Capacitors and Inductors

Capacitors and Inductors

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- Capacitors
- Series and Parallel Capacitors
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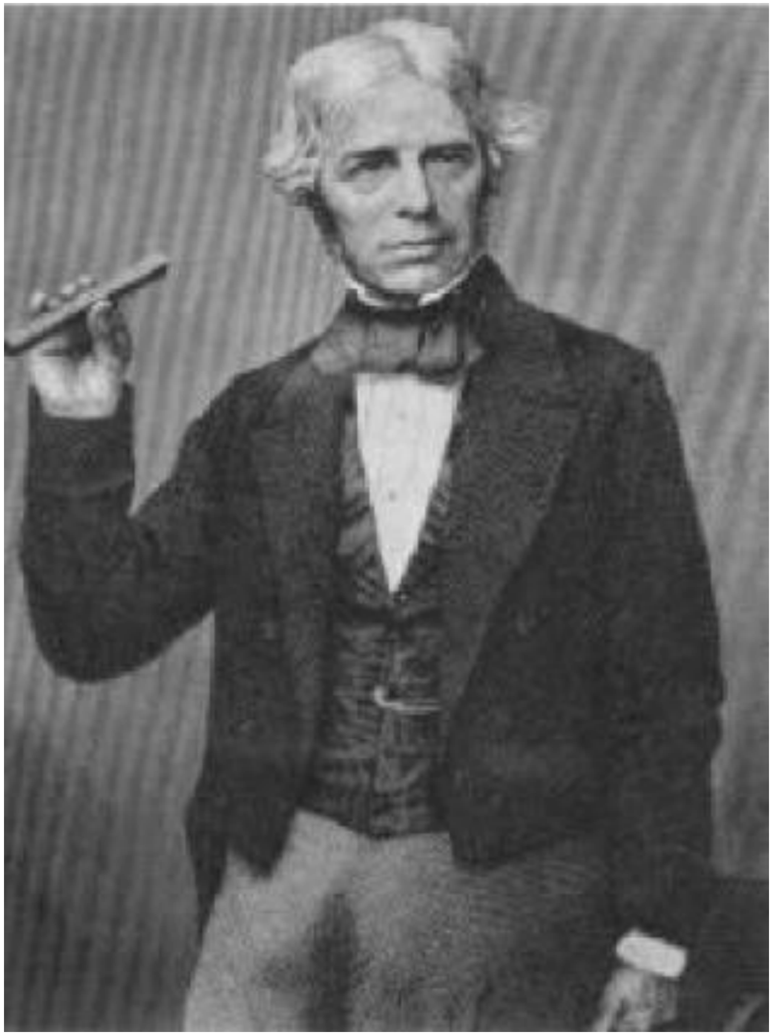
- Series and Parallel Inductors

Introduction

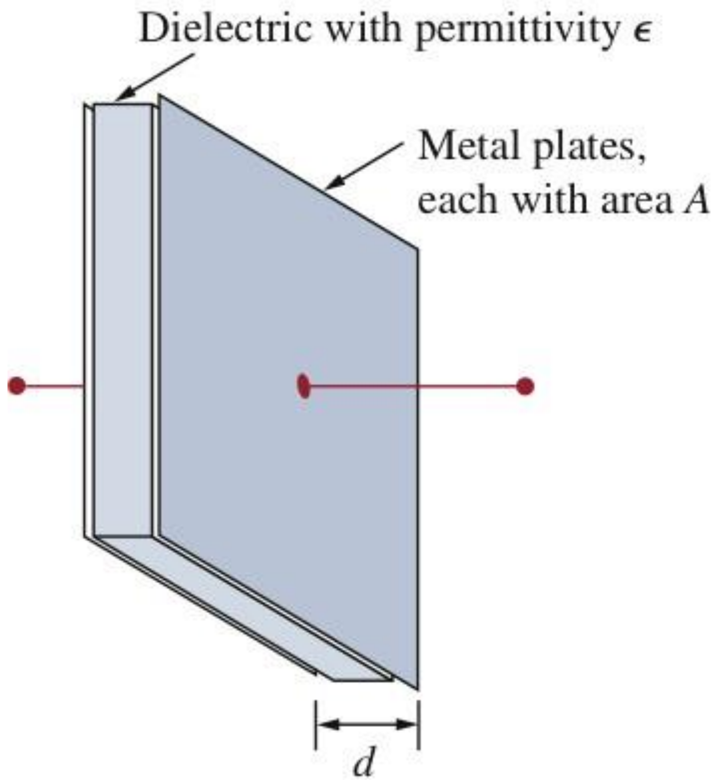
- Resistor: a passive element which dissipates energy only
- Two important **passive** linear circuit elements:
  1) Capacitor
  2) Inductor
- Capacitor and inductor can **store energy** only and they can **neither generate nor dissipate energy**.

Michael Faraday (1971-1867)



6.2 Capacitors

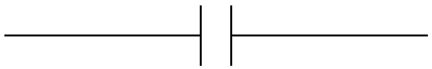
- A **capacitor** consists of two conducting plates separated by an insulator (or dielectric).



$$C = \frac{\varepsilon A}{d}$$

$$\varepsilon = \varepsilon_r \varepsilon_0$$

$$\varepsilon_0 = 8.854 \times 10^{-12} \, \text{(F/m)}$$



$$C = \frac{\varepsilon A}{d}$$

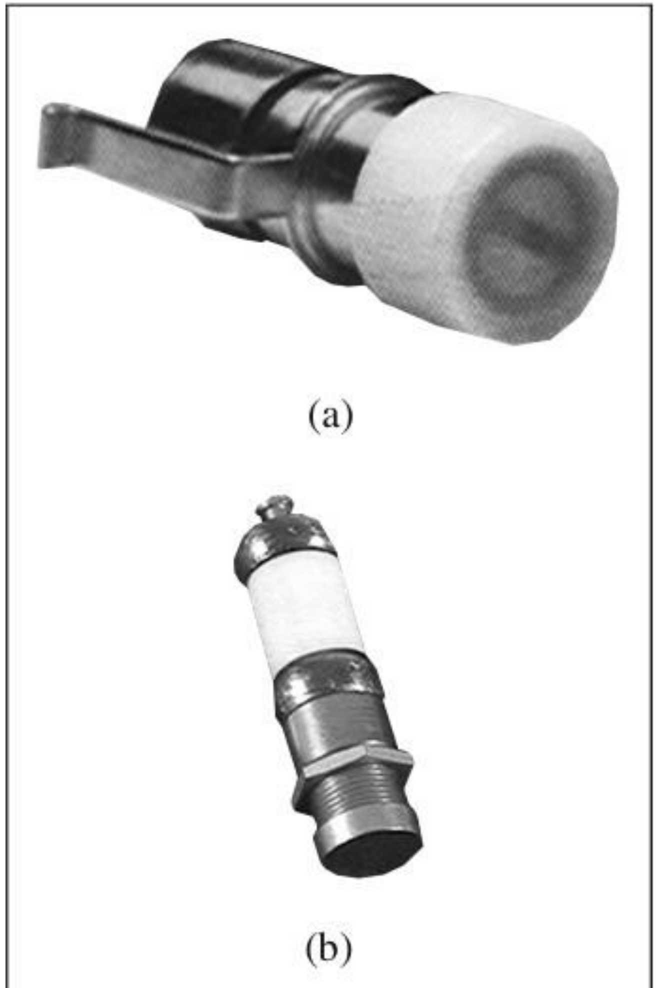
- Three factors affecting the value of capacitance:
  1. Area: the larger the area, the greater the capacitance.
  2. Spacing between the plates: the smaller the spacing, the greater the capacitance.
  3. Material permittivity: the higher the permittivity, the greater the capacitance.

Fig 6.4

(a) (b) (c)

(a) Polyester capacitor, (b) Ceramic capacitor, (c) Electrolytic capacitor

Fig 6.5

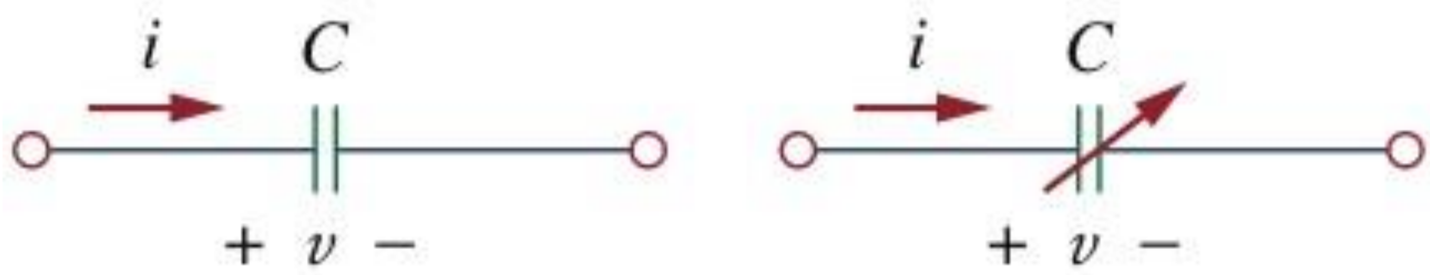


(a)

(b)

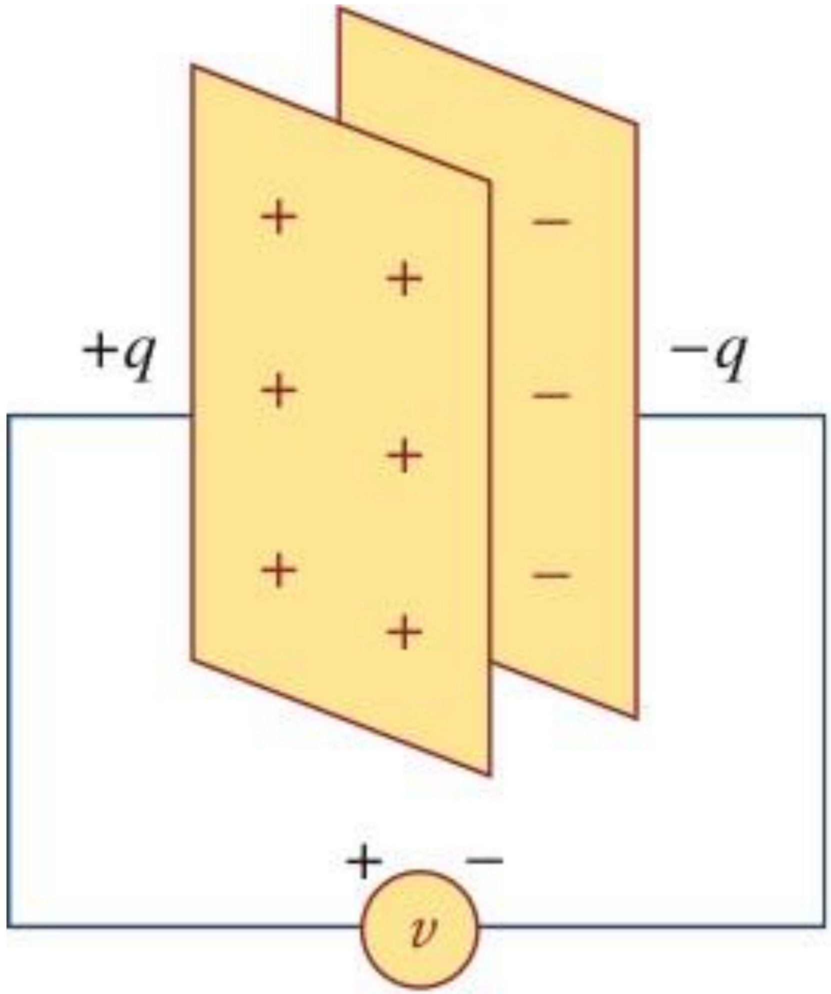
Variable capacitors

Fig 6.3



(a) (b)

Fig 6.2

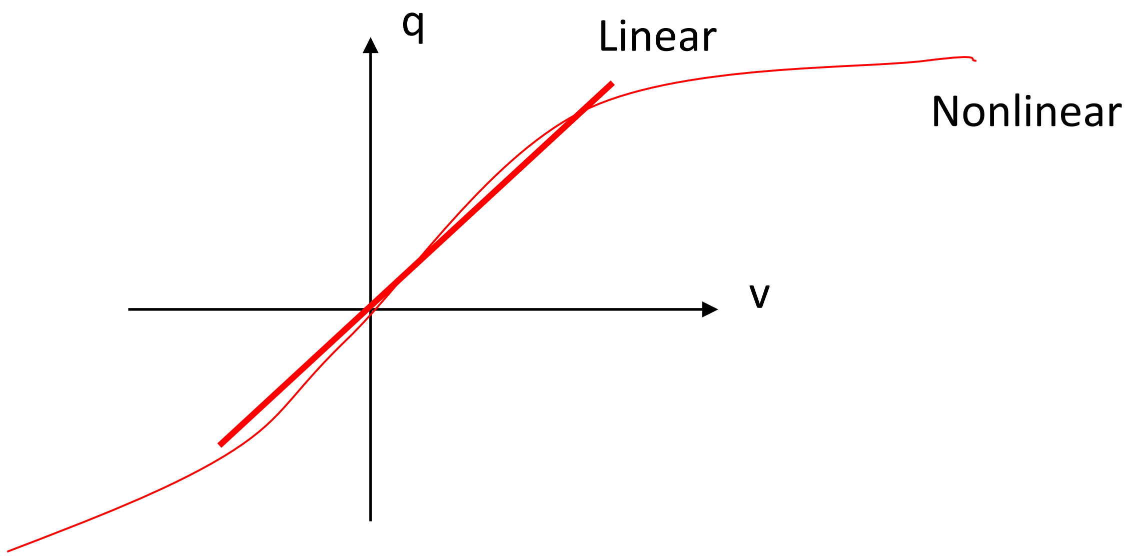


Charge in Capacitors

- The relation between the charge in plates and the voltage across a capacitor is given below.

$$q = Cv$$

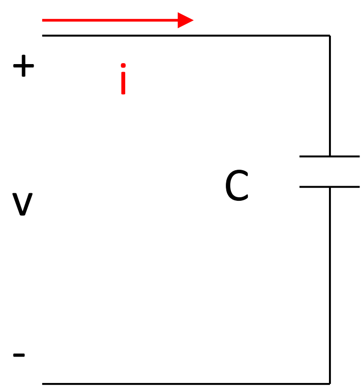
$$1\text{F} = 1\ \text{C/V}$$



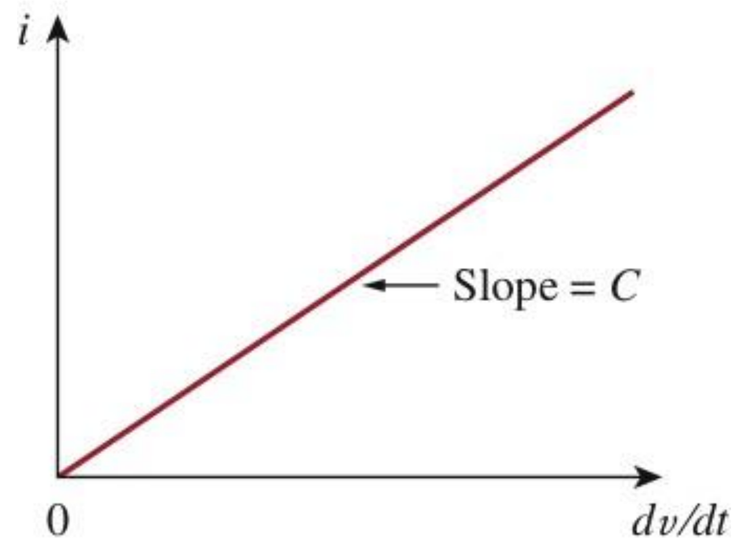
Voltage Limit on a Capacitor

- Since $q=Cv$, the plate charge increases as the voltage increases. The electric field intensity between two plates increases. If the voltage across the capacitor is so large that the field intensity is large enough to break down the insulation of the dielectric, the capacitor is out of work. Hence, every practical capacitor has a maximum limit on its operating voltage.

I-V Relation of Capacitor

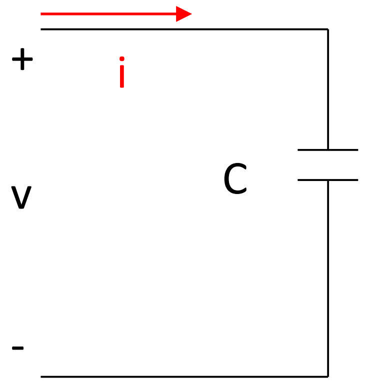


$$q = Cv, i = \frac{dq}{dt} = C\frac{dv}{dt}$$



Physical Meaning

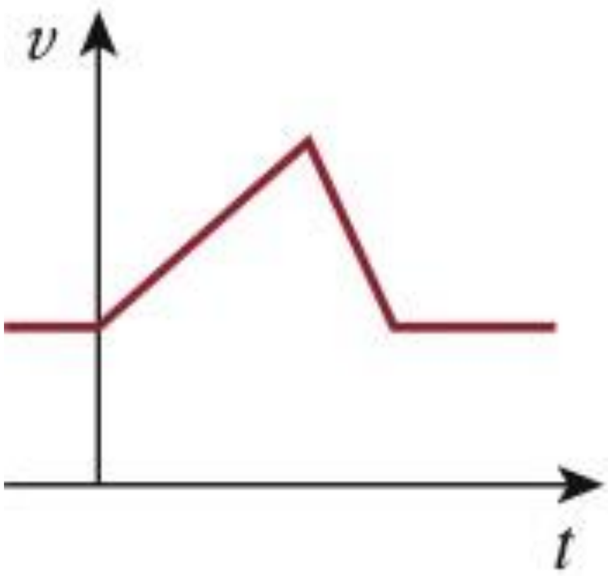
$$i = C\frac{dv}{dt}$$



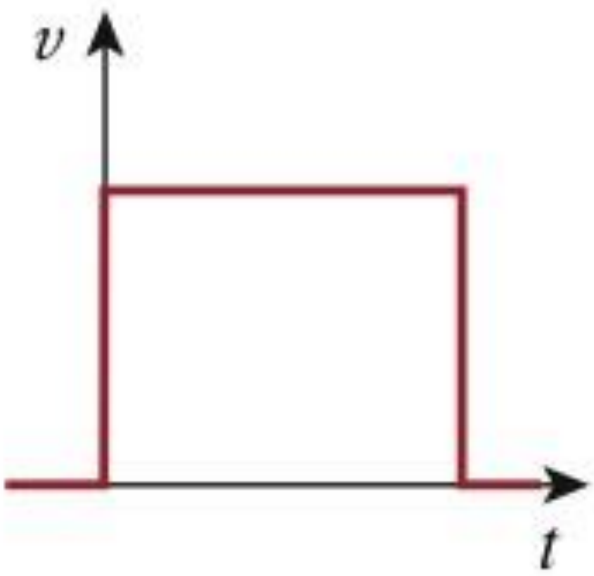
- when **v is a constant voltage**, then i=0; a constant voltage across a capacitor creates no current through the capacitor, the capacitor in this case is the same as an **open circuit**.
- If v is abruptly changed, then the current will have an infinite value that is practically impossible. Hence, a capacitor is impossible to have **an abrupt change in its voltage** except an infinite current is applied.

Fig 6.7

- A capacitor is an open circuit to dc.
- The voltage on a capacitor cannot change abruptly.



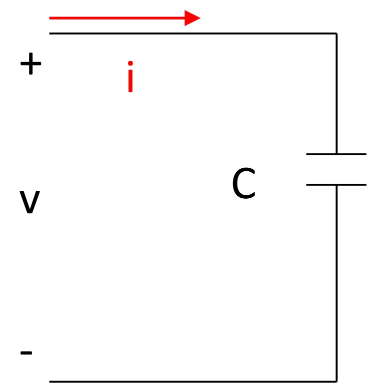
(a)



(b) Abrupt change

$$i = C\frac{dv}{dt} \qquad v(t) = \frac{1}{C}\int_{-\infty}^{t} i\,dt \qquad \left(v(-\infty) = 0 \right)$$

$$v(t) = \frac{1}{C}\int_{t_o}^{t} i\,dt + v(t_o) \qquad \left(v(t_o) = q(t_o)/C \right)$$



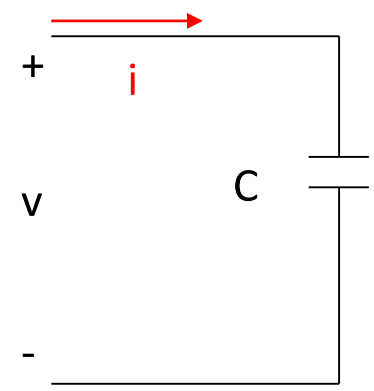
- The charge on a capacitor is an integration of current through the capacitor. Hence, the **memory effect** counts.

Energy Storing in Capacitor

$$p = vi = Cv\frac{dv}{dt}$$

$$w = \int_{-\infty}^{t} p\,dt = C\int_{-\infty}^{t} v\frac{dv}{dt}dt = C\int_{v(-\infty)}^{v(t)} v\,dv = \frac{1}{2}Cv^2\Big|_{v(-\infty)}^{v(t)}$$

$$w(t) = \frac{1}{2}Cv^2(t) \qquad (v(-\infty) = 0)$$



$$w(t) = \frac{q^2(t)}{2C}$$

Example 6.1

(a) Calculate the charge stored on a 3-pF capacitor with 20V across it.

(b) Find the energy stored in the capacitor.

Example 6.1

Solution:

(a) Since $q = Cv,$

$$q = 3 \times 10^{-12} \times 20 = 60\text{pC}$$

(b) The energy stored is

$$w = \frac{1}{2} C v^2 = \frac{1}{2} \times 3 \times 10^{-12} \times 400 = 600\text{pJ}$$

Example 6.2

- The voltage across a 5- μF capacitor is

$$v(t) = 10\cos 6000t \ \ \text{V}$$

Calculate the current through it.

Solution:

- By definition, the current is

$$i = C\frac{dv}{dt} = 5\times10^{-6}\frac{d}{dt}(10\cos 6000t)$$

$$= -5\times10^{-6}\times6000\times10\sin 6000t = -0.3\sin 6000t \ \ \text{A}$$

Example 6.3

- Determine the voltage across a 2-μF capacitor if the current through it is

$$i(t) = 6e^{-3000t}\,\text{mA}$$

Assume that the initial capacitor voltage is zero.

Solution:

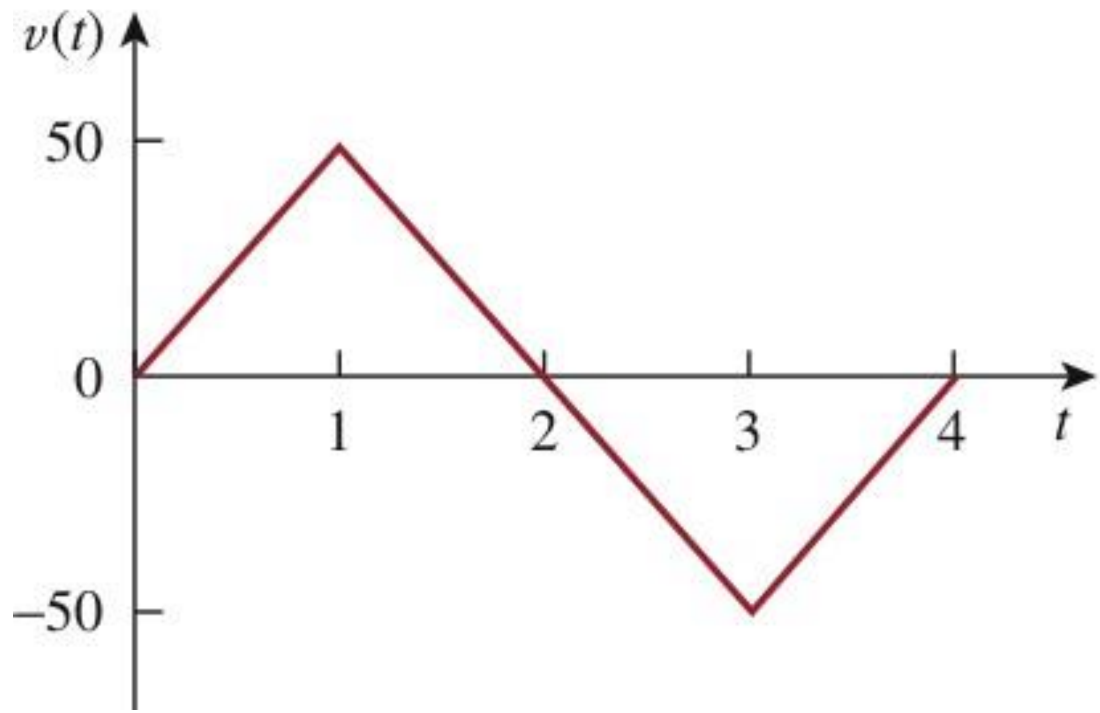
- Since $v = \dfrac{1}{C}\int_0^t i\,dt + v(0)$ and $v(0) = 0,$

$$v = \frac{1}{2\times 10^{-6}}\int_0^t 6e^{-3000t}\;dt \cdot 10^{-3} = \frac{3\times 10^{3}}{-3000}e^{-3000t}\Big|_0^t$$

$$= (1 - e^{-3000t})\text{V}$$

Example 6.4

- Determine the current through a 200- μF capacitor whose voltage is shown in Fig 6.9.



Example 6.4

Solution:

- The voltage waveform can be described mathematically as

$$v(t) = \begin{cases} 50t \text{ V} & 0 < t < 1 \\ 100 - 50t \text{ V} & 1 < t < 3 \\ -200 + 50t \text{ V} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

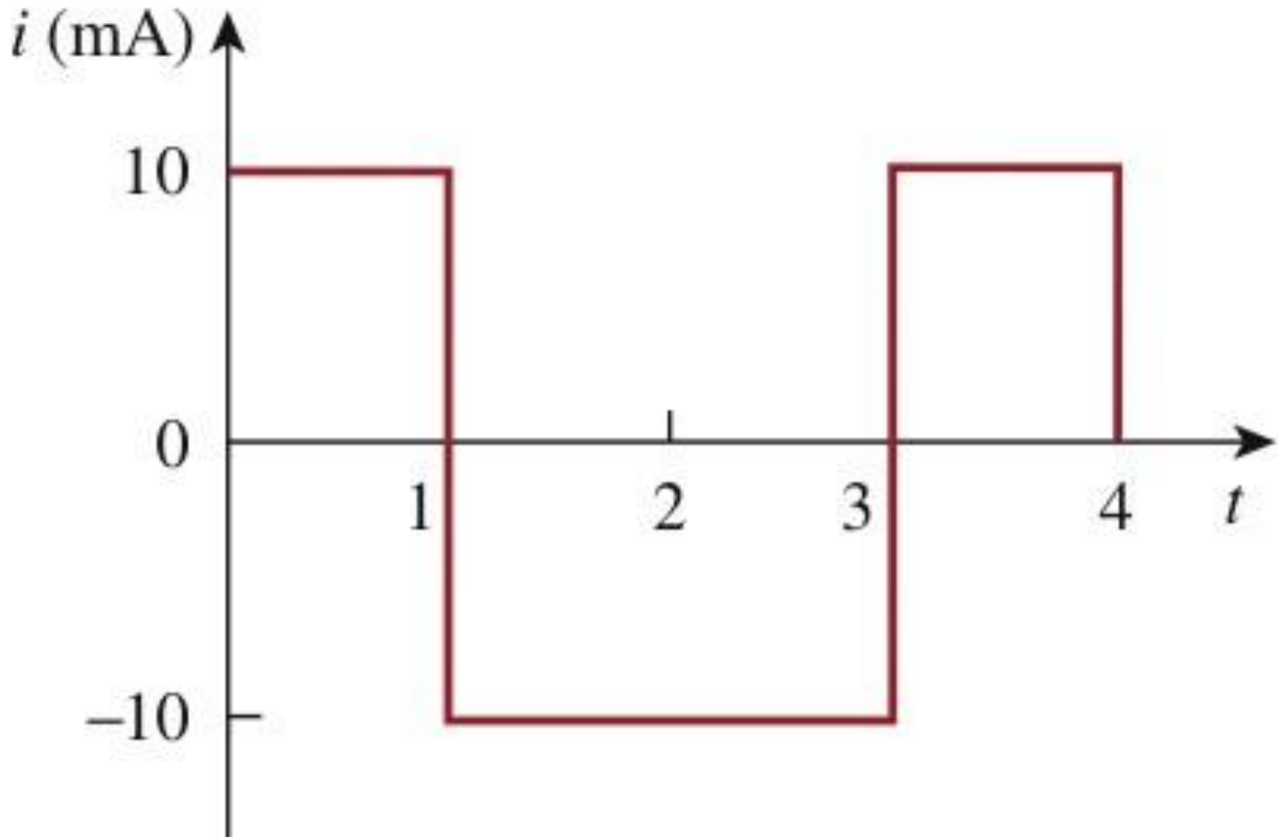
Example 6.4

- Since $i = C$ dv/dt and $C = 200\ \mu$F, we take the derivative of to obtain

$$i(t) = 200\times 10^{-6} \times \begin{cases} 50 & 0<t<1 \\ -50 & 1<t<3 \\ 50 & 3<t<4 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 10\text{mA} & 0<t<1 \\ -10\text{mA} & 1<t<3 \\ 10\text{mA} & 3<t<4 \\ 0 & \text{otherwise} \end{cases}$$

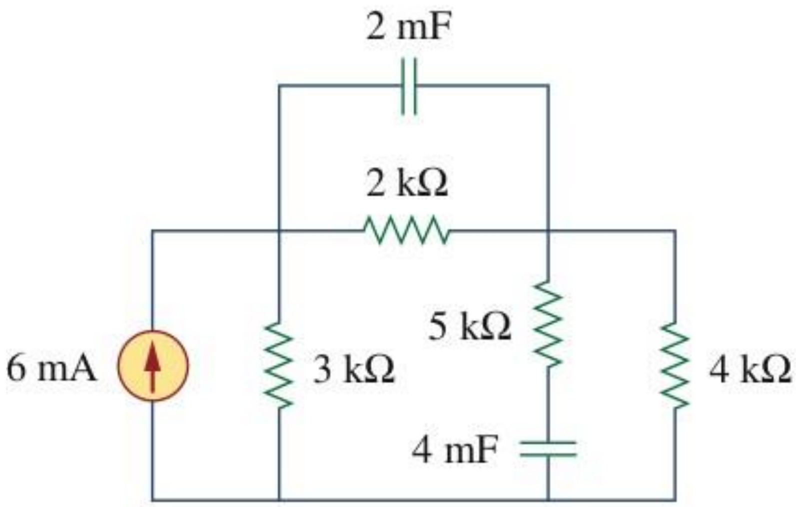
- Thus the current waveform is shown in Fig.6.10.

Example 6.4

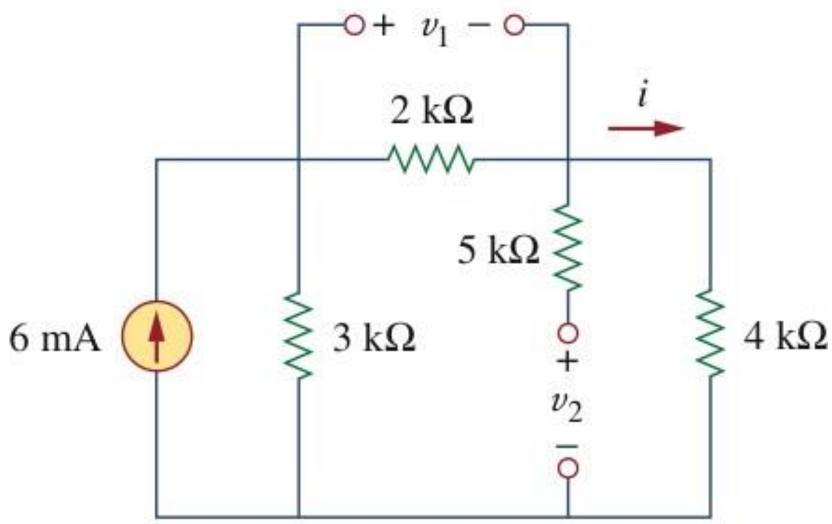


Example 6.5

- Obtain the energy stored in each capacitor in Fig. 6.12(a) under dc condition.



(a)



(b)

Example 6.5

Solution:

- Under dc condition, we replace each capacitor with an open circuit. By current division,

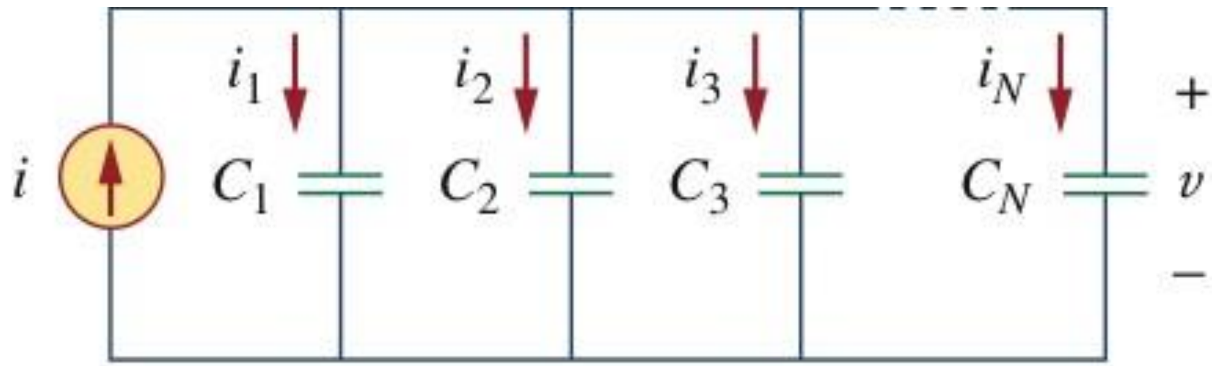
$$i = \frac{3}{3+2+4}(6\text{mA}) = 2\text{mA}$$

$$\therefore v_1 = 2000i = 4\,\text{V}, \quad v_2 = 4000i = 8\,\text{V}$$

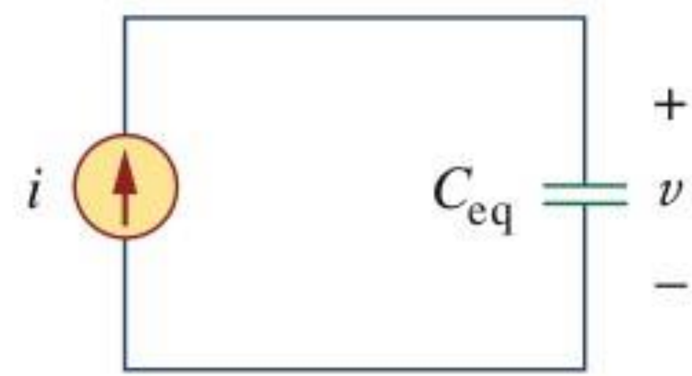
$$\therefore w_1 = \frac{1}{2}C_1 v_1^2 = \frac{1}{2}(2\times 10^{-3})(4)^2 = 16\text{mJ}$$

$$w_2 = \frac{1}{2}C_2 v_2^2 = \frac{1}{2}(4\times 10^{-3})(8)^2 = 128\text{mJ}$$

Fig 6.14



(a)



(b)

6.3 Series and Parallel Capacitors

$$i = i_1 + i_2 + i_3 + \ldots + i_N$$

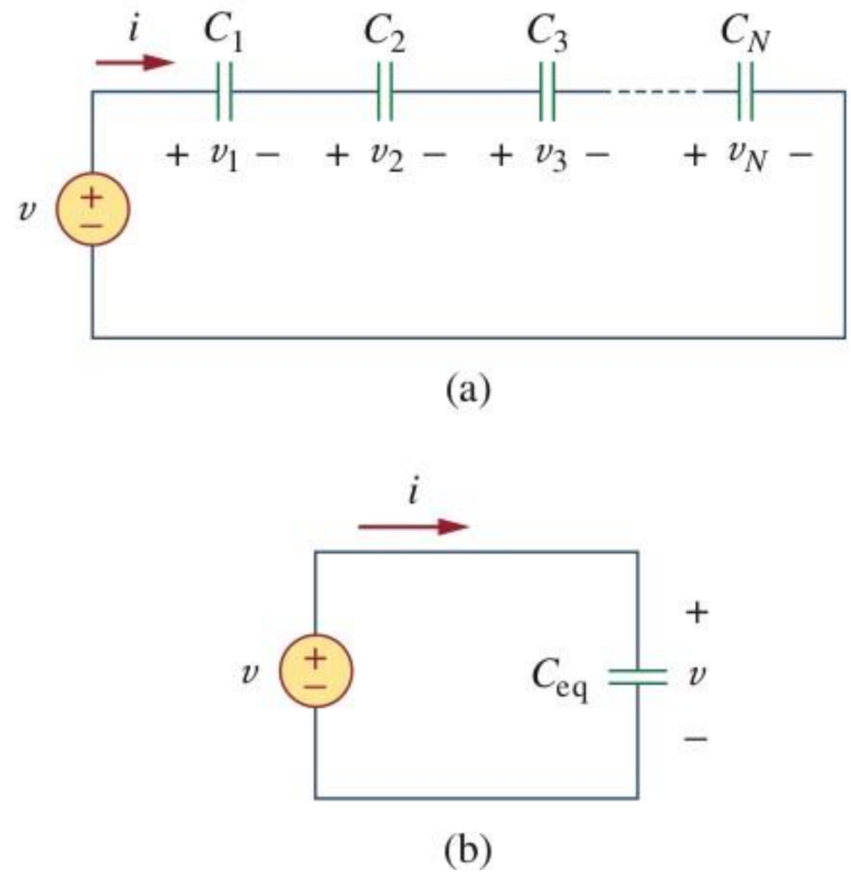
$$i = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt} + \ldots + C_N \frac{dv}{dt}$$

$$= \left(\sum_{k=1}^{N} C_K \right) \frac{dv}{dt} = C_{eq} \frac{dv}{dt}$$

$$C_{eq} = C_1 + C_2 + C_3 + \ldots. + C_N$$

- The **equivalent capacitance** of *N* parallel-connected capacitors is the sum of the individual capacitance.

Fig 6.15



$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots + \frac{1}{C_N}$$

Series Capacitors

$$v(t) = v_1(t) + v_2(t) + \ldots + v_N(t)$$

$$\frac{1}{C_{eq}} \int_{-\infty}^{t} i d\tau = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots + \frac{1}{C_N}\right) \int_{-\infty}^{t} i d\tau$$

$$\frac{q(t)}{C_{eq}} = \frac{q(t)}{C_1} + \frac{q(t)}{C_2} + \cdots + \frac{q(t)}{C_N}$$

- The **equivalent capacitance** of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

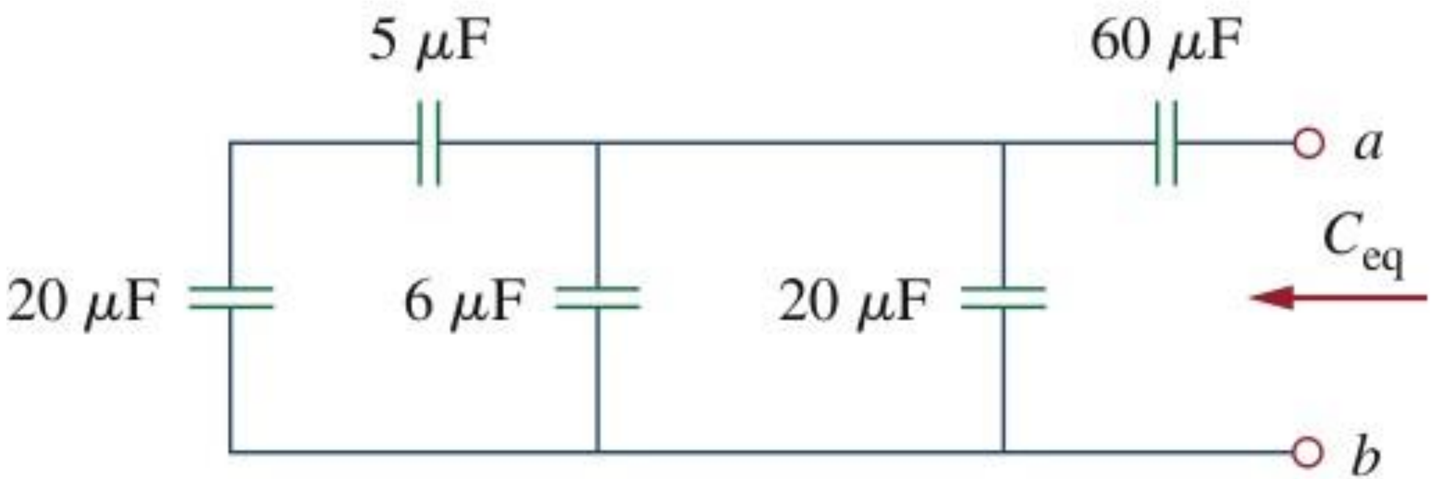
$$\boxed{C_{eq} = \frac{C_1 C_2}{C_1 + C_2}}$$

Summary

- These results enable us to look the capacitor in this way: **1/C has the equivalent effect as the resistance.** The equivalent capacitor of capacitors connected in parallel or series can be obtained via this point of view, so is the Y-△ connection and its transformation

Example 6.6

- Find the equivalent capacitance seen between terminals a and b of the circuit in Fig 6.16.



Example 6.6

Solution:

- $20-\mu\text{F}$ and $5-\mu\text{F}$ capacitors are in series:

$$\therefore \frac{20\times 5}{20+5} = 4\mu\text{F}$$

- $4-\mu\text{F}$ capacitor is in parallel with the $6-\mu\text{F}$ and $20-\mu\text{F}$ capacitors:

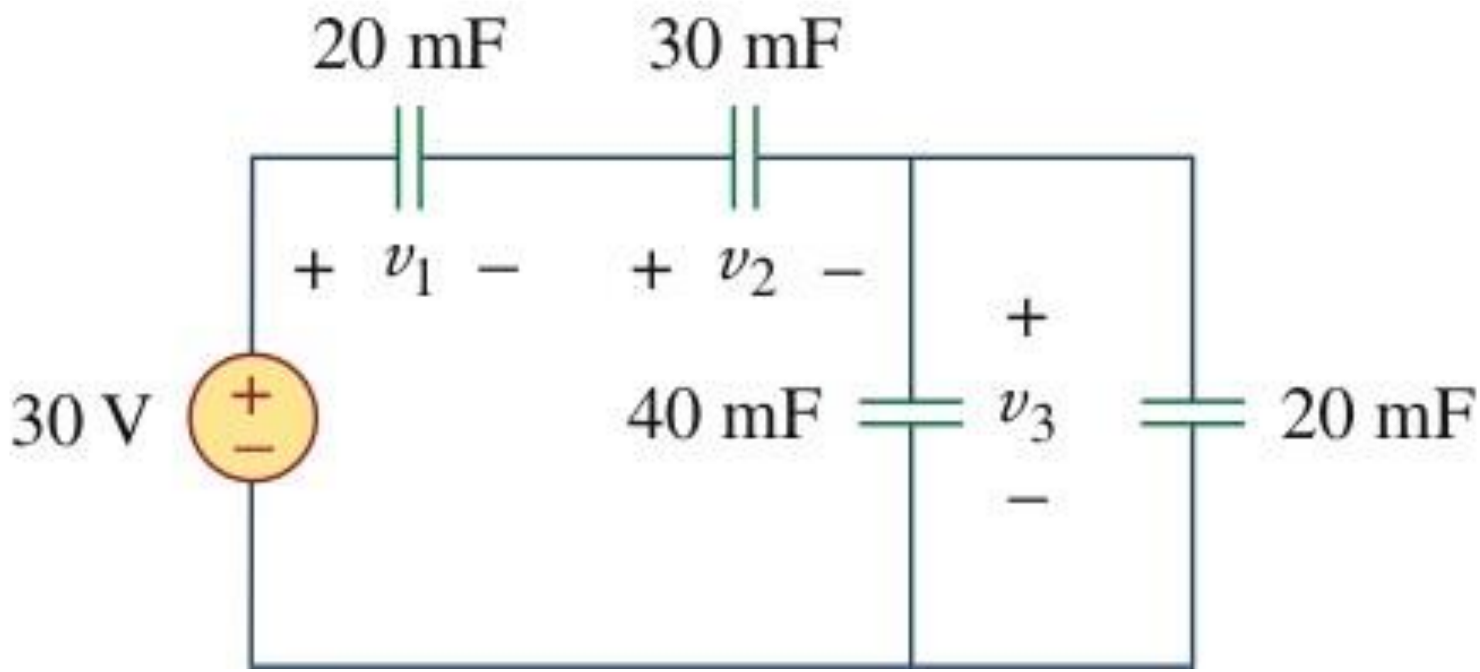
$$\therefore 4+6+20 = 30\mu\text{F}$$

- $30-\mu\text{F}$ capacitor is in series with the $60-\mu\text{F}$ capacitor.

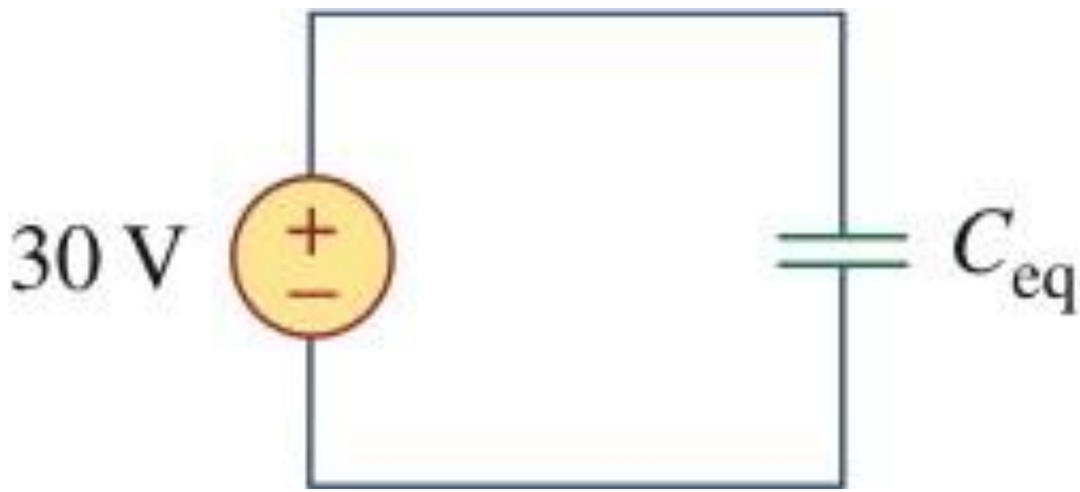
$$C_{eq} = \frac{30\times 60}{30+60}\mu\text{F} = 20\mu\text{F}$$

Example 6.7

- For the circuit in Fig 6.18, find the voltage across each capacitor.



Example 6.7



Example 6.7

Solution:

- Two parallel capacitors:

$$\therefore C_{eq} = \frac{1}{\frac{1}{60}+\frac{1}{30}+\frac{1}{20}} \mathrm{mF} = 10\mathrm{mF}$$

- Total charge

$$q = C_{eq}v = 10\times 10^{-3}\times 30 = 0.3\mathrm{C}$$

- This is the charge on the 20-mF and 30-mF capacitors, because they are in series with the 30-v source. (A crude way to see this is to imagine that charge acts like current, since $i = dq/dt$)

Example 6.7

- Therefore,

$$v_1 = \frac{q}{C_1} = \frac{0.3}{20\times10^{-3}} = 15\text{ V},$$

$$v_2 = \frac{q}{C_2} = \frac{0.3}{30\times10^{-3}} = 10\text{ V}$$

- Having determined v_1 and v_2, we now use KVL to determine v_3 by

$$v_3 = 30 - v_1 - v_2 = 5\text{V}$$

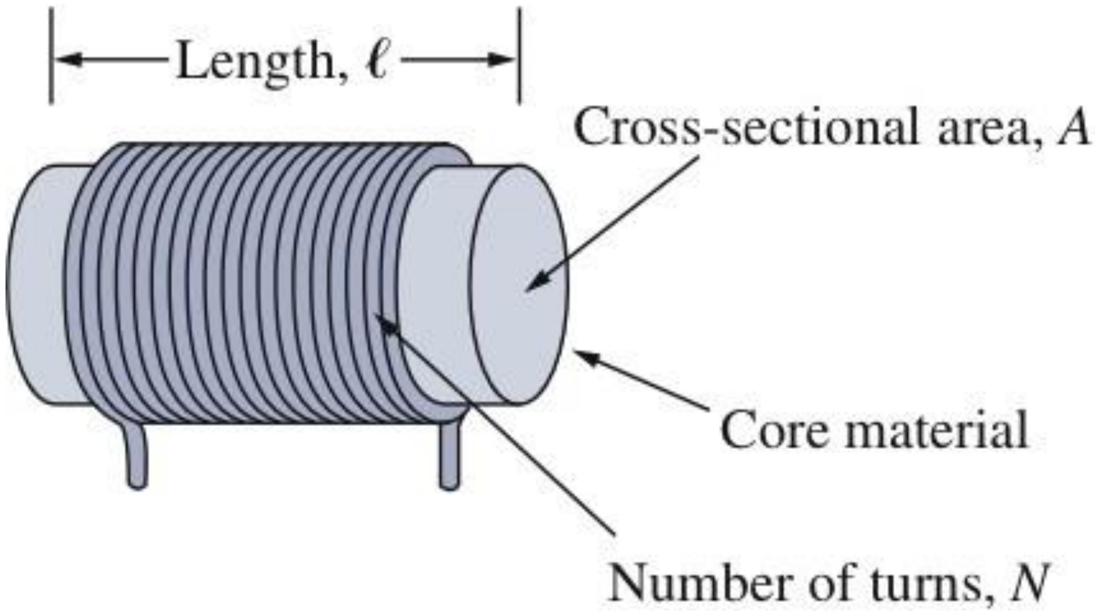
- Alternatively, since the 40-mF and 20-mF capacitors are in parallel, they have the same voltage v_3 and their combined capacitance is 40+20=60mF.

$$\therefore v_3 = \frac{q}{60\text{mF}} = \frac{0.3}{60\times10^{-3}} = 5\text{V}$$

Joseph Henry (1979-1878)

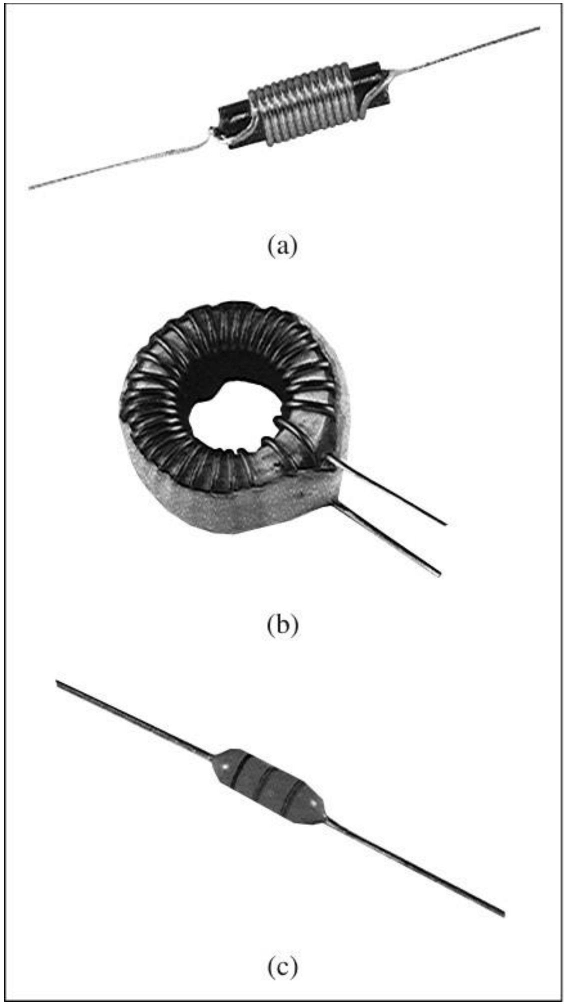
6.4 Inductors

- An inductor is made of a coil of conducting wire



$$L = \frac{N^2 \mu A}{l}$$

Fig 6.22



$$L = \frac{N^2 \mu A}{l}$$

$$\mu = \mu_r \mu_0$$

$$\mu_0 = 4\pi \times 10^{-7} \, (\mathrm{H/m})$$

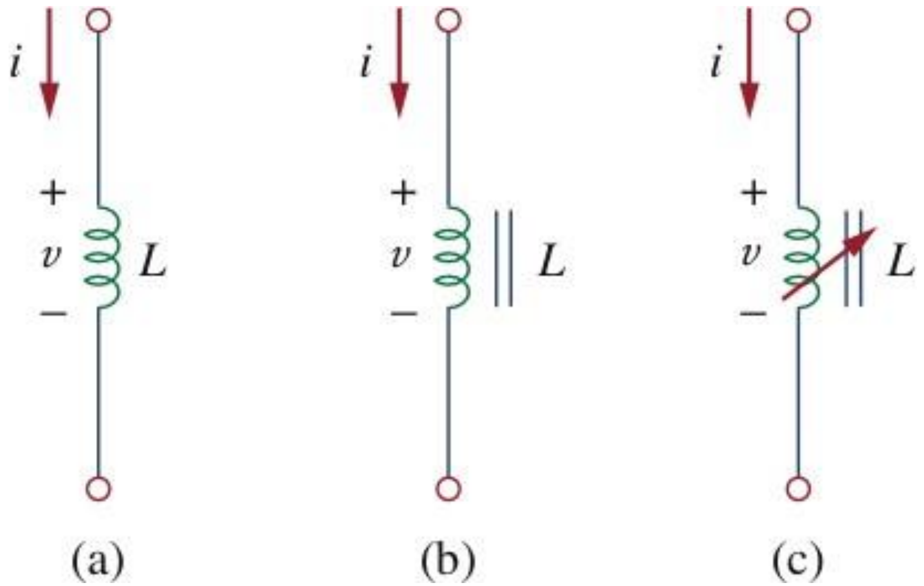
N : number of turns.

l : length.

A : cross – sectional area.

μ : permeability of the core

Fig 6.23



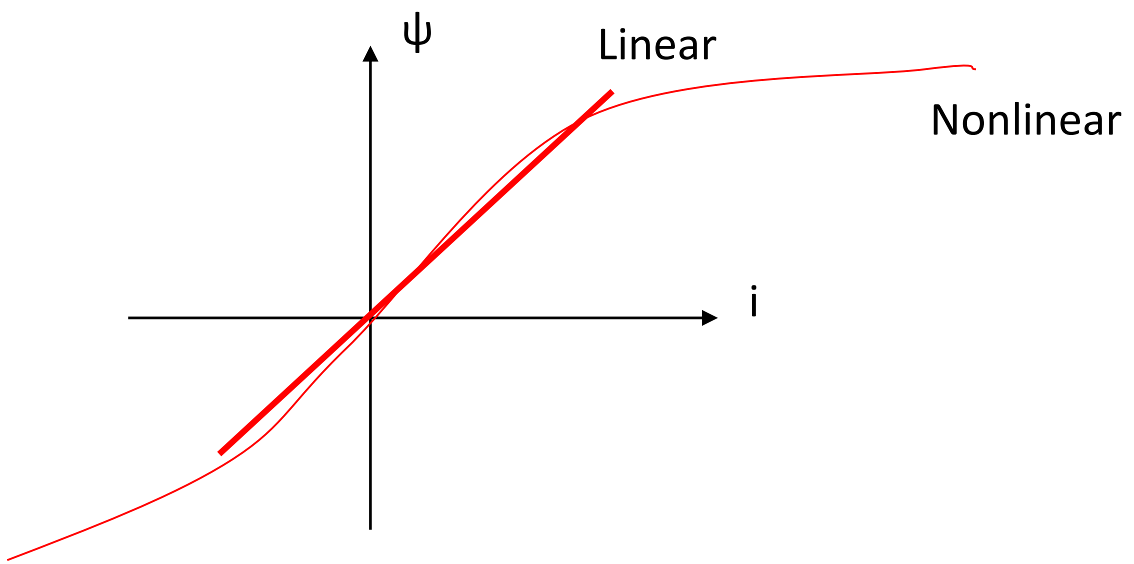
(a) air-core
(b) iron-core
(c) variable iron-core

Flux in Inductors

- The relation between the flux in inductor and the current through the inductor is given below.

$$\phi = Li$$

$$1\text{H} = 1\ \text{Weber/A}$$



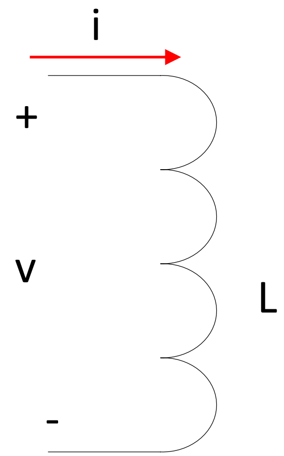
Energy Storage Form

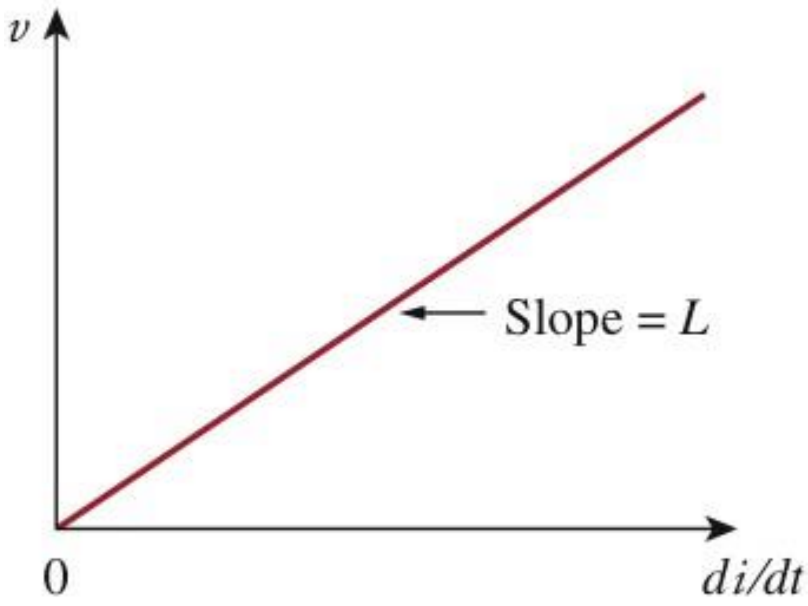
- An inductor is a passive element designed to store energy in the **magnetic** field while a capacitor stores energy in the **electric** field.

I-V Relation of Inductors

- An **inductor** consists of a coil of conducting wire.

$$v = \frac{d\phi}{dt} = L\frac{di}{dt}$$





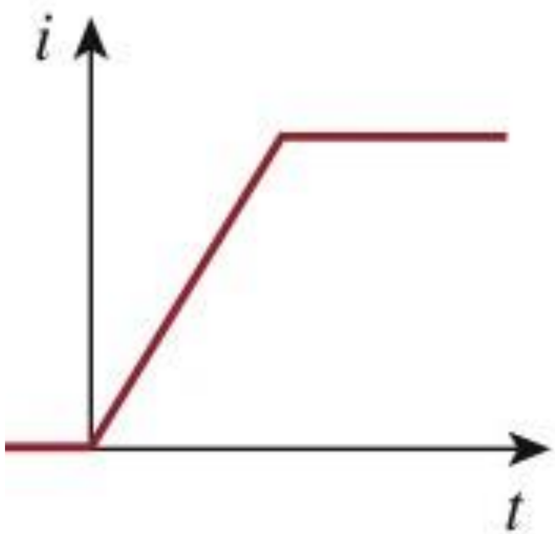
Physical Meaning

$$v = \frac{d\phi}{dt} = L\frac{di}{dt}$$

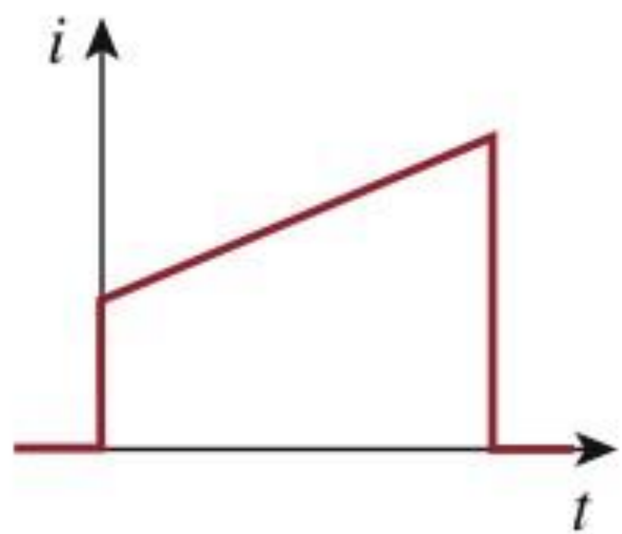
- When the **current through an inductor is a constant,** then the voltage across the inductor is zero, same as **a short circuit**.
- No **abrupt change of the current** through an inductor is possible except an infinite voltage across the inductor is applied.
- The inductor can be used to generate a high voltage, for example, used as an igniting element.

Fig 6.25

- An inductor are like a short circuit to dc.
- The current through an inductor cannot change instantaneously.

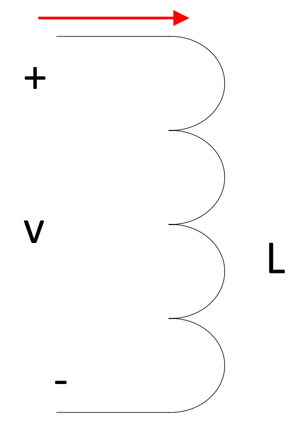


(a)



(b)

$$di = \frac{1}{L} v dt \qquad i = \frac{1}{L}\int_{-\infty}^{t} v(t)dt$$

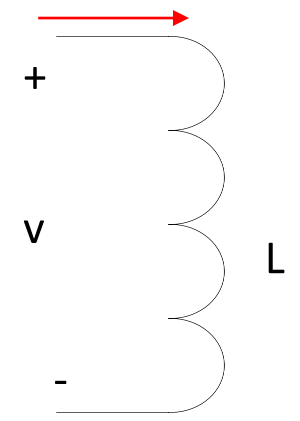


$$i = \frac{1}{L}\int_{t_o}^{t} v(t)dt + i(t_o)$$

The inductor has memory.

Energy Stored in an Inductor

$$P = vi = \left(L\frac{di}{dt} \right) i$$



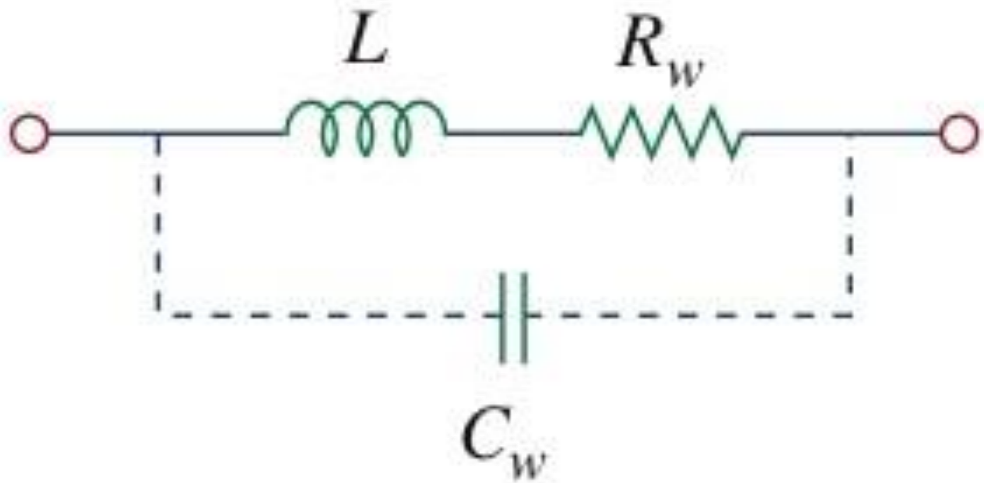
$$w = \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t} \left(L\frac{di}{dt} \right) i\,dt$$

$$= L\int_{i(-\infty)}^{i(t)} i\,di = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(-\infty) \quad i(-\infty) = 0,$$

- The energy stored in an inductor

$$w(t) = \frac{1}{2}Li^2(t)$$

Model of a Practical Inductor



Example 6.8

- The current through a 0.1-H inductor is $i(t) = 10te^{-5t}$ A. Find the voltage across the inductor and the energy stored in it.

Solution:

$$\text{Since } v = L\frac{di}{dt} \text{ and } L = 0.1\text{H},$$

$$v = 0.1\frac{d}{dt}(10te^{-5t}) = e^{-5t} + t(-5)e^{-5t} = e^{-5t}(1-5t)\text{V}$$

The energy stored is

$$w = \frac{1}{2}Li^2 = \frac{1}{2}(0.1)100t^2e^{-10t} = 5t^2e^{-10t}\text{J}$$

Example 6.9

- Find the current through a 5-H inductor if the voltage across it is

$$v(t) = \begin{cases} 30t^2, & t > 0 \\ 0, & t < 0 \end{cases}$$

- Also find the energy stored within 0 < t < 5s. Assume $i(0)$=0.

Solution:

$$\text{Since } i = \frac{1}{L}\int_{t_0}^{t} v(t)dt + i(t_0) \text{ and } L = 5\text{H}.$$

$$i = \frac{1}{5}\int_0^t 30t^2 dt + 0 = 6 \times \frac{t^3}{3} = 2t^3 \text{ A}$$

Example 6.9

The power $p = vi = 60t^5$, and the energy stored is then

$$w = \int p\,dt = \int_0^5 60t^5\,dt = 60\frac{t^6}{6}\bigg|_0^5 = 156.25\,\text{kJ}$$

Alternatively, we can obtain the energy stored using Eq.(6.13), by writing

$$w(5) - w(0) = \frac{1}{2}Li^2(5) - \frac{1}{2}Li(0)$$

$$= \frac{1}{2}(5)(2\times 5^3)^2 - 0 = 156.25\,\text{kJ}$$

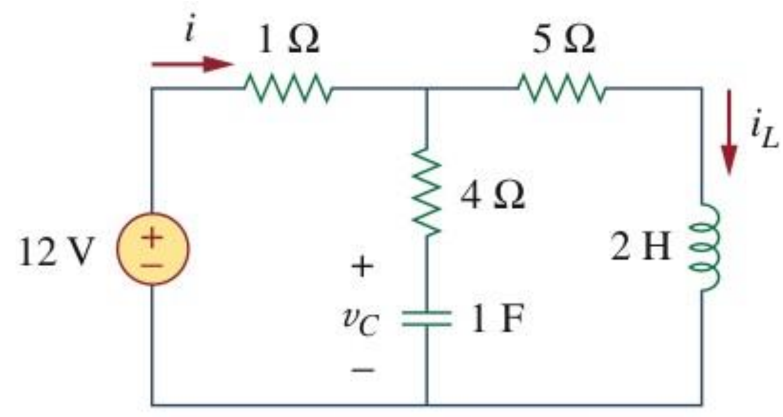
as obtained before.

Example 6.10

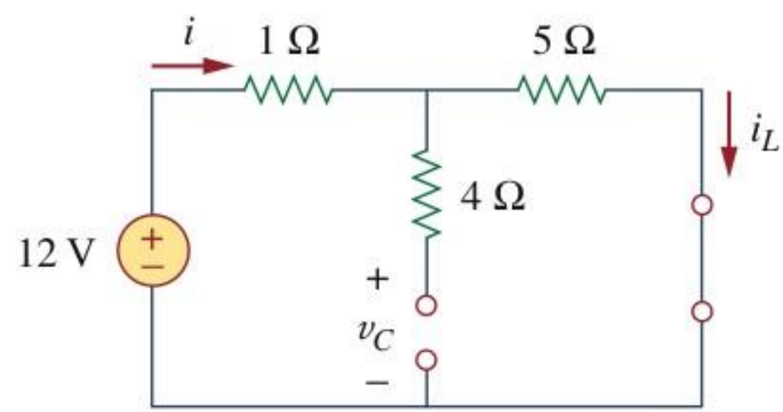
- Consider the circuit in Fig 6.27(a). Under dc conditions, find:

  (a) i, v_C, and i_L.

  (b) the energy stored in the capacitor and inductor.



(a)



(b)

Example 6.10

Solution:

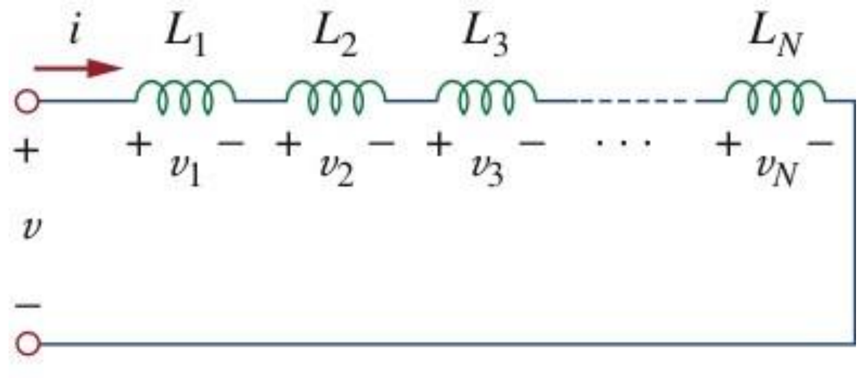
$$(a)\,\text{Under dc condition: capacitor} \rightarrow \text{open circuit}$$
$$\text{inductor} \rightarrow \text{short circuit}$$

$$i = i_L = \frac{12}{1+5} = 2A,\ v_c = 5i = 10\,\text{V}$$

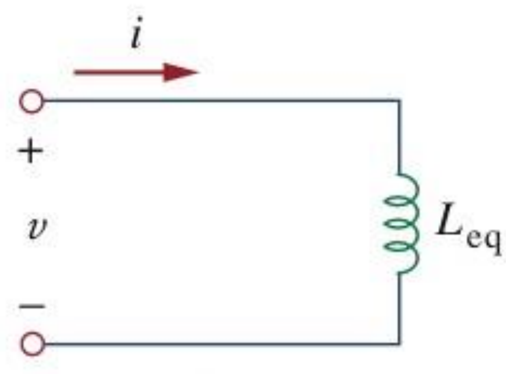
$$(b)\quad w_c = \frac{1}{2}Cv_c{}^2 = \frac{1}{2}(1)(10^2) = 50\text{J},$$

$$w_L = \frac{1}{2}L_i{}^2 = \frac{1}{2}(2)(2^2) = 4\text{J}$$

Inductors in Series



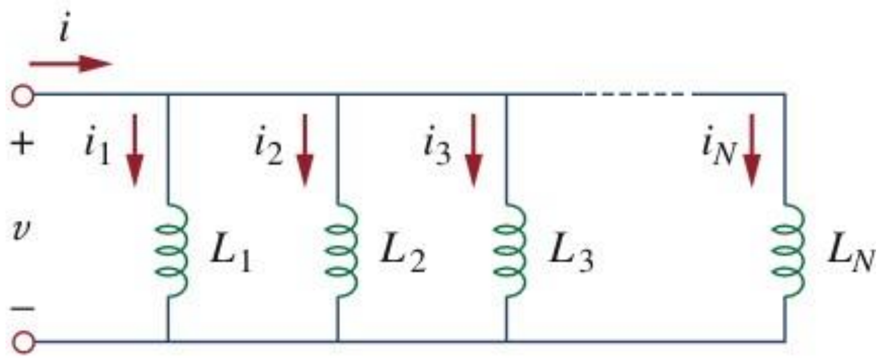
(a)



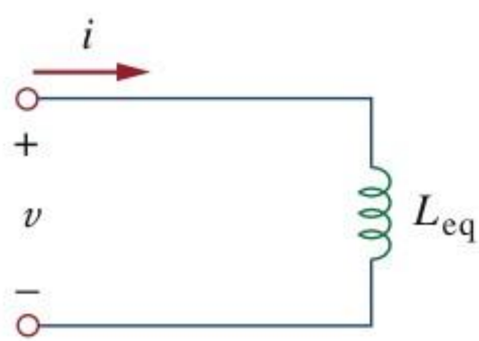
(b)

$$L_{eq} = L_1 + L_2 + L_3 + \ldots + L_N$$

Inductors in Parallel



(a)



(b)

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}$$

6.5 Series and Parallel Inductors

- Applying KVL to the loop,

$$v = v_1 + v_2 + v_3 + \ldots + v_N$$

- Substituting $v_k = L_k\, di/dt$ results in

$$\begin{aligned} v &= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt} + \ldots + L_N \frac{di}{dt} \\ &= (L_1 + L_2 + L_3 + \ldots + L_N) \frac{di}{dt} \\ &= \left(\sum_{K=1}^{N} L_K \right) \frac{di}{dt} = L_{eq} \frac{di}{dt} \end{aligned}$$

$$L_{eq} = L_1 + L_2 + L_3 + \ldots + L_N$$

Parallel Inductors

$$i = i_1 + i_2 + i_3 + \ldots + i_N$$

- Using KCL,
- But

$$i_k = \frac{1}{L_k}\int_{to}^{t} vdt + i_k(t_0)$$

$$\therefore\ i = \frac{1}{L_k}\int_{t_0}^{t} vdt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} vdt + i_s(t_0)\ + \ldots + \frac{1}{L_N}\int_{t_0}^{t} vdt + i_N(t_0)$$

$$= \left(\frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N}\right)\int_{t_0}^{t} vdt + i_1(t_0) + i_2(t_0) + \ldots + i_N(t_0)$$

$$= \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_0}^{t} vdt + \sum_{k=1}^{N} i_k(t_0) = \frac{1}{L_{eq}}\int_{t_0}^{t} vdt + i(t_0)$$

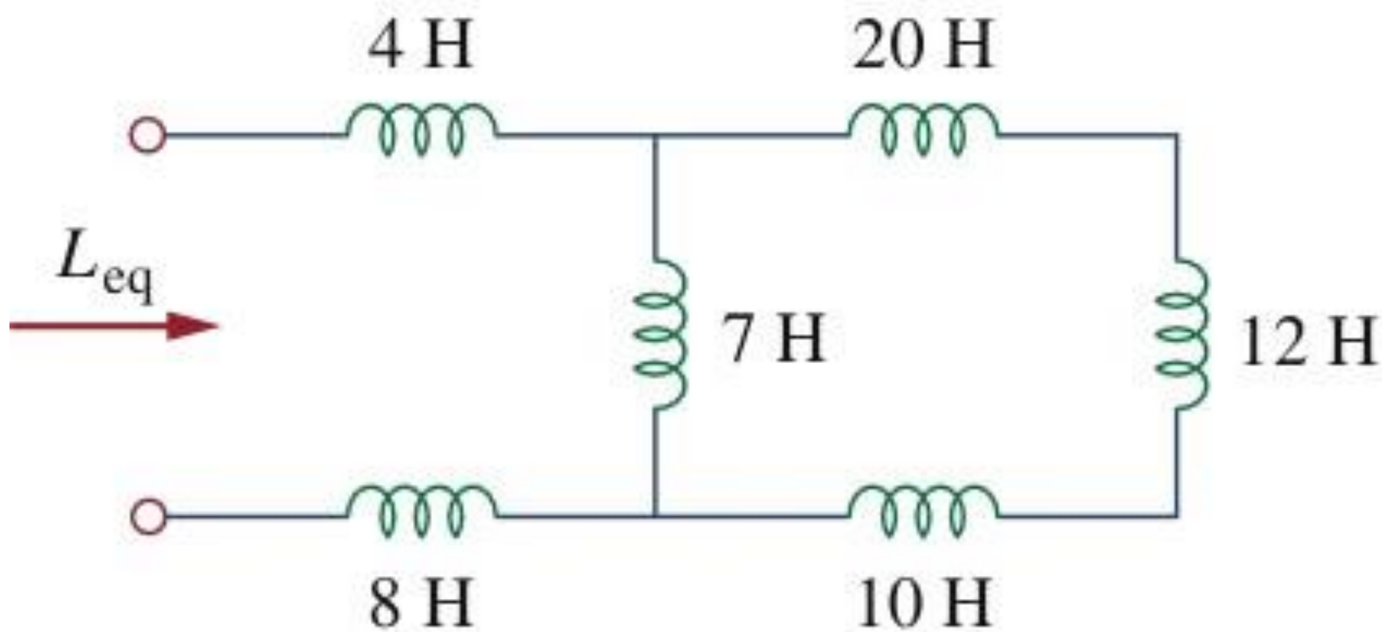
- The inductor in various connection has the same effect as the resistor. Hence, the Y-Δ transformation of inductors can be similarly derived.

Table 6.1

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} i\,dt + i(t_0)$
p or w:	$p = i^2 R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

Example 6.11

- Find the equivalent inductance of the circuit shown in Fig. 6.31.



Example 6.11

- **Solution:**

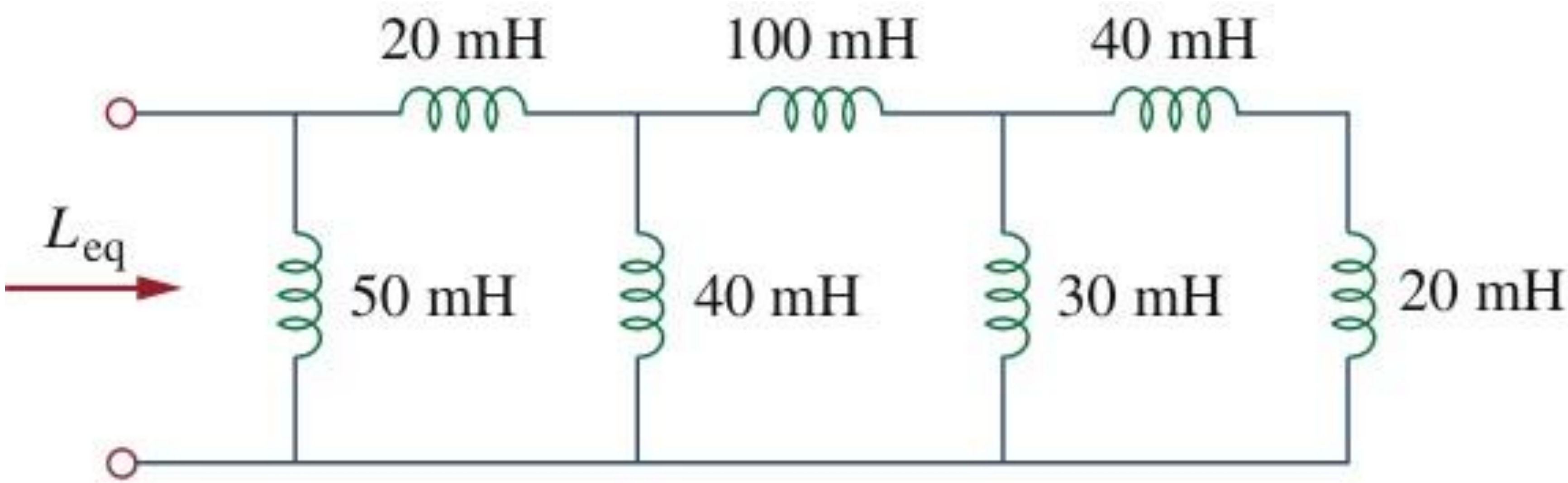
$$\text{Series}: \; 20\text{H}, 12\text{H}, 10\text{H}$$

$$\rightarrow 42\text{H}$$

$$\text{Parallel}: \; \frac{7 \times 42}{7 + 42} = 6\text{H}$$

$$\therefore L_{eq} = 4 + 6 + 8 = 18\text{H}$$

Practice Problem 6.11



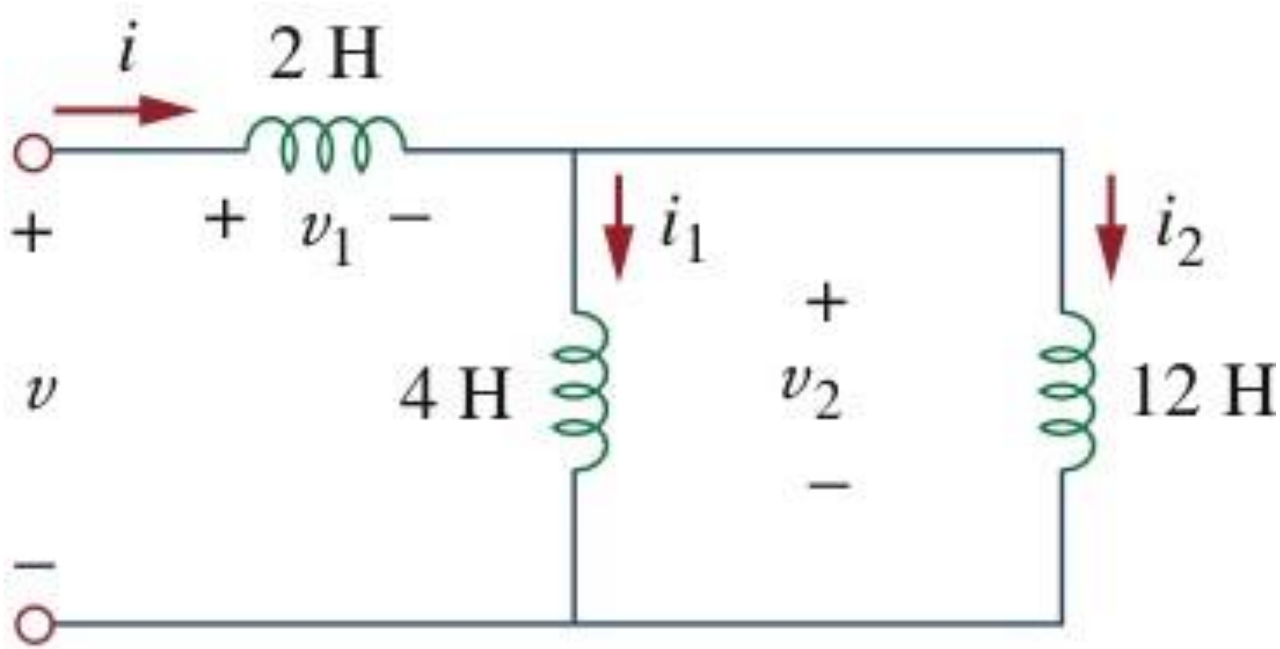
Example 6.12

$$i(t) = 4(2 - e^{-10t})\text{mA}.$$

- Find the circuit in Fig. 6.33,

If $i_2(0) = -1\,\text{mA},$ find : (a) $i_1(0)$

(b) $v(t)$, $v_1(t)$, and $v_2(t)$; (c) $i_1(t)$ and $i_2(t)$



Example 6.12

Solution:

(a) $i(t) = 4(2 - e^{-10t})\text{mA} \rightarrow i(0) = 4(2-1) = 4\text{mA}.$

$$\therefore i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5\text{mA}$$

(b) The equivalent inductance is

$$L_{eq} = 2 + 4 \parallel 12 = 2 + 3 = 5\text{H}$$

$$\therefore v(t) = L_{eq}\frac{di}{dt} = 5(4)(-1)(-10)e^{-10t}\text{mV} = 200e^{-10t}\text{mV}$$

$$v_1(t) = 2\frac{di}{dt} = 2(-4)(-10)e^{-10t}\text{mV} = 80e^{-10t}\text{mV}$$

$$\therefore v_2(t) = v(t) - v_1(t) = 120e^{-10t}\text{mV}$$

Example 6.12

$$\text{(c) } i = \frac{1}{L}\int_0^t v(t)\,dt + i(0) \Rightarrow$$

$$i_1(t) = \frac{1}{4}\int_0^t v_2\,dt + i_1(0) = \frac{120}{4}\int_0^t e^{-10t}\,dt + 5\,\text{mA}$$

$$= -3e^{-10t}\Big|_0^t + 5\,\text{mA} = -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t}\ \text{mA}$$

$$i_2(t) = \frac{1}{12}\int_0^t v_2\,dt + i_2(0) = \frac{120}{12}\int_0^t e^{-10t}\,dt - 1\text{mA}$$

$$= -e^{-10t}\Big|_0^t - 1\text{mA} = -e^{-10t} + 1 - 1 = -e^{-10t}\ \text{mA}$$

$$\text{Note that } i_1(t) + i_2(t) = i(t)$$