Differential equation: An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

For example of differential equations we consider the following:

(i) $\dfrac{dy}{dx}+5y=0$

(ii) $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}+6y=0$

(iii) $\dfrac{\partial \upsilon}{\partial s}+\dfrac{\partial \upsilon}{\partial t}=\upsilon$

(iv) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$

(v) $\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0$

Ordinary differential equation: A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.

Examples (i) and (ii) are ordinary differential equations.

Partial differential equation: A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation.

Examples (iii) and (iv) are partial differential equations.

Order of a differential equation: The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

For example,

(1) $\dfrac{dy}{dx}+5y=0$ First order

(2) $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}+6y=0$ Second order

(3) $\dfrac{d^3y}{dx^3}-3\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}-y=0$ Third order

Degree of a differential equation: The degree of a differential equation is the degree of the highest differential coefficient which occurs in it after the differential equation has been cleared of radicals and functions.

Let us consider

$$\rho = \frac{\left(1+y'^2\right)^{3/2}}{y''}$$

$$\Rightarrow \rho^2 = \frac{\left(1+y'^2\right)^3}{y''^2}$$

$$\therefore \ \rho^2 y''^2 = \left(1+y'^2\right)^3$$

So this equation is of second degree.

Solution of a differential equation: Any function ϕ defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an n-th order ordinary differential equation reduces the equation to an identity, is said to be a solution of the equation on the interval.

Linear ordinary differential equation: A linear ordinary differential equation of order n, in the dependent variable y and the independent variable x, is an equation that is in, or can be expressed in, the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x),$$

where a_0 is not identically zero.

Non-linear:

$$\left(\frac{d^2y}{dx^2}\right)^2 + y = 0$$

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} + xy = 0$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy^2 = 0$$

Linear:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$$

Separable equation: An equation of the form

$$f(x)g(y)dx + l(x)m(y)dy = 0$$

is called an equation with variables separable or simply a separable equation.

Example 1: $\left(1+y^2\right)dx + \left(1+x^2\right)dy = 0$

Solution: Given

$$\left(1+y^2\right)dx+\left(1+x^2\right)dy=0$$

Dividing throughout by $\left(1+x^2\right)\left(1+y^2\right)$, we get

$$\frac{dx}{1+x^2}+\frac{dy}{1+y^2}=0$$

Integrating, we have

$$\int\frac{dx}{1+x^2}+\int\frac{dy}{1+y^2}=c'$$

$$\Rightarrow \tan^{-1}x+\tan^{-1}y=c'$$

$$\Rightarrow \tan^{-1}\frac{x+y}{1-xy}=c'$$

$$\Rightarrow \frac{x+y}{1-xy}=\tan c'$$

$$\Rightarrow \frac{x+y}{1-xy}=C \ \ (\text{say})$$

$$\therefore x+y=C\left(1-xy\right)$$

which is the required solution.

Example 2: Solve the initial value problem $2x\left(1+y\right)dx-ydy=0$ when $x=0,\ y=-2.$

Solution: Given

$$2x\left(1+y\right)dx-ydy=0 \quad \text{when} \quad x=0,\ y=-2.$$

Dividing by $\left(1+y\right)$, we get

$$2x\,dx-\frac{y}{1+y}dy=0$$

Integrating, we have

$$2\int x\,dx-\int\frac{y}{1+y}dy=0$$

$$\Rightarrow 2\int x\,dx-\int\frac{\left(1+y\right)-1}{1+y}dy=0$$

$$\Rightarrow 2\int x\,dx-\int dy+\int\frac{1}{1+y}dy=0$$

$$\Rightarrow x^2 - y + \ln(1+y) = c \qquad (1)$$

Now using the initial condition $x = 0$ and $y = -2$, we obtain

$0^2 + 2 + \ln|1-2| = c$

$\therefore \quad c = 2$

Thus equation (1) gives

$x^2 - y + \ln(1+y) = 2$

$\therefore \quad x^2 = y - \ln(1+y) + 2$

which is the required solution of the given initial value problem.

Exercises:

1. $(4+x)\,y' = y^3$
2. $e^{y^2} dx + x^2 y dy = 0$
3. $\cos x \cos y\, dx + \sin x \sin y\, dy = 0$
4. $e^x (y-1)\, dx + 2(e^x + 4)\, dy = 0$
5. $(xy - x)\, dx + (y + xy)\, dy = 0$
6. $(y+1)\, dx = 2xy\, dy$
7. $x \cos^2 y\, dx + \tan y\, dy = 0$
8. $(xy + x)\, dx = (x^2 y^2 + x^2 + y^2 + 1)\, dy$
9. $x^2 y y' = e^y$
10. $\tan^2 y\, dy = \sin^3 x dx$
11. $(1 + \ln x)\, dx + (1 + \ln y)\, dy = 0$