Exact or total differential: Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential dF of the function F is defined by the formula

$$dF(x,y) = \frac{\partial F(x,y)}{\partial x}dx + \frac{\partial F(x,y)}{\partial y}dy$$

for all $(x,y) \in D$.

Exact differential equation:

The expression

$$M(x,y)\,dx + N(x,y)\,dy \qquad (1)$$

is called an exact differential in a domain D if there exists a function F of two real variables such that this expression equals the total differential $dF(x, y)$ for all $(x,y) \in D$.

That is, expression (1) is an exact differential in D if there exists a function F such that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

for all $(x,y) \in D$.

If $M(x, y)\,dx + N(x, y)\,dy$ is an exact differential, then the differential equation

$M(x, y)\,dx + N(x, y)\,dy = 0$

is called an exact differential equation.

Example: Let $F(x,y) = xy$

$$\therefore \quad \frac{\partial F}{\partial x} = y, \quad \frac{\partial F}{\partial y} = x$$

Then

$$dF = y\,dx + x\,dy = d(xy)$$

for all real (x, y).

Theorem 1: Consider the differential equation

$$M\,dx + N\,dy = 0 \qquad (1)$$

where M and N have continuous first partial derivatives at all points (x, y) in a rectangular domain D.

1. If the differential equation (1) is exact in D, then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{for all } (x, y) \in D.$$

2. Conversely, if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

for all $(x, y) \in D,$ then the differential equation (1) is exact in *D*.

Problem: Solve $3x(xy-2)\,dx + (x^3 + 2y)\,dy = 0.$

Solution: $3x(xy-2)\,dx + (x^3 + 2y)\,dy = 0.$

Here

$$M = 3x(xy-2)$$

$$N = x^3 + 2y$$

$$\therefore \quad \frac{\partial M}{\partial y} = 3x^2$$

and $\dfrac{\partial N}{\partial x} = 3x^2$

Therefore, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Hence the equation is exact.

Now

$$3x^2 y\,dx - 6x\,dx + x^3\,dy + 2y\,dy = 0$$

$$\Rightarrow 3x^2 y\,dx + x^3\,dy - 6x\,dx + 2y\,dy = 0$$

$$\Rightarrow d\left(x^3 y\right) - d\left(3x^2\right) + d\left(y^2\right) = 0$$

$$\Rightarrow d\left(x^3 y - 3x^2 + y^2\right) = 0$$

Integrating, we get

$x^3 y - 3x^2 + y^2 = C,$ where *C* is an integrating constant.

Theorem 2: Suppose the differential equation $M\,dx + N\,dy = 0$ satisfies the differentiability requirements of Theorem 1 and is exact in a rectangular domain *D*. Then a one parameter family of solutions of this differential equation is given by $F(x, y) = C,$ where *F* is a function such that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N \quad \text{for all } (x, y) \in D$$

and C is an arbitrary constant.

Problem: Solve $3x(xy-2)dx+(x^3+2y)dy=0.$

Solution: $3x(xy-2)dx+(x^3+2y)dy=0.$

Here

$M=3x(xy-2)$

$N=x^3+2y$

$$\therefore \quad \frac{\partial M}{\partial y}=3x^2$$

and $\dfrac{\partial N}{\partial x}=3x^2$

Therefore, $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$

Hence the equation is exact and its solution is given by $F(x,y)=C,$ where

$$\frac{\partial F}{\partial x}=M=3x^2y-6x \tag{1}$$

and

$$\frac{\partial F}{\partial y}=N=x^3+2y \tag{2}$$

Integrating equation (1) with respect to *x*, we get

$$F(x,y)=x^3y-3x^2+g(y) \tag{3}$$

Now from (3), we find

$$\frac{\partial F}{\partial y}=x^3+g'(y) \tag{4}$$

Comparing (3) and (4), we have

$g'(y)=2y$

Integrating,

$g(y)=y^2$

Therefore, the general solution is

$$x^3y - 3x^2 + y^2 = C.$$