LECTURE NO - 23

Linear approximation and differentials

Local linear approximation:

The line that best approximates the graph of f in the vicinity of $P(x_0, f(x_0))$ is the tangent line to the graph of f at x_0, given by the equation

$$y = f(x_0) + f'(x_0)(x - x_0) \quad \text{---------------} \ (i)$$

Thus, for values of x near x_0 we can approximate values of $f(x)$ by

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

This is called the ***local linear approximation*** of f at x_0.

This formula can also be expressed in terms of the increment $\Delta x = x - x_0$ as

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x \quad \text{-----------} \ (ii)$$

Example:

(a) Find the local linear approximation of $f(x) = \sqrt{x}$ at $x_0 = 1$.

(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{1.1}$, and compare your approximation to the result produced directly by a calculating utility.

Solution: Given that $f(x) = \sqrt{x}$

$$\therefore f'(x) = \frac{1}{2\sqrt{x}}$$

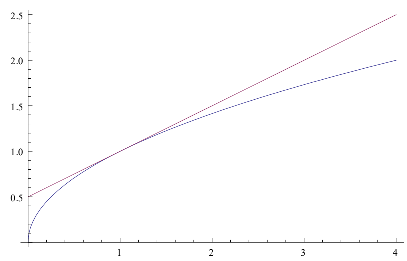
The local linear approximation of $\sqrt{x}$ at a point x_0 is

$$\sqrt{x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}(x - x_0)$$

Thus, the local linear approximation at $x_0 = 1$

$$\sqrt{x} \approx 1 + \frac{1}{2}(x - 1) \qquad (i)$$

(b) The graph of $y = \sqrt{x}$ and the local linear approximation $y \approx 1 + \frac{1}{2}(x - 1)$ are shown in the following figure.



Applying (i) with $x = 1.1$ yields

$$\sqrt{1.1} \approx 1 + \frac{1}{2}(1.1 - 1) = 1.05 \quad \text{Ans.}$$

By using calculator $\sqrt{1.1} = 1.04881$

Example:

(a) Find the local linear approximation of $f(x) = sinx$ at $x_0 = 0$.

(b) Use the local linear approximation obtained in part (a) to approximate $sin2°$, and compare your approximation to the result produced directly by your calculating device.

Solution:

Given that $f(x) = sinx$

$$\therefore f'(x) = cosx$$

The local linear approximation of $sinx$ at a point x_0 is

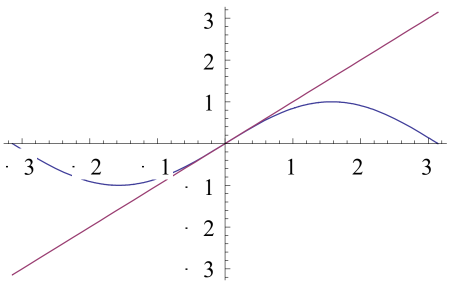
$$sinx \approx sinx_0 + cosx_0(x - x_0)$$

Thus, the local linear approximation at $x_0 = 0$ is

$$sinx \approx sin0 + cos0(x - 0)$$

Which implies $\quad sinx \approx x$

(b) The graph of $y = sinx$ and the local linear approximation $sinx \approx x$ are shown in the following figure.



First we convert 2° to radians before we can apply this approximation.

$$2° = 2\left(\frac{\pi}{180}\right) = \frac{\pi}{90} \approx 0.0349066 \text{ radian}$$

Again, by calculator $sin2° \approx 0.0348995$

Differentials:

We have interpreted *dy/dx* as a single entity representing the derivative of *y* with respect to *x*; the symbols "dy" and "dx", which are called ***differentials***, have had no meanings attached to them.

Now we define *dy* by the formula

$$dy = f'(x)dx \text{ --------- (1)}$$

If$dx \neq 0$, then we can define both sides of the above equation

$$\frac{dy}{dx} = f'(x) \text{ ------------ (2)}$$

Formula (1) is said to express (2) in ***differentials form.***

Note: It is important to understand the distinction between the increment Δy and the differential *dy*. To see the difference, let us assign the independent variables *dx* and Δx the same value, so *dx* = Δx.Then Δy represents the change in *y* that occurs when we start at *x* and travel *along the curve* $y = f(x)$ until we have moved $\Delta x (= dx)$ units in the x-direction, while *dy* represents the change in *y* that occurs if we start at x and travel *along the tangent line* until we have moved $dx (= \Delta x)$ units in the x-direction.

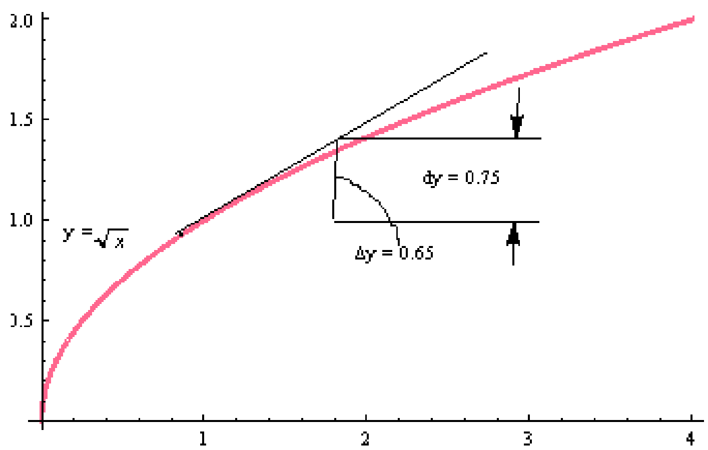
Example:

Let$y = \sqrt{x}$. Find *dy* and Δy at *x* = *4* with $dx = \Delta x = 3$. Then make a sketch of $y = \sqrt{x}$, showing *dy* and Δy in the picture.

Solution: With $f(x) = \sqrt{x}$ we obtain

$$\Delta y = f(x + \Delta x) - f(x) = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{7} - \sqrt{4} \approx 0.65$$

If $y = \sqrt{x}$, then $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, so $dy = \frac{1}{2\sqrt{x}} dx = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4} = 0.75$



Problem: Evaluate $\sqrt[3]{25}$ approximately by using of differentials.

Solution: If Δx is small.

$$\Delta y = f(x + \Delta x) - f(x) = f'(x)\, \Delta x \text{ approximately}$$

Let $f(x) = \sqrt[3]{x}$, then

$$\sqrt[3]{x + \Delta x} - \sqrt[3]{x} \approx \frac{1}{3} x^{-\frac{2}{3}} \Delta x$$

If x = 27 and $\Delta x = -2$ then we have

$$\sqrt[3]{27 - 2} - \sqrt[3]{27} \approx \frac{1}{3}(27)^{-\frac{2}{3}}(-2)$$

$$\Rightarrow \sqrt[3]{25} - 3 \approx \frac{1}{3} . \frac{1}{9}(-2)$$

$$\Rightarrow \sqrt[3]{25} \approx -\frac{2}{27} + 3$$

$$\Rightarrow \sqrt[3]{25} \approx 2.926 \text{ Ans.}$$

Problem:

1) $y = \frac{1}{x-1}$; x decreases from 2 to1.5

2) $y = \sqrt{25 - x^2}$; x increases from 0 to 3.