LECTURE NO-20

Rules and Techniques of Differentiation

Trigonometry Formulae:

1. $\sin(A+B) = sinAcosB + cosAsinB$

2. $\sin(A-B) = sinAcosB - cosAsinB$

3. $\cos(A+B) = cosAcosB - sinAsinB$

4. $\cos(A-B) = cosAcosB + sinAsinB$

5. $\tan(\mathrm{A}+\mathrm{B}) = \frac{tanA+tanB}{1-tanAtanB}$

6. $\tan(\mathrm{A}-\mathrm{B}) = \frac{tanA-tanB}{1+tanAtanB}$

7. $\cot(\mathrm{A}+\mathrm{B}) = \frac{cotAcotB-1}{cotB+cotA}$

8. $\cot(\mathrm{A}-\mathrm{B}) = \frac{cotAcotB+1}{cotB-cotA}$

9. $\sin(A+B) + \sin(A-B) = 2sinAcosB$

10. $\sin(A+B) - \sin(A-B) = 2cosAsinB$

11. $\sin C + \sin \mathrm{D} = 2sin\frac{C+D}{2}cos\frac{C-D}{2}$

12. $\sin C - \sin \mathrm{D} = 2cos\frac{C+D}{2}sin\frac{C-D}{2}$

13. $\cos(A+B) + cos(A-B) = 2cosAcosB$

14. $\cos(A+B) - cos(A-B) = 2sinAsinB$

15. $\cos C + cos\,\mathrm{D} = 2cos\frac{C+D}{2}cos\frac{C-D}{2}$

16. $\cos C - \cos \mathrm{D} = 2sin\frac{C+D}{2}sin\frac{D-C}{2}$

17. $sin2A = 2sinAcosA = \frac{2tanA}{1+tan^2A}$

18. $cos2A = cos^2A - sin^2A = 2cos^2A - 1 = 1 - 2sin^2A = \frac{1-tan^2A}{1+tan^2A}$

19. $1 + cos2A = 2cos^2A \quad 1 - cos2A = 2sin^2A$

20. $\sin^{-1}x \pm \sin^{-1}x = \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$

21. $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\{xy \mp \sqrt{(1-x^2)(1-y^2)}\}$

22. $2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2} = \sin^{-1}\frac{2x}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}$

Differentiation Formulae:

1. $\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\frac{d}{dx}(c) = 0$

3. $\frac{d}{dx}(cx^n) = c\frac{d}{dx}(x^n) = cnx^{n-1}$

4. $\frac{d}{dx}\left(af(x) \pm cg(x)\right) = a\frac{d}{dx}(f(x)) \pm c\frac{d}{dx}(g(x))$

5. $\frac{d}{dx}(u.v) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

6. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

7. $\frac{d}{dx}(lnx) = \frac{1}{x}$

8. $\frac{d}{dx}(e^x) = e^x$

9. $\frac{d}{dx}(a^x) = a^x lna$

10. $\frac{d}{dx}(sinx) = cosx$

11. $\frac{d}{dx}(cosx) = -sinx$

12. $\frac{d}{dx}(tanx) = sec^2x$

13. $\frac{d}{dx}(cosecx) = -cosecx\ cotx$

14. $\frac{d}{dx}(secx) = secx\ tanx$

15. $\frac{d}{dx}(cotx) = -cosec^2x$

16. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

17. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

18. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

19. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

20. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

21. $\frac{d}{dx}(\mathrm{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

Change Rule or Derivatives of Composition:

$y = f(v)$, Where $v = \varphi(x)$, $f(v)$ and $\varphi(x)$ are continuous. Then

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$$

Similarly if $y = f(u), u = g(v)$ and $v = h(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Example:

i) $y = \{1 + \tan(1+\sqrt{x})\}^{1/3}$

2 ii) $\tan(\log_e \sin e^{\sqrt{x}})$

iii) $(x^2+1)\sin^{-1}x + e^{\sqrt{1+x^2}}$

iv) $y = \sin^{-1}\left(\frac{a + b\cos x}{b + a\cos x}\right)$ 5

v) $y = \tan(\log_e x^2)$

vi) $y = \sqrt{\frac{1-\sin x}{1+\sin x}}$

vii) $y = \sqrt{\frac{1-\tan x}{1+\tan x}}$

viii) $y = \log_{\cos x} \tan x$

Solution: i)

$$y = \{1 + \tan(1+\sqrt{x})\}^{1/3}$$

$$\therefore \frac{dy}{dx} = \frac{1}{3}\{1+\tan(1+\sqrt{x})\}^{\frac{1}{3}-1} \cdot \frac{d}{dx}\left[1 + \tan(1+\sqrt{x})\right]$$

$$= \frac{1}{3}\{1 + \tan(1+\sqrt{x})\}^{-\frac{2}{3}} \cdot \left[0 + \sec^2(1+\sqrt{x}) \cdot \frac{d}{dx}(1+\sqrt{x})\right]$$

$$= \frac{1}{3}\{1 + \tan(1+\sqrt{x})\}^{-\frac{2}{3}} \sec^2(1+\sqrt{x}) \left(0 + \frac{1}{2\sqrt{x}}\right)$$

$$= \frac{1}{3}\{1 + \tan(1+\sqrt{x})\}^{-\frac{2}{3}} \sec^2(1+\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{6\sqrt{x}}\{1 + \tan(1+\sqrt{x})\}^{-\frac{2}{3}} \sec^2(1+\sqrt{x})$$ Ans.

ii) let $y = \tan\left(\log_e \sin e^{\sqrt{x}}\right)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\tan\left(\log_e \sin e^{\sqrt{x}}\right)\right]$$

$$= \sec^2\left(\log_e \sin e^{\sqrt{x}}\right) \cdot \frac{d}{dx}\left(\log_e \sin e^{\sqrt{x}}\right)$$

$$= \sec^2\left(\log_e \sin e^{\sqrt{x}}\right) \frac{1}{\sin e^{\sqrt{x}}} \cdot \frac{d}{dx}\left(\sin e^{\sqrt{x}}\right)$$

$$= \sec^2\left(\log_e \sin e^{\sqrt{x}}\right) \frac{1}{\sin e^{\sqrt{x}}} \cos e^{\sqrt{x}} \cdot \frac{d}{dx}\left(e^{\sqrt{x}}\right)$$

$$= \sec^2\left(\log_e \sin e^{\sqrt{x}}\right) \frac{\cos e^{\sqrt{x}}}{\sin e^{\sqrt{x}}} e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \sec^2\left(\log_e \sin e^{\sqrt{x}}\right) \cot e^{\sqrt{x}} \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$$= \frac{e^{\sqrt{x}}}{2\sqrt{x}} \sec^2\left(\log_e \sin e^{\sqrt{x}}\right) \cot e^{\sqrt{x}} \quad \text{Ans.}$$

iii) let $y = (x^2+1)\sin^{-1}x + e^{\sqrt{1+x^2}}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[(x^2+1)\sin^{-1}x\right] + \frac{d}{dx}\left(e^{\sqrt{1+x^2}}\right)$$

$$= (x^2+1)\frac{d}{dx}(\sin^{-1}x) + \sin^{-1}x \frac{d}{dx}(x^2+1) + e^{\sqrt{1+x^2}} \cdot \frac{d}{dx}\left(\sqrt{1+x^2}\right)$$

$$= (x^2+1)\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x \cdot (2x) + e^{\sqrt{1+x^2}} \cdot \frac{1}{\cancel{2}\sqrt{1+x^2}} \cdot \cancel{2}x$$

$$= \frac{x^2+1}{\sqrt{1-x^2}} + 2x\sin^{-1}x + \frac{xe^{\sqrt{1+x^2}}}{\sqrt{1+x^2}} \quad \text{Ans.}$$

iv) $y = \sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)\right]$$

$$= \frac{1}{\sqrt{1-\left(\frac{a+b\cos x}{b+a\cos x}\right)^2}} \cdot \frac{d}{dx}\left(\frac{a+b\cos x}{b+a\cos x}\right)$$

$$y' = \frac{1}{\sqrt{1-\frac{(a+b\cos x)^2}{(b+a\cos x)^2}}} \cdot \frac{(b+a\cos x)\frac{d}{dx}(a+b\cos x)-(a+b\cos x)\frac{d}{dx}(b+a\cos x)}{(b+a\cos x)^2}$$

$$= \frac{1}{\sqrt{\frac{(b+a\cos x)^2-(a+b\cos x)^2}{(b+a\cos x)^2}}} \cdot \frac{(b+a\cos x)(-b\sin x)-(a+b\cos x)(-a\sin x)}{(b+a\cos x)^2}$$

$$= \frac{b+a\cos x}{\sqrt{b^2+\cancel{2ab\cos x}+a^2\cos^2 x-a^2-\cancel{2ab\cos x}-b^2\cos^2 x}} \quad \frac{-b^2\sin x-ab\sin x\cos x+a^2\sin x+ab\sin x\cos x}{(b+a\cos x)^2}$$

$$= \frac{(a^2-b^2)\sin x}{\sqrt{(b^2-a^2)-(b^2-a^2)\cos^2 x} \quad (b+a\cos x)}$$

$$= \frac{(a^2-b^2)\sin x}{(b+a\cos x)\sqrt{(b^2-a^2)-(a^2+b^2)\cos^2 x}}$$

$$y' = \frac{-(b^2-a^2)\sin x}{(b+a\cos x)\sqrt{b^2-a^2}\cdot\sqrt{1-\cos^2 x}}$$

$$= \frac{-\sqrt{b^2-a^2}\,\sin x}{(b+a\cos x)\sin x}$$

$$= \frac{-\sqrt{b^2-a^2}}{b+a\cos x} \quad \text{Ans.}$$

v) $y = \tan(\log_e x^2)$

$\Rightarrow y = \tan(2\ln x)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\tan(2\ln x)\right]$$

$$= \sec^2(2\ln x)\cdot\frac{d}{dx}(2\ln x)$$

$$= \sec^2(2\ln x)\cdot 2\frac{1}{x}$$

$$= \frac{2\sec^2(2\ln x)}{x} \quad \text{Ans.}$$

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vi) $y = \sqrt{\frac{1-\sin x}{1+\sin x}}$

$$\Rightarrow y = \sqrt{\frac{1-\cos(\frac{\pi}{2}-x)}{1+\cos(\frac{\pi}{2}-x)}}$$

$$\Rightarrow y = \sqrt{\frac{1-\cos 2\cdot(\frac{\pi}{4}-\frac{x}{2})}{1+\cos 2\cdot(\frac{\pi}{4}-\frac{x}{2})}}$$

$$\Rightarrow y = \sqrt{\frac{2\sin^2(\frac{\pi}{4}-\frac{x}{2})}{2\cos^2(\frac{\pi}{4}-\frac{x}{2})}}$$

$\Rightarrow y = \tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$

$$\therefore \frac{dy}{dx} = \sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)\cdot\frac{d}{dx}\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

$$= \frac{1}{\cos^2(\frac{\pi}{4}-\frac{x}{2})}\cdot\frac{-1}{2}$$

$$= \frac{-1}{2\cos^2(\frac{\pi}{4}-\frac{x}{2})}$$

$$= \frac{-1}{1+\cos 2(\frac{\pi}{4}-\frac{x}{2})}$$

$$= \frac{-1}{1+\cos(\frac{\pi}{2}-x)} = \frac{-1}{1+\sin x} \quad \text{Ans.}$$

vii) $$y = \sqrt{\frac{1-\tan x}{1+\tan x}}$$

$$\Rightarrow y = \sqrt{\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}}$$

$$\Rightarrow y = \sqrt{\frac{\cos x - \sin x}{\cos x + \sin x}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{\frac{\cos x - \sin x}{\cos x + \sin x}}} \frac{d}{dx}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$$

$$= \frac{\sqrt{\cos x + \sin x}}{2\sqrt{\cos x - \sin x}} \frac{(\cos x + \sin x)\frac{d}{dx}(\cos x - \sin x) - (\cos x - \sin x)\frac{d}{dx}(\cos x + \sin x)}{(\cos x + \sin x)^2}$$

$$= \frac{\sqrt{\cos x + \sin x}}{2\sqrt{\cos x - \sin x}} \cdot \frac{(\cos x + \sin x)(-\sin x - \cos x) - (\cos x - \sin x)(-\sin x + \cos x)}{(\cos x + \sin x)(\cos x + \sin x)}$$

$$= \frac{-(\cos x + \sin x)^2 - (\cos x - \sin x)^2}{2\sqrt{\cos x - \sin x}\ \sqrt{\cos x + \sin x}\ (\cos x + \sin x)}$$

$$= \frac{-\cos^2 x - 2\sin x \cos x - \sin^2 x - \cos^2 x + 2\sin x \cos x - \sin^2 x}{2\sqrt{(\cos x - \sin x)(\cos x + \sin x)}\ (\cos x + \sin x)}$$

$$= \frac{-\cancel{2}(\cos^2 x + \sin^2 x)}{\cancel{2}\sqrt{\cos^2 x - \sin^2 x}\ (\cos x + \sin x)}$$

$$= \frac{-1}{\sqrt{\cos 2x}\ (\cos x + \sin x)}$$ Ans.

(viii)

$$y = \log_{\cos x} \tan x$$

$$\Rightarrow y = \log_{e} \tan x \, \log_{\cos x} e$$

$$= \frac{\ln(\tan x)}{\log_e \cos x}$$

$$= \frac{\ln(\tan x)}{\ln(\cos x)}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\frac{\ln \tan x}{\ln \cos x}\right]$$

$$= \frac{\ln \cos x \frac{d}{dx}(\ln \tan x) - (\ln \tan x)\frac{d}{dx}(\ln \cos x)}{(\ln \cos x)^2}$$

$$= \frac{(\ln \cos x)\frac{1}{\tan x}\cdot \sec^2 x - (\ln \tan x)\frac{1}{\cos x}\cdot(-\sin x)}{(\ln \cos x)^2}$$

$$= \frac{(\ln \cos x)\cdot \frac{\cancel{\cos x}}{\sin x}\cdot\frac{1}{\cancel{\cos^2 x}\cos x} + (\ln \tan x)\frac{\sin x}{\cos x}}{(\ln \cos x)^2}$$

$$= \frac{\frac{\ln \cos x + (\ln \tan x)\sin^2 x}{\sin x \cos x}}{(\ln \cos x)^2}$$

$$= \frac{\ln \cos x + \ln(\tan x)\sin^2 x}{\sin x \cos x \, (\ln \cos x)^2}$$

Ex

$$y = \log_e\left(\tan e^{x^{\frac{1}{3}}}\right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{\tan e^{x^{\frac{1}{3}}}} \frac{d}{dx}\left(\tan e^{x^{\frac{1}{3}}}\right)$$

$$= \frac{1}{\tan e^{x^{\frac{1}{3}}}}\cdot \sec^2 e^{x^{\frac{1}{3}}}\cdot \frac{d}{dx}\left(e^{x^{\frac{1}{3}}}\right)$$

$$= \frac{1}{\tan e^{x^{1/3}}}\cdot \sec^2 e^{x^{\frac{1}{3}}}\cdot e^{x^{\frac{1}{3}}}\cdot \frac{1}{3}x^{\frac{1}{3}-1} = \frac{e^{x^{\frac{1}{3}}}\cdot x^{-\frac{2}{3}}}{3\sin e^{x^{\frac{1}{3}}}\cos e^{x^{\frac{1}{3}}}}$$ Ans.

Ex: $y = \log\{\sqrt{1+\log x} - \sin x\}$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1+\log x} - \sin x} \frac{d}{dx}\left(\sqrt{1+\log x} - \sin x\right)$$

$$= \frac{1}{\sqrt{1+\log x} - \sin x}\left(\frac{1}{2\sqrt{1+\log x}} \cdot \frac{1}{x} - \cos x\right)$$

$$= \frac{1}{\sqrt{1+\log x} - \sin x}\left(\frac{1}{2x\sqrt{1+\log x}} - \cos x\right) \quad \text{Ans.}$$

$$= \frac{1}{\sqrt{1+\log x} - \sin x} \times \frac{1 - 2x\cos x\sqrt{1+\log x}}{2x\sqrt{1+\log x}}$$

$$= \frac{1 - 2x\cos x\sqrt{1+\log x}}{2x\sqrt{1+\log x}\left(\sqrt{1+\log x} - \sin x\right)}$$

Ex. $(1+x^2)\tan x + (2-\sin x)\log x$

Soln $y = (1+x^2)\tan x + (2-\sin x)\log x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left((1+x^2)\tan x\right) + \frac{d}{dx}\left((2-\sin x)\log x\right)$$

$$= (1+x^2)\frac{d}{dx}(\tan x) + \tan x\frac{d}{dx}(1+x^2) + (2-\sin x)\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(2-\sin x)$$

$$= (1+x^2)\sec^2 x + (\tan x)2x + (2-\sin x)\frac{1}{x} + \log x(-\cos x)$$

$$= (1+x^2)\sec^2 x + 2x\tan x - \frac{2-\sin x}{x} - \cos x\log x$$

Ans.

Ex. Diff. $(1+x)\tan^{-1}\sqrt{x} - \sqrt{x}$

Sol^n

let $y = (1+x)\tan^{-1}\sqrt{x} - \sqrt{x}$

$$\therefore \frac{dy}{dx} = (1+x)\frac{d}{dx}(\tan^{-1}\sqrt{x}) + \tan^{-1}\sqrt{x}\,\frac{d}{dx}(1+x) - \frac{d}{dx}(\sqrt{x})$$

$$= (1+x)\frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}} + 1.\tan^{-1}\sqrt{x} - \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} + \tan^{-1}\sqrt{x} - \frac{1}{2\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = \tan^{-1}\sqrt{x}$$ Ans.

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Ex. $(x^2+1)\sqrt{1-x^2} - (\sin^{-1}x)^2$

Sol^n let $y = (x^2+1)\sqrt{1-x^2} - (\sin^{-1}x)^2$

$$\therefore \frac{dy}{dx} = (x^2+1)\frac{d}{dx}(\sqrt{1-x^2}) + (\sqrt{1-x^2})\frac{d}{dx}(x^2+1) - 2(\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}$$

$$= (x^2+1)\frac{1}{2\sqrt{1-x^2}}\cdot(-2x) + \sqrt{1-x^2}\cdot 2x - \frac{2\sin^{-1}x}{\sqrt{1-x^2}}$$

$$= \frac{-x(x^2+1) + 2x(\sqrt{1-x^2})^2 - 2\sin^{-1}x}{\sqrt{1-x^2}}$$

$$= \frac{-x^3 - x + 2x\ 1-x^2 - 2\sin^{-1}x}{\sqrt{1-x^2}}$$

$$= \frac{x - 3x^3 - 2\sin^{-1}x}{\sqrt{1-x^2}}$$ Ans.

Ex. $y = \dfrac{\sin\sqrt{x}}{\sqrt{x}}$

Sol^n $y = \dfrac{\sin\sqrt{x}}{\sqrt{x}}$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{x}\frac{d}{dx}(\sin\sqrt{x}) - \sin\sqrt{x}\,\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$$

$$= \frac{\sqrt{x} \cdot \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}} - \sin\sqrt{x} \cdot \frac{1}{2\sqrt{x}}}{x}$$

$$= \frac{\frac{\cos\sqrt{x}}{2} - \frac{\sin\sqrt{x}}{2\sqrt{x}}}{x}$$

$$= \frac{\frac{\sqrt{x}\cos\sqrt{x} - \sin\sqrt{x}}{2\sqrt{x}}}{x}$$

$$= \frac{\sqrt{x}\cos\sqrt{x} - \sin\sqrt{x}}{2x\sqrt{x}} \quad \text{Ans.}$$

Exercise:

i) $y = \ln \sec(ax+b)^3$

ii) $y = \ln(\sec x + \tan x)$