in\mathcal{E}_{\max}(\mathbf{x})} \mathcal{F}'_{ij}(\mathbf{x};-\mathbf{d}) + \sum_{(i,j)\in\mathcal{E}_{\min}(\mathbf{x})} \mathcal{F}'_{ij}(\mathbf{x};-\mathbf{d}) + \sum_{(i,j)\in\mathcal{E}_{\mathrm{nm}}(\mathbf{x})} \mathcal{F}'_{ij}(\mathbf{x};-\mathbf{d}) = \mathcal{F}'(\mathbf{x};-\mathbf{d}),$$

which implies that $-\mathbf{d}$ is a descent direction of $\mathcal{F}$ at $\mathbf{x}$, as required. □

An immediate consequence of Theorem 1, is that any optimal solution of Problem (3) is necessarily a stationary point, i.e., a differentiable point with a gradient of all zeros. As mentioned in Section 1, this result is already known for DC programming problems [6], which as discussed above also applies to our case. However, the major motivation to develop our results is that Theorem 1 also gives an easy-to-find and *explicit* descent direction, that can be used to escape non-differentiable points, as we discuss next.

# 4 Escaping Non-Differentiable Saddle Points

As mentioned in Section 1, in this paper we aim at finding locally optimal solutions for Problem (3). Theorem 1 asserts that every non-differentiable point possesses an easy-to-find descent direction, hence such points cannot be optimal solutions for Problem (3). With this information in mind, one can evade any non-differentiable point encountered by an algorithm and reach a differentiable point with a lower function value by applying a simple backtracking procedure. This escape procedure, abbreviated as EP, is recorded in Procedure 1.

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**Procedure 1** Escape Procedure (EP)

---

1: **Initialization:** $\mathbf{x} \in \mathbb{R}^{nK}$ a non-differentiable point of $\mathcal{F}$ and $t = 1$.
2: Pick $\mathbf{d} \in \mathbb{R}^{nK}$ such that $\mathbf{d}_i \neq \mathbf{d}_j$ for all $(i,j) \in \mathcal{E}_{\max}(\mathbf{x})$.
3: *Double backtracking procedure:* **do in parallel**
  $\rightarrow$ **while** $\mathcal{F}(\mathbf{x}) \leq \mathcal{F}(\mathbf{x} + t\mathbf{d})$ or $\mathbf{x}_i + t\mathbf{d}_i = \mathbf{x}_j + t\mathbf{d}_j$ for some $(i,j) \in \mathcal{E}$ **then** set $t \coloneqq t/2$.
  $\rightarrow$ **while** $\mathcal{F}(\mathbf{x}) \leq \mathcal{F}(\mathbf{x} - t\mathbf{d})$ or $\mathbf{x}_i - t\mathbf{d}_i = \mathbf{x}_j - t\mathbf{d}_j$ for some $(i,j) \in \mathcal{E}$ **then** set $t \coloneqq t/2$.
4: Set the output as $\mathbf{z} = \mathbf{x} \pm t\mathbf{d}$ according to the first while loop that breaks.

---

We recall that *saddle points* are those points that satisfy the criticality condition, but also have a non-ascent direction.

*Remark* 2. The literature makes a distinction between *strict* saddle points and *non-strict* saddle points [41]. Strict saddle points are saddles that have a descent direction, while non-strict saddle points are saddles that have a non-ascent direction but are not local minimum points. It is worth noting that in the perspective of critical points, while it is possible to distinguish between strict saddles and local minima using first-order information, distinguishing between non-strict saddles and local minima requires *at least* second-order information, specifically the Hessian matrix, Therefore, we postpone this discussion to Section 5. Here, using our results above, we established that all non-differentiable points have a descent direction, implying that all non-differentiable saddles are strict saddles.

The importance of the procedure EP comes from the fact that incorporating this procedure can prevent any optimization algorithm from being trapped in non-differentiable points, which are all non-optimal solutions, and by that may lead to the desired convergence to differentiable points. This phenomena is very important in the case of Problem (3), since the Hessian matrix of the function $\mathcal{F}$ is continuous around differentiable points, and therefore one can utilize its eigenvalues to deduce whether the differentiable point at hand is a (local) minimum point or not. This topic will be further discussed in Section 5.

Now, we are ready to prove that the procedure EP indeed leads to a differentiable point with a lower function value.

**Proposition 1.** *Let $\mathbf{x} \in \mathbb{R}^{nK}$ be the non-differentiable input point of EP. Then, the output point $\mathbf{z} \in \mathbb{R}^{nK}$ is a differentiable point of $\mathcal{F}$ for which $\mathcal{F}(\mathbf{z}) < \mathcal{F}(\mathbf{x})$.*

*Proof.* In order to prove the result, we show that the while loop in step 3 of the procedure EP terminates after a finite number of iterations.

Let $\mathbf{d} \in \mathbb{R}^{nK}$ be a direction picked according to step 2 in of EP. That is, $\mathbf{d}_i \neq \mathbf{d}_j$ for all $(i,j) \in \mathcal{E}_{\max}(\mathbf{x})$. Note that such $\mathbf{d}$ surely exists as the set $\mathcal{E}_{\max}(\mathbf{x})$ is finite by its definition. Now, since $\mathbf{x}$ is a non-differentiable point, it follows from Theorem 1 that either $\mathbf{d}$ or $-\mathbf{d}$ is a descent direction of $\mathcal{F}$ at $\mathbf{x}$. We assume without the loss of generality that $\mathbf{d}$ is a descent direction. Therefore, there exists $\bar{\epsilon} > 0$ such that $\mathcal{F}(\mathbf{x}) > \mathcal{F}(\mathbf{x} + t\mathbf{d})$ for all $t \in (0, \bar{\epsilon}]$. Notice that if $\mathbf{x}_i = \mathbf{x}_j$

then $(i,j) \in \mathcal{E}_{\max}(\mathbf{x})$, hence $\mathbf{d}_i \neq \mathbf{d}_j$ and therefore $\mathbf{x}_i + t\mathbf{d}_i \neq \mathbf{x}_j + t\mathbf{d}_j$ for all $t \in (0, \bar{\epsilon}]$. If $\mathbf{x}_i \neq \mathbf{x}_j$ then we can set $\mathbf{d}_i = \mathbf{d}_j$ and therefore $\mathbf{x}_i + t\mathbf{d}_i \neq \mathbf{x}_j + t\mathbf{d}_j$ for all $t \in (0, \bar{\epsilon}]$. This means that $\mathbf{x}_i + t\mathbf{d}_i \neq \mathbf{x}_j + t\mathbf{d}_j$ for all $(i,j) \in \mathcal{E}$. Therefore, $\mathbf{z} \equiv \mathbf{x} + t\mathbf{d}$ is a differentiable point with a lower function value than $\mathbf{x}$, as required. □

Now, we would like to discuss a computational aspect of the procedure EP. The main computational effort in the procedure is the evaluation of the function $\mathcal{F}$. This effort depends on the computational setting of the function $\mathcal{F}$. To simplify the discussion, we consider Problem (3) using the terminology the SNL problem (see Problem (5) and Section 2). In this case, evaluating the $\mathcal{F}$ requires gathering information from all sensors in the network. As a result, this procedure can only be implemented in centralized network architectures, that is, networks with a central computing unit responsible for data collection from all sensors. However, in certain practical situations the architecture of the network has no centralized computing unit, like in distributed architectures. Instead, each sensor (or cluster of some sensors) performs its own calculations using information available locally, and this information is collected from neighboring sensors (or clusters) through communication (for further information about centralized and distributed network architectures in the context of the SNL problem, we refer the reader to [19]). This limitation has inspired us to also design a distributed version of the procedure EP, named EP-D. Due to the similarity of the arguments, we develop and state it in Appendix A.

## 4.1 Convergence to Stationary Points

Equipped with the procedure EP, or its distributed version EP-D, that escapes non-differentiable points of the function $\mathcal{F}$, one can incorporate it into any minimization algorithm that converges to critical points of $\mathcal{F}$, to possibly obtain a stationary point of $\mathcal{F}$. Recall that any optimal solution of Problem (3) is necessarily a stationary point (see Theorem 1). Therefore, by using the procedure EP we easily transform algorithms which are guaranteed to converge to critical points, into algorithms that converge to stationary points. This can be done by executing the following process:

1. Run an algorithm $\mathcal{A}$ with some starting point to obtain a critical point $\mathbf{x} \in \mathbb{R}^{nK}$ of $\mathcal{F}$.
2. Run EP (or EP-D) to obtain a differentible point $\mathbf{z} \in \mathbb{R}^{nK}$ with a lower function value.
3. Repeat the process with $\mathbf{z} \in \mathbb{R}^{nK}$ as the starting point of $\mathcal{A}$.

**Illustration of escaping a non-differentibale saddle point.** Here we give a simple one-dimensional numerical example that illustrates the convergence to a stationary point by utilizing the process described above. We show how a convergent algorithm escapes a non-differentiable saddle point and converges to a stationary point by using the procedure EP. We note that the description provided here is applicable to any dimension, but we choose to focus on the one-dimensional setting for clarity. This decision enables us to plot the function, facilitating a clear visualization of the saddle point, which is more challenging to illustrate in higher dimensions.

Using again the terminology of SNL, we consider a one-dimensional network with three sensors, where sensor #3 is an anchor, that is, has a known location. In this example, $x_3 = 2 \in \mathbb{R}$. The unknown locations of sensors #1 and #2 are denoted by $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$, respectively. In addition, we take $d_{12} = d_{13} = 1$. Under this setting, the objective function of Problem (5) takes the form

$$\mathcal{F}(x_1, x_2) = (|x_1 - x_2| - 1)^2 + (|x_1 - 2| - 1)^2 .$$

Easy calculations show that the set of critical points of $\mathcal{F}$ is classified as follows:

- $(2,2)$ is a non-differentiable local maximum point with value 2.
- $(1,1)$, $(3,3)$, $(2,1)$ and $(2,3)$ are non-differentiable saddle points with value 1.
- $(1,0)$, $(1,2)$, $(3,2)$ and $(3,4)$ are differentiable global minimum points with value 0. By definition, these points are stationary points.

We apply iterations of the classical Sub-Gradient (SG) method to minimize $\mathcal{F}$, and we will show that this method (given a specific starting point) converges to a non-differentiable saddle point. Then, we will invoke procedure EP to establish convergence of SG to a stationary point. This process is illustrated below in Figure 1, and is discussed now.

Notice that the directional derivative of $\mathcal{F}$ at any point satisfying $x_1 = x_2$ and $x_1 \neq 2$, at the direction $\mathbf{d} \in \left\{\mathbf{d} \in \mathbb{R}^2 \colon d_1 = d_2\right\}$ is given by

$$\begin{aligned} g\left(x_1\right) \equiv \mathcal{F}'\left(\left(x_1, x_1\right); \mathbf{d}\right) &= \lim_{\epsilon \to 0^+} \frac{\mathcal{F}\left(x_1 + \epsilon d_1, x_1 + \epsilon d_1\right) - \mathcal{F}\left(x_1, x_1\right)}{\epsilon} \\ &= 2d_1 \cdot \operatorname{sign}\left(x_1 - 2\right) \cdot \left(\left|x_1 - 2\right| - 1\right). \end{aligned}$$

Hence, for a starting point $\left(x_1^0, x_2^0\right)$ satisfying $x_1^0 = x_2^0$ and $x_1 \neq 2$, SG update steps take the form

$$\left(x_1^{k+1}, x_2^{k+1}\right) = \left(x_1^k, x_2^k\right) - t^k \cdot \left(g\left(x_1^k\right), g\left(x_1^k\right)\right),$$

for any iteration $k \geq 0$ and for some step-sizes $t^k > 0$, assuming that $x_1^k \neq 2$ for all $k \geq 0$.

Initializing the above update steps with $\left(x_1^0, x_2^0\right) = (-1,-1)$, $t^k \equiv 1/2$ and $\mathbf{d} = (1,1)$ for any $k \geq 0$, it can be proved that SG converges to the critical point $(1,1)$, which is a non-differentiable saddle. We now invoke procedure EP, and obtain a random differentiable point $\left(\bar{x}_1, \bar{x}_2\right)$ satisfying $\mathcal{F}(1,1) > \mathcal{F}\left(\bar{x}_1, \bar{x}_2\right)$. Restarting SG with $\left(\bar{x}_1, \bar{x}_2\right)$ for additional iterations, yields convergence to one of the stationary global minimum points.

Figure 1: The trajectory of SG is depicted on the contour plots of the function $\mathcal{F}$, where the black points represent iterations. The method starts from the point $(-1,-1)$ and after 100 iterations, it converges to the non-differentiable saddle point $(1,1)$. By invoking EP and performing an additional 100 iterations of SG, the method eventually converges to the stationary point $(1,0)$.

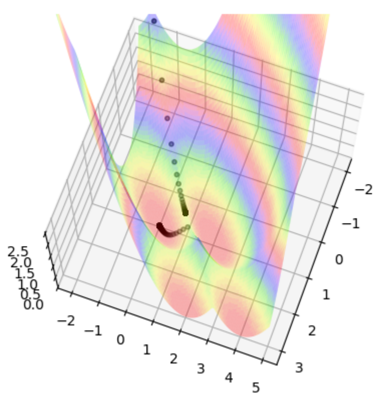
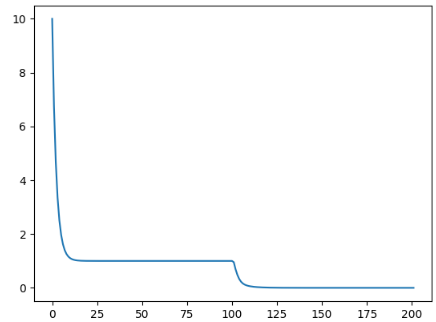



Figure 2 illustrates the values of $\mathcal{F}$ plotted against the iterations $k \geq 0$ of SG. It is evident that the method converged to the value 1 after 100 iterations, which corresponds to the non-differentiable saddle point $(1,1)$. Subsequently, after invoking EP and resuming SG for an additional 100 iterations, the function value decreased, ultimately converging to the optimal value of 0.

Figure 2: Values of $\mathcal{F}$ ($y$-axis) plotted against the iterations ($x$-axis) generated by SG with EP.




# 5 Classifying Stationary Points

In this section, our main goal is to identify the local minimum points of Problem (3). To achieve this goal, we utilized the first-order optimality condition, which asserts that any local minimum point must be a critical point. As proved in Section 3, we established that any non-differentiable critical point is not a local minimum point. Consequently, our focus shifts to the differentiable critical points, namely stationary points. In Section 4, we introduced the procedure EP designed to navigate away from non-differentiable critical points. This procedure effectively enables us to concentrate solely on stationary points.

Now that we have narrowed our focus to the stationary points, we can proceed to distinguish the local minimum points from the stationary points. It is worth noting that for the Hessian matrix of a function to be well-defined at a point, that point must be differentiable. Considering that the function $\mathcal{F}$ is smooth at its stationary points, we can leverage second-order information, particularly the eigenvalues of the Hessian matrix at these points, to further classify these stationary points.

We denote the Hessian matrix of $\mathcal{F}$ at a differentiable point $\mathbf{x} \in \mathbb{R}^{nK}$ as $\nabla^2 \mathcal{F}(\mathbf{x}) \in \mathbb{R}^{nK \times nK}$. We recall that a necessary second-order optimality condition applied to a stationary point $\mathbf{x} \in \mathbb{R}^{nK}$, states that if $\lambda_{\min}\left(\nabla^2 \mathcal{F}(\mathbf{x})\right) < 0$ (where $\lambda_{\min}$ denotes the minimal eigenvalue), then $\mathbf{x}$ is not a local minimum point [4] (in particular, it is not an optimal solution). Otherwise, if $\lambda_{\min}\left(\nabla^2 \mathcal{F}(\mathbf{x})\right) \geq 0$, one cannot conclusively determine whether this point is a local minimum point using second-order information alone, and higher-order derivatives must be considered.

This implies that upon obtaining a stationary point of $\mathcal{F}$, such as by employing our suggested procedure EP, one can compute the minimal eigenvalue of its Hessian matrix (which is guaranteed to exist as this point as discussed above) to further classify this point. At this juncture, as we discussed above in the context of procedure EP, also when discussing the computation of the eigenvalues of the Hessian $\nabla^2 \mathcal{F}(\mathbf{x})$, it is essential to differentiate between centralized and distributed computational settings. In the centralized setup, computationally intensive calculations, such as direct eigenvalue computations, are feasible. However, even in centralized scenarios, accurately computing the minimal eigenvalue can sometimes be challenging, especially for large matrices. In distributed architectures, direct computation of eigenvalues becomes impractical due to the limited information available at each node.

To overcome these challenges, we propose an alternative approach that approximates the eigenvalues by obtaining lower and upper bounds on the minimal eigenvalue in a distributed fashion. By calculating these bounds based on local information, we can still make meaningful assessments regarding the optimality of stationary points. This approach enables us to evaluate the optimality conditions for stationary points in distributed settings while circumventing the need for a central computational unit and the computationally demanding eigenvalue calculations associated with it.

The goal of the rest of this section is to derive bounds on the minimal eigenvalue that can be calculated in centralized and distributed computational settings. These bounds will enable us to establish a stricter necessary condition (see Section 5.2) for a stationary point to qualify as a (local) minimum point in the distributed setting. Before developing this necessary condition, we calculate the eigenvalues of the Hessian matrix of the functions $\mathcal{F}_{ij}$, $(i,j) \in \mathcal{E}$, as defined in (6).

## 5.1 Eigenvalues of the Hessian of $\mathcal{F}_{ij}$

We denote by $\mathbf{e}_i \in \mathbb{R}^K$ the $i$-th unit vector, where $K$ is the number of nodes. That is, $\mathbf{e}_i$ is the vector of all zeros, except for the $i$-th coordinate which is 1. For any $(i,j) \in \mathcal{E}$, $i < j$, we denote by $\mathbf{A}_{ij} \in \mathbb{R}^{n \times nK}$ the matrix

$$\mathbf{A}_{ij} \equiv (\mathbf{e}_i - \mathbf{e}_j)^T \otimes \mathbf{I}_n = \begin{bmatrix} \mathbf{0}_{n\times n(i-1)} & \mathbf{I}_n & \mathbf{0}_{n\times n(j-i)} & -\mathbf{I}_n & \mathbf{0}_{n\times n(K-j)} \end{bmatrix}, \tag{15}$$

where $\otimes$ denotes the Kronecker matrix product, $\mathbf{I}_n$ is the $n \times n$ identity matrix, and $\mathbf{0}_{p\times q}$ is the $p \times q$ zero matrix. Under this notation we can rewrite (recall (6))

$$\mathcal{F}_{ij}(\mathbf{x}) \equiv (\|\mathbf{x}_i - \mathbf{x}_j\| - d_{ij})^2 = (\|\mathbf{A}_{ij}\mathbf{x}\| - d_{ij})^2, \quad \forall (i,j) \in \mathcal{E}.$$

Now, recall that for any differentiable point $\mathbf{x} \in \mathbb{R}^{nK}$ of $\mathcal{F}_{ij}$ (i.e., a point satisfying $\mathbf{x}_i \neq \mathbf{x}_j$ and hence $\|\mathbf{A}_{ij}\mathbf{x}\| > 0$), the Hessian matrix of $\mathcal{F}_{ij}$ at $\mathbf{x}$ exists and is continuous. To compute the Hessian matrix $\nabla^2 \mathcal{F}_{ij}(\mathbf{x})$, we use the following technical lemma that its proof simply follows by applying the chain rule for multi-variate functions, and is therefore skipped.

**Lemma 3.** *For any matrix $\mathbf{A} \in \mathbb{R}^{p\times q}$ and vector $\mathbf{x} \in \mathbb{R}^q$ satisfying $\|\mathbf{A}\mathbf{x}\| > 0$, it holds that*

*(i)* $\nabla(\|\mathbf{A}\mathbf{x}\|) = \frac{\mathbf{A}^T\mathbf{A}\mathbf{x}}{\|\mathbf{A}\mathbf{x}\|}$.

*(ii)* $\nabla^2(\|\mathbf{A}\mathbf{x}\|) = \frac{\mathbf{A}^T\mathbf{A} - \nabla(\|\mathbf{A}\mathbf{x}\|)\nabla(\|\mathbf{A}\mathbf{x}\|)^T}{\|\mathbf{A}\mathbf{x}\|}$.

Following Lemma 3, we immediately obtain explicit formulas for the gradient and Hessian matrix of the functions $\mathcal{F}_{ij}$. To this end, for any $(i,j) \in \mathcal{E}$, we define the scalars $\epsilon_{ij}(\mathbf{x}) \equiv \|\mathbf{A}_{ij}\mathbf{x}\| - d_{ij}$ and matrices

$$\mathbf{X}_{ij} = (\mathbf{x}_i - \mathbf{x}_j)(\mathbf{x}_i - \mathbf{x}_j)^T \in \mathbb{R}^{n\times n}. \tag{16}$$

**Corollary 1.** *Let $(i,j) \in \mathcal{E}$ and let $\mathbf{x} \in \mathbb{R}^{nK}$ such that $\mathbf{x}_i \neq \mathbf{x}_j$. Then,*

*(i)* $\nabla \mathcal{F}_{ij}(\mathbf{x}) = \frac{2}{\|\mathbf{A}_{ij}\mathbf{x}\|}\epsilon_{ij}(\mathbf{x})\mathbf{A}_{ij}^T\mathbf{A}_{ij}\mathbf{x}$.

*(ii)* $\nabla^2 \mathcal{F}_{ij}(\mathbf{x}) = 2\mathbf{A}_{ij}^T\left(\frac{\epsilon_{ij}(\mathbf{x})}{\|\mathbf{A}_{ij}\mathbf{x}\|}\mathbf{I}_n + \frac{d_{ij}}{\|\mathbf{A}_{ij}\mathbf{x}\|^3}\mathbf{X}_{ij}\right)\mathbf{A}_{ij}$.

To write $\nabla^2 \mathcal{F}_{ij}(\mathbf{x})$ compactly, we define for any $(i,j) \in \mathcal{E}$ and $\mathbf{x} \in \mathbb{R}^{nK}$ satisfying $\mathbf{x}_i \neq \mathbf{x}_j$, the symmetric matrices $\mathbf{B}_{ij}(\mathbf{x}), \mathbf{C}_{ij}(\mathbf{x}), \mathbf{G}_{ij}(\mathbf{x}) \in \mathbb{R}^{n\times n}$ as (recall the definition of $\mathbf{X}_{ij}$ in (16))

$$\mathbf{B}_{ij}(\mathbf{x}) \equiv \frac{\epsilon_{ij}(\mathbf{x})}{\|\mathbf{A}_{ij}\mathbf{x}\|}\mathbf{I}_n, \quad \mathbf{C}_{ij}(\mathbf{x}) \equiv \frac{d_{ij}}{\|\mathbf{A}_{ij}\mathbf{x}\|^3}\mathbf{X}_{ij} \quad \text{and} \quad \mathbf{G}_{ij}(\mathbf{x}) \equiv \mathbf{B}_{ij}(\mathbf{x}) + \mathbf{C}_{ij}(\mathbf{x}), \tag{17}$$

and we get that

$$\nabla^2 \mathcal{F}_{ij}(\mathbf{x}) = 2\mathbf{A}_{ij}^T\mathbf{G}_{ij}(\mathbf{x})\mathbf{A}_{ij}. \tag{18}$$

Using (7) it immediately follows that

$$\nabla^2 \mathcal{F}(\mathbf{x}) = 2 \sum_{(i,j)\in\mathcal{E}} \mathbf{A}_{ij}^T \mathbf{G}_{ij}(\mathbf{x}) \mathbf{A}_{ij}. \tag{19}$$

Simple calculations show that (19) is given explicitly as

$$\nabla^2 \mathcal{F}(\mathbf{x}) = 2 \begin{bmatrix} \sum_{j\in\mathcal{E}_1} \mathbf{G}_{1j}(\mathbf{x}) & -\bar{\mathbf{G}}_{12}(\mathbf{x}) & \cdots & -\bar{\mathbf{G}}_{1K}(\mathbf{x}) \\ -\bar{\mathbf{G}}_{12}(\mathbf{x}) & \sum_{j\in\mathcal{E}_2} \mathbf{G}_{2j}(\mathbf{x}) & \cdots & -\bar{\mathbf{G}}_{2K}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\mathbf{G}}_{1K}(\mathbf{x}) & -\bar{\mathbf{G}}_{2K}(\mathbf{x}) & \cdots & \sum_{j\in\mathcal{E}_K} \mathbf{G}_{Kj}(\mathbf{x}) \end{bmatrix} \in \mathbb{R}^{nK\times nK}, \tag{20}$$

where we set

$$\bar{\mathbf{G}}_{ij}(\mathbf{x}) \equiv \begin{cases} \mathbf{G}_{ij}(\mathbf{x}), & (i,j)\in\mathcal{E}, \\ \mathbf{0}_{n\times n}, & (i,j)\notin\mathcal{E}. \end{cases}$$

Now, recall that we are interested in calculating a lower bound on the minimal eigenvalue of the $\nabla^2\mathcal{F}(\mathbf{x})$ in a distributed fashion. Such lower bound can be obtained by calculating the eigenvalues of principal sub-matrices of $\nabla^2\mathcal{F}(\mathbf{x})$ (see exact definition and statement below in Theorem 2). Hence, following (20), we first calculate the eigenvalues of the matrices $\mathbf{G}_{ij}(\mathbf{x})$, $(i,j)\in\mathcal{E}$. To this end, we begin with finding the eigenvalues of the matrix $\mathbf{C}_{ij}(\mathbf{x})$ as defined in (17).

**Lemma 4.** *Let $(i,j)\in\mathcal{E}$ and $\mathbf{x}\in\mathbb{R}^{nK}$ satisfying $\mathbf{x}_i \neq \mathbf{x}_j$. Then,*

*(i)* $\operatorname{rank}(\mathbf{X}_{ij}) = 1$ *and* $\lambda_{\max}(\mathbf{X}_{ij}) = \|\mathbf{x}_i - \mathbf{x}_j\|^2 = \|\mathbf{A}_{ij}\mathbf{x}\|^2$.

*(ii) The eigenvalues of the matrix $\mathbf{C}_{ij}(\mathbf{x})$ are $d_{ij}/\|\mathbf{A}_{ij}\mathbf{x}\|$ with multiplicity 1, and 0 with multiplicity $n-1$.*

*Proof.* We first prove item (i). Notice that for any two vectors $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$ we have $\mathbf{u}\mathbf{u}^T\mathbf{v} = (\mathbf{u}^T\mathbf{v})\,\mathbf{u}$, which means that the matrix $\mathbf{u}\mathbf{u}^T$ maps any vector $\mathbf{v}$ to a vector in the space $\operatorname{span}(\mathbf{u})$. If $\mathbf{u}\neq\mathbf{0}_n$, then $\operatorname{rank}(\mathbf{u}\mathbf{u}^T) = \dim(\operatorname{image}(\mathbf{u}\mathbf{u}^T)) = 1$. This means that all eigenvalues of $\mathbf{u}\mathbf{u}^T$ are 0 except one of them. Moreover, taking $\mathbf{v}=\mathbf{u}$ we see that $\mathbf{u}$ is an eigenvector of $\mathbf{u}\mathbf{u}^T$ corresponding to $\lambda_{\max}(\mathbf{u}\mathbf{u}^T) = \mathbf{u}^T\mathbf{u} = \|\mathbf{u}\|^2$. Hence, item (i) now follows by taking $\mathbf{u} = \mathbf{x}_i - \mathbf{x}_j$ and recalling that $\mathbf{X}_{ij} = (\mathbf{x}_i - \mathbf{x}_j)(\mathbf{x}_i - \mathbf{x}_j)^T$.

Now we prove item (ii). First, from item (i) it follows that $\operatorname{rank}(\mathbf{C}_{ij}(\mathbf{x})) = 1$, which implies that 0 is an eigenvalue of $\mathbf{C}_{ij}(\mathbf{x})$ with multiplicity $n-1$. Last, since the only non-zero eigenvalue of $\mathbf{X}_{ij}$ is $\|\mathbf{A}_{ij}\mathbf{x}\|^2$, it follows that $d_{ij}/\|\mathbf{A}_{ij}\mathbf{x}\|$ is an eigenvalue of $\mathbf{C}_{ij}(\mathbf{x})$ with multiplicity 1. $\square$

Now, we are ready to explicitly find the eigenvalues of the matrix $\mathbf{G}_{ij}(\mathbf{x})$, a result that is formalized in the next lemma.

**Lemma 5.** *Let $(i,j)\in\mathcal{E}$ and $\mathbf{x}\in\mathbb{R}^{nK}$ satisfying $\mathbf{x}_i\neq\mathbf{x}_j$. Then, the eigenvalues of $\mathbf{G}_{ij}(\mathbf{x})$ are $\lambda_{\max}(\mathbf{G}_{ij}(\mathbf{x})) = 1$ with multiplicity 1, and $\lambda_{\min}(\mathbf{G}_{ij}(\mathbf{x})) = \epsilon_{ij}(\mathbf{x})/\|\mathbf{A}_{ij}\mathbf{x}\|$ with multiplicity $n-1$.*

*Proof.* First, we denote by $\mathbf{D}_{\mathbf{C}_{ij}}(\mathbf{x})$ the diagonal matrix containing the eigenvalues of $\mathbf{C}_{ij}(\mathbf{x})$. Then, there exists an orthogonal matrix $\mathbf{U}$ such that

$$\mathbf{B}_{ij}(\mathbf{x}) + \mathbf{C}_{ij}(\mathbf{x}) = \mathbf{B}_{ij}(\mathbf{x}) + \mathbf{U}^T\mathbf{D}_{\mathbf{C}_{ij}}(\mathbf{x})\,\mathbf{U} = \mathbf{U}^T\left(\mathbf{B}_{ij}(\mathbf{x}) + \mathbf{D}_{\mathbf{C}_{ij}}(\mathbf{x})\right)\mathbf{U},$$

where we used the fact that $\mathbf{B}_{ij}(\mathbf{x})$ is a scalar multiplication of the identity matrix. Since the matrix $\mathbf{B}_{ij}(\mathbf{x}) + \mathbf{D}_{\mathbf{C}_{ij}}(\mathbf{x})$ is diagonal, it follows that the eigenvalues of $\mathbf{G}_{ij}(\mathbf{x}) = \mathbf{B}_{ij}(\mathbf{x}) + \mathbf{C}_{ij}(\mathbf{x})$

are exactly the sum of eigenvalues of $\mathbf{B}_{ij}(\mathbf{x})$ and $\mathbf{C}_{ij}(\mathbf{x})$. Since the eigenvalues of $\mathbf{B}_{ij}(\mathbf{x})$ are $\epsilon_{ij}(\mathbf{x})/\|\mathbf{A}_{ij}\mathbf{x}\|$ with multiplicity $n$, and since $(\epsilon_{ij}(\mathbf{x})+d_{ij})/\|\mathbf{A}_{ij}\mathbf{x}\|=1$, the required result now follows from Lemma 4(ii).

Finally, we have

$$\frac{\epsilon_{ij}(\mathbf{x})}{\|\mathbf{A}_{ij}\mathbf{x}\|}=1-\frac{d_{ij}}{\|\mathbf{A}_{ij}\mathbf{x}\|}<1,$$

and therefore 1 is indeed the maximal eigenvalue of $\mathbf{G}_{ij}(\mathbf{x})$. □

We conclude this part with a full characterization of eigenvalues of the Hessian $\nabla^2\mathcal{F}_{ij}(\mathbf{x})$.

**Lemma 6.** *Let $(i,j)\in\mathcal{E}$ and $\mathbf{x}\in\mathbb{R}^{nK}$ such that $\mathbf{x}_i\neq\mathbf{x}_j$. Then, $\lambda\neq 0$ is an eigenvalue of $\nabla^2\mathcal{F}_{ij}(\mathbf{x})$ if and only if $\lambda/4$ is an eigenvalue of $\mathbf{G}_{ij}(\mathbf{x})$.*

*Proof.* Let $\mathbf{0}_{nK}\neq\mathbf{y}\in\mathbb{R}^{nK}$ be an eigenvector of $\nabla^2\mathcal{F}_{ij}(\mathbf{x})$ corresponding to the eigenvalue $\lambda\neq 0$. Hence, $\nabla^2\mathcal{F}_{ij}(\mathbf{x})\mathbf{y}=\lambda\mathbf{y}$, and by writing in an explicit form using (18) we get

$$\begin{cases}2\mathbf{G}_{ij}(\mathbf{x})\mathbf{y}_i-2\mathbf{G}_{ij}(\mathbf{x})\mathbf{y}_j=\lambda\mathbf{y}_i,\\ -2\mathbf{G}_{ij}(\mathbf{x})\mathbf{y}_i+2\mathbf{G}_{ij}(\mathbf{x})\mathbf{y}_j=\lambda\mathbf{y}_j.\end{cases}\tag{21}$$

Summing (21) we get $\frac{\lambda}{2}(\mathbf{y}_i+\mathbf{y}_j)=0$. Since $\lambda\neq 0$ then $\mathbf{y}_i=-\mathbf{y}_j$. If $\mathbf{y}_i=\mathbf{0}_n$ then it follows that $\mathbf{y}=\mathbf{0}_{nK}$ which is a contradiction. Plugging $\mathbf{y}_i=-\mathbf{y}_j$ in the first equation of (21) we get $\mathbf{G}_{ij}(\mathbf{x})\mathbf{y}_i=\frac{\lambda}{4}\mathbf{y}_i$, which implies that $\lambda/4$ is an eigenvalue of $\mathbf{G}_{ij}(\mathbf{x})$.

Conversely, assume that $\lambda/4$ is an eigenvalue of $\mathbf{G}_{ij}(\mathbf{x})$ with eigenvector $\mathbf{0}_n\neq\tilde{\mathbf{z}}\in\mathbb{R}^n$. Let $\mathbf{0}_{nK}\neq\mathbf{z}\in\mathbb{R}^{nK}$ such that $\mathbf{z}_i=-\mathbf{z}_j=\tilde{\mathbf{z}}$. Now, it immediately follows from the LHS of (21) that $\nabla^2\mathcal{F}_{ij}(\mathbf{x})\mathbf{z}=\lambda\mathbf{z}$, and therefore $\lambda$ is an eigenvalue of $\nabla^2\mathcal{F}_{ij}(\mathbf{x})$, as required. □

The following result is an immediate consequence.

**Corollary 2.** *Let $(i,j)\in\mathcal{E}$ and $\mathbf{x}\in\mathbb{R}^{nK}$ such that $\mathbf{x}_i\neq\mathbf{x}_j$. Then,*

*(i) The eigenvalues of $\nabla^2\mathcal{F}_{ij}(\mathbf{x})$ are $\lambda_{\max}\left(\nabla^2\mathcal{F}_{ij}(\mathbf{x})\right)=4$ with multiplicity $1$, $0$ with multiplicity $n(K-1)$, and $4(\|\mathbf{x}_i-\mathbf{x}_j\|-d_{ij})/\|\mathbf{x}_i-\mathbf{x}_j\|$ with multiplicity $n-1$.*

*(ii) If $\|\mathbf{x}_i-\mathbf{x}_j\|\geq d_{ij}$ then $\nabla^2\mathcal{F}_{ij}(\mathbf{x})$ is positive semi-definite, and otherwise it is indefinite.*

## 5.2 Necessary Condition for a Locally Optimal Solution

We recall that given a stationary point $\mathbf{x}\in\mathbb{R}^{nK}$ of $\mathcal{F}$, if the Hessian $\nabla^2\mathcal{F}(\mathbf{x})$ has a negative eigenvalue, then $\mathbf{x}$ is *not* a minimum point. Therefore, in this sub-section, we find lower and upper bounds on the minimal eigenvalue of $\nabla^2\mathcal{F}(\mathbf{x})$ that can be calculated in a distributed fashion. This is accomplished using the Eigenvalue Interlacing theorem and Weyl's theorem (see, for example, [20]), which are also stated below. First, we recall that for any square matrix $\mathbf{A}\in\mathbb{R}^{q\times q}$, then a square matrix $\mathbf{B}\in\mathbb{R}^{p\times p}$ for some $p<q$ is called a *principal sub-matrix* of $\mathbf{A}$, if there exists an orthogonal matrix $\mathbf{P}\in\mathbb{R}^{q\times p}$ such that $\mathbf{P}^T\mathbf{A}\mathbf{P}=\mathbf{B}$. In addition, in this paper, we index the eigenvalues of a symmetric matrix $\mathbf{A}\in\mathbb{R}^{q\times q}$ in a non-decreasing order, i.e.,

$$\lambda_{\max}(\mathbf{A})\equiv\lambda_q(\mathbf{A})\geq\ldots\geq\lambda_2(\mathbf{A})\geq\lambda_1(\mathbf{A})\equiv\lambda_{\min}(\mathbf{A}).$$

**Theorem 2** (Eigenvalue Interlacing theorem)**.** *Let $\mathbf{A}\in\mathbb{R}^{q\times q}$ be a symmetric matrix. Let $\mathbf{B}\in\mathbb{R}^{p\times p}$ for some $p<q$ be a principal sub-matrix of $\mathbf{A}$. Then, it holds that*

$$\lambda_s(\mathbf{A})\leq\lambda_s(\mathbf{B})\leq\lambda_{s+q-p}(\mathbf{A}),\quad\forall s=1,2,\ldots,p.$$

**Theorem 3** (Weyl's theorem)**.** *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{q \times q}$ *be two symmetric matrices. Then, for any* $p = 1, 2, \ldots, q$ *it holds that*

$$\lambda_p (\mathbf{A} + \mathbf{B}) \leq \lambda_{p+s} (\mathbf{A}) + \lambda_{q-s} (\mathbf{B}), \quad \forall s = 0, 1, \ldots, q - p.$$

Now, we are ready to provide the main result of this section, which is an explicit necessary fully-distributed condition for a stationary point of $\mathcal{F}$ to be a minimum point.

**Theorem 4.** *Let* $\mathbf{x} \in \mathbb{R}^{nK}$ *be a stationary point of* $\mathcal{F}$*. If* $\mathbf{x}$ *is a local minimum point of* $\mathcal{F}$*, then* $|\mathcal{E}_i| \geq d_{ij} / \|\mathbf{x}_i - \mathbf{x}_j\|$ *for any* $(i, j) \in \mathcal{E}$*.*

*Proof.* We will prove that if there exists some $(i, j) \in \mathcal{E}$ such that $|\mathcal{E}_i| - d_{ij} / \|\mathbf{x}_i - \mathbf{x}_j\| < 0$, then $\mathbf{x}$ is not a minimum point of $\mathcal{F}$. More precisely, we will show that $\lambda_{\min} \left(\nabla^2 \mathcal{F} (\mathbf{x})\right) < 0$.

Since $\mathbf{x}$ is a stationary point of $\mathcal{F}$, then in particular $\mathcal{F}$ is smooth at $\mathbf{x}$ and $\nabla^2 \mathcal{F} (\mathbf{x})$ exists. Plugging $\mathbf{A} = \nabla^2 \mathcal{F} (\mathbf{x})$, $\mathbf{B} = 2 \sum_{j \in \mathcal{E}_i} \mathbf{G}_{ij} (\mathbf{x})$, $q = nK$, $p = n$ and $s = 1$ in Theorem 2, we obtain

$$\lambda_{\min} \left(\nabla^2 \mathcal{F} (\mathbf{x})\right) \leq 2 \lambda_{\min} \left( \sum_{j \in \mathcal{E}_i} \mathbf{G}_{ij} (\mathbf{x}) \right). \tag{22}$$

Now, plugging $\mathbf{A} = \mathbf{G}_{ij} (\mathbf{x})$, $\mathbf{B} = \sum_{l \in \mathcal{E}_i, l \neq j} \mathbf{G}_{il} (\mathbf{x})$, $q = n$, $p = 1$ and $s = 0$ in Theorem 3, we get

$$\begin{aligned} \lambda_{\min} \left( \sum_{l \in \mathcal{E}_i} \mathbf{G}_{il} (\mathbf{x}) \right) &\leq \lambda_{\min} (\mathbf{G}_{ij} (\mathbf{x})) + \lambda_{\max} \left( \sum_{l \in \mathcal{E}_i, l \neq j} \mathbf{G}_{il} (\mathbf{x}) \right) \\ &\leq \lambda_{\min} (\mathbf{G}_{ij} (\mathbf{x})) + \sum_{l \in \mathcal{E}_i, l \neq j} \lambda_{\max} (\mathbf{G}_{il} (\mathbf{x})), \end{aligned} \tag{23}$$

where the second inequality follows by applying Theorem 3 with $p = q = n$ and $s = 0$.

Now, from Lemma 5 we know that $\lambda_{\min} (\mathbf{G}_{ij} (\mathbf{x})) = (\|\mathbf{x}_i - \mathbf{x}_j\| - d_{ij}) / \|\mathbf{x}_i - \mathbf{x}_j\|$ and that $\lambda_{\max} (\mathbf{G}_{il} (\mathbf{x})) = 1$ for any $(i, l) \in \mathcal{E}$. Therefore, by combining (22) and (23), we obtain that

$$\lambda_{\min} \left(\nabla^2 \mathcal{F} (\mathbf{x})\right) \leq \frac{2 (\|\mathbf{x}_i - \mathbf{x}_j\| - d_{ij})}{\|\mathbf{x}_i - \mathbf{x}_j\|} + 2 (|\mathcal{E}_i| - 1) = 2 \left( |\mathcal{E}_i| - \frac{d_{ij}}{\|\mathbf{x}_i - \mathbf{x}_j\|} \right) < 0,$$

and hence $\mathbf{x}$ is not a minimum point of $\mathcal{F}$ (specifically, it is a strict saddle point). □

*Remark* 3. By inspecting the proof of Theorem 4, we see that it can be stated in the following equivalent way: given a stationary point $\mathbf{x} \in \mathbb{R}^{nK}$ of $\mathcal{F}$, if $\min_{(i,j) \in \mathcal{E}} \{|\mathcal{E}_i| - d_{ij} / \|\mathbf{x}_i - \mathbf{x}_j\|\} < 0$, then $\mathbf{x}$ is not a local minimum point of $\mathcal{F}$. Hence, this point is a strict (differentiable) saddle point, that can be escaped by applying a backtracking procedure in the direction of the eigenvector corresponding to the minimal eigenvalue.

Moreover, it is worth noting that the condition $|\mathcal{E}_i| < d_{ij} / \|\mathbf{x}_i - \mathbf{x}_j\|$ for some $(i, j) \in \mathcal{E}$ (indicating that $\mathbf{x}$ is not a local minimum) is anticipated to be more prevalent in networks with high measurement noise. In such networks, the measurements $d_{ij}$ tend to be larger, increasing the likelihood of encountering this condition.

*Remark* 4. Theorem 4 can be generalized to any network that is divided into clusters. In such a configuration, the network is divided into clusters, each containing a central processor called a *clusterhead* (see [19] for more details). For a given cluster represented by the index set $\mathcal{C} \subseteq [K]$, we can construct a principal sub-matrix $\left[\nabla^2 \mathcal{F}(\mathbf{x})\right]_{\mathcal{C}} \in \mathbb{R}^{n|\mathcal{C}| \times n|\mathcal{C}|}$ of $\nabla^2 \mathcal{F}(\mathbf{x})$ by selecting the rows and columns corresponding to the indices in $\mathcal{C}$. Similar to the previous analysis, it follows that

$$\lambda_{\min}\left(\nabla^2 \mathcal{F}(\mathbf{x})\right) \leq \lambda_{\min}\left(\left[\nabla^2 \mathcal{F}(\mathbf{x})\right]_{\mathcal{C}}\right).$$

In the case of cluster architectures, each cluster's central processor collects data from the sensors within the cluster, enabling explicit distributed computation of $\lambda_{\min}\left(\left[\nabla^2 \mathcal{F}(\mathbf{x})\right]_{\mathcal{C}}\right)$.

# 6 Conclusion

In this paper, we delved into the mathematical geometry of a popular class of non-linear, non-convex and non-smooth least squares problems motivated by two challenging applications: the Wireless Sensor Network Localization and Multi-Dimensional Scaling. Our study led to several key findings. Firstly, we analyzed the extremum points of this class of problems and proved that any non-differentiable critical point corresponds to a saddle point. Building upon this result, we devised a procedure to identify an easy-to-find and explicit descent direction, enabling efficient escape from non-differentiable saddles. Importantly, this procedure is applicable to both centralized and distributed computational settings. Furthermore, we leveraged our understanding of the stationary points by examining the eigenvalues of the corresponding Hessian matrix. Building on this second-order information, we established a distributed necessary condition for local optimality. This condition allows us to assess the quality of stationary points in a distributed fashion, even when direct eigenvalue computations are infeasible due to limited information exchange or large matrix sizes.

# A Appendix

Before we present the distributed escape procedure, we need the following notations, that will enable us to treat each sensor separately in a distributed manner. For any $i \in \{1, 2, \ldots, K\}$, we define $\mathcal{E}_i$ as the set containing the indices of all neighbors of sensor $i$. That is, $j \in \mathcal{E}_i$ if and only if $(i, j) \in \mathcal{E}$ or $(j, i) \in \mathcal{E}$. Now, for any $i \in \{1, 2, \ldots, K\}$, we define the function $\mathcal{F}_i \colon \mathbb{R}^{nK} \to \mathbb{R}$ as

$$\mathcal{F}_i(\mathbf{x}) \equiv \sum_{j \in \mathcal{E}_i} \mathcal{F}_{ij}(\mathbf{x}) = \sum_{j \in \mathcal{E}_i} \left(\|\mathbf{x}_i - \mathbf{x}_j\| - d_{ij}\right)^2, \tag{24}$$

where we set $\mathcal{F}_{ij} \equiv \mathcal{F}_{ji}$ if $(j, i) \in \mathcal{E}$.

In addition, in the context of (24), by $\mathcal{F}_i(\mathbf{x}_i; \mathbf{z}) : \mathbb{R}^n \to \mathbb{R}$ for some $\mathbf{z} \in \mathbb{R}^{nK}$ we denote the partial function $\mathbf{x}_i \mapsto \mathcal{F}_i(\mathbf{z})$. That is, $\mathcal{F}_i(\mathbf{x}_i; \mathbf{z})$ treats $\mathbf{x}_i \in \mathbb{R}^n$ as the variable, while all $\mathbf{z}_j \in \mathbb{R}^n$, $j \neq i$, are fixed. Notice that for any $i \in \{1, 2, \ldots, K\}$, evaluating the function $\mathcal{F}_i(\mathbf{x}_i; \mathbf{z})$ at sensor $i$, only requires the collection of vectors $\mathbf{z}_j \in \mathbb{R}^n$, $j \in \mathcal{E}_i$, from the neighbors of $i$. Hence, for each sensor $i$, evaluating $\mathcal{F}_i(\mathbf{x})$ only requires information that is locally available at sensor $i$.

In Procedure 2 below, we present the distributed version of EP, designed to escape non-differentiable points in distributed computational settings. It is important to note that EP-D operates using locally available information at each node, making it a distributed procedure.

The input point $\mathbf{x} \in \mathbb{R}^{nK}$ of EP-D can be any point, not necessarily a non-differentiable point. It is important to note that since determining the non-differentiability of a point requires full network information, which is not available in distributed architectures. However, EP-D guarantees that the output point is a differentiable point of the function $\mathcal{F}$ and has a lower function value than the input point (see Proposition 2).

**Procedure 2** Escape Procedure – Distributed (EP-D)

1: **Initialization:** $\mathbf{x} \in \mathbb{R}^{nK}$ and set $\mathbf{z} = \mathbf{x}$.
2: **for** $i = 1, 2, \ldots, K$ **do**
3: **if** $\mathbf{x}_i = \mathbf{z}_j$ for some $j \in \mathcal{E}_i$ **then** set $t = 1$ and pick $\mathbf{0}_n \neq \mathbf{d} \in \mathbb{R}^n$.
4: *Double backtracking procedure:* **do in parallel**
$\rightarrow$ **while** $\mathcal{F}_i(\mathbf{x}_i; \mathbf{z}) \leq \mathcal{F}_i(\mathbf{x}_i + t\mathbf{d}; \mathbf{z})$ or $\mathbf{x}_i + t\mathbf{d} = \mathbf{z}_j$ for some $j \in \mathcal{E}_i$ **then** set $t \coloneqq t/2$.
$\rightarrow$ **while** $\mathcal{F}_i(\mathbf{x}_i; \mathbf{z}) \leq \mathcal{F}_i(\mathbf{x}_i - t\mathbf{d}; \mathbf{z})$ or $\mathbf{x}_i - t\mathbf{d} = \mathbf{z}_j$ for some $j \in \mathcal{E}_i$ **then** set $t \coloneqq t/2$.
5: Update $\mathbf{z}_i \coloneqq \mathbf{x}_i \pm t\mathbf{d}$ according to the first while loop that breaks.
6: **end if**
7: **end for**
8: Return $\mathbf{z}$ as the output.

To prove the above assertions about EP-D, it is required to derive a distributed variant of Theorem 1. To establish this, we prove a variant of Theorem 1 that considers only the sub-network consisting of sensor $i$ and its neighbors.

**Lemma 7.** *Let $i \in \{1, 2, \ldots, K\}$ and let some $\mathbf{z} \in \mathbb{R}^{nK}$. Assume that $\mathbf{x}_i \in \mathbb{R}^n$ is a non-differentiable point of the function $\mathcal{F}_i(\mathbf{x}_i; \mathbf{z})$. Then, for any $\mathbf{0}_n \neq \mathbf{d} \in \mathbb{R}^n$ such that $\mathbf{d} \neq \mathbf{z}_j$ for all $j \in \mathcal{E}_i$, either $\mathbf{d}$ or $-\mathbf{d}$ is a descent direction of $\mathcal{F}_i(\mathbf{x}_i; \mathbf{z})$ at $\mathbf{x}_i$.*

*Proof.* Denote by $\tilde{\mathbf{x}} \in \mathbb{R}^{nK}$ the point $\tilde{\mathbf{x}} \equiv (\mathbf{z}_1, \ldots, \mathbf{z}_{i-1}, \mathbf{x}_i, \mathbf{z}_{i+1}, \ldots, \mathbf{z}_K)$. Since $\mathbf{x}_i \in \mathbb{R}^n$ is a non-differentiable point of $\mathcal{F}_i(\mathbf{x}_i; \mathbf{z})$, then there exist some $j \in \mathcal{E}_i$ such that $\tilde{\mathbf{x}}_i = \tilde{\mathbf{x}}_j$, and hence the function $\mathcal{F}_i$ (see (24)), is also non-differentiable at $\tilde{\mathbf{x}}$. Now, we denote by $\tilde{\mathbf{d}} \in \mathbb{R}^{nK}$ the vector satisfying $\tilde{\mathbf{d}}_i = \mathbf{d} \neq \mathbf{0}_n$ and $\tilde{\mathbf{d}}_j = \mathbf{0}_n$ for any $j \neq i$, and in particular $\tilde{\mathbf{d}}_i \neq \tilde{\mathbf{d}}_j$ for any $(i, j) \in \mathcal{E}$.

Notice that Theorem 1, which holds true for *any* network, considers the function $\mathcal{F}$ of Problem (5), which in turn is the sum of all functions $\mathcal{F}_{ij}$, for all $(i, j) \in \mathcal{E}$. Therefore, by taking the (sub)network that is composed of the sensor $i$ and all it neighboring sensors, it immediately follows from Theorem 1 that either $\tilde{\mathbf{d}}$ or $-\tilde{\mathbf{d}}$ is a descent direction of the function $\mathcal{F}_i$ (as defined in (24)), at the point $\tilde{\mathbf{x}}$. We assume without the loss of generality that $\tilde{\mathbf{d}}$ is a descent direction. Hence, there exists $t > 0$ such that

$$\mathcal{F}_i(\mathbf{x}_i; \mathbf{z}) = \mathcal{F}_i(\tilde{\mathbf{x}}) > \mathcal{F}_i\left(\tilde{\mathbf{x}} + t\tilde{\mathbf{d}}\right) = \mathcal{F}_i(\mathbf{x}_i + t\mathbf{d}; \mathbf{z}),$$

and we obtain that $\mathbf{d}$ is a descent direction of $\mathcal{F}_i(\mathbf{x}_i; \mathbf{z})$ at $\mathbf{x}_i$, as required. □

Next, we prove that EP-D indeed yields a differentiable point of the function $\mathcal{F}$ with a lower function value.

**Proposition 2.** *Let $\mathbf{x} \in \mathbb{R}^{nK}$ and $\mathbf{z} \in \mathbb{R}^{nK}$ be the input and output points, respectively, of EP-D. Then, $\mathbf{z}$ is a differentiable point of $\mathcal{F}$ for which $\mathcal{F}(\mathbf{z}) < \mathcal{F}(\mathbf{x})$.*

*Proof.* Initially we set $\mathbf{z} = \mathbf{x}$ (see step 1 in EP-D). We focus on the case in which $\mathbf{x}$ is a non-differentiable point of $\mathcal{F}$. In particular, there exist $i$ and some $j \in \mathcal{E}_i$ such that $\mathbf{x}_i = \mathbf{z}_j$. For any $\mathbf{d} \neq \mathbf{0}_n$, it follows from Lemma 7 that there exists $\bar{\epsilon}_i > 0$ such that either $\mathcal{F}_i(\mathbf{x}_i; \mathbf{z}) > \mathcal{F}_i(\mathbf{x}_i + t\mathbf{d}; \mathbf{z})$ or $\mathcal{F}_i(\mathbf{x}_i; \mathbf{z}) > \mathcal{F}_i(\mathbf{x}_i - t\mathbf{d}; \mathbf{z})$ for all $t \in (0, \bar{\epsilon}_i]$. For the sake of simplicity, we assume without the loss of generality that $\mathbf{d}$ is a descent direction.

We now derive that (recall that initially $\mathbf{z} = \mathbf{x}$)

$$\mathcal{F}(\mathbf{x}) = \mathcal{F}_i(\mathbf{x}_i; \mathbf{z}) + \sum_{\substack{(k,j) \in \mathcal{E} \\ k,j \neq i}} \mathcal{F}_{kj}(\mathbf{z}) > \mathcal{F}_i(\mathbf{x}_i + t\mathbf{d}; \mathbf{z}) + \sum_{\substack{(k,j) \in \mathcal{E} \\ k,j \neq i}} \mathcal{F}_{kj}(\mathbf{z}) = \mathcal{F}(\mathbf{z}),$$

where the last eqaulity follows from the fact that we set $\mathbf{z}_i := \mathbf{x}_i + t\mathbf{d}$ (see step 5 in EP-D). Since the above process holds true for any $\mathbf{x}$, then indeed the output point of EP-D has a lower function value, as required.

Last, since the set $\mathcal{E}$ is finite, one can pick $t \in (0, \bar{\epsilon}_i]$ such that $\mathbf{x}_i + t\mathbf{d} \neq \mathbf{z}_j$ for all $j \in \mathcal{E}_i$ (see step 4 in EP-D). Therefore, the output of EP-D is a differentiable point of $\mathcal{F}$. $\square$

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