Relations

PRODUCT SETS

Consider two arbitrary sets A and B. The set of all ordered pairs: (a, b) where $a \in A$ and $b \in B$ is called the *product*, or *Cartesian product*, of A and B. A short designation of this product is $A \times B$, which is read "A cross B." By definition,

$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

One frequently writes A^2 instead of $A \times A$.

Example-1

Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

$B \times A = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$

Also, $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

Definition: Let A and B be sets. A binary relation or, simply, relation from A to B is a subset of $A \times B$.

Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair a ∈ A and b ∈ B, exactly one of the following is true:

(i) $(a, b) \notin R$; we then say "*a* is R-related to *b*", written aRb.
(ii) $(a, b) \in R$; we then say "*a* is not R-related to *b*", written $a \not{R} b$.

Example-2

Let *A* be the set of cities in the Bangladesh., and let *B* be the set of the 8 divisions in the Bangladesh. Define the relation *R* by specifying that *(a, b)* belongs to *R* if a city with name *a* is in the division *b*. For instance, (Uttrra, Dhaka), (Jessore, Khulna), (Sirajgonj, Rajshahi), (Feni, Chittagong), (Comilla, Chittagong), (Sunamgonj, Shylet), and (Bhola, Barisal) are in *R*.

The ***domain*** of a relation *R* is the set of all first elements of the ordered pairs which belong to *R*, and the ***range*** is the set of second elements.

Example-3

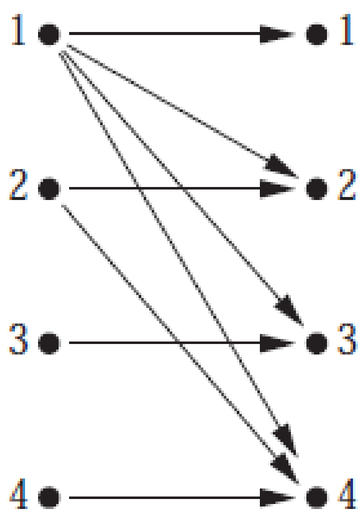
$A = (1, 2, 3)$ and $B = \{x, y, z\}$, and let $R = \{(1, y), (1, z), (3, y)\}$. Then R is a relation from A to B since R is a subset of $A \times B$. With respect to this relation, $1Ry$, $1Rz$, $3Ry$, but $1\not{R}x$, $2\not{R}x$, $2\not{R}y$, $2\not{R}z$, $3\not{R}x$, $3\not{R}z$

The domain of R is $\{1, 3\}$ and the range is $\{y, z\}$

DEFINITION 1 Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

EXAMPLE 4 Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that
$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Displaying the Ordered Pairs in the Relation R

Example

A = {eggs, milk, corn}

B = {cows, goat, hens}

Obtain relation R from A to B by $(a, b) \in R$ if a is produced by b.

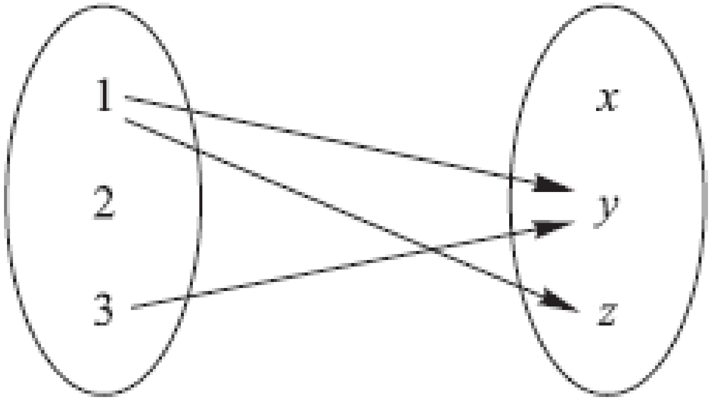
Ans. R={(eggs, hens), (milk, cows), (milk, goats)}

Example-5

$A = (1, 2, 3)$ and $B = \{x, y, z\}$, and $R = \{(1, y), (1, z), (3, y)\}$.
Show the relation with table and diagram.

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0

(i)



(ii)

$R = \{(1, y),\ (1, z),\ (3, y)\}$

Example-6

How many relations are there on a set with n elements?

Solution: A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of $A \times A$. Thus, there are 2^{n^2} relations on a set with n elements

EXAMPLE 7

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain

$R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$,

$R_1 \cap R_2 = \{(1, 1)\}$,

$R_1 - R_2 = \{(2, 2), (3, 3)\}$,

$R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}$.

Inverse Relation

Let R be any relation from a set A to a set B. The *inverse* of R, denoted by R^{-1}, is the relation from B to A
which consists of those ordered pairs which, when reversed, belong to R; that is,

$R^{-1} = \{(b, a) \mid (a, b) \in R\}$

For example, let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. Then the inverse of $R = \{(1, y), (1, z), (3, y)\}$ is $R^{-1} = \{(y, 1), (z, 1), (y, 3)\}$

Let $R = \{(a, b) \mid a \text{ divides } b\}$ on the set of positive integer. Find (a) R^{-1} and (b) $\overline{R}$

(a) $R^{-1} = \{(a, b) \mid b \text{ divides } a\}$
(b) $\overline{R} = \{(a, b) \mid a \text{ does not divide } b\}$

COMPOSITION OF RELATIONS

Let A, B and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of $A \times B$ and S is a subset of $B \times C$. Then R and S give rise to a relation from A to C denoted by $R \circ S$ and defined by:

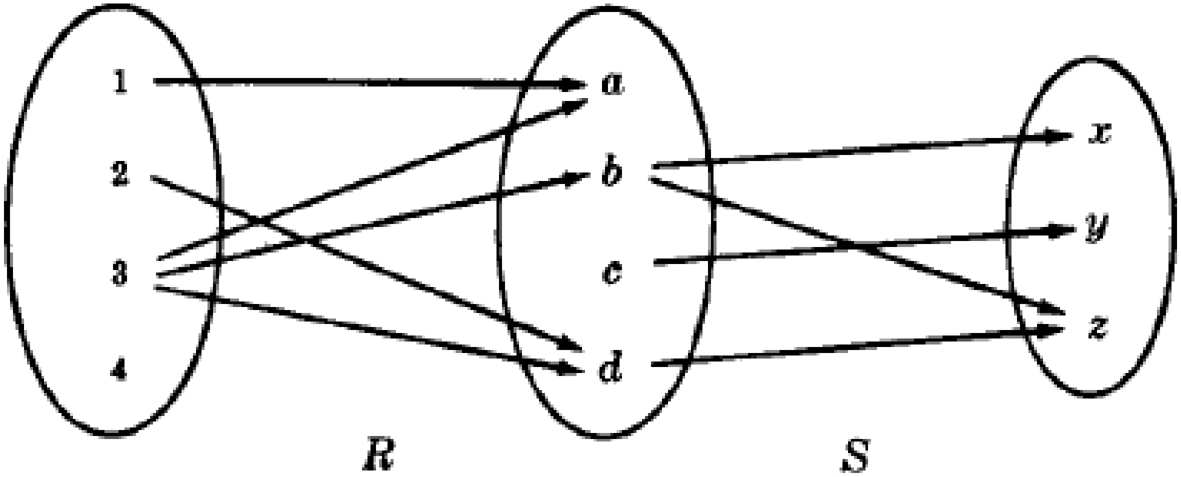
$a(R \circ S)c$ if for some $b \in B$ we have aRb and bSc.

That is ,

$R \circ S = \{(a, c) \mid$ there exists $b \in B$ for which $(a, b) \in R$ and $(b, c) \in S\}$

The relation $R \circ S$ is called the *composition* of R and S;

EXAMPLE 8 Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$ and let $R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}$ and $S = \{(b, x), (b, z), (c, y), (d. z)\}$ Consider the arrow diagrams of R and S as in Fig. below.



Observe that there is an arrow from 2 to d which is followed by an arrow from d to z. We can view these two arrows as a "path" which "connects" the element $2 \in A$ to the element $z \in C$. Thus:

$2(R \circ S)z$ since $2Rd$ and dSz

Similarly there is a path from 3 to x and a path from 3 to z. Hence $3(R\circ S)x$ and $3(R\circ S)z$

No other element of A is connected to an element of C. Accordingly,

$R \circ S = \{(2, z), (3, x), (3, z)\}$

Definition

Suppose R is a relation on a set A, that is, R is a relation from a set A to itself. Then $R \circ R$, the composition of R with itself, is always defined. Also, $R \circ R$ is sometimes denoted by R^2. Similarly, $R^3 = R^2 \circ R = R \circ R \circ R$, and so on. Thus R^n is defined for all positive n.

EXAMPLE 9

Consider the relation $R = \{(1, 2), (2, 3), (3, 3)\}$ on $A = \{1, 2, 3\}$. Then:
$R^2 = R \circ R = \{(1, 3), (2, 3), (3, 3)\}$ and $R^3 = R^2 \circ R = \{(1, 3), (2, 3), (3, 3)\}$

Example-10

Let R = {(1, 1), (2, 1), (3, 2), (4, 3)}. Find the powers R^n, n = 2, 3, 4, ...

Solution: Because R2 = R $\circ$R, we find that R_2 = {(1, 1), (2, 1), (3, 1), (4, 2)}. Furthermore, because $R^3 = R^2 \circ R$, R^3 = {(1, 1), (2, 1), (3, 1), (4, 1)}. Additional computation shows that R^4 is the same as R^3, so R^4 = {(1, 1), (2, 1), (3, 1), (4, 1)}. It also follows that $R^n = R^3$

for n = 5, 6, 7, ... The reader should verify this.

Example-10

Given $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Let R be the following relation from A to B:

$R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$

(a) Determine the matrix of the relation.

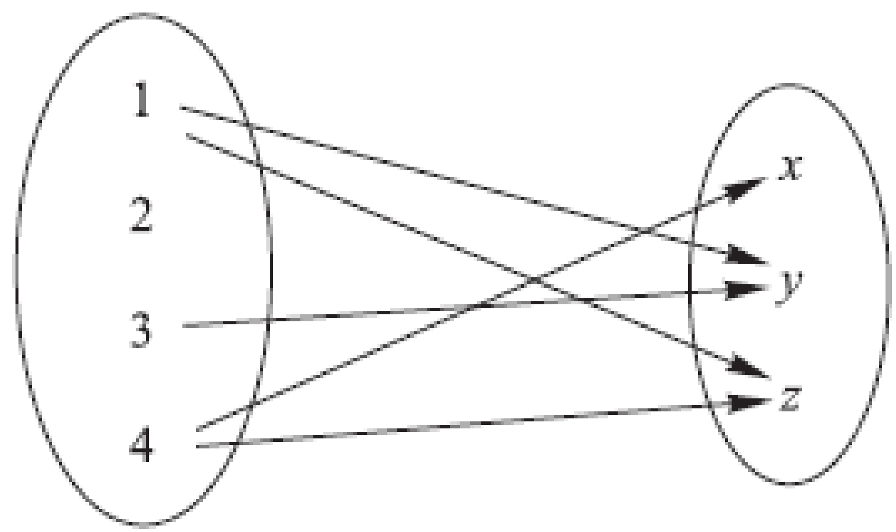
(b) Draw the arrow diagram of R.

(c) Find the inverse relation $R-1$ of R.

(d) Determine the domain and range of R.

$$
\begin{array}{c c}
 & \begin{array}{ccc} x & y & z \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}
\end{array}
$$

(a)



(b)

(c) Reverse the ordered pairs of R to obtain R^{-1}:
$R^{-1} = \{(y, 1), (z, 1), (y, 3), (x, 4), (z, 4)\}$
Observe that by reversing the arrows in Fig. 2.6(b), we obtain the arrow diagram of R^{-1}.
(d) The domain of R, Dom(R), consists of the first elements of the ordered pairs of R, and the range of R, Ran(R),
consists of the second elements. Thus,
Dom$(R) = \{1, 3, 4\}$ and Ran$(R) = \{x, y, z\}$

Example-11

Let $A = \{1, 2, 3\}$,$B = \{a, b, c\}$, and $C = \{x, y, z\}$. Consider the following relations R and S from A to B
and from B to C, respectively.

$R = \{(1, b), (2, a), (2, c)\}$ and $S = \{(a, y), (b, x), (c, y), (c, z)\}$

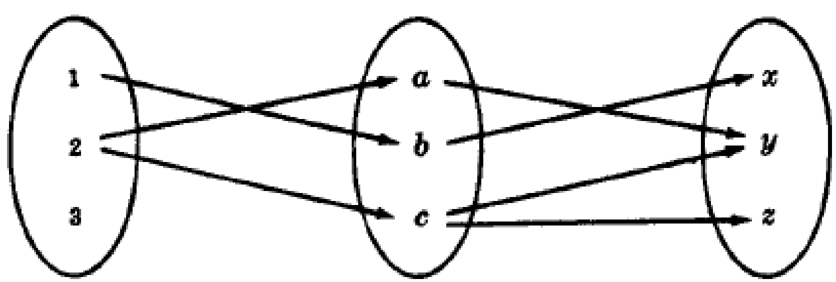
(a) Find the composition relation $R \circ S$.

(b) Find the matrices MR,MS, and $MR \circ S$ of the respective relations R, S, and $R \circ S$, and compare $MR \circ S$ to
the product $MRMS$.

Draw the arrow diagram of the relations R and S as in Fig. 2-7(a). Observe that 1 in A is "connected" to x in C by the path $1 \rightarrow b \rightarrow x$; hence $(1, x)$ belongs to $R \circ S$. Similarly, $(2, y)$ and $(2, z)$ belong to $R \circ S$.

We have

$R \circ S = \{(1, x), (2, y), (2, z)\}$



(a)

(b) The matrices of M_R, M_S, and $M_{R\circ S}$ follow:

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad M_S = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \quad M_{R\circ S} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Multiplying M_R and M_S we obtain

$$M_R M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Observe that $M_{R\circ S}$ and $M_R M_S$ have the same zero entries.

Example-12

Consider the relation $R = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)\}$ on $A = \{1, 2, 3, 4\}$.

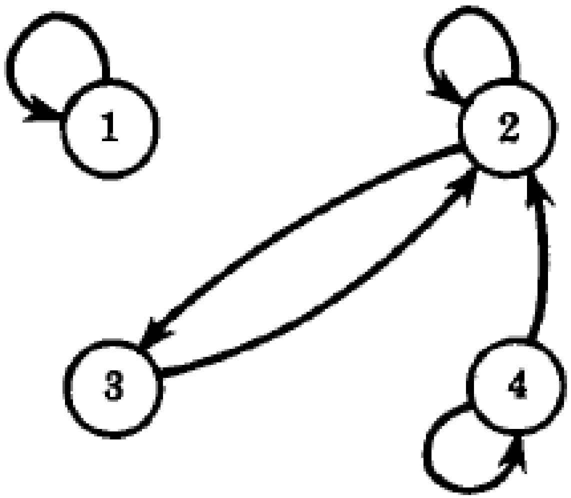
(*a*) Draw its directed graph. (*b*) Find $R^2 = R \circ R$.

(*a*) For each $(a, b) \in R$, draw an arrow from a to b as in Fig. 2-7(*b*).

(*b*) For each pair $(a, b) \in R$, find all $(b, c) \in R$. Then $(a, c) \in R^2$.

Thus

$R2 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)\}$



EXAMPLE 13 What is the composite of the relations R and S, where R is the relation from **{1, 2, 3} to {1, 2, 3, 4}** with R = {(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)} and S is the relation from {1, 2, 3, 4} to {0, 1, 2} with S = {(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}?

Solution: $S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. For example, the ordered pairs (2, 3) in R and (3, 1) in S produce the ordered pair (2, 1) in $S \circ R$. Computing all the ordered pairs in the composite, we find $S \circ R$ = {(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)}.

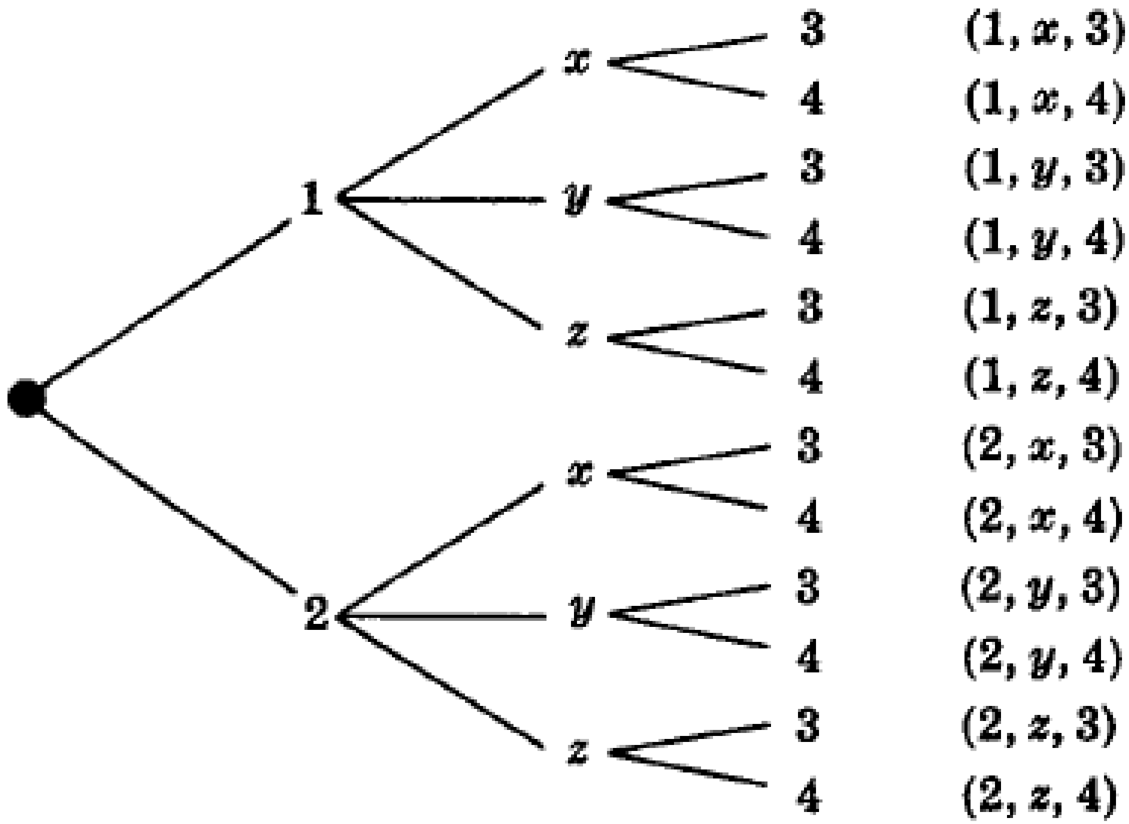
EXAMPLE 14

Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$. Find the powers R^n, $n = 2, 3, 4, \ldots$.

Solution: Because $R^2 = R \circ R$, we find that $R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$. Furthermore, because $R^3 = R^2 \circ R$, $R^3 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$. Additional computation shows that R4 is the same as R^3, so $R^4 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$. It also follows that $R^n = R^3$ for $n = 5, 6, 7, \ldots$. The reader should verify this.

Example 15

Given: A = {1, 2},B = {x, y, z}, and C = {3, 4}. Find: A × B × C using tree diagram.



Example-16

Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution: The matrices of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Formula

$$\mathbf{M}_{S \circ R} = \boldsymbol{M}_{\boldsymbol{R}} \bigoplus \boldsymbol{M}_{\boldsymbol{S}}$$

Example-17

Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution: The matrix for $S \circ R$ is

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

DEFINITION 3 A relation R *on a set* A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

EXAMPLE 18 Consider the following relations on $\{1, 2, 3, 4\}$:

$R1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R6 = \{(3, 4)\}$.

Which of these relations are reflexive?

Solution: The relations R3 and R5 are reflexive because they both contain all pairs of the form (a, a), namely, (1, 1), (2, 2), (3, 3), and (4, 4). The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, R1, R2, R4, and R6 are not reflexive because (3, 3) is not in any of these relations.

EXAMPLE 19 Is the "divides" relation on the set of positive integers reflexive?

Solution: Because $a \mid a$ whenever a is a positive integer, the "divides" relation is reflexive.

DEFINITION 4 A relation R on a set A is called symmetric if ***(b, a) ∈ R*** whenever ***(a, b) ∈ R,*** for all ***a, b ∈ A.*** A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

EXAMPLE 20 Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

Solution: This relation is not symmetric because 1| 2, but 2 | 1. It is antisymmetric, for if a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$.

DEFINITION 5 A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

EXAMPLE 21 Is the "divides" relation on the set of positive integers transitive?

Solution: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. It follows that this relation is transitive.

DEFINITION 1

A relation R on a set S is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S,R). Members of S are called elements of the poset.

Example-22

Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

Solution: Because $a \geq a$ for every integer a, $\geq$ is reflexive. If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric. Finally, $\geq$ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$. It follows that $\geq$ is a partial ordering on the set of integers and **($\mathbf{Z}$, $\geq$)** is a **poset.**

EXAMPLE 23 The divisibility relation | is a partial ordering on the set of positive integers, because it is reflexive, antisymmetric, and transitive, as was shown in Section 9.1. We see that ($Z+$, |*) is a* poset. Recall that ($Z+$ denotes the set of positive integers.)

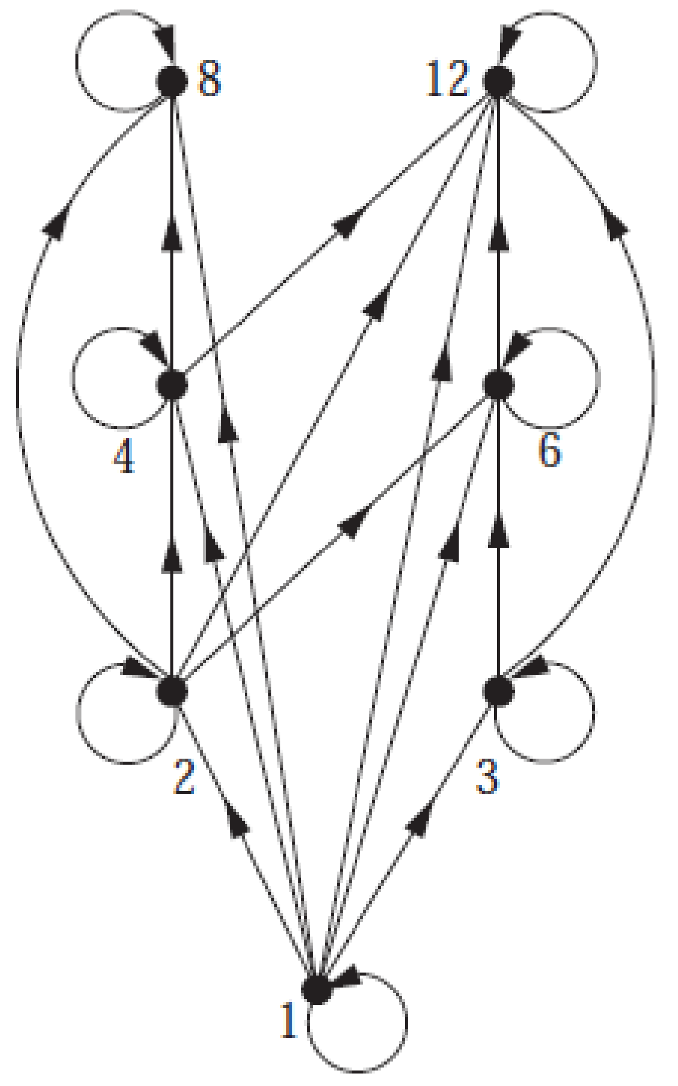
EXAMPLE 24 Show that the inclusion relation ⊆ is a partial ordering on the power set of a set *S*.

Solution: BecauseA ⊆ AwheneverAis a subset of S,⊆is reflexive. It is antisymmetric because A ⊆ B and B ⊆ A imply that A = B. Finally, ⊆ is transitive, because A ⊆ B and B ⊆ C imply that A ⊆ C. Hence, ⊆ is a partial ordering on P(S), and (P (S),⊆) is a poset.

Examle-25

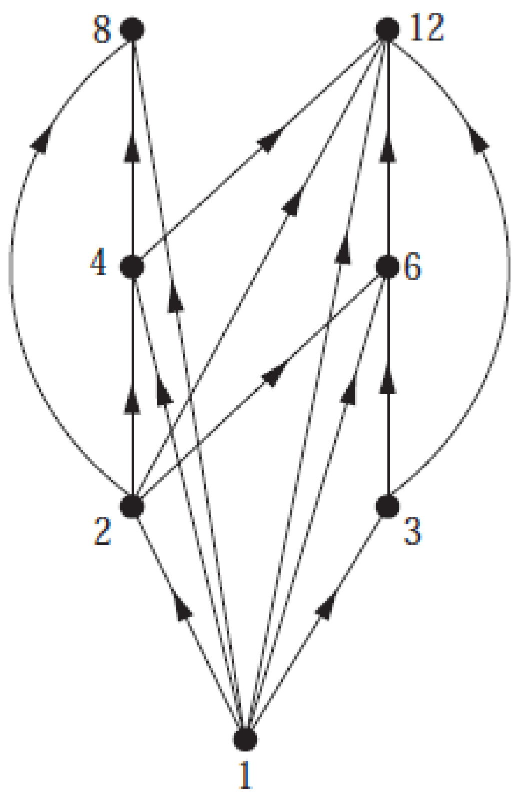
Draw the Hasse diagram representing the partial ordering {*(a, b)* |*a* divides *b}* on {1, *2, 3, 4, 6, 8, 12}*.

Solution: All the possible ordered pairs: (1,1), (1,2), (1,3), (1,4), (1,6), (1,8), (1,12), (2,2), (2,4), (2,6), (2,8), (2,12), (3,3), (3,6), (3, 12), (4,4), (4,8), (4,12), (6,6), (6,12), (8,8), (12,12). The directed graph of the relation is shown in fig.(a)

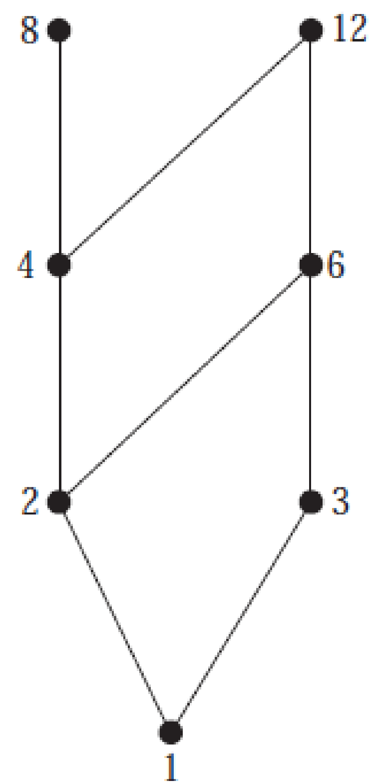


(a)

Remove all loops, as shown in Figure (b). Then delete all the edges implied by the transitive property. These are *(1, 4)*, *(1, 6)*, *(1, 8)*, *(1, 12)*, *(2, 8)*, *(2, 12)*, *and (3, 12)*. Arrange all edges to point upward, and delete all arrows to obtain the Hasse diagram. The resulting Hasse diagram is shown in Figure (c).



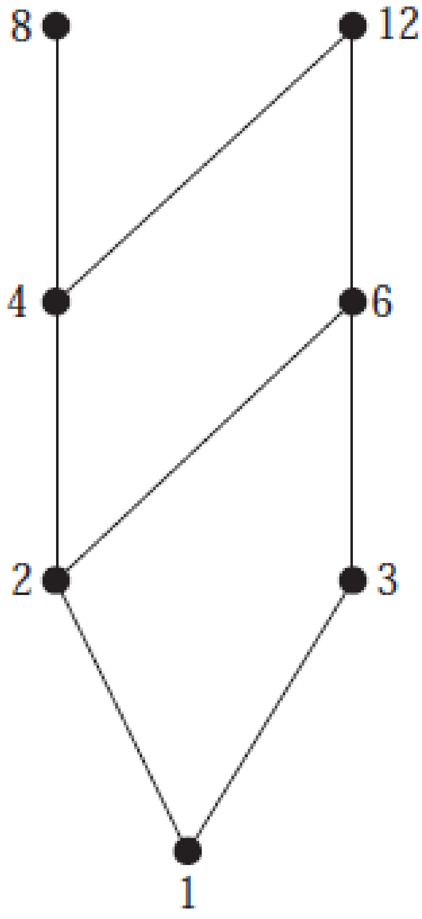
(b)



(c)

✓There is no element *a* like (8, *a*) (12, *a*) on the poset*(S, ≤)* i.e. 8 and 12 is not connected to any element greater than them. These two elements are on the top of the Hasse diagram and is not connected to any element greater them any of them. 8 and 12 are the **maximal** of the relation.

✓Similarly there is no element *b* like (*b*, *1*) i.e. . 1 is not connected to any element less then it called **minimal** shown at the bottom of the Hasse diagram.



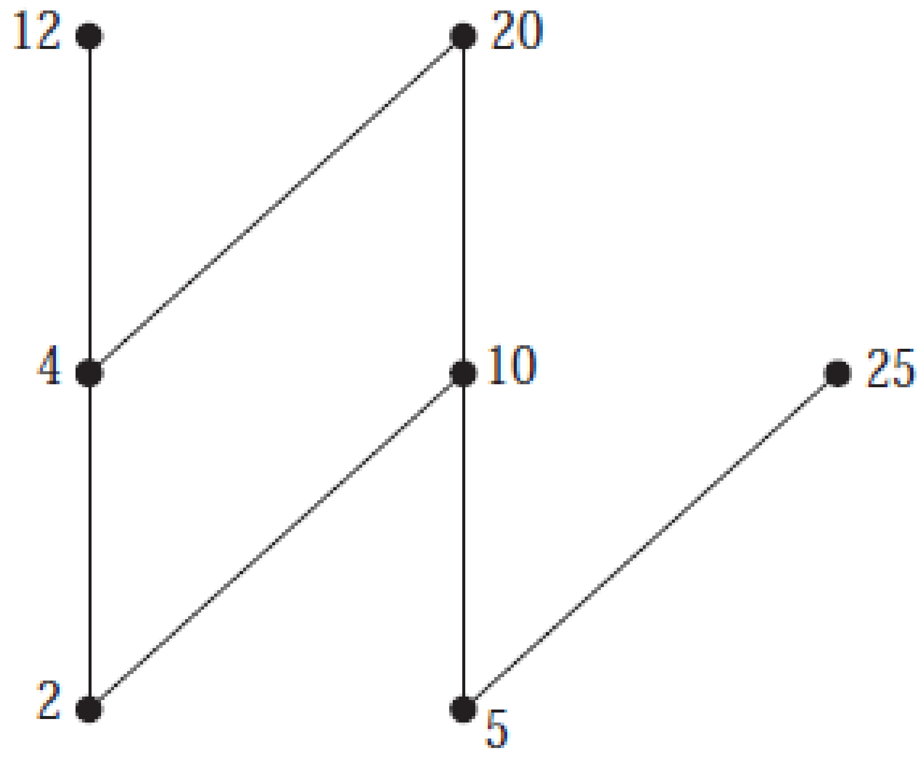
(c)

If there is only one maximal then called **greatest element.**

If there is only one minimal then called **least element.**

EXAMPLE 26 Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal, and which are minimal?

Solution: The Hasse diagram in Figure below for this poset shows that the maximal elements are 12, 20, and 25, and the minimal elements are 2 and 5.



EXAMPLE 27 Let S be a set. Determine whether there is a greatest element and a least element in the poset *(P (S),⊆)*.

Solution: The least element is the empty set, because $\emptyset \subseteq T$ for any subset T of S. The set S is the greatest element in this poset, because $T \subseteq S$ whenever T is a subset of S.

EXAMPLE 28 Is there a greatest element and a least element in the poset $(Z+, |)$?

Solution: The integer 1 is the least element because $1|n$ whenever n is a positive integer. Because there is no integer that is divisible by all positive integers, there is no greatest element.

Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset A of a poset $(S, \leq)$. If u is an element of S such that $a \leq u$ for all elements $a \in A$, then u is called an **upper bound of *A***. Likewise, there may be an element less than or equal to all the elements in A. If l is an element of S such that l a for all elements $a \in$ A, then l is called a **lower bound of A**.

Let the poset$(\{1,2,3,4,5,6,7, 8\}, \leq)$; where S= {1,2,3,4,5,6,7,8}, let us assume a subset of S is $A=\{3,4, 5\}$.

(3,3), (3,4), (3,5), (3,6), (3,7) , (3,8) $\in S$

(4,4), (4,5) , (4,6) (4,7) , (4,8) $\in S$

(5,5), (5,6) , (5,7) (5,8) $\in S$

The **upper bound of *A*** is5, 6, 7, 8 and the **least upper bound** is 5.

$(1,3), (2,3), (3,3) \in S$
$(1,4), (2,4), (3,4), (4,4) \in S$
$(1,5), (2,5)\ (3,5)\ (4,5)\ (5,5) \in S$

The **lower bound of *A*** is 1,2 and 3 and the **greatest lower bound of A** is 3.

Example-29

Answer these questions for the poset *({3, 5, 9, 15*, 24, *45}*, *|)*.

a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of {3, *5}*.

f) Find the least upper bound of {3, *5}*, *if it exists.*

g) Find all lower bounds of {15, *45}*.

h) Find the greatest lower bound of {15, *45*}, if it exists.

Ans. The relations, R={(3,9),(3,15), (3, 24), (3, 45), (5, 15), (5,45), (9, 45), (15, 45)}.

(a) 24, 45

(b) 3, 5

(c) No

(d) No

(e) A={3, *5}*

$(3,3),(3,9),(3,15),(3,24),\ (3,45) \in R$

$(5,15),\ (5,45) \in R$

The upper bounds 15, 45

(f) 15

(g) (3, 15), (5, 15), (15, 15) $\in R$

(3, 45), (5, 45), (15, 45), (45, 45) $\in R$

The lower bound 3,5, 15

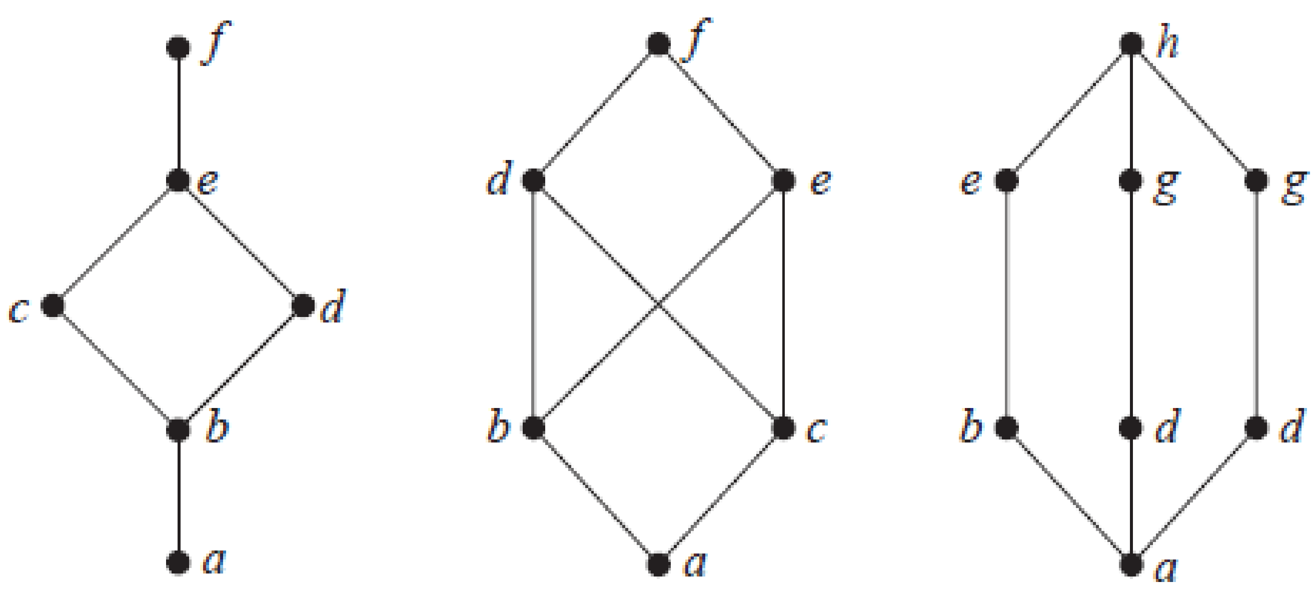
(h) 15

Lattices

A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**. Lattices have many special properties. Furthermore, lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra.

EXAMPLE 30 Determine whether the posets represented by each of the Hasse diagrams in Figure 8 are lattices.

Solution: The posets represented by the Hasse diagrams in (a) and (c) are both lattices because in each poset every pair of elements has both a least upper bound and a greatest lower bound, as the reader should verify. On the other hand, the poset with the Hasse diagram shown in (b) is not a lattice, because the elements b and c have no least upper bound. To see this, note that each of the elements d, e, and f is an upper bound, but none of these three elements precedes the other two with respect to the ordering of this poset



(a) (b) (c)

EXAMPLE 31 Is the poset *(**Z**+ , |)* a lattice?

Solution: Let *a* and *b* be two positive integers. The least upper bound and greatest lower bound of these two integers are the least common multiple and the greatest common divisor of these integers, respectively, as the reader should verify. It follows that this poset is a lattice.

EXAMPLE 32 Determine whether the posets *(*{1, 2, 3, 4, 5}, |*)* and *(*{1, 2, 4, 8, 16}, |*)* are lattices.

Solution: Because 2 and 3 have no upper bounds in *(*{1, 2, 3, 4, 5}, |*)*, they certainly do not have a least upper bound. Hence, the first poset is not a lattice. Every two elements of the second poset have both a least upper bound and a greatest lower bound. The least upper bound of two elements in this poset is the larger of the elements and the greatest lower bound of two elements is the smaller of the elements, as the reader should verify. Hence, this second poset is a lattice

EXAMPLE 33 Determine whether $(P(S), \subseteq)$ is a lattice where S is a set.

Solution: Let A and B be two subsets of S. The least upper bound and the greatest lower bound of A and B are $A \cup B$ and $A \cap B$, respectively, as the reader can show. Hence, $(P(S), \subseteq)$ is a lattice