Lecture 20

Average, RMS value, Phasors, Solving with AC input.

The Average Value

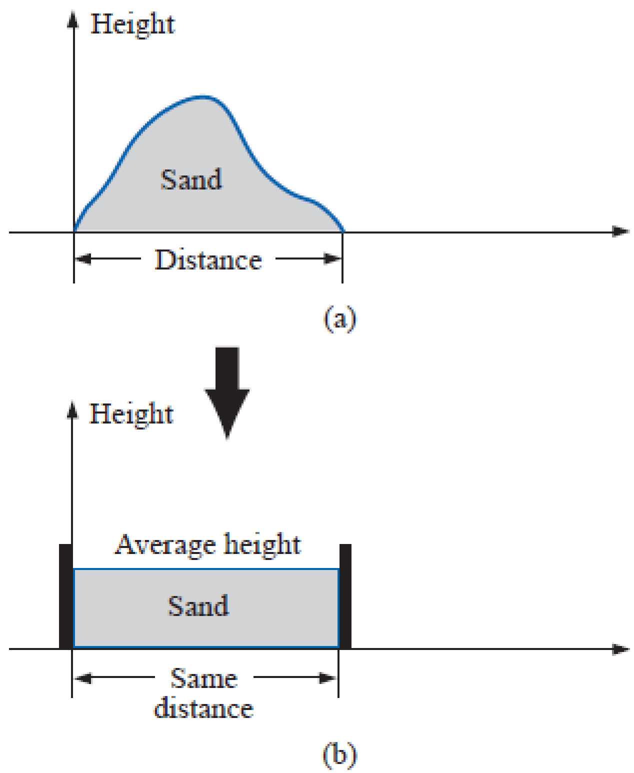


FIG. 13.33
Defining average value.

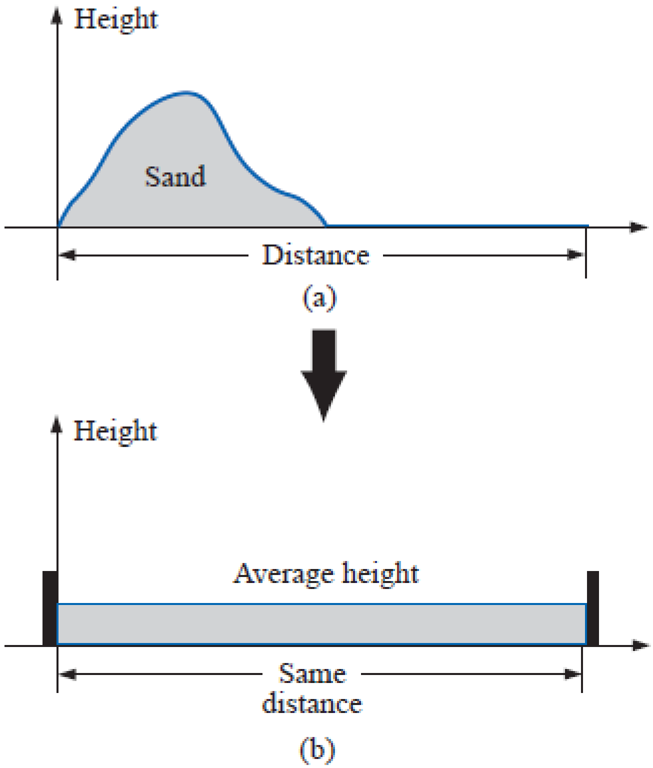


FIG. 13.34

Effect of distance (length) on average value.

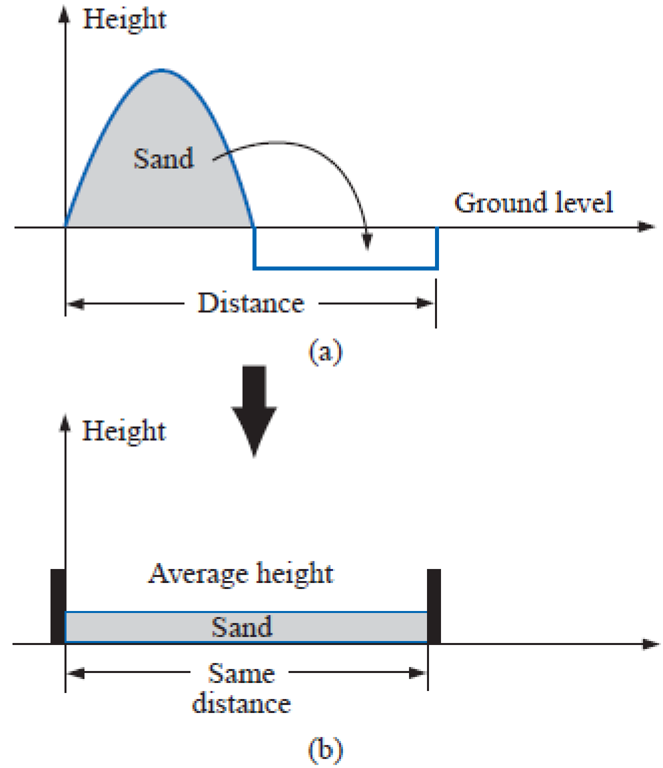


FIG. 13.35

Effect of depressions (negative excursions) on average value.

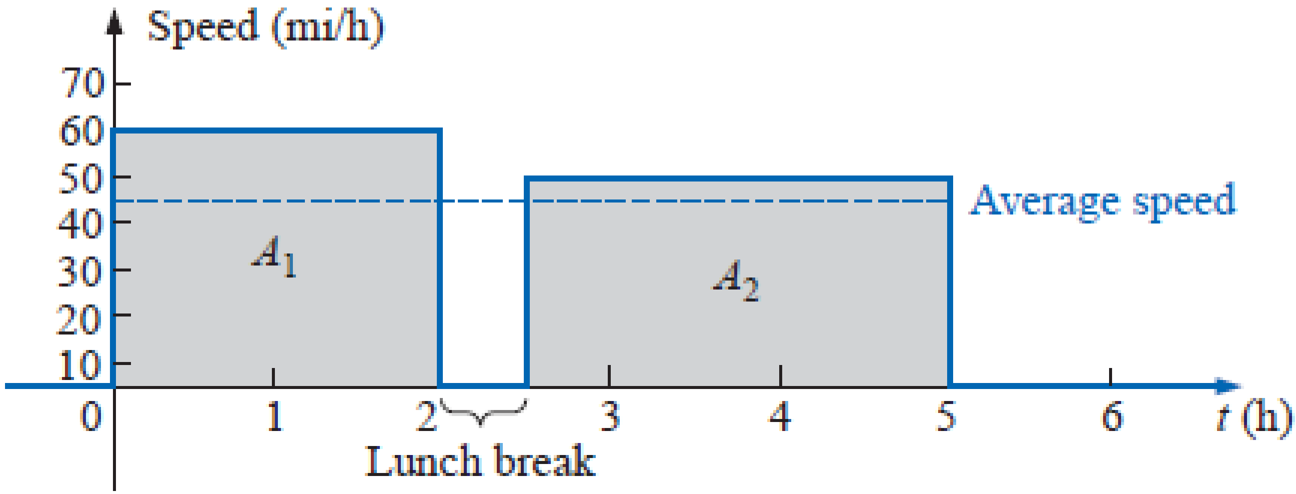


FIG. 13.36

Plotting speed versus time for an automobile excursion.

EXAMPLE 13.13 Determine the average value of the waveforms of Fig. 13.37.

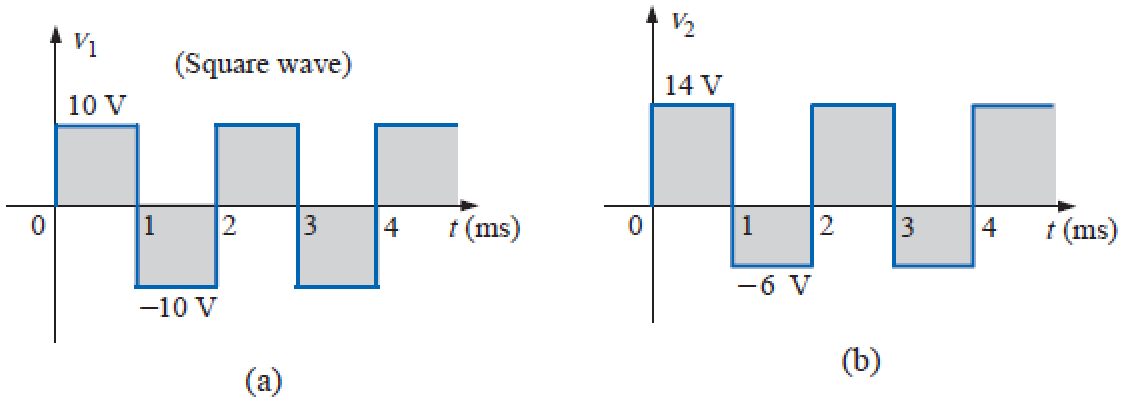


FIG. 13.37
Example 13.13.

Solutions:

a. By inspection, the area above the axis equals the area below over one cycle, resulting in an average value of zero volts. Using Eq. (13.26):

$$G = \frac{(10\text{ V})(1\text{ ms}) - (10\text{ V})(1\text{ ms})}{2\text{ ms}}$$

$$= \frac{0}{2\text{ ms}} = \mathbf{0\ V}$$

b. Using Eq. (13.26):

$$G \text{ (average value)} = \frac{\text{algebraic sum of areas}}{\text{length of curve}} \qquad \textbf{(13.26)}$$

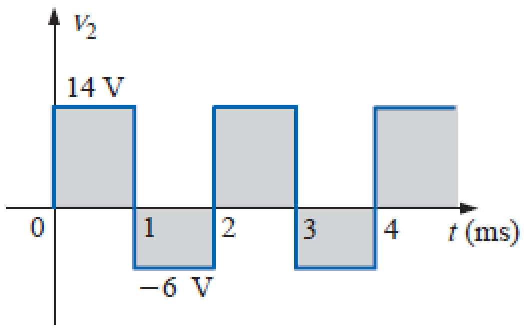
$$G = \frac{(14\text{ V})(1\text{ ms}) - (6\text{ V})(1\text{ ms})}{2\text{ ms}}$$

$$= \frac{14\text{ V} - 6\text{ V}}{2} = \frac{8\text{ V}}{2} = \mathbf{4\ V}$$

as shown in Fig. 13.38.

In reality, the waveform of Fig. 13.37(b) is simply the square wave of Fig. 13.37(a) with a dc shift of 4 V; that is,

$$v_2 = v_1 + 4\text{ V}$$



(b)

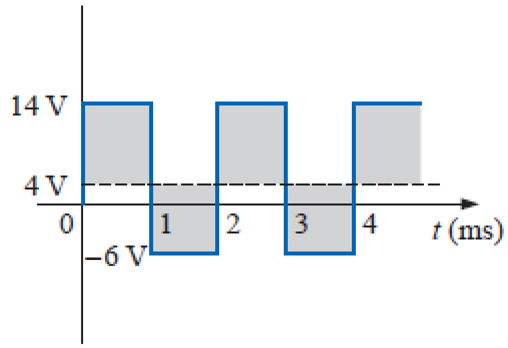


FIG. 13.38
Defining the average value for the waveform of Fig. 13.37(b).

EXAMPLE 13.14 Find the average values of the following waveforms over one full cycle:

a. Fig. 13.39.
b. Fig. 13.40.

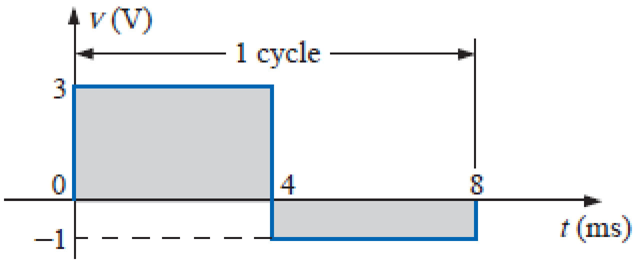


FIG. 13.39
Example 13.14, part (a).

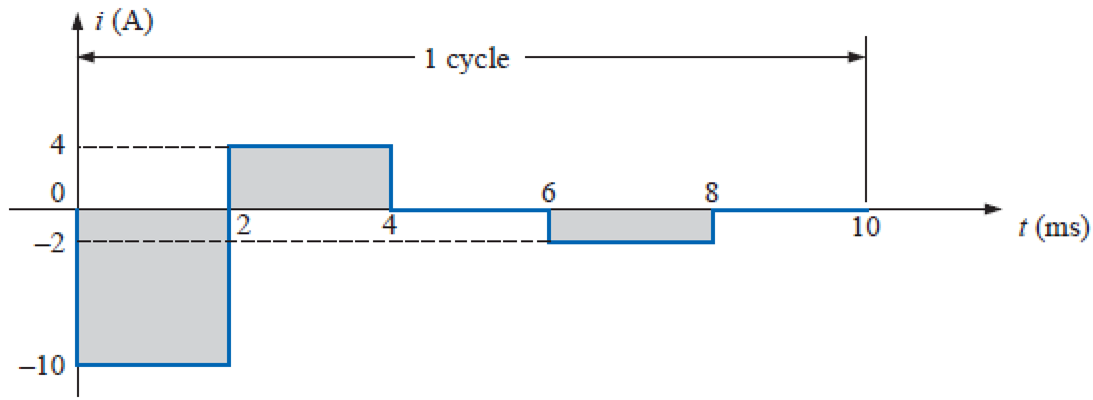


FIG. 13.40
Example 13.14, part (b).

Solutions:

a. $G = \dfrac{+(3\text{ V})(4\text{ ms}) - (1\text{ V})(4\text{ ms})}{8\text{ ms}} = \dfrac{12\text{ V} - 4\text{ V}}{8} = \mathbf{1\ V}$

Note Fig. 13.41.

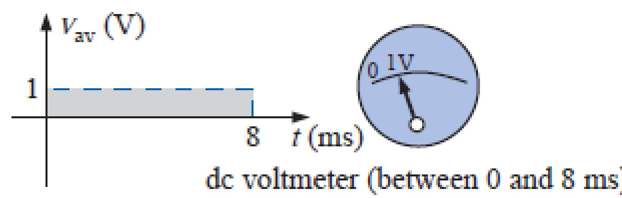


FIG. 13.41
The response of a dc meter to the waveform of Fig. 13.39.

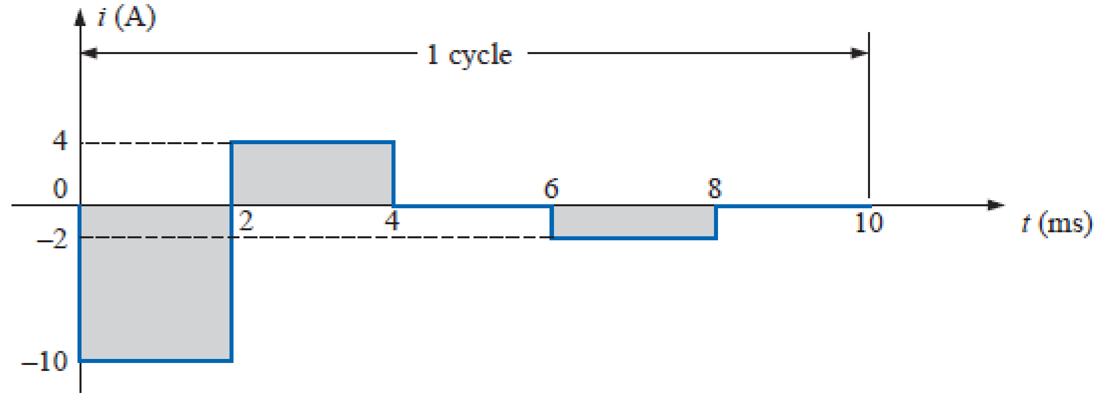


FIG. 13.40
Example 13.14, part (b).

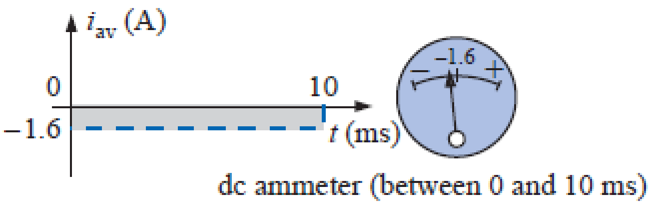


FIG. 13.42
The response of a dc meter to the waveform of Fig. 13.40.

b. $$G = \frac{-(10\text{ V})(2\text{ ms}) + (4\text{ V})(2\text{ ms}) - (2\text{ V})(2\text{ ms})}{10\text{ ms}}$$

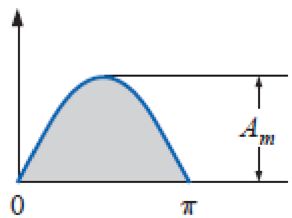
$$= \frac{-20\text{ V} + 8\text{ V} - 4\text{ V}}{10} = -\frac{16\text{ V}}{10} = \mathbf{-1.6\text{ V}}$$

Note Fig. 13.42.

$$\text{Area} = \int_0^{\pi} A_m \sin \alpha \, d\alpha$$

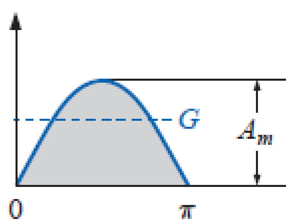
$$\begin{aligned}\text{Area} &= A_m[-\cos \alpha]_0^{\pi} \\ &= -A_m(\cos \pi - \cos 0°) \\ &= -A_m[-1 - (+1)] = -A_m(-2)\end{aligned}$$

$$\boxed{\text{Area} = 2A_m}$$



$$G = \frac{2A_m}{\pi}$$

$$\boxed{G = 0.637A_m}$$



For the waveform of Fig. 13.45,

$$G = \frac{(2A_m/2)}{\pi/2} = \frac{2A_m}{\pi} \qquad \text{(average the same as for a full pulse)}$$

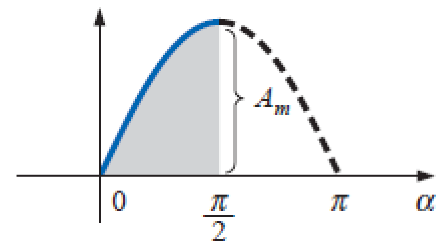


FIG. 13.45

Finding the average value of one-half the positive pulse of a sinusoidal waveform.

EXAMPLE 13.15 Determine the average value of the sinusoidal waveform of Fig. 13.46.

Solution: By inspection it is fairly obvious that

the average value of a pure sinusoidal waveform over one full cycle is zero.

Eq. (13.26):

$$G = \frac{+2A_m - 2A_m}{2\pi} = \mathbf{0\ V}$$

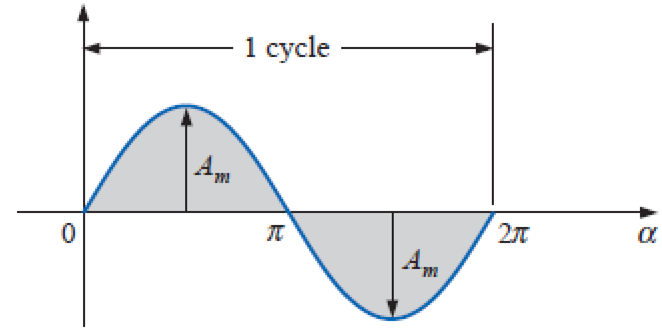


FIG. 13.46
Example 13.15.

EXAMPLE 13.16 Determine the average value of the waveform of Fig. 13.47.

Solution: The peak-to-peak value of the sinusoidal function is 16 mV + 2 mV = 18 mV. The peak amplitude of the sinusoidal waveform is, therefore, 18 mV/2 = 9 mV. Counting down 9 mV from 2 mV (or 9 mV up from −16 mV) results in an average or dc level of −7 mV, as noted by the dashed line of Fig. 13.47.

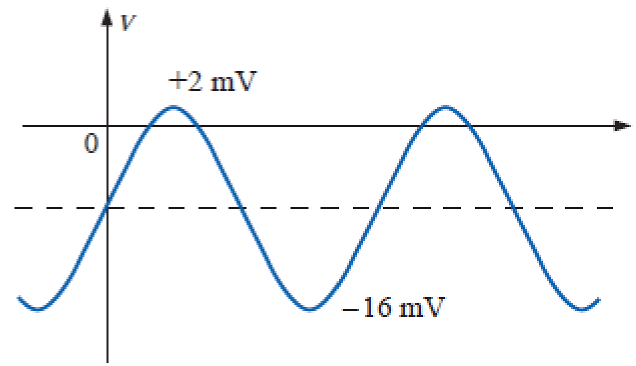
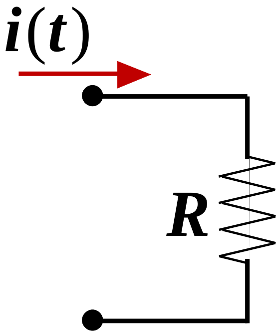


FIG. 13.47
Example 13.16.

EFFECTIVE OR RMS VALUES



Instantaneous power

$$p(t) = i^2(t)R$$

If the current is sinusoidal the average pow to be

$$P_{av} = \frac{1}{2} I_M^2 R$$

$$\therefore I_{eff}^2 = \frac{1}{2} I_M^2$$

The effective value is the equivalent DC value that supplies the same average power

For a sinusoidal signal

$$x(t) = X_M \cos(\omega t + \theta)$$

the effective value is

$$X_{eff} = \frac{X_M}{\sqrt{2}}$$

If current is periodic with period T

$$P_{av} = \frac{1}{T} \int_{t_0}^{t_0+T} p(t)dt = R\left(\frac{1}{T} \int_{t_0}^{t_0+T} i^2(t)dt \right)$$

$$I_{eff} : P_{av} = P_{dc}$$

If current is DC ($i(t) = I_{dc}$) then

$$P_{dc} = RI_{dc}^2$$

$$I_{eff}^2 = \frac{1}{T} \int_{t_0}^{t_0+T} i^2(t)dt$$

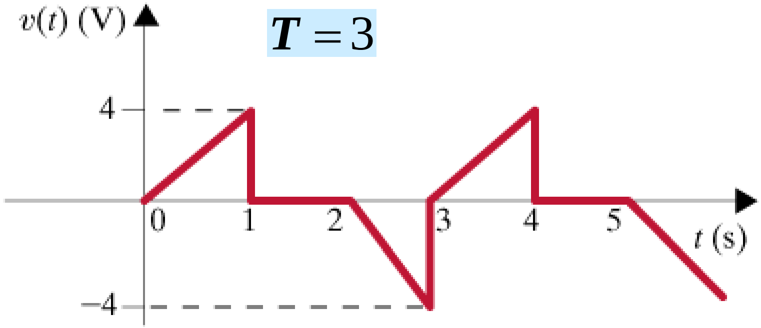
$$I_{eff} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} i^2(t)dt}$$

effective $\approx$ rms (root mean square)

Definition is valid for ANY periodic signal with period T

LEARNING EXAMPLE

Compute the rms value of the voltage waveform



$$T = 3$$

$$X_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)dt}$$

$$v(t) = \begin{cases} 4t & 0 < t \le 1 \\ 0 & 1 < t \le 2 \\ -4(t-2) & 2 < t \le 3 \end{cases}$$

The two integrals have the same value

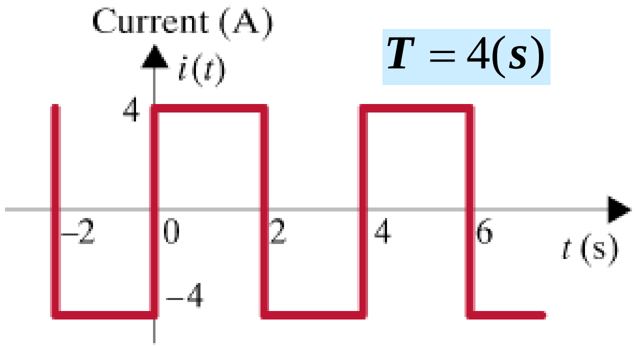
$$\int_0^T v^2(t)dt = \int_0^1 (4t)^2 dt + \int_2^3 (4(t-2))^2 dt$$

$$\int_0^3 v^2(t)dt = 2 \times \left[\frac{16}{3}t^3\right]_0^1 = \frac{32}{3}$$

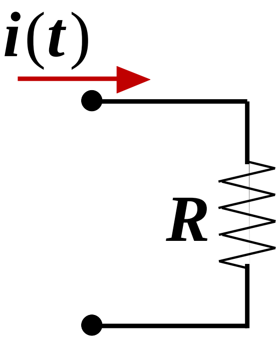
$$V_{rms} = \sqrt{\frac{1}{3} \times \frac{32}{3}} = 1.89(V)$$

LEARNING EXAMPLE

Compute the rms value of the voltage waveform and use it to determine the average power supplied to the resistor



$T = 4(s)$



$R = 2\Omega$

$$X_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)dt}$$

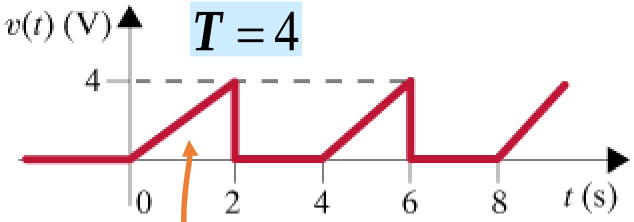
$i^2(t) = 16;\ 0 \le t < 4$

$I_{rms} = 4(A)$

$P_{av} = RI_{rms}^2 = 32(W)$

LEARNING EXTENSION

Compute rms value of the voltage waveform



$T = 4$

$v = 2t$

$$X_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)dt}$$

$$V_{rms} = \sqrt{\frac{1}{4}\int_0^2 (2t)^2 dt}$$

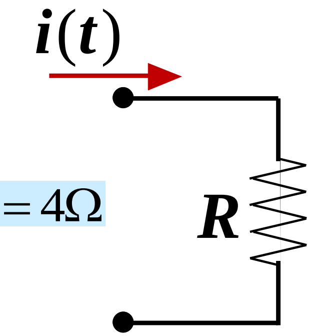
$$= \left[\frac{1}{3}t^3\right]_0^2 = \frac{8}{3}(V)$$

LEARNING EXTENSION

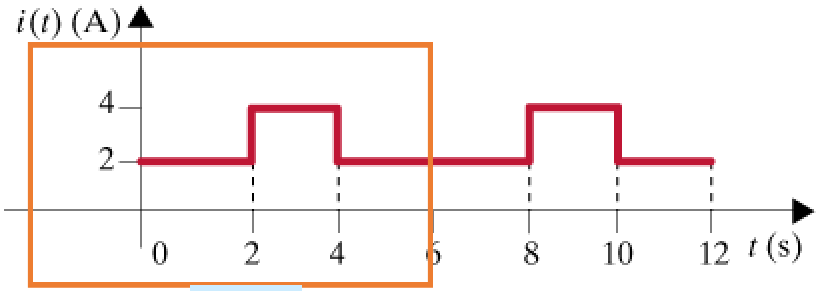
Compute the rms value for the current waveforms and use them to determine average power supplied to the resistor

$$X_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\,dt}$$

$$P_{av} = I_{rms}^2 R$$



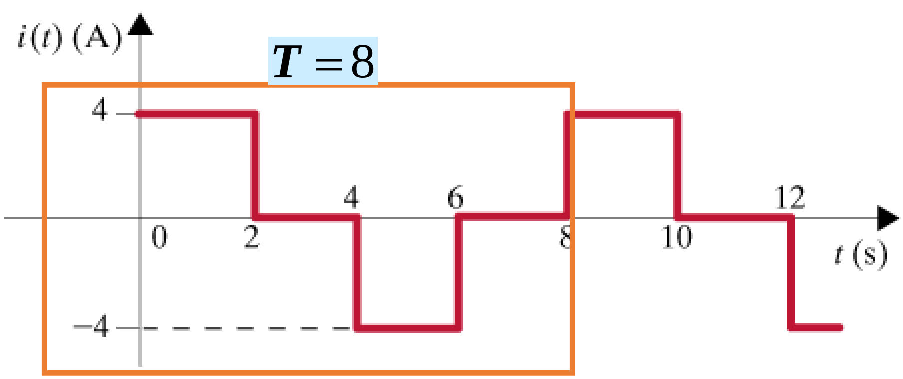
$R = 4\Omega$



$T = 6$

$$I_{rms}^2 = \frac{1}{6}\left[\int_0^2 4dt + \int_2^4 16dt + \int_4^6 4dt\right] = \frac{8+32+8}{6} = 8$$

$$P = 8 \times 4 = 32(W)$$



$T = 8$

$$I_{rms}^2 = \frac{1}{8}\left[\int_0^2 16dt + \int_4^6 16dt\right] = 8$$

$$P = 32(W)$$

Complex Numbers

- A powerful method for representing sinusoids is the phasor.
- But in order to understand how they work, we need to cover some complex numbers first.
- A complex number *z* can be represented in rectangular form as:

$$z = x + jy$$

- It can also be written in polar or exponential form as:

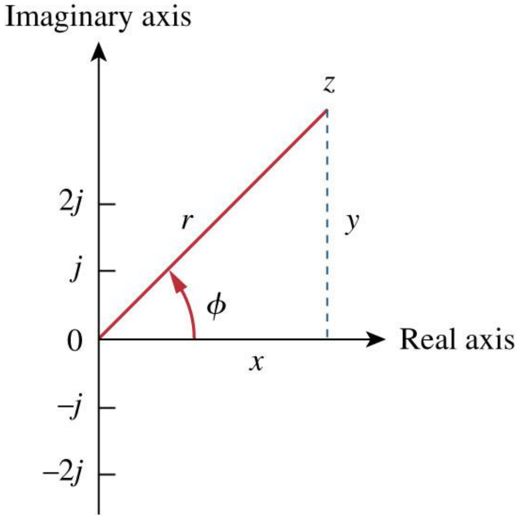
$$z = r\angle\phi = re^{j\phi}$$

Complex Numbers

- The different forms can be interconverted.
- Starting with rectangular form, one can go to polar:
- Likewise, from polar to rectangular form goes as follows:

$$r = \sqrt{x^2 + y^2} \quad \phi = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\phi \quad y = r\sin\phi$$



Complex Numbers

- The following mathematical operations are important

Addition

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication

$$z_1 z_2 = r_1 r_2 \angle(\phi_1 + \phi_2)$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle(\phi_1 - \phi_2)$$

Reciprocal

$$\frac{1}{z} = \frac{1}{r} \angle(-\phi)$$

Square Root

$$\sqrt{z} = \sqrt{r} \angle(\phi / 2)$$

Complex Conjugate

$$z^* = x - jy = r\angle -\phi = re^{-j\phi}$$

Phasors

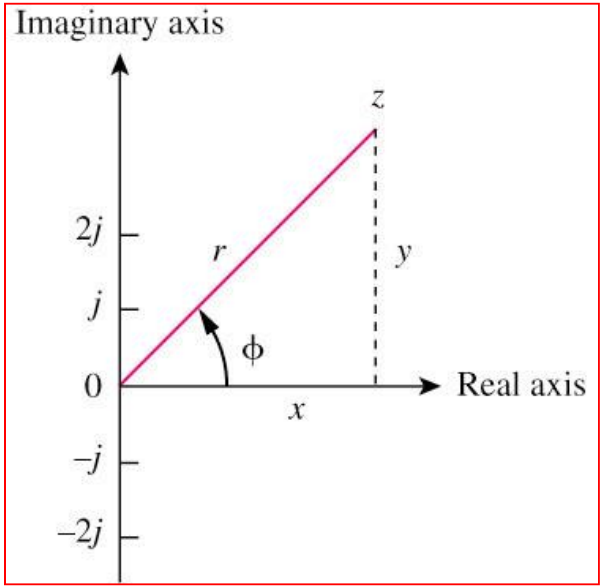
- The idea of a phasor representation is based on Euler's identity:

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

- From this we can represent a sinusoid as the real component of a vector in the complex plane.
- The length of the vector is the amplitude of the sinusoid.
- The vector, *V*, in polar form, is at an angle ϕ with respect to the positive real axis.

- A phasor is a complex number that represents the amplitude and phase of a sinusoid.

- It can be represented in one of the following three forms:



a. Rectangular $z = x + jy = r(\cos\phi + j\sin\phi)$

b. Polar $z = r\angle\phi$

c. Exponential $z = re^{j\phi}$

where $r = \sqrt{x^2 + y^2}$

$\phi = \tan^{-1}\dfrac{y}{x}$

Example 3

- Evaluate the following complex numbers:

a. $$[(5+j2)(-1+j4)-5\angle 60^{o}]$$

b. $$\frac{10+j5+3\angle 40^{o}}{-3+j4}+10\angle 30^{o}$$

Solution:

a. –15.5 + j13.67

8.293 + j2.2b.

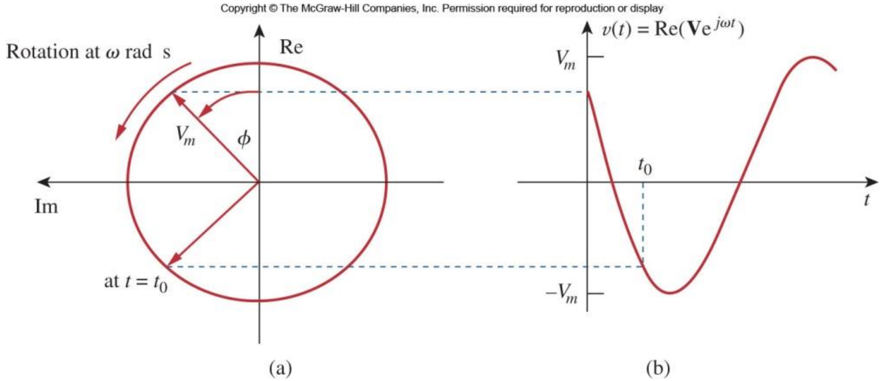
Phasors

- Phasors are typically represented at *t=0*.
- As such, the transformation between time domain to phasor domain is:

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad V = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)

- They can be graphically represented as shown here.



Transform the following sinusoids to phasors:

$i = 6\cos(50t - 40^o)$ A

$v = -4\sin(30t + 50^o)$ V

$\cos(wt + 90^o) = -\sin wt$, $\cos(wt - 90^o) = \sin wt$

Solution:

a. $\mathbf{I} = 6\angle -40^\circ$ A

b. Since $-\sin(A) = \cos(A+90^o)$;

$v(t) = 4\cos(30t+50^o+90^o) = 4\cos(30t+140^o)$ V

Transform to phasor => $\mathbf{V} = 4\angle 140^\circ$ V

Sinusoid-Phasor Transformation

- Here is a handy table for transforming various time domain sinusoids into phasor domain:

TABLE 9.1

Sinusoid-phasor transformation.

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90^\circ$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90^\circ$

Sinusoid-Phasor Transformation

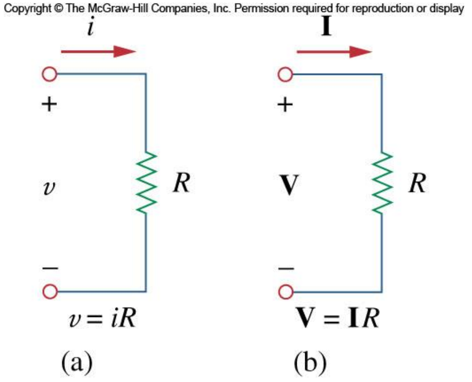
- Note that the frequency of the phasor is not explicitly shown in the phasor diagram
- For this reason phasor domain is also known as frequency domain.
- Applying a derivative to a phasor yields:
- Applying an integral to a phasor yeilds:

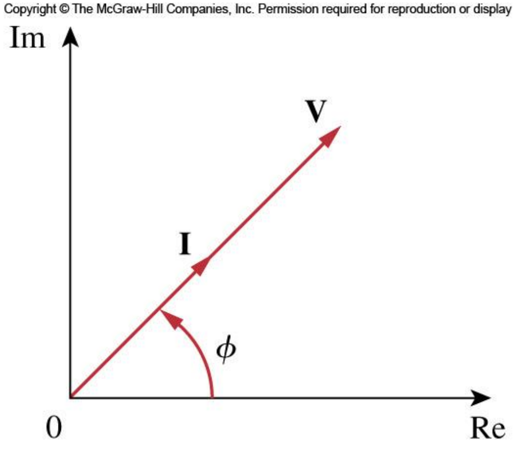
$$\underset{\text{(Time domain)}}{\frac{dv}{dt}} \quad \Leftrightarrow \quad \underset{\text{(Phasor domain)}}{j\omega V}$$

$$\underset{\text{(Time domain)}}{\int v dt} \quad \Leftrightarrow \quad \underset{\text{(Phasor domain)}}{\frac{V}{j\omega}}$$

Phasor Relationships for Resistors

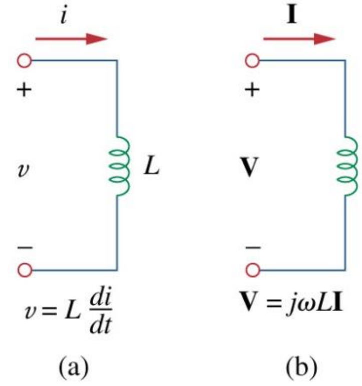
- Each circuit element has a relationship between its current and voltage.
- These can be mapped into phasor relationships very simply for resistors capacitors and inductor.
- For the resistor, the voltage and current are related via Ohm's law.
- As such, the voltage and current are in phase with each other.

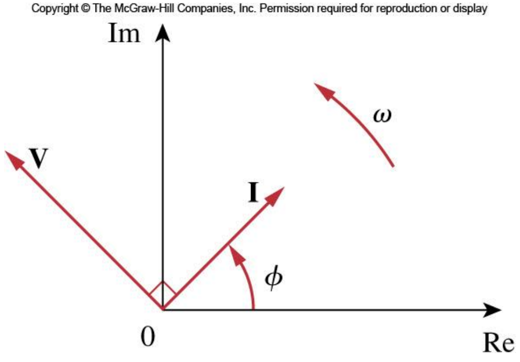




Phasor Relationships for Inductors

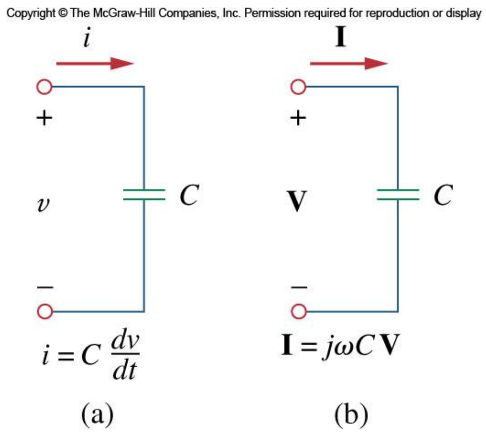
- Inductors on the other hand have a phase shift between the voltage and current.
- In this case, the voltage leads the current by 90°.
- Or one says the current lags the voltage, which is the standard convention.
- This is represented on the phasor diagram by a positive phase angle between the voltage and current.

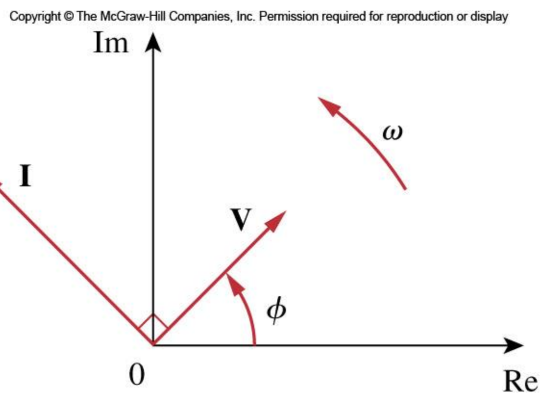




Phasor Relationships for Capacitors

- Capacitors have the opposite phase relationship as compared to inductors.
- In their case, the current leads the voltage.
- In a phasor diagram, this corresponds to a negative phase angle between the voltage and current.





Voltage current relationships

TABLE 9.2

Summary of voltage-current relationships.

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

Find the voltage $v(t)$ in a circuit described by the integrodiffl equation

$$2\frac{dv}{dt} + 5v + 10\int v\,dt = 50\cos(5t - 30^\circ)$$

using the phasor approach.

Given that

$$2\frac{dv}{dt} + 5v + 10\int v\,dt = 50\cos(5t - 30^\circ)$$

we take the phasor of each term to get

$$2j\omega\,\mathbf{V} + 5\,\mathbf{V} + \frac{10}{j\omega}\mathbf{V} = 50\angle\text{-}30^\circ, \qquad \omega = 5$$

$$\mathbf{V}\,[j10 + 5 - j(10/5)] = \mathbf{V}\,(5 + j8) = 50\angle\text{-}30^\circ$$

$$\mathbf{V} = \frac{50\angle -30^\circ}{5 + j8} = \frac{50\angle -30^\circ}{9.434\angle 58^\circ}$$

$$\mathbf{V} = 5.3\angle\text{-}88^\circ$$

Converting **V** to the time domain

$$v(t) = \mathbf{5.3\cos(5t - 88^\circ)V}$$

Impedance and Admittance

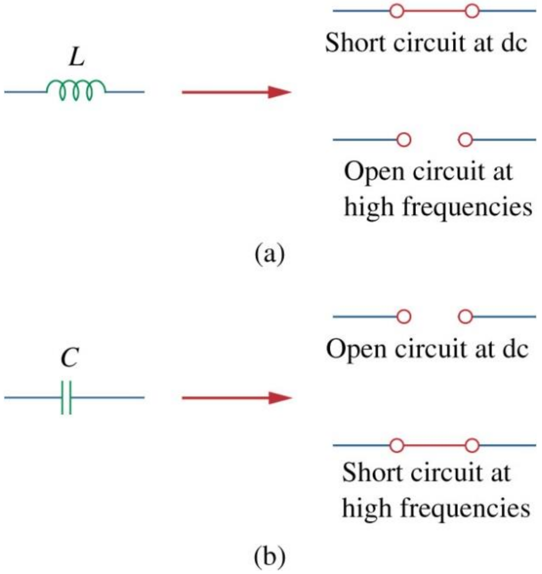
- It is possible to expand Ohm's law to capacitors and inductors.
- In time domain, this would be tricky as the ratios of voltage and current and always changing.
- But in frequency domain it is straightforward
- The impedance of a circuit element is the ratio of the phasor voltage to the phasor current.

$$Z = \frac{V}{I} \quad \text{or} \quad V = ZI$$

- Admittance is simply the inverse of impedance.

Impedance and Admittance

- It is important to realize that in frequency domain, the values obtained for impedance are only valid at that frequency.
- Changing to a new frequency will require recalculating the values.
- The impedance of capacitors and inductors are shown here:



Impedance and Admittance

- As a complex quantity, the impedance may be expressed in rectangular form.
- The separation of the real and imaginary components is useful.
- The real part is the resistance.
- The imaginary component is called the reactance, X.
- When it is positive, we say the impedance is inductive, and capacitive when it is negative.

Impedance and Admittance

- Admittance, being the reciprocal of the impedance, is also a complex number.
- It is measured in units of Siemens
- The real part of the admittance is called the conductance, *G*
- The imaginary part is called the susceptance, *B*
- These are all expressed in Siemens or (mhos)
- The impedance and admittance components can be related to each other:

$$G = \frac{R}{R^2 + X^2} \qquad B = -\frac{X}{R^2 + X^2}$$

Impedance and Admittance

TABLE 9.3

Impedances and admittances of passive elements.

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \dfrac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \dfrac{1}{j\omega L}$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Example 9.9

Find $v(t)$ and $i(t)$ in the circuit shown in Fig. 9.16.

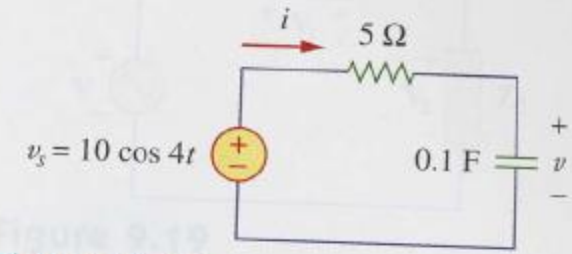


Figure 9.16
For Example 9.9.

Solution:

From the voltage source $10\cos 4t$, $\omega = 4$,

$$\mathbf{V}_s = 10\underline{/0^\circ} \text{ V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current

$$\begin{aligned}\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} &= \frac{10\underline{/0^\circ}}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2} \\ &= 1.6 + j0.8 = 1.789\underline{/26.57^\circ} \text{ A}\end{aligned} \tag{9.9.1}$$

The voltage across the capacitor is

$$\begin{aligned}\mathbf{V} = \mathbf{I}\mathbf{Z}_C &= \frac{\mathbf{I}}{j\omega C} = \frac{1.789\underline{/26.57^\circ}}{j4 \times 0.1} \\ &= \frac{1.789\underline{/26.57^\circ}}{0.4\underline{/90^\circ}} = 4.47\underline{/-63.43^\circ} \text{ V}\end{aligned} \tag{9.9.2}$$

onverting **I** and **V** in Eqs. (9.9.1) and (9.9.2) to the time domain, we get

$$i(t) = 1.789\cos(4t + 26.57^\circ) \text{ A}$$

$$v(t) = 4.47\cos(4t - 63.43^\circ) \text{ V}$$

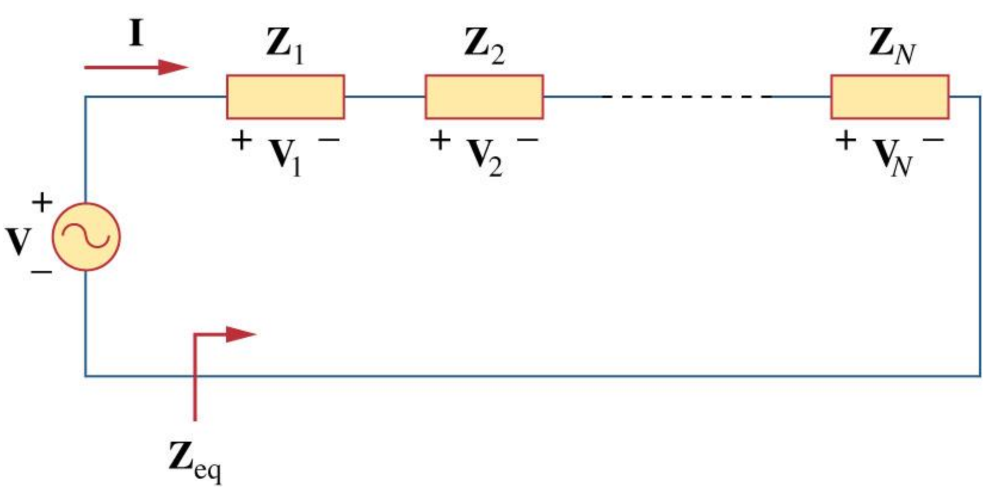
tice that $i(t)$ leads $v(t)$ by 90° as expected.

Kirchoff's Laws in Frequency Domain

- A powerful aspect of phasors is that Kirchoff's laws apply to them as well.
- This means that a circuit transformed to frequency domain can be evaluated by the same methodology developed for KVL and KCL.
- One consequence is that there will likely be complex values.

Impedance Combinations

- Once in frequency domain, the impedance elements are generalized.
- Combinations will follow the rules for resistors:



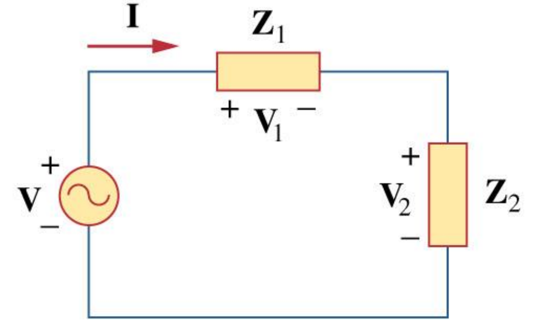
Impedance Combinations

- Series combinations will result in a sum of the impedance elements:

$$Z_{eq} = Z_1 + Z_2 + Z_3 + \cdots + Z_N$$

- Here then two elements in series can act like a voltage divider

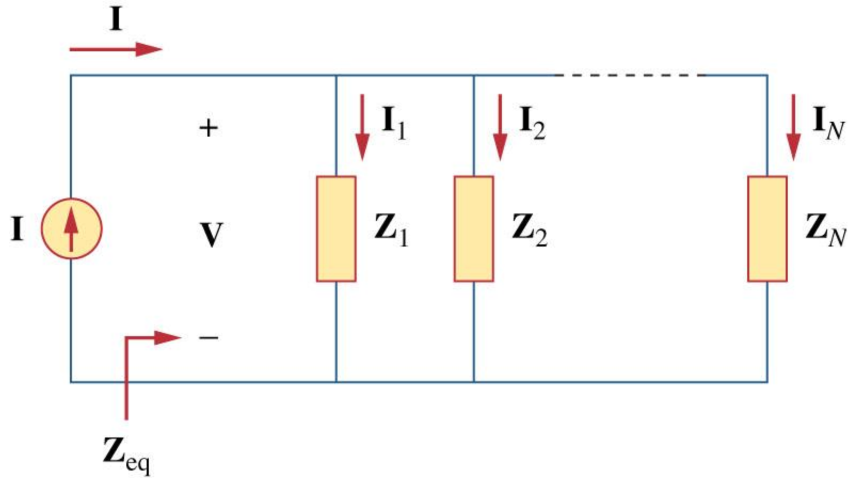
$$V_1 = \frac{Z_1}{Z_1 + Z_2} V \quad V_2 = \frac{Z_2}{Z_1 + Z_2} V$$



Parallel Combination

- Likewise, elements combined in parallel will combine in the same fashion as resistors in parallel:

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots + \frac{1}{Z_N}$$



Example 9.10

Find the input impedance of the circuit in Fig. 9.23. Assume that the circuit operates at $\omega = 50$ rad/s.

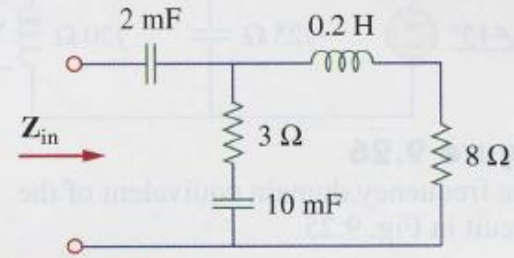


Figure 9.23
For Example 9.10.

Solution:

Let

$\mathbf{Z}_1$ = Impedance of the 2-mF capacitor

$\mathbf{Z}_2$ = Impedance of the 3-Ω resistor in series with the10-mF capacitor

$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

Then

$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

The input impedance is

$$\mathbf{Z}_{in} = \mathbf{Z}_1 + \mathbf{Z}_2 \| \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07\ \Omega$$

Thus,

$$\mathbf{Z}_{in} = 3.22 - j11.07\ \Omega$$

Admittance

- Expressed as admittance, though, they are again a sum:

$$Y_{eq} = Y_1 + Y_2 + Y_3 + \cdots + Y_N$$

- Once again, these elements can act as a current divider:

$$I_1 = \frac{Z_2}{Z_1 + Z_2} I \qquad I_2 = \frac{Z_1}{Z_1 + Z_2} I$$

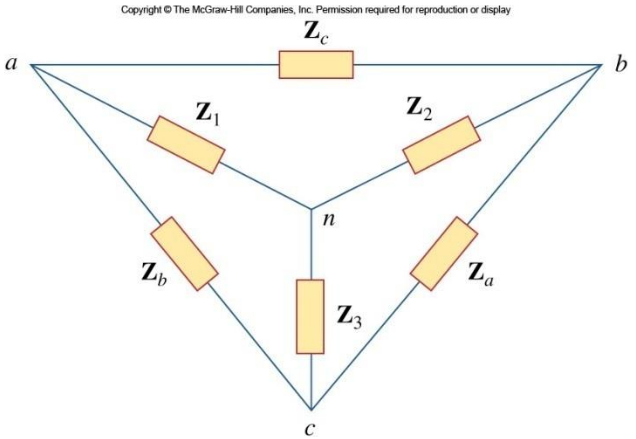
Impedance Combinations

- The Delta-Wye transformation is:

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c} \qquad Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c} \qquad Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c} \qquad Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$$



Phasor algebra for sinusoidal quantities is applicable only for waveforms having the same frequency.

EXAMPLE 14.30 Write the sinusoidal expression for the following phasors if the frequency is 60 Hz:

Phasor Domain	Time Domain
a. $\mathbf{I} = 10 \angle 30°$	$i = \sqrt{2}(10) \sin(2\pi 60t + 30°)$ and $i = \mathbf{14.14 \sin(377t + 30°)}$
b. $\mathbf{V} = 115 \angle -70°$	$v = \sqrt{2}(115) \sin(377t - 70°)$ and $v = \mathbf{162.6 \sin(377t - 70°)}$

EXAMPLE 14.31 Find the input voltage of the circuit of Fig. 14.65 if

$$\left.\begin{aligned} v_a &= 50 \sin(377t + 30°) \\ v_b &= 30 \sin(377t + 60°) \end{aligned}\right\} f = 60 \text{ Hz}$$

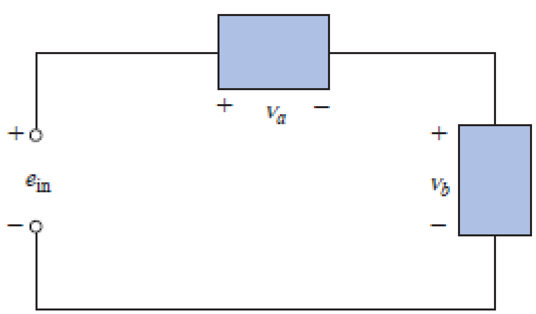


FIG. 14.65
Example 14.31.

Solution: Applying Kirchhoff's voltage law, we have

$$e_{in} = v_a + v_b$$

Converting from the time to the phasor domain yields

$$v_a = 50 \sin(377t + 30°) \Rightarrow \mathbf{V}_a = 35.35 \text{ V} \angle 30°$$
$$v_b = 30 \sin(377t + 60°) \Rightarrow \mathbf{V}_b = 21.21 \text{ V} \angle 60°$$

Converting from polar to rectangular form for addition yields

$$\mathbf{V}_a = 35.35 \text{ V} \angle 30° = 30.61 \text{ V} + j\,17.68 \text{ V}$$
$$\mathbf{V}_b = 21.21 \text{ V} \angle 60° = 10.61 \text{ V} + j\,18.37 \text{ V}$$

Then

$$\begin{aligned} \mathbf{E}_{in} = \mathbf{V}_a + \mathbf{V}_b &= (30.61 \text{ V} + j\,17.68 \text{ V}) + (10.61 \text{ V} + j\,18.37 \text{ V}) \\ &= 41.22 \text{ V} + j\,36.05 \text{ V} \end{aligned}$$

Converting from rectangular to polar form, we have

$$\mathbf{E}_{in} = 41.22 \text{ V} + j\,36.05 \text{ V} = 54.76 \text{ V} \angle 41.17°$$

Converting from the phasor to the time domain, we obtain

$$\mathbf{E}_{in} = 54.76 \text{ V} \angle 41.17° \Rightarrow e_{in} = \sqrt{2}(54.76) \sin(377t + 41.17°)$$

and $$e_{in} = \mathbf{77.43 \sin(377t + 41.17°)}$$

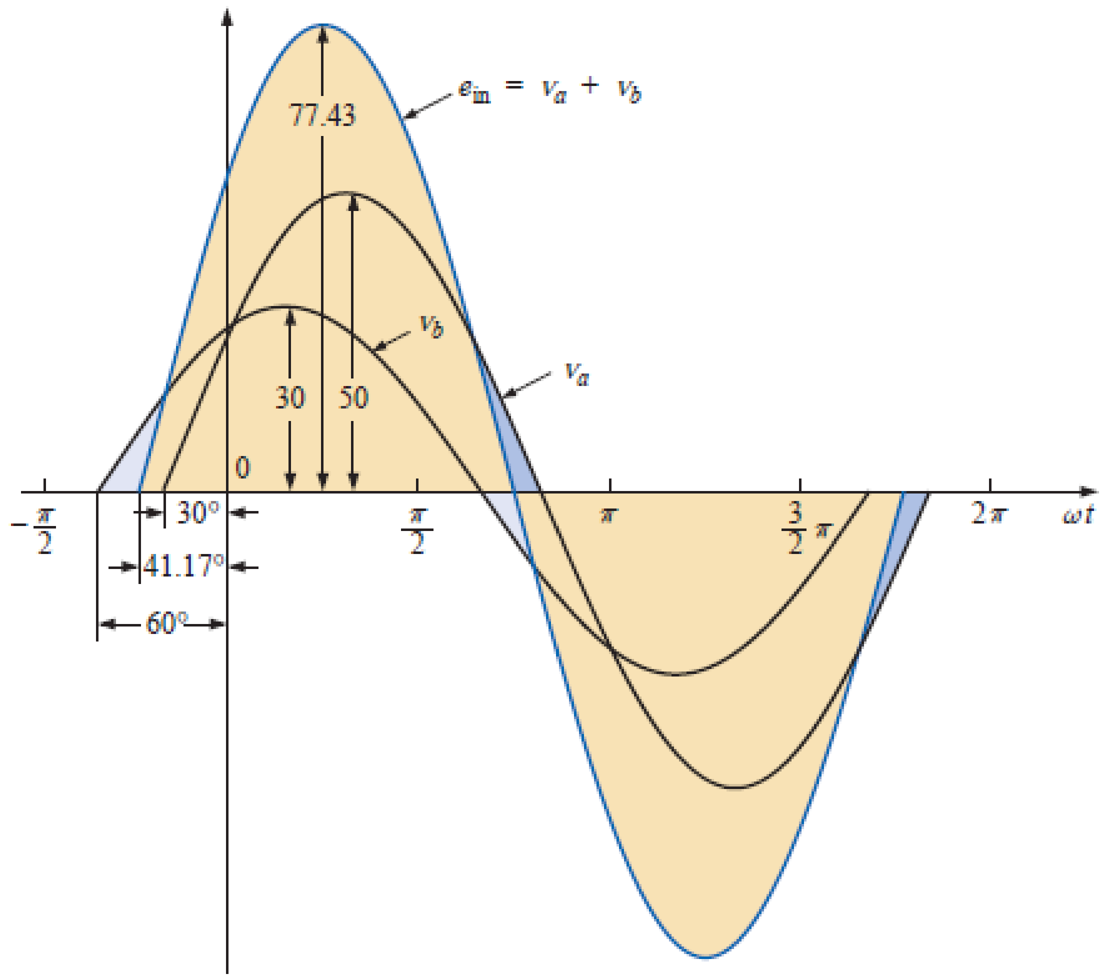


FIG. 14.66
Solution to Example 14.31.

EXAMPLE 14.32 Determine the current i_2 for the network of Fig. 14.67.

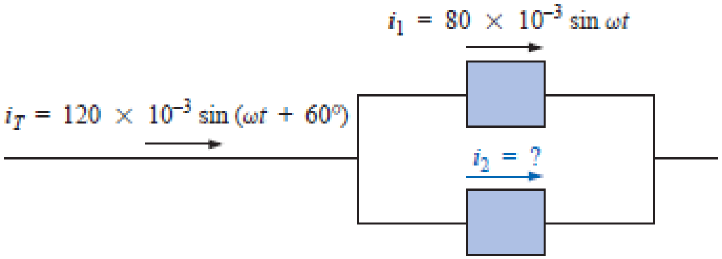


FIG. 14.67
Example 14.32.

Solution: Applying Kirchhoff's current law, we obtain

$$i_T = i_1 + i_2 \quad \text{or} \quad i_2 = i_T - i_1$$

Converting from the time to the phasor domain yields

$$i_T = 120 \times 10^{-3} \sin(\omega t + 60°) \Rightarrow 84.84 \text{ mA} \angle 60°$$
$$i_1 = 80 \times 10^{-3} \sin \omega t \Rightarrow 56.56 \text{ mA} \angle 0°$$

Converting from polar to rectangular form for subtraction yields

$$\mathbf{I}_T = 84.84 \text{ mA} \angle 60° = 42.42 \text{ mA} + j\,73.47 \text{ mA}$$
$$\mathbf{I}_1 = 56.56 \text{ mA} \angle 0° = 56.56 \text{ mA} + j\,0$$

Then

$$\mathbf{I}_2 = \mathbf{I}_T - \mathbf{I}_1$$
$$= (42.42 \text{ mA} + j\,73.47 \text{ mA}) - (56.56 \text{ mA} + j\,0)$$

and $$\mathbf{I}_2 = -14.14 \text{ mA} + j\,73.47 \text{ mA}$$

Converting from rectangular to polar form, we have

$$\mathbf{I}_2 = 74.82 \text{ mA} \angle 100.89°$$

Converting from the phasor to the time domain, we have

$$\mathbf{I}_2 = 74.82 \text{ mA} \angle 100.89° \Rightarrow$$
$$i_2 = \sqrt{2}(74.82 \times 10^{-3}) \sin(\omega t + 100.89°)$$

and $$\boldsymbol{i_2 = 105.8 \times 10^{-3} \sin(\omega t + 100.89°)}$$

A plot of the three waveforms appears in Fig. 14.68. The waveforms clearly indicate that $i_T = i_1 + i_2$.

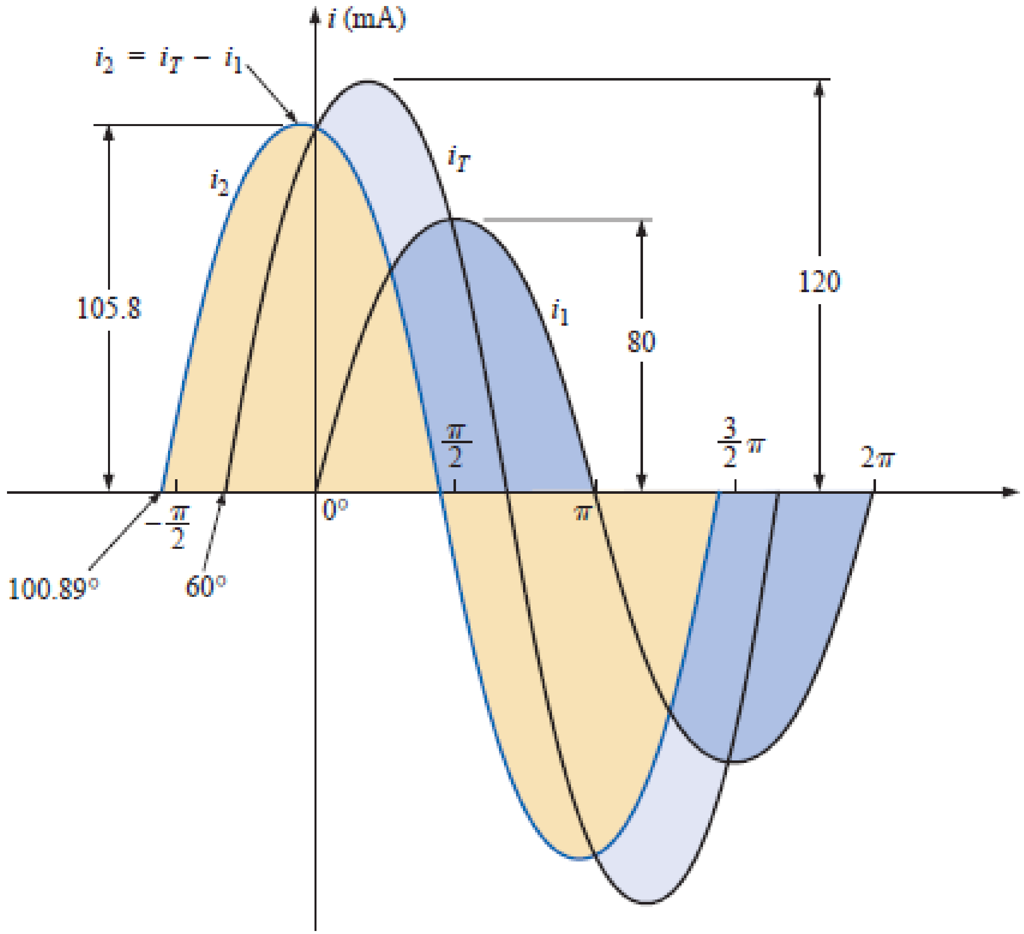


FIG. 14.68
Solution to Example 14.32.

IMPEDANCE AND THE PHASOR DIAGRAM

Resistive Elements

In phasor form,

$$v = V_m \sin \omega t \Rightarrow \mathbf{V} = V \angle 0°$$

where $V = 0.707V_m$.

Applying Ohm's law and using phasor algebra, we have

$$\mathbf{I} = \frac{V \angle 0°}{R \angle \theta_R} = \frac{V}{R} \underline{/0° - \theta_R}$$

Since i and v are in phase, the angle associated with i also must be 0°. To satisfy this condition, θ_R must equal 0°. Substituting $\theta_R = 0°$, we find

$$\mathbf{I} = \frac{V \angle 0°}{R \angle 0°} = \frac{V}{R} \underline{/0° - 0°} = \frac{V}{R} \angle 0°$$

so that in the time domain,

$$i = \sqrt{2}\left(\frac{V}{R}\right) \sin \omega t$$

The fact that $\theta_R = 0°$ will now be employed in the following polar format to ensure the proper phase relationship between the voltage and current of a resistor:

$$\boxed{\mathbf{Z}_R = R \angle 0°} \qquad \textbf{(15.1)}$$

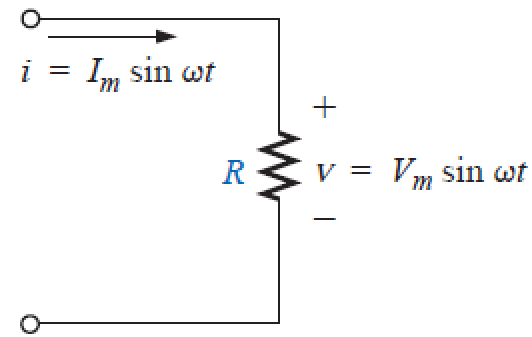


FIG. 15.1
Resistive ac circuit.

$$I_m = \frac{V_m}{R} \quad \text{or} \quad V_m = I_m R$$

The angle $\angle 0°$ of the resistance means that "the resistance in an AC circuit causes a zero phase shift between the current passing through it and the supplied voltage".

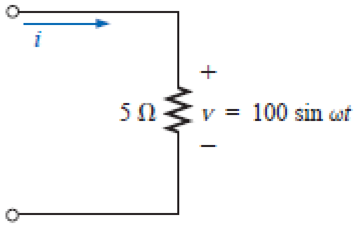


FIG. 15.2
Example 15.1.

EXAMPLE 15.1 Using complex algebra, find the current i for the circuit of Fig. 15.2. Sketch the waveforms of v and i.

Solution: Note Fig. 15.3:

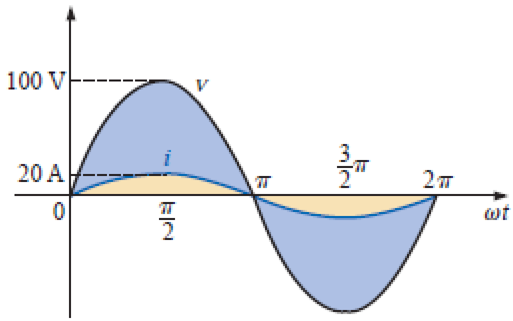


FIG. 15.3
Waveforms for Example 15.1.

$$v = 100 \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = 70.71 \text{ V} \angle 0°$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_R} = \frac{V \angle \theta}{R \angle 0°} = \frac{70.71 \text{ V} \angle 0°}{5\ \Omega \angle 0°} = 14.14 \text{ A} \angle 0°$$

and
$$i = \sqrt{2}(14.14) \sin \omega t = \mathbf{20 \sin \omega t}$$

Note:

The physical meaning of representing the resistance as a vector of magnitude $|R|$ and of angle $(\angle 0^{\circ})$ is due to a truth that “the current passes through the resistance is in-phase with the voltage of the source supplying it ”. This representation is suitable to treat the sinusoidal quantities (voltage and current) using vector algebra.

EXAMPLE 15.2 Using complex algebra, find the voltage v for the circuit of Fig. 15.4. Sketch the waveforms of v and i.

Solution: Note Fig. 15.5:

$$i = 4\sin(\omega t + 30°) \Rightarrow \text{phasor form } \mathbf{I} = 2.828\text{ A} \angle 30°$$

$$\begin{aligned}\mathbf{V} = \mathbf{I}\mathbf{Z}_R &= (I\angle\theta)(R\angle 0°) = (2.828\text{ A}\angle 30°)(2\ \Omega\angle 0°)\\ &= 5.656\text{ V}\angle 30°\end{aligned}$$

and $$v = \sqrt{2}(5.656)\sin(\omega t + 30°) = \mathbf{8.0\ sin(\omega t + 30°)}$$

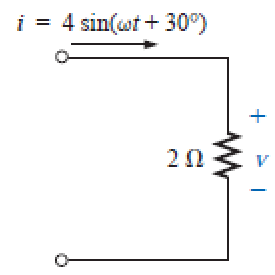


FIG. 15.4
Example 15.2.

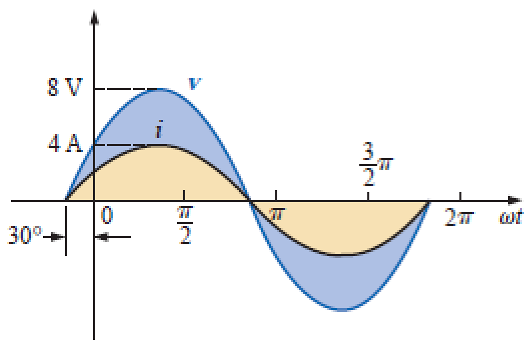


FIG. 15.5
Waveforms for Example 15.2.

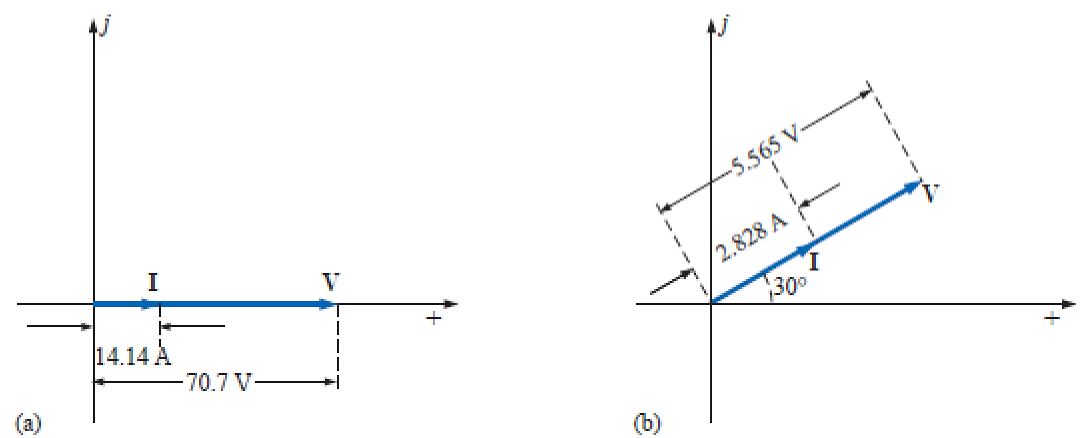


FIG. 15.6
Phasor diagrams for Examples 15.1 and 15.2.

Inductive Reactance

It was learned in Chapter 13 that for the pure inductor of Fig. 15.7, the voltage leads the current by 90° and that the reactance of the coil X_L is determined by ωL.

$$v = V_m \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = V \angle 0°$$

By Ohm's law,

$$\mathbf{I} = \frac{V \angle 0°}{X_L \angle \theta_L} = \frac{V}{X_L} \underline{/ 0° - \theta_L}$$

Since v leads i by 90°, i must have an angle of $-90°$ associated with it. To satisfy this condition, θ_L must equal $+90°$. Substituting $\theta_L = 90°$, we obtain

$$\mathbf{I} = \frac{V \angle 0°}{X_L \angle 90°} = \frac{V}{X_L} \underline{/ 0° - 90°} = \frac{V}{X_L} \angle -90°$$

so that in the time domain,

$$i = \sqrt{2}\left(\frac{V}{X_L}\right) \sin(\omega t - 90°)$$

The fact that $\theta_L = 90°$ will now be employed in the following polar format for inductive reactance to ensure the proper phase relationship between the voltage and current of an inductor.

$$\boxed{\mathbf{Z}_L = X_L \angle 90°} \qquad \textbf{(15.2)}$$

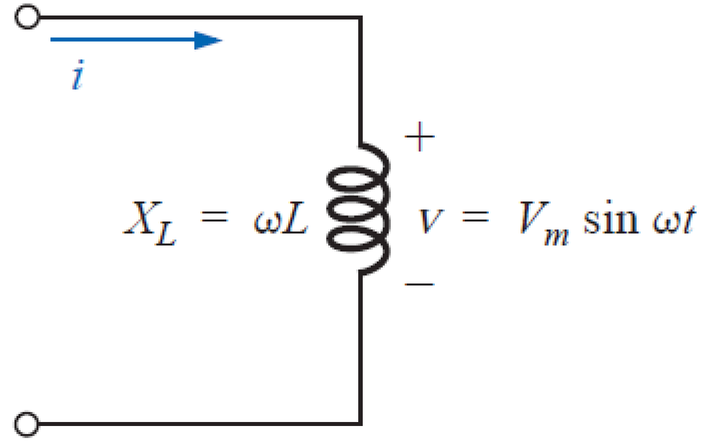


FIG. 15.7
Inductive AC circuit.

EXAMPLE 15.3 Using complex algebra, find the current i for the circuit of Fig. 15.8. Sketch the v and i curves.

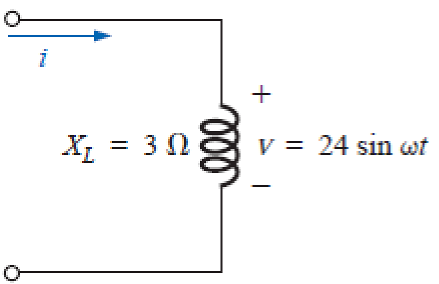


FIG. 15.8
Example 15.3.

Solution: Note Fig. 15.9:

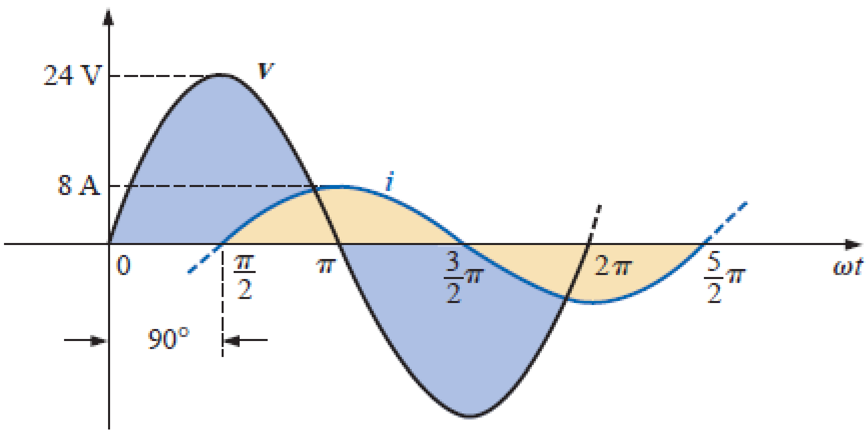


FIG. 15.9
Waveforms for Example 15.3.

$$v = 24 \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = 16.968\ \text{V} \angle 0°$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_L} = \frac{V \angle \theta}{X_L \angle 90°} = \frac{16.968\ \text{V} \angle 0°}{3\ \Omega \angle 90°} = 5.656\ \text{A} \angle -90°$$

and $$i = \sqrt{2}(5.656) \sin(\omega t - 90°) = \mathbf{8.0 \sin(\omega t - 90°)}$$

EXAMPLE 15.4 Using complex algebra, find the voltage v for the circuit of Fig. 15.10. Sketch the v and i curves.

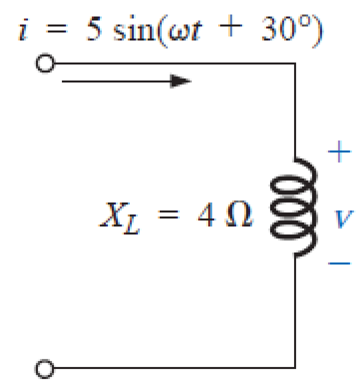


FIG. 15.10
Example 15.4.

Solution: Note Fig. 15.11:

$$i = 5\sin(\omega t + 30°) \Rightarrow \text{phasor form } \mathbf{I} = 3.535\text{ A}\angle 30°$$

$$\begin{aligned}\mathbf{V} = \mathbf{I}\mathbf{Z}_L &= (I\angle\theta)(X_L\angle 90°) = (3.535\text{ A}\angle 30°)(4\ \Omega\angle +90°)\\ &= 14.140\text{ V}\angle 120°\end{aligned}$$

and $$v = \sqrt{2}(14.140)\sin(\omega t + 120°) = \mathbf{20\sin(\omega t + 120°)}$$

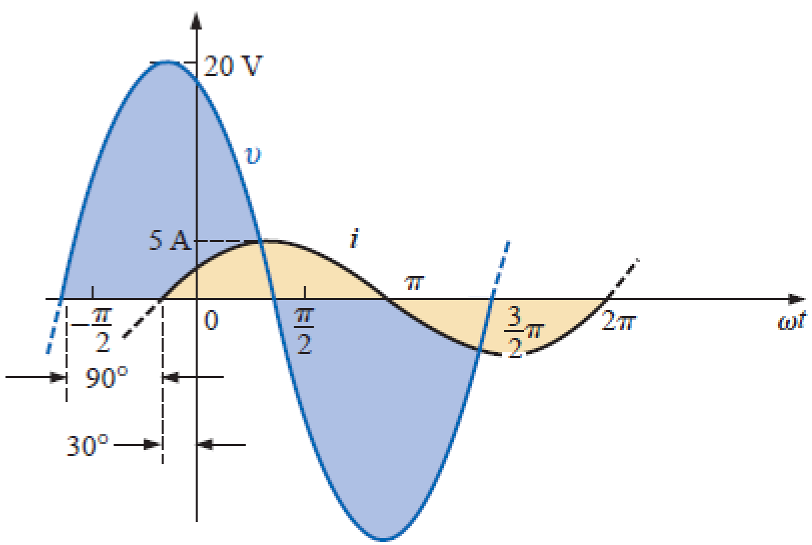


FIG. 15.11
Waveforms for Example 15.4.

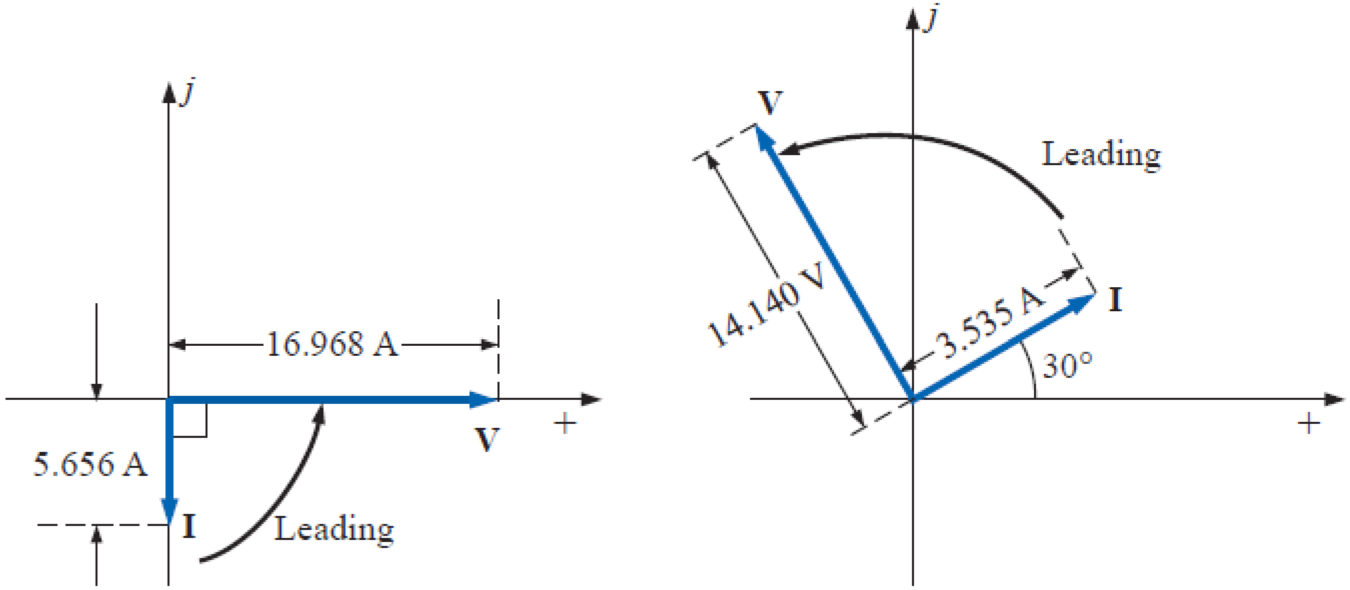


FIG. 15.12
Phasor diagrams for Examples 15.3 and 15.4.

Capacitive Reactance

It was learned in Chapter 13 that for the pure capacitor of Fig. 15.13, the current leads the voltage by 90° and that the reactance of the capacitor X_C is determined by $1/\omega C$.

$$v = V_m \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = V \angle 0°$$

Applying Ohm's law and using phasor algebra, we find

$$\mathbf{I} = \frac{V \angle 0°}{X_C \angle \theta_C} = \frac{V}{X_C} \underline{/0° - \theta_C}$$

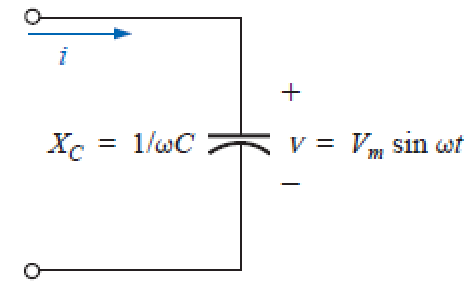


FIG. 15.13
Capacitive ac circuit.

Since i leads v by 90°, i must have an angle of +90° associated with it. To satisfy this condition, θ_C must equal −90°. Substituting $\theta_C = -90°$ yields

$$\mathbf{I} = \frac{V \angle 0°}{X_C \angle -90°} = \frac{V}{X_C} \underline{/0° - (-90°)} = \frac{V}{X_C} \angle 90°$$

so, in the time domain,

$$i = \sqrt{2}\left(\frac{V}{X_C}\right) \sin(\omega t + 90°)$$

The fact that $\theta_C = -90°$ will now be employed in the following polar format for capacitive reactance to ensure the proper phase relationship between the voltage and current of a capacitor.

$$\boxed{\mathbf{Z}_C = X_C \angle -90°} \qquad \textbf{(15.3)}$$

EXAMPLE 15.5 Using complex algebra, find the current i for the circuit of Fig. 15.14. Sketch the v and i curves.

Solution: Note Fig. 15.15:

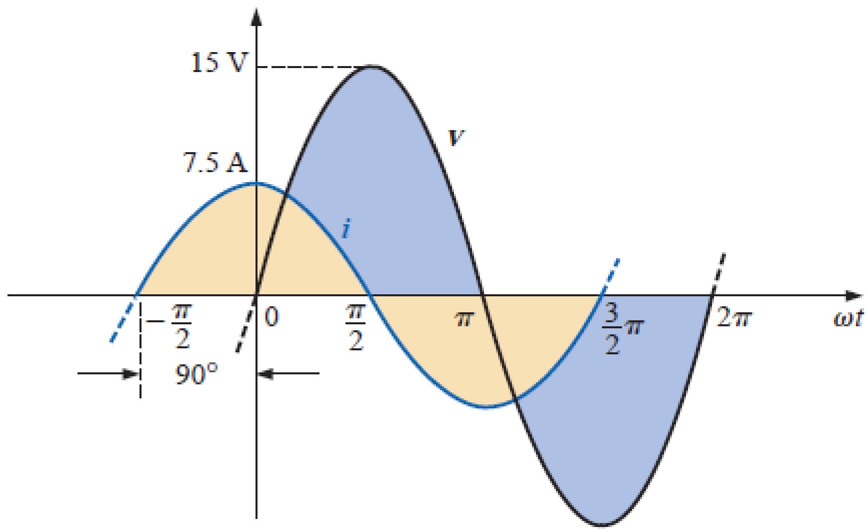


FIG. 15.15
Waveforms for Example 15.5.

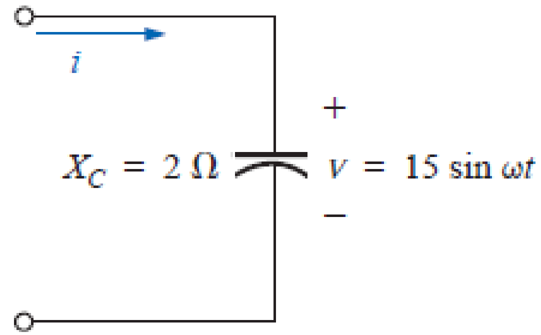


FIG. 15.14
Example 15.5.

$$v = 15 \sin \omega t \Rightarrow \text{phasor notation } \mathbf{V} = 10.605 \text{ V} \angle 0°$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_C} = \frac{V \angle \theta}{X_C \angle -90°} = \frac{10.605 \text{ V} \angle 0°}{2\ \Omega \angle -90°} = 5.303 \text{ A} \angle 90°$$

and $$i = \sqrt{2}(5.303) \sin(\omega t + 90°) = \mathbf{7.5 \sin(\omega t + 90°)}$$

EXAMPLE 15.6 Using complex algebra, find the voltage v for the circuit of Fig. 15.16. Sketch the v and i curves.

Solution: Note Fig. 15.17:

$$i = 6 \sin(\omega t - 60°) \Rightarrow \text{phasor notation } \mathbf{I} = 4.242 \text{ A} \angle -60°$$

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = (I \angle \theta)(X_C \angle -90°) = (4.242 \text{ A} \angle -60°)(0.5\ \Omega \angle -90°)$$
$$= 2.121 \text{ V} \angle -150°$$

and $$v = \sqrt{2}(2.121) \sin(\omega t - 150°) = \mathbf{3.0 \sin(\omega t - 150°)}$$

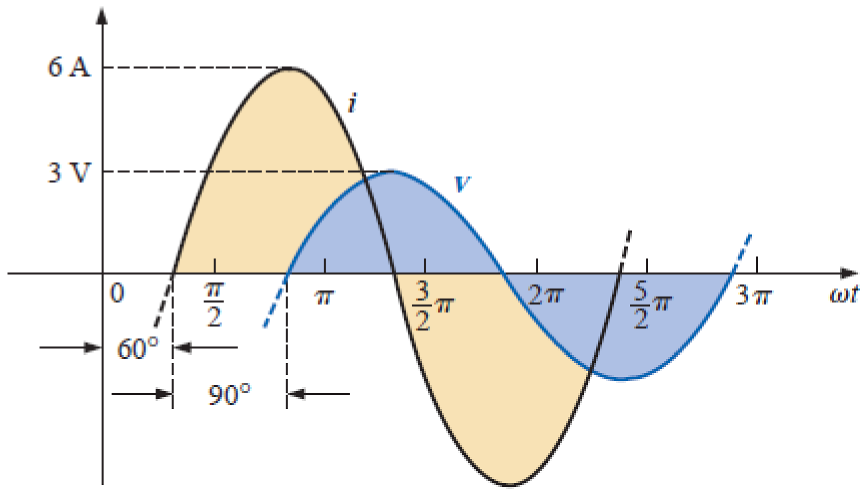


FIG. 15.17
Waveforms for Example 15.6.

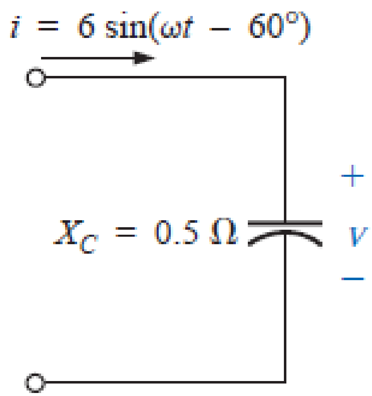


FIG. 15.16
Example 15.6.

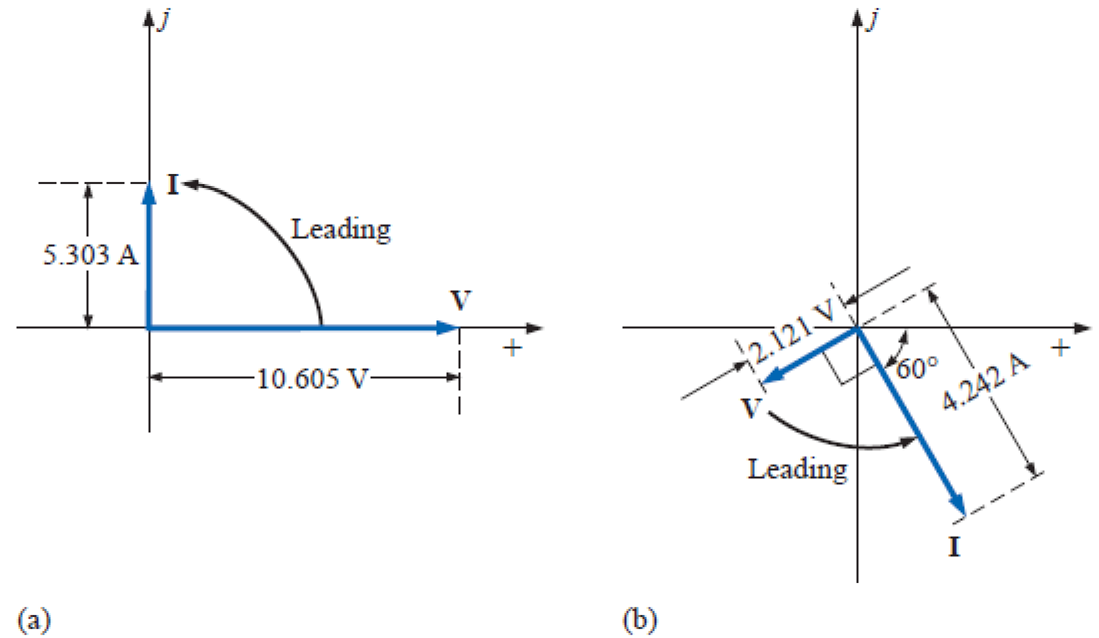


FIG. 15.18

Phasor diagrams for Examples 15.5 and 15.6.

EXAMPLE 15.8 Determine the input impedance to the series network of Fig. 15.23. Draw the impedance diagram.

Solution:

$$\begin{aligned}
\mathbf{Z}_T &= \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 \\
&= R \angle 0^\circ + X_L \angle 90^\circ + X_C \angle -90^\circ \\
&= R + jX_L - jX_C \\
&= R + j(X_L - X_C) = 6\ \Omega + j(10\ \Omega - 12\ \Omega) = 6\ \Omega - j2\ \Omega
\end{aligned}$$

$$\mathbf{Z}_T = \mathbf{6.325\ \Omega \angle -18.43^\circ}$$

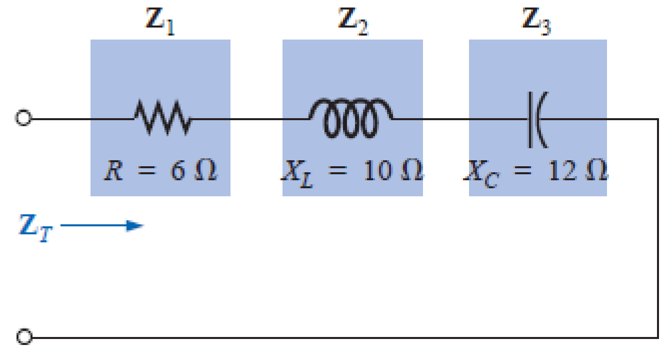


FIG. 15.23
Example 15.8

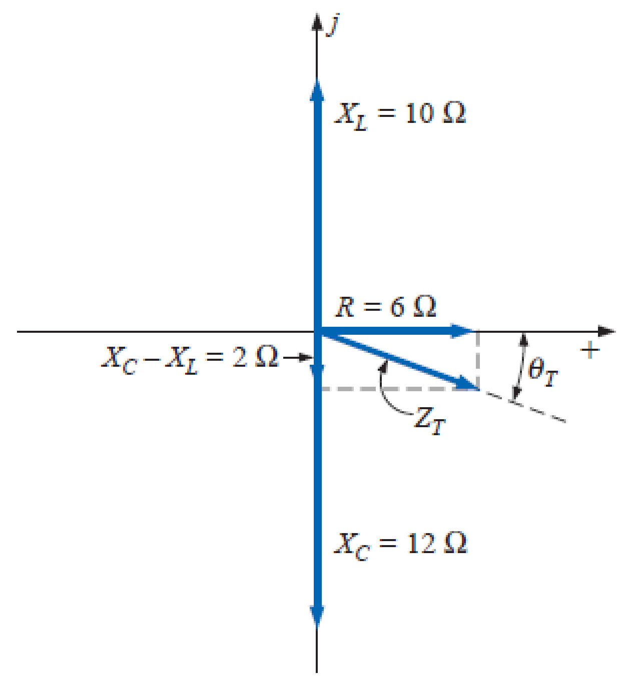


FIG. 15.24
Impedance diagram for Example 15.8.

R-L-C

Refer to Fig. 15.35.

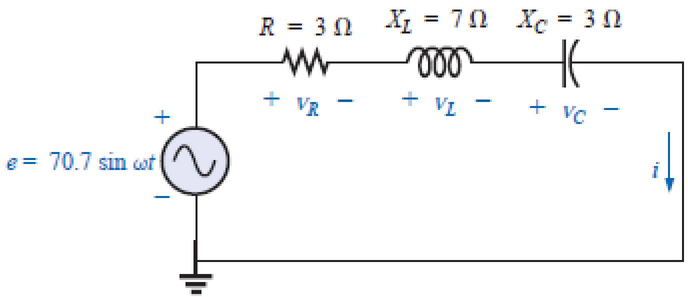


FIG. 15.35
Series R-L-C ac circuit.

Phasor Notation As shown in Fig. 15.36.

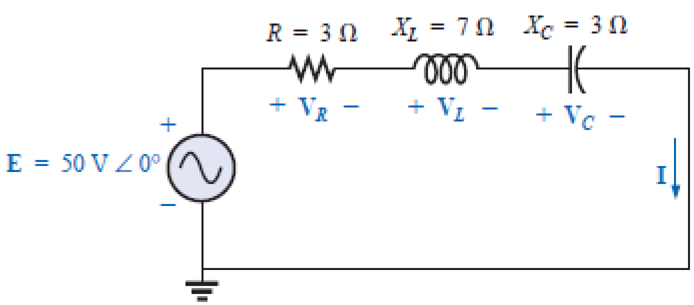


FIG. 15.36
Applying phasor notation to the circuit of Fig. 15.35.

$\mathbf{Z}_T$

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 = R \angle 0^\circ + X_L \angle 90^\circ + X_C \angle -90^\circ$$
$$= 3\ \Omega + j\,7\ \Omega - j\,3\ \Omega = 3\ \Omega + j\,4\ \Omega$$

and
$$\mathbf{Z}_T = \mathbf{5\ \Omega \angle 53.13^\circ}$$

Impedance diagram: As shown in Fig. 15.37.

I

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \frac{50\ \text{V} \angle 0^\circ}{5\ \Omega \angle 53.13^\circ} = \mathbf{10\ A \angle -53.13^\circ}$$

$\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$

$$\mathbf{V}_R = \mathbf{I}\mathbf{Z}_R = (I \angle \theta)(R \angle 0^\circ) = (10\ \text{A} \angle -53.13^\circ)(3\ \Omega \angle 0^\circ)$$
$$= \mathbf{30\ V \angle -53.13^\circ}$$

$$\mathbf{V}_L = \mathbf{I}\mathbf{Z}_L = (I \angle \theta)(X_L \angle 90^\circ) = (10\ \text{A} \angle -53.13^\circ)(7\ \Omega \angle 90^\circ)$$
$$= \mathbf{70\ V \angle 36.87^\circ}$$

$$\mathbf{V}_C = \mathbf{I}\mathbf{Z}_C = (I \angle \theta)(X_C \angle -90^\circ) = (10\ \text{A} \angle -53.13^\circ)(3\ \Omega \angle -90^\circ)$$
$$= \mathbf{30\ V \angle -143.13^\circ}$$

Kirchhoff's voltage law:

$$\Sigma_{\circlearrowright} \mathbf{V} = \mathbf{E} - \mathbf{V}_R - \mathbf{V}_L - \mathbf{V}_C = 0$$

The sinusoidal waveform is the only alternating waveform whose shape is unaffected by the response characteristics of ***R, L, and C*** elements.

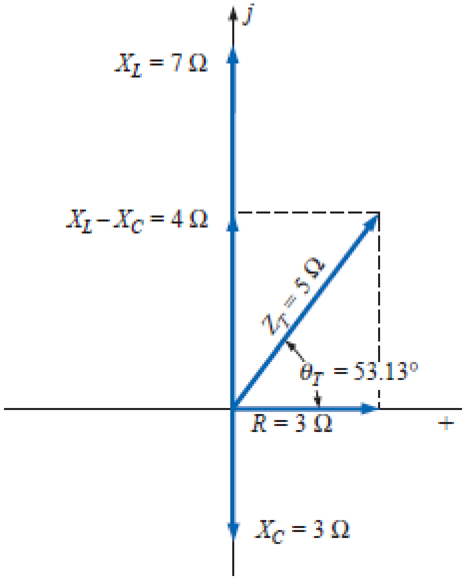


FIG. 15.37
Impedance diagram for the series R-L-C circuit of Fig. 15.35.

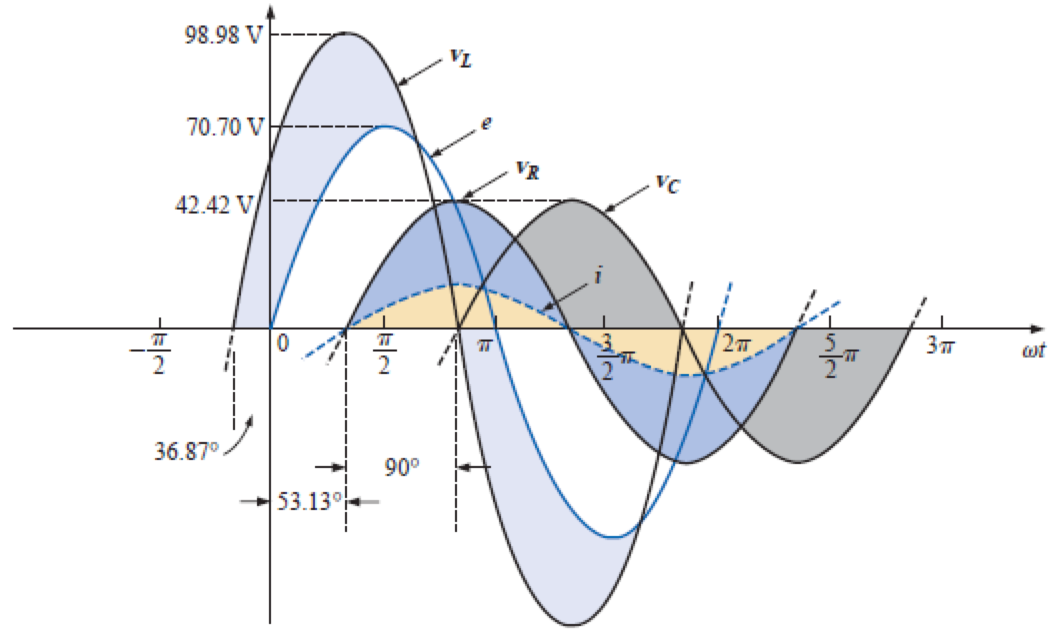


FIG. 15.39
Waveforms for the series R-L circuit of Fig. 15.35.

or
$$\mathbf{E} = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C$$

which can also be verified through vector algebra.

Phasor diagram: The phasor diagram of Fig. 15.38 indicates that the current **I** is in phase with the voltage across the resistor, lags the voltage across the inductor by 90°, and leads the voltage across the capacitor by 90°.

Time domain:

$$i = \sqrt{2}(10)\sin(\omega t - 53.13°) = \mathbf{14.14\ sin(\omega t - 53.13°)}$$
$$v_R = \sqrt{2}(30)\sin(\omega t - 53.13°) = \mathbf{42.42\ sin(\omega t - 53.13°)}$$
$$v_L = \sqrt{2}(70)\sin(\omega t + 36.87°) = \mathbf{98.98\ sin(\omega t + 36.87°)}$$
$$v_C = \sqrt{2}(30)\sin(\omega t - 143.13°) = \mathbf{42.42\ sin(\omega t - 143.13°)}$$

A plot of all the voltages and the current of the circuit appears in Fig. 15.39.

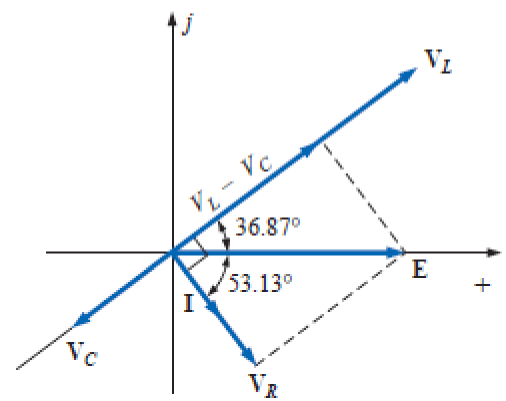


FIG. 15.38
Phasor diagram for the series R-L-C circuit of Fig. 15.35.