Linear equation: A first order ordinary differential equation is linear in the dependent variable y and the independent variable x if it is, or can be written in the form

$$\frac{dy}{dx}+P(x)y=Q(x).$$
$$\frac{dx}{dy}+P(y)x=Q(y). \tag{1}$$

For example, the equation

$$x\frac{dy}{dx}+(x+1)y=x^3$$

is a first order linear differential equation, for it can be written as

$$\frac{dy}{dx}+\left(1+\frac{1}{x}\right)y=x^2,$$

which is of the form (1) with $P(x)=1+\dfrac{1}{x}$ and $Q(x)=x^2$.

Integrating factor: If the differential equation

$$M(x,y)dx+N(x,y)dy=0 \tag{1}$$

is not exact in a domain D but the differential equation

$$\mu(x,y)M(x,y)dx+\mu(x,y)N(x,y)dy=0 \tag{2}$$

is exact in D, then $\mu(x, y)$ is called an integrating factor of the differential equation (2).

Theorem:

The linear differential equation

$$\frac{dy}{dx}+P(x)y=Q(x). \tag{1}$$

has an integrating factor of the form

$$e^{\int P(x)dx}$$

A one-parameter family of solutions of this equation is

$$ye^{\int P(x)dx}=\int e^{\int P(x)dx}Q(x)\,dx+c$$

that is,

$$y = e^{-\int P(x)dx}\left[\int e^{\int P(x)dx} Q(x)\,dx + c\right]$$

Problem: Solve the differential equation

$$x\frac{dy}{dx} + (x+1)y = x^3$$

$$\Rightarrow \left\{(x+1)y - x^3\right\}dx + xdy = 0$$

$$M = (x+1)y - x^3, \quad N = x$$

Solution: Given the differential equation

$$x\frac{dy}{dx} + (x+1)y = x^3$$

$$\therefore \ \frac{dy}{dx} + \left(1+\frac{1}{x}\right)y = x^2 \qquad (1)$$

So the integrating factor is

$$I.F. = e^{\int\left(1+\frac{1}{x}\right)dx} = e^{x+\ln x} = xe^x$$

Multiplying both sides of equation (1) by xe^x, we have

$$xe^x\frac{dy}{dx} + xe^x\left(1+\frac{1}{x}\right)y = x^3e^x$$

$$\Rightarrow \frac{d}{dx}\left(xe^x y\right) = x^3e^x$$

Integrating,

$$\Rightarrow xe^x y = \int x^3e^x dx + C, \quad \text{where } C \text{ is an integrating constant.}$$

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$\Rightarrow xe^x y = x^3\int e^x dx - \int\left\{\frac{d}{dx}\left(x^3\right)\int e^x dx\right\}dx + C$$

$$= x^3\int e^x dx - 3\int x^2e^x dx + C$$

$$= x^3e^x - 3\left[x^2\int e^x dx - \int\left\{\frac{d}{dx}\left(x^2\right)\int e^x dx\right\}dx\right] + C$$

$$= x^3e^x - 3\left[x^2e^x - 2\int xe^x dx\right] + C$$

$$= x^3e^x - 3x^2e^x + 6\left[x\int e^x dx - \int\left\{\frac{d}{dx}(x)\int e^x dx\right\}dx\right] + C$$

$$= x^3e^x - 3x^2e^x + 6\left[xe^x - \int e^x dx\right] + C$$

$$= x^3e^x - 3x^2e^x + 6xe^x - 6\int e^x dx + C$$

$$\Rightarrow xe^x y = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

$$\therefore\ y = x^2 - 3x + 6 - \frac{6}{x} + \frac{C}{x}e^{-x}$$

which is the required solution.

Problem: Solve $\left(x+2y^3\right)\frac{dy}{dx} = y$.

Solution: Given the differential equation,

$$\left(x+2y^3\right)\frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x+2y^3}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x+2y^3}{y}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x}{y} + 2y^2$$

$$\therefore\ \frac{dx}{dy} - \frac{x}{y} = 2y^2$$

So the integrating factor is

$$I.F. = e^{-\int \frac{1}{y}dy} = e^{-\ln y} = e^{\ln y^{-1}} = \frac{1}{y}$$

Multiplying both sides of equation (1) by *I.F*, we have

$$\frac{1}{y}\frac{dx}{dy} - \frac{x}{y^2} = 2y$$

$$\Rightarrow \frac{d}{dy}\left(\frac{x}{y}\right) = 2y^2$$

Integrating,

$\frac{x}{y} = 2\int y^2 dy + C,$ where C is an integrating factor.

$\Rightarrow \frac{x}{y} = 2 \times \frac{y^3}{3} + C,$

$\therefore x = \frac{2}{3} y^4 + Cy,$

which is the required solution.

Exercises:

(1) $\left(1+x^2\right)\frac{dy}{dx} + 2xy = 4x^2$

(2) $\left(x+2y^3\right)\frac{dy}{dx} = y$

(3) $\cos^2 x \frac{dy}{dx} + y = \tan x$

(4) $\frac{dy}{dx} + xy = x$

(5) $y \ln y \, dx + \left(x - \ln y\right) dy = 0$

(6) $\frac{dy}{dx} + y \tan x - \sec x = 0$

(7) $\left(x+y+1\right)\frac{dy}{dx} = 1$

(8) $\frac{dy}{dx} + 2xy = 2x\left(1+x^2\right)$

(9) $x\frac{dy}{dx} - 2y = \left(x-2\right)e^x$

Bernouli differential equation:

(1) $\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3 \cos^2 y$

(2) $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

(3) $\frac{dy}{dx} + y = y^2 e^x$

(4) $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

(5) $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$

(6) $\frac{dy}{dx} + \frac{y}{x}\ln y = \frac{y}{x^2}(\ln y)^2$

(7) $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

(8) $x\frac{dy}{dx} + y = y^2 \ln x$

(9) $\frac{dy}{dx} - \frac{y}{x} = y^2 \sin x^2$

(10) $\frac{dy}{dx} + x\sin 2y = x\cos^2 y$

(11) $\left(1-x^2\right)\frac{dy}{dx} + xy = xy^2 \quad (x<1)$

(12) $\left(1-x^2\right)\frac{dy}{dx} + xy = xy^2 \quad (x>1)$

(13) $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

(14) $\left(x^2y^3 + xy\right)\frac{dy}{dx} = 1$

(15) $\left(x^3y^2 + xy\right)dx = dy$

(16) $\frac{dy}{dx} - \frac{y}{x} = -\frac{y^2}{x}$

(17) $\frac{dy}{dx} + y = xy^3$

Bernoulli Equation:

An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad n \neq 0 \text{ or} 1$$

is called a Bernoulli differential equation.

Theorem: Suppose n ≠ 0 or 1. Then the transformation $\upsilon = y^{1-n}$ reduces the Bernoulli equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad n \neq 0 \text{ or} 1$$

to a linear equation in υ.

Proof: Given the differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad n \neq 0 \text{ or} 1 \tag{1}$$

We can rewrite it as

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \tag{2}$$

If we let $\upsilon = y^{1-n}$ then we have

$$\frac{d\upsilon}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

or, $y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{d\upsilon}{dx}$

Therefore, Eq. **Error! Reference source not found.** reduces to

$$\frac{1}{(1-n)}\frac{d\upsilon}{dx} + P(x)\upsilon = Q(x)$$

or, $\frac{d\upsilon}{dx} + (1-n)P(x)\upsilon = (1-n)Q(x)$ (3)

Assuming

$$P_1(x) = (1-n)P(x)$$

and

$$Q_1(x) = (1-n)Q(x)$$

equation **Error! Reference source not found.** can be written as

$$\frac{d\upsilon}{dx} + P_1(x)\upsilon = Q_1(x) \tag{4}$$

which is a linear equation in υ.

Example: Solve $\frac{dy}{dx} + y = xy^3$.

Solution: Given differential equation

$$\frac{dy}{dx} + y = xy^3$$

Rewriting the equation, we obtain

$$y^{-3}\frac{dy}{dx} + y^{-2} = x \tag{5}$$

Let $\upsilon = y^{-2}$. Then we find

$$\frac{d\upsilon}{dx} = -2y^{-3}\frac{dy}{dx}$$

or, $y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{d\upsilon}{dx}$

Thus equation **Error! Reference source not found.** becomes

$$-\frac{1}{2}\frac{d\upsilon}{dx} + \upsilon = x$$

$$\therefore \quad \frac{d\upsilon}{dx} - 2\upsilon = -2x \tag{6}$$

Integrating factor of this equation

$$\begin{aligned} I.F. &= e^{-2\int dx} \\ &= e^{-2x} \end{aligned}$$

Multiplying both sides of Eq. **Error! Reference source not found.** by e^{-2x}, we have

$$e^{-2x}\frac{d\upsilon}{dx} - 2e^{-2x}\upsilon = -2xe^{-2x}$$

$$\Rightarrow \frac{d}{dx}\left(e^{-2x}\upsilon\right) = -2xe^{-2x}$$

Integrating, we get

$e^{-2x}\upsilon = -2\int xe^{-2x}dx + C$ where C is an integrating constant.

$$\Rightarrow \frac{1}{y^2}e^{-2x} = -2x \times \frac{e^{-2x}}{-2} - \int e^{-2x}dx + C$$

$$\Rightarrow \frac{1}{y^2}e^{-2x} = xe^{-2x} + \frac{1}{2}e^{-2x} + C$$

$$\Rightarrow \frac{1}{y^2}e^{-2x} = xe^{-2x} + \frac{1}{2}e^{-2x} + C$$

$$\therefore \quad \frac{1}{y^2} = x + \frac{1}{2} + Ce^{2x}$$

which is the required solution.