UNIVERSITY OF CALIFORNIA BERKELEY
MEC180/CEE133

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Due: 2/21/2025

# Project 1

1. (10 points) Solve the following boundary value problem analytically (This will give you the true solution, $u_{true}$, to the problem allowing you to check your FEM solution):

$$\boxed{\frac{d}{dx}\left(E\frac{du}{dx}\right) = k^2 sin(\frac{2\pi kx}{L}) + 2x^2 \qquad (1)}$$

First, we'll find the general form of $u(x)$. We'll treat Equation (1) as a separable ODE and integrate twice.

$$\int d\left(E\frac{du}{dx}\right) = \int \left(k^2 \sin(\frac{2\pi kx}{L}) + 2x^2\right) dx$$

$$E\frac{du}{dx} = -\frac{kL}{2\pi}\cos(\frac{2\pi kx}{L}) + \frac{2}{3}x^3 + C_1$$

$$E\int du = \int \left(-\frac{kL}{2\pi}\cos(\frac{2\pi kx}{L}) + \frac{2}{3}x^3 + C_1\right) dx$$

$$Eu = -\frac{L^2}{4\pi^2}\sin(\frac{2\pi kx}{L}) + \frac{1}{6}x^4 + C_1x + C_2$$

Thus:

$$u(x) = -\frac{L^2}{4E\pi^2}\sin(\frac{2\pi kx}{L}) + \frac{1}{6E}x^4 + C_1x + C_2$$

Now, since we know our boundary conditions $u(0) = 0$, and $u(L) = 1$, we can find $C_1$ and $C_2$. Starting with $u(0) = 0$, we can inspect:

$$u(0) = 0 \ \therefore \ C_2 = 0$$

And $C_2$:

$$u(L) = 1 \ \therefore \ 1 = -\frac{L^2}{4E\pi^2}\sin(2\pi k) + \frac{1}{6E}L^4 + C_1L + 0$$

$$C_1 = 1 + \frac{L}{4E\pi^2}\sin(2\pi k) - \frac{L^3}{6E}$$

Thus, combining together, we have $u_{true}$!

$$\boxed{u_{true}(x) = -\frac{L^2}{4E\pi^2}\sin(\frac{2\pi kx}{L}) + \frac{1}{6E}x^4 + x + \frac{Lx}{4E\pi^2}\sin(2\pi k) - \frac{L^3x}{6E}}$$

2. (10 points) Use Galerkin's method to obtain the weak form of the differential equation given above. Fill in the blanks by following the derivation in Section 2.2 of the textbook.

Starting with the strong form of the differential equation:

$$\frac{d\sigma}{dx} + f(x) = 0 \tag{2}$$

Where:

$$f(x) = -\left(k^2 sin(\frac{2\pi kx}{L}) + 2x^2\right) \tag{3}$$

$$\sigma(x) = E(x)\frac{du}{dx} \tag{4}$$

Set 2 equal to the residual $r$, then multiply by a smooth test function $\nu = \nu(x)$ and integrate over the domain:

$$\boxed{\int_\Omega \left(\frac{d\sigma}{dx}\nu + f(x)\nu\right) dx = \int_\Omega r\nu dx} \tag{5}$$

Apply the product rule to $\sigma\nu$:

$$\boxed{\begin{aligned} \frac{d}{dx}(\sigma\nu) &= (\frac{d\sigma}{dx})\nu + \sigma(\frac{d\nu}{dx}) \\ \frac{d\sigma}{dx}\nu &= \frac{d(\sigma\nu)}{dx} - \sigma\frac{d\nu}{dx} \end{aligned}} \tag{6}$$

Plug 6 into 5:

$$\int_\Omega \left[\frac{d(\sigma\nu)}{dx} - \sigma\frac{d\nu}{dx}\right] dx + \int_\Omega f\nu dx = \int_\Omega r\nu dx \tag{7}$$

Let the result hold $\forall\nu$, making $r(x) = 0$ since the test function $\nu(x)$ will "find" the solution and force the residual to be $r = \frac{d\sigma}{dx} + f = 0$ at all points in the domain:

$$\int_\Omega \frac{d(\sigma\nu)}{dx}dx - \int_\Omega \sigma\frac{d\nu}{dx}dx + \int_\Omega f\nu dx = 0 \ , \ \forall\nu \tag{8}$$

$$\int_\Omega \sigma\frac{d\nu}{dx}dx = \int_\Omega f\nu dx + \sigma\nu\Big|_{\partial\Omega} \ , \ \forall\nu \tag{9}$$

Force $\nu = 0$ on the sections of the boundary where displacement $u$ is specified, $\Gamma_u$: $\nu\big|_{\Gamma_u} = 0 \ , \ \forall \nu$.
Since the displacement is specified on the entire boundary, $\partial\Omega = \Gamma_u = \{0, L\}$ and $\nu = 0$ on $\Gamma_u$, meaning: $\sigma \cancelto{0}{\nu(L)} - \sigma \cancelto{0}{\nu(0)} = 0$

Lastly, to constrain the solution to have finite energy, let $u, \nu \in H^1(\Omega)$ so that the energy norm of the solution is finite.

Leaving the weak form to be the following:

$$\int_\Omega \sigma \frac{d\nu}{dx} dx = \int_\Omega f \nu dx \tag{10}$$

Where the solution $u \in H^1(\Omega)$ is chosen such that $u\big|_{\Gamma_u} = u^*$ and $\forall \nu \in H^1(\Omega)$, $\nu\big|_{\Gamma_u} = 0$
Where:

$$\begin{aligned}
\Omega &= (0, L) \ , \text{ where } L = 1 \\
f(x) &= -\left(k^2 sin(\frac{2\pi k x}{L}) + 2x^2\right) \\
\sigma &= E(x)\frac{du}{dx} \\
u(0) &= u_0^* = 0 \\
u(L) &= u_L^* = 1
\end{aligned}$$

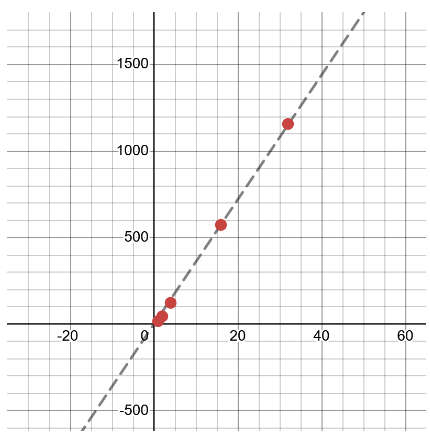
3. Now write a 1D finite element method program using linear equal-sized elements to solve Part 1's BVP.

   (a) (35 points) How many finite elements (N) are needed for k = 1, 2, 4, 16, 32. Put in a table and discuss the results.

| $k$ | $N$ |
|---|---|
| 1 | 18 |
| 2 | 46 |
| 4 | 124 |
| 16 | 574 |
| 32 | 1157 |

Table 1: k & N for $e^N \leq 0.05$

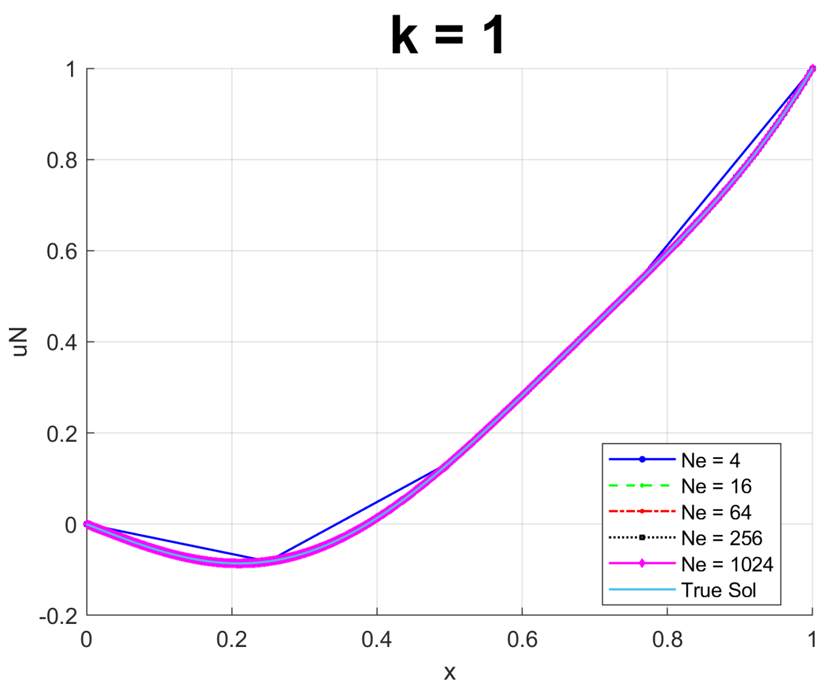
Unsurprisingly, it seems that $k \propto N$. That is to say, as we increase the value of $k$ (adding more wiggly details to our solution function), the amount of nodes required to capture all of this detail increases. This makes intuitive sense.

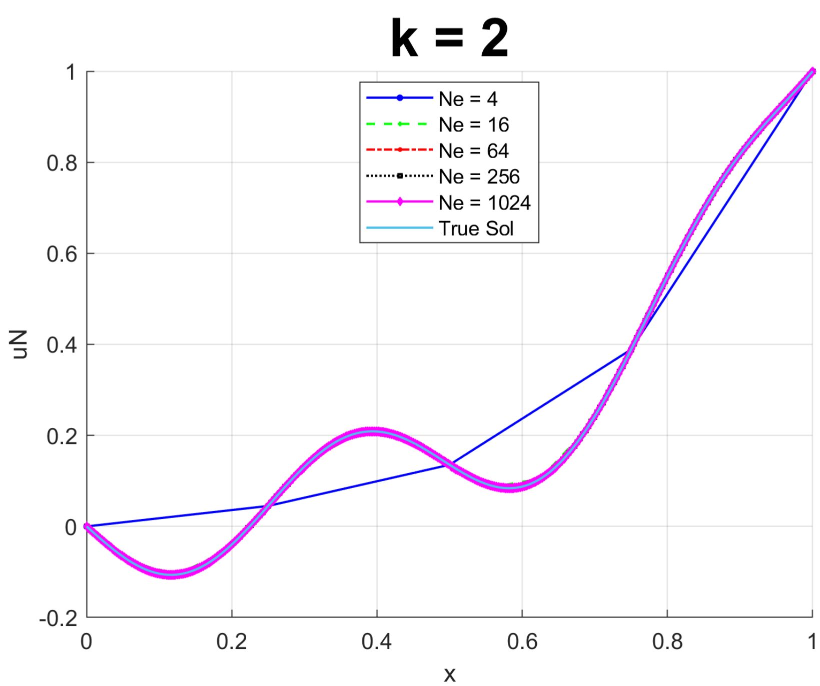


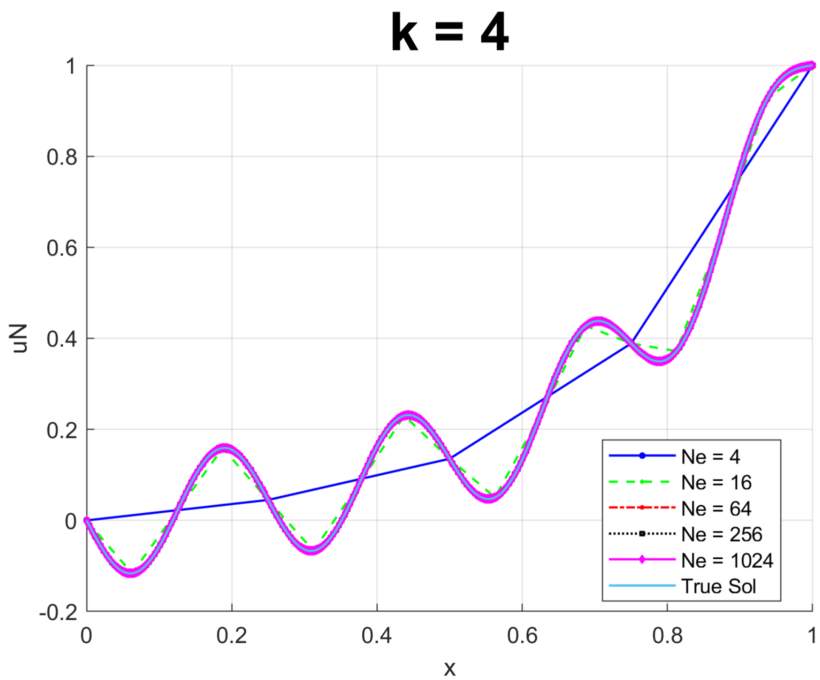

Plotting out $\vec{k}$ vs $\vec{N}$, we find that the minimum number of elements grows linearly ($R^2 = 0.9985$) to the amount of detail (i.e. $k$) on the solution curve.

Notably, because of how `myFEM1D` was written, it also took longer to compute the minimum number of elements. Because we only added one element per check the larger $k$'s required orders more calculations to solve for a suitable minimum. As we know from lecture, FEM has a computational cost roughly on the order of $O(N^3)$ to solve, so it makes sense that the larger $N$'s took more time to solve.

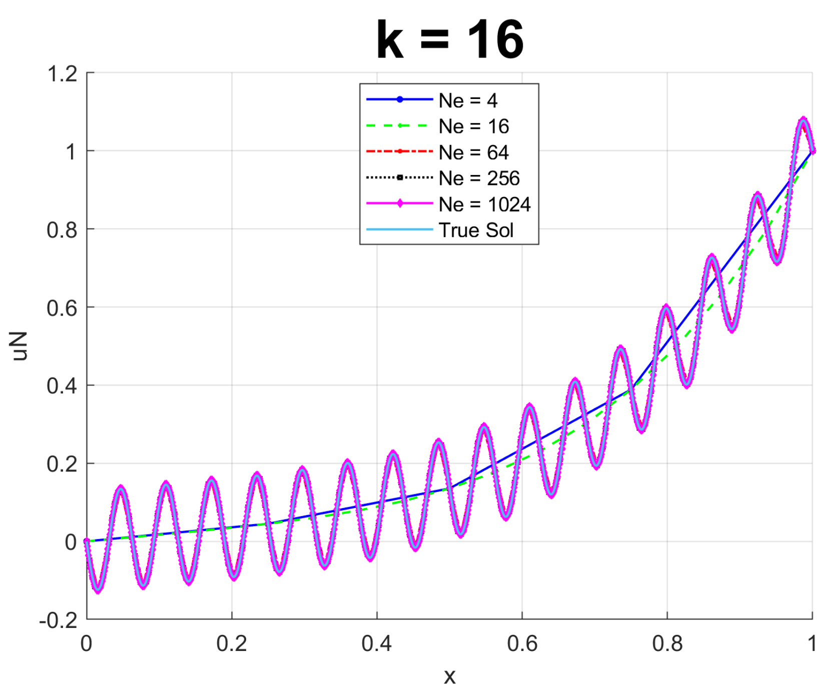
(b) (35 points) Plot the numerical solutions for $N = 4, 16, 64, 256, 1024$, for each k, along with the true solution (make a plot $u$ vs $x$ for each $k$, with the true solution and the numerical solutions visible and discuss).



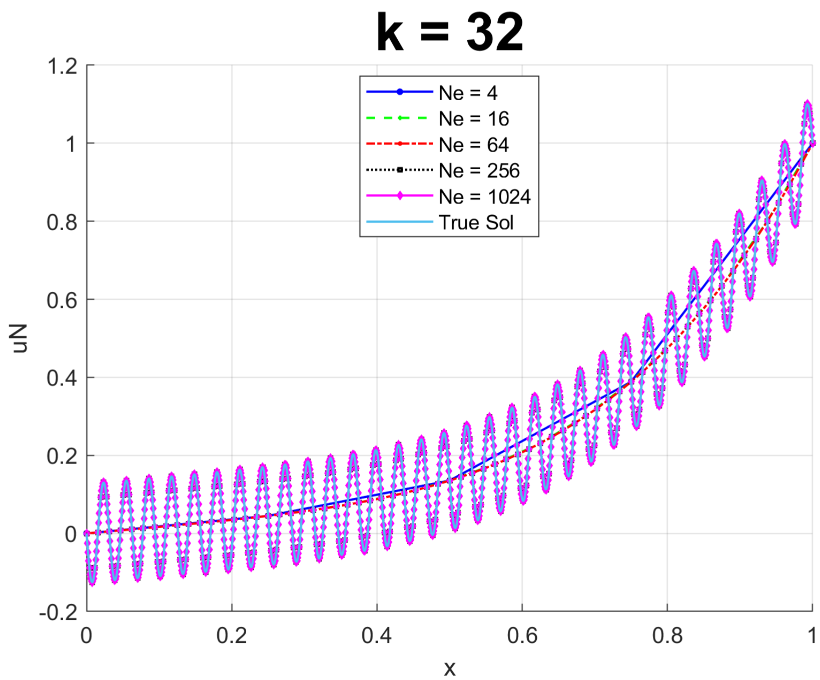





k = 4
uN
x
Ne = 4
Ne = 16
Ne = 64
Ne = 256
Ne = 1024
True Sol

k = 16
uN
x
Ne = 4
Ne = 16
Ne = 64
Ne = 256
Ne = 1024
True Sol




After plotting the numerical solutions for all values of $\vec{N}$ alongside the true solution $u^{true}$, I observed that the coarser meshes (except $N = 4$) performed reasonably well in approximating the true solution for smaller $k$ values (e.g., $k = 1, 2$). Because the lower-order $k$ values represent simpler functions, most of the numerical solution curves fall on top of each other, making them somewhat difficult to distinguish visually.

However, as $k$ increased (e.g., $k = 4, 16, 32$), only the denser meshes were able to accurately capture the increased complexity. At this order, it's easy to pick apart the different curves for different value sof $Ne$.

While the higher-density meshes provided more precise approximations, they required more computational resources and were unnecessary for the simpler functions. In the future, it could be helpful to implement adaptive meshing that can automatically stop adding nodes.

Overall, it's impressive that this method captured meaningful detail regardless of mesh size. While the accuracy wasn't always perfect, each approximation of $u^{true}$ (regardless of mesh density) preserved key features and trends, with higher mesh densities always improving precision. This highlights one of the main advantages of FEM.

(c) (10 points) Plot $e^N$ vs $\frac{1}{N}$ for each k (make this a log-log plot).

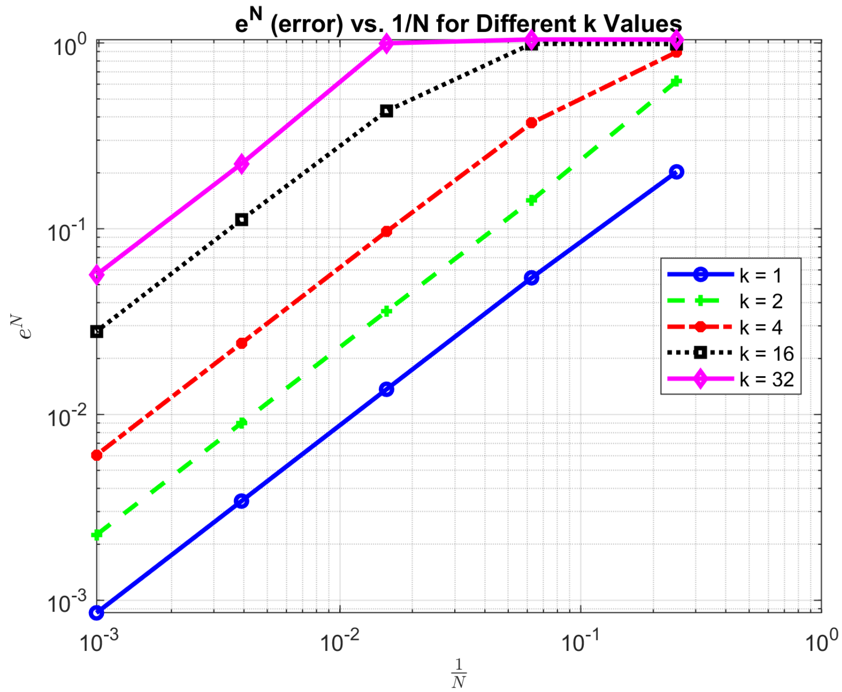



Figure 1: My plot of $e^N$ vs. $\frac{1}{N}$. As expected, as $N \to \infty$, $e^N \to 0$.

As we observed in 3.b., higher and higher values of $N$ results in lower and lower error. Therefore, as expected, our solution curve converges to zero as the number of elements increases.

Interestingly, the error seems to converge almost linearly when plotted in this log-log space. This suggests a power-law or expenontial relationship between $e^N$ and $N$.