Path Planning

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We assume without the loss of generality that the starting point $\mathcal{A}$ is the point $(1,1) \in \mathbb{R}^2$ and that the end point $\mathcal{C}$ is $(-1,1) \in \mathbb{R}^2$. Next, we assume that the point $\mathcal{B}$ is the point $(0,0) \in \mathbb{R}^2$. We point out that even though that the assumption on the point $\mathcal{B}$ (given the initial assumption on $\mathcal{A}$ and $\mathcal{C}$) *does* restrict the generality, it renders the derivations below much simpler. In the case where the point $\mathcal{B}$ is given by any other point, the derivations below should be revised accordingly.

Given the structure of the three points, a parameterization for a path from $\mathcal{A}$ to $\mathcal{C}$ through $\mathcal{B}$ with minimal length is the curve

$$\gamma\left(t\right) = \left(t, \left|t\right|\right) \subset \mathbb{R}^2, \quad \forall t \in \left[-1,1\right],$$

where t can be viewed as the time variable (under some linear transformation). Notice that γ is simply the absolute value curve. We point out that the speed along this curve is constant, since it holds that

$$\left\|\dot{\gamma}\left(t\right)\right\| = \left\|\left(1, \operatorname{sign}\left(t\right)\right)\right\| = \sqrt{2}, \quad t \neq 0.$$

Unfortunately, this curve in non-differentiable at the point $\mathcal{B}$, and we need to obtain a smooth (i.e., continuously differentiable) approximation for this curve.

One suitable choice (among others) is the function $\sqrt{t^2+\epsilon^2}$, for some smoothing constant $\epsilon > 0$, which is a differentiable function of any order (i.e., smooth) and it also approaches the function $|t|$ as $\epsilon \to 0$. This function gives rise to the following approximating curve

$$\eta\left(t\right) = \left(x\left(t\right), \sqrt{t^2+\epsilon^2} + C\right), \quad \forall t \in \left[-1,1\right],$$

where $x\left(t\right)$ is a function that needs to be determined, and $\epsilon > 0$ and $C \in \mathbb{R}$ are constants to be determined.

Formulating an initial optimization problem.

Since we want to move at a constant speed (up to a possibly scalar multiplication) along the curve, then the task of finding a suitable curve η that best approximates the curve γ can be formulated as the following minimization problem

$$\begin{aligned}
\underset{\epsilon>0, C\in\mathbb{R}, x(t)}{\text{minimize}} \quad & \text{distance}\left(\eta\left(t\right), \gamma\left(t\right)\right) \\
\text{subject to} \quad & \left\|\dot{\eta}\left(t\right)\right\| = \left\|\dot{\gamma}\left(t\right)\right\|, \quad \forall t \in \left[-1, 1\right], \\
& \eta\left(1\right) = \left(1, 1\right), \\
& \eta\left(-1\right) = \left(-1, 1\right),
\end{aligned} \tag{1}$$

where the objective function guarantees that the two curves are as close as possible under some metric, the first constraint guarantees that η has a constant speed, and the last two constraints guarantee that η starts at $\mathcal{A}$ and ends at $\mathcal{C}$.

We aim at finding a solution for this minimization problem. We mention that the variables of this problem are $\epsilon > 0$, $C \in \mathbb{R}$ and the function $x\left(t\right)$.

Rewriting the optimization problem.

- *Objective function (minimal distance):* In order to write an objective function explicitly, we notice that $\gamma\left(0\right) = \left(0, 0\right)$ (the point $\mathcal{B}$) and that $\eta\left(0\right) = \left(0, \epsilon\right)$. Hence, the two curves deviate at the point B by ϵ, so one possible objective function is simply ϵ.

- *First constraint (constant speed):* We notice that the constraint can be written as (after taking the square on both sides)

  $$2 = \left(x'\left(t\right)\right)^2 + \frac{t^2}{t^2+\epsilon^2},$$

  and solving this equation for $x'\left(t\right)$ yields

  $$x'\left(t\right) = \sqrt{\frac{t^2+2\epsilon^2}{t^2+\epsilon^2}}.$$

  Taking the integral to find $x\left(t\right)$ we get the real function $\bar{x}_\epsilon \colon \mathbb{R} \to \mathbb{R}$ defined as

  $$\bar{x}_\epsilon\left(t\right) \equiv -i\sqrt{2}\epsilon \cdot E\left(i \cdot \sinh^{-1}\left(\frac{t}{\epsilon}\right) \middle| \frac{1}{2}\right), \tag{2}$$

  where $E\left(\cdot \middle| \frac{1}{2}\right)$ is the Elliptic integral of the second kind with parameter $1/2$. Hence, the curve η takes the form

  $$\eta\left(t\right) = \left(\bar{x}_\epsilon\left(t\right) + D, \sqrt{t^2+\epsilon^2} + C\right), \quad t \in \left[-1, 1\right],$$

  where $D \in \mathbb{R}$ is some constant to be determined. We point out that such curve η is *smooth* and has a constant speed of $\sqrt{2}$ (this constant can be modified by applying some normalization).

- *Last two constraints (starting and end points):* Notice that

$$(1,1) = \eta(1) = \left(\bar{x}_\epsilon(1) + D, \sqrt{1+\epsilon^2} + C\right),$$

and hence we take the constant C and D as

$$\bar{C}_\epsilon \equiv 1 - \sqrt{1+\epsilon^2} \quad \text{and} \quad \bar{D}_\epsilon \equiv 1 - \bar{x}_\epsilon(1). \tag{3}$$

With this choice of the constants C and D we get that indeed $\eta(1) = (1,1) = \mathcal{A}$, as required. In addition, we get

$$\eta(-1) = \left(\bar{x}_\epsilon(-1) + 1 - \bar{x}_\epsilon(1), \sqrt{1+\epsilon^2} + 1 - \sqrt{1+\epsilon^2}\right) = (1 - 2\bar{x}_\epsilon(1), 1).$$

Unfortunately, we see that $\eta(-1)$ does not return the point $\mathcal{C}$ exactly, but since $\bar{x}_\epsilon(1) \to 1$ as $\epsilon \to 0$, we get that the point $\eta(-1)$ approaches the point $\mathcal{C}$ by taking small enough values of $\epsilon > 0$.

Finding the optimal smoothing constant $\epsilon > 0$.

Now, after we have dealt with the variables C and $x(t)$ in the minimization problem, we still need to find the best choice for $\epsilon > 0$. Following the above derivations, rewriting the minimization problem in (1) we obtain

$$\begin{aligned} \underset{\epsilon>0}{\text{minimize}} \quad & \epsilon \\ \text{subject to} \quad & \eta(-1) \approx 1 - 2\bar{x}_\epsilon(1). \end{aligned}$$

The optimal solution for this problem is of course taking $\epsilon > 0$ as small as possible (approaching 0). However, in practical terms, taking such small values implies that the acceleration along the curve in the vicinity of the point $\mathcal{B}$ approaches infinity (we stress that the speed remains constant), which is unlikely. Hence, we need to add a constraint to the minimization problem in (1) that limits the acceleration.

Limiting the acceleration: assume that the allowed acceleration/deceleration along the x-axis is bounded by some $\alpha > 0$, and along the y-axis by some $\beta > 0$, for some known and given threshold constants α, β. I.e,

$$-\alpha \le x''(t) \le \alpha \quad \text{and} \quad -\beta \le \left(\sqrt{t^2+\epsilon^2}\right)'' \le \beta, \quad \forall t \in [-1,1].$$

Writing the above inequalities explicitly, the minimization problem in (1) now takes the form

$$\begin{aligned} \underset{\epsilon>0}{\text{minimize}} \quad & \epsilon \\ \text{subject to} \quad & \max_{t\in[-1,1]} \frac{|t|\,\epsilon^2}{\sqrt{t^2+2\epsilon^2}\left(t^2+\epsilon^2\right)^{1.5}} \le \alpha, \\ & \max_{t\in[-1,1]} \frac{\epsilon^2}{\left(t^2+\epsilon^2\right)^{1.5}} \le \beta. \end{aligned} \tag{4}$$

The two inner maximization problems in the constraints of problem (4) are 1-dimensional and can be solved analytically by finding stationary points. After some algebraic calculations, we obtain that their solution is

$$\max_{t\in[-1,1]} \frac{|t|\,\epsilon^2}{\sqrt{t^2+2\epsilon^2}\left(t^2+\epsilon^2\right)^{1.5}} = \begin{cases} \frac{3\sqrt{3\sqrt{10}-9}}{\left(1+\sqrt{10}\right)^{1.5}}\cdot\frac{1}{\epsilon}, & \epsilon \le \sqrt{\frac{3}{\sqrt{10}-2}} \\ \frac{\epsilon^2}{\sqrt{1+2\epsilon^2}\left(1+\epsilon^2\right)^{1.5}}, & \text{otherwise,} \end{cases}$$

and

$$\max_{t\in[-1,1]} \frac{\epsilon^2}{\left(t^2+\epsilon^2\right)^{1.5}} = \frac{1}{\epsilon}.$$

Hence, the minimization problem in (4) can now be written as

$$\begin{aligned} &\underset{\epsilon>0}{\text{minimize}} && \epsilon \\ &\text{subject to} && \frac{3\sqrt{3\sqrt{10}-9}}{\left(1+\sqrt{10}\right)^{1.5}}\cdot\frac{1}{\epsilon} \le \alpha \quad \text{if} \quad \epsilon \le \sqrt{\frac{3}{\sqrt{10}-2}}, \\ &&& \frac{\epsilon^2}{\sqrt{1+2\epsilon^2}\left(1+\epsilon^2\right)^{1.5}} \le \alpha \quad \text{if} \quad \epsilon > \sqrt{\frac{3}{\sqrt{10}-2}}, \\ &&& \frac{1}{\epsilon} \le \beta. \end{aligned} \tag{5}$$

Formulating an algorithm for finding the optimal curve η.

We notice that the minimization problem in (5) is a 1-dimensional problem with constraints that are closed and bounded from below, hence we can devise an algorithm for finding its optimal solution. The algorithm is given below in Algorithm 1.

For a simple presentation of the algorithm, we mention that

$$\frac{3\sqrt{3\sqrt{10}-9}}{\left(1+\sqrt{10}\right)^{1.5}} \approx 0.2465 \quad \text{and} \quad \sqrt{\frac{3}{\sqrt{10}-2}} \approx 1.6066.$$

In addition, we define the function $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ as

$$\phi\left(\delta\right) = \frac{\delta^2}{\sqrt{1+2\delta^2}\left(1+\delta^2\right)^{1.5}},$$

and notice that $\max\phi\left(\delta\right) \approx 0.2056$.

Finally, summing the above derivations, we get that a smooth curve of the form of $\eta\left(t\right)$ over time t that best approximates the initial curve γ and starts at the point $\mathcal{A}$ is given by

$$\eta\left(t\right) = \left(\bar{x}_\epsilon\left(t\right) + \bar{D}_\epsilon, \sqrt{t^2+\epsilon^2} + \bar{C}_\epsilon\right), \quad t \in [-1,1], \tag{6}$$

where $\epsilon > 0$ is the output of Algorithm 1, the function $\bar{x}_\epsilon\left(t\right)$ is defined in (2), and the constants $\bar{C}_\epsilon$ and $\bar{D}_\epsilon$ are defined in (3).

Algorithm 1 Finding optimal (minimal) smoothing constant $\epsilon > 0$

Input: Acceleration bounding constants $\alpha, \beta > 0$ for the x and y axes, respectively.
if $\max\left\{\frac{1}{\beta}, \frac{0.2465}{\alpha}\right\} \leq 1.6066$ **then** return $\epsilon = \max\left\{\frac{1}{\beta}, \frac{0.2465}{\alpha}\right\}$.
else
    if $\alpha \geq 0.2056$ **then** return $\epsilon = \frac{1}{\beta}$.
    else
        Use bisection method to find δ^* the unique root of $\phi(\delta) - \alpha = 0$ over $\delta > 1.6066$.
        Return $\epsilon = \max\left\{\frac{1}{\beta}, \delta^*\right\}$.
    end if
end if

Numerical properties.

In Figure 1, we plot four representing plots of the smooth approximating curve η as defined in (6) for different values of $\alpha = \beta$, along with the original curve γ. As we can see, η approaches γ as the values of α, β increase (equivalently, as the value of the smoothing constant ϵ approach 0). As developed above, the curve η always starts at the point $\mathcal{A} = (1, 1)$, and approaches $\mathcal{B}$ and $\mathcal{C}$ as ϵ decreases.

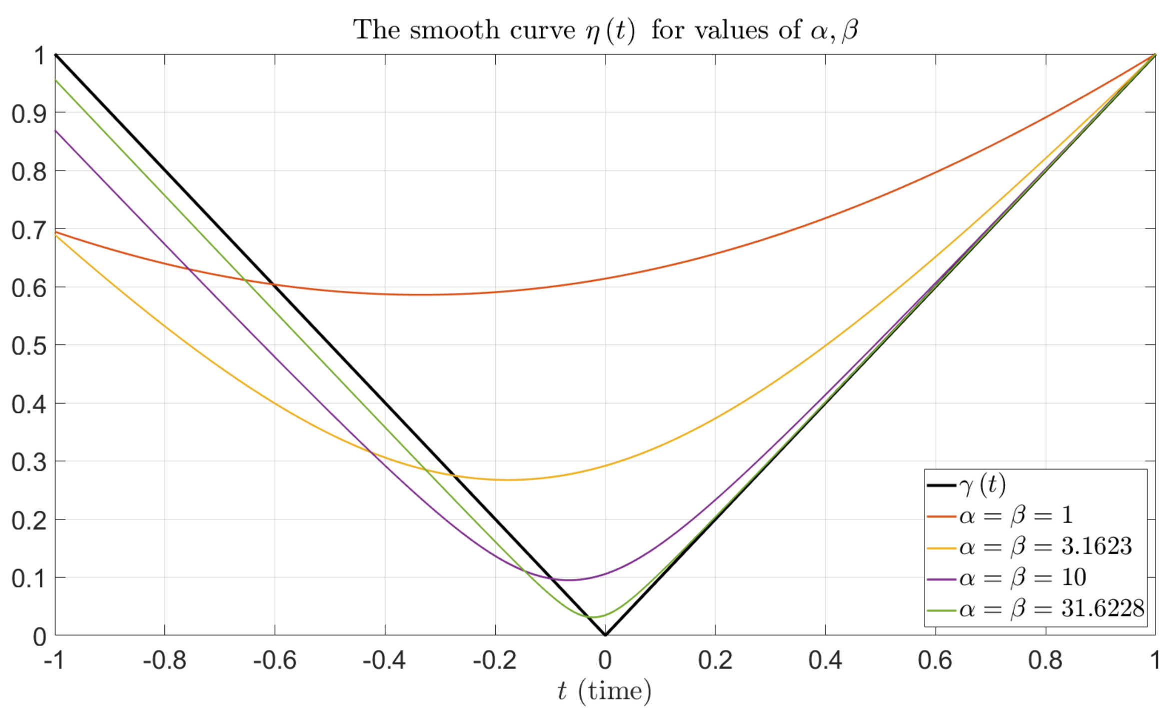

Figure 1

In Figure 2, we plot the differences between the curve γ and its smooth approximation

η, as a function of $\epsilon > 0$ in a logarithmic scale (the x-axis).

As discusses above, the deviation from the point $\mathcal{C}$ is upper bounded by the term $2\left|\bar{x}_{\epsilon}(1) - 1\right|$ (the blue line). We can see that this upper bound approaches 0 as $\epsilon \to 0$, and it approaches ≈ 0.8284 as $\epsilon \to \infty$.

The deviation from the point $\mathcal{B}$ (the red line) is calculated by

$$\min_{t \in [-1,1]} \sqrt{t^2 + \bar{y}_{\epsilon}^2(t)},$$

where $\bar{y}_{\epsilon}(t) = \sqrt{t^2 + \epsilon^2} + \bar{C}_{\epsilon}$. It can be seen that the deviation approaches 0 as $\epsilon \to 0$ and it approaches 1 as $\epsilon \to \infty$.

The total deviation between the two curves (the yellow line) is calculated by

$$\left| \int_{-1}^{1} \left(\bar{y}_{\epsilon}(t) - |t| \right) dt \right|,$$

which is the total area bounded between the two curves. It can be seen that this difference approaches 0 as $\epsilon \to 0$ and it approaches 1 as $\epsilon \to \infty$.

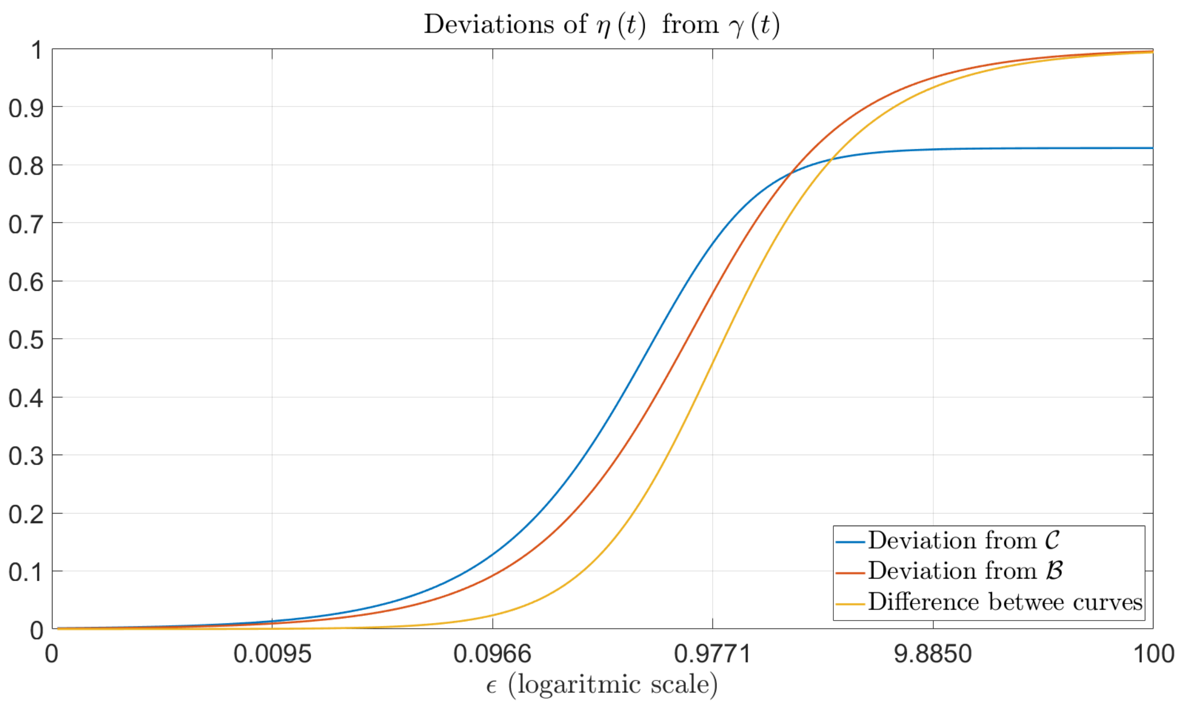

Figure 2

Last, in Figure 3 we plot an example that displays the acceleration profile of $\eta(t)$ along its x-axis (blue line) and its y-axis (red line), as a function of the time t. In this example, we set $\alpha = 2$ and $\beta = 8$. It can be seen that indeed, as developed above, the optimal ϵ generated by Algorithm 1 gives a curve η for which the acceleration along its x-axis is bounded from below and from above by $\alpha = 2$, and along its y-axis by $\beta = 8$.

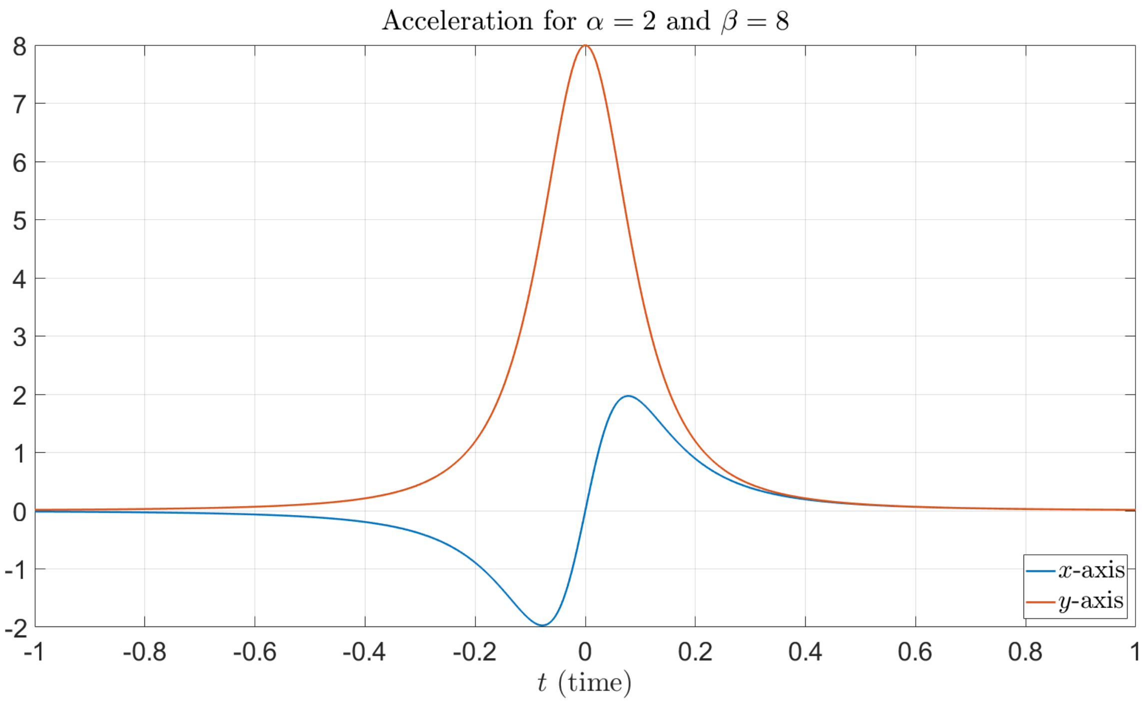

Figure 3

Future analysis.

In this assignment, I limited the discussion only to one type of smooth approximating curves, namely the curves that rely on the smooth approximation $\sqrt{t^2+\epsilon^2}$ of the absolute value function, which are dictated by a suitable choice of a smoothing constant $\epsilon > 0$. However, there are other smooth approximations of the absolute value function, such as:

- $\sqrt{t^2+\mu\epsilon^2}$, for $\mu > 0$ and $\epsilon > 0$.
- $\frac{2}{k}\log\left(1+e^{kt}\right) - t - \frac{2}{k}\log(2)$, for $k > 0$.
- The Huber function defined as $H_\delta(t) = \frac{1}{2}t^2$ for $|t| \leq \delta$ and $\delta\left(|t| - \frac{1}{2}\delta\right)$ otherwise, for some $\delta > 0$.
- $t \cdot \text{erf}\left(\frac{t}{\mu}\right)$ and $t \cdot \tanh\left(\frac{t}{\mu}\right)$, for $\mu > 0$ where $\text{erf}(\cdot)$ is the error function.

Surely, there are additional suitable functions, and one can try and follow the derivations given here for any of these functions, and inspect their results for better/worse performance.

But how can we choose the optimal curve out of *all* possible approximating smooth curves? This is a very challenging problem (if one can even find its solution), as it requires to solve an optimization problem over an infinite-dimensional function space.