LECTURE NO - 17

DERIVATIVES

Formula of first principles/First principles law:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Relevant formulae:

(a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ----------\infty$

(b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - ----------------\infty$

(c) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ----------------\infty$

Example:

1. Differentiate from first principles of $\tan^{-1} x$, $\sec^{-1} x$, $\cot^{-1} x$,

   $x \times lnx, lncosx,\ cos(lnx), \ln(sinx), x\tan^{-1} x,\ x \times sinx, x^3 lnx.$

2. Find from first principle the differential co-efficient of e^{sinx} at the point

   $x = a$.

Sol^n For $\tan^{-1}x$:

Let $f(x) = \tan^{-1}x$.

$\therefore f(x+h) = \tan^{-1}(x+h)$

$$\therefore f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\tan^{-1}(x+h) - \tan^{-1}x}{h}$$

Again let, $y = \tan^{-1}x$ $\quad \therefore x = \tan y$

$\therefore y+k = \tan^{-1}(x+h) \quad \therefore x+h = \tan(y+k)$

$\therefore k = y+k-y = \tan^{-1}(x+h) - \tan^{-1}x$

and $h = x+h-\cancel{x} = \tan(y+k) - \tan y$.

when $h\to 0$ then $k\to 0$.

$$\therefore f'(x) = \lim_{k\to 0} \frac{k}{\tan(y+k) - \tan y}$$

$$= \lim_{k\to 0} \frac{k}{\frac{\sin(y+k)}{\cos(y+k)} - \frac{\sin y}{\cos y}}$$

$$= \lim_{k\to 0} \frac{k}{\frac{\sin(y+k)\cos y - \cos(y+k)\sin y}{\cos(y+k)\cos y}}$$

$$= \lim_{k\to 0} \frac{k \cos(y+k)\cdot\cos y}{\sin(\cancel{y}+k-\cancel{y})}$$

$$= \lim_{k\to 0} \frac{k}{\sin k}\cdot \cos(y+k)\cos y$$

$$= \lim_{k \to 0} \frac{k}{\sin k} \cdot \lim_{k \to 0} \cos(y+k) \cos y.$$

$$= 1 \cdot \cos y \cdot \cos y$$

$$= \cos^2 y$$

$$= \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\therefore f'(x) = \frac{1}{1+x^2} \quad \underline{\underline{\text{Ans.}}}$$

For $\sec^{-1}x$

$f(x) = \sec^{-1}x \qquad \therefore f(x+h) = \sec^{-1}(x+h).$

$$\therefore f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sec^{-1}(x+h) - \sec^{-1}x}{h}$$

Let $y = \sec^{-1}x \qquad \therefore x = \sec y.$

$\therefore y+k = \sec^{-1}(x+h) \qquad \therefore x+h = \sec(y+k)$

$\therefore k = y+k-y = \sec^{-1}(x+h) - \sec^{-1}x$

$h = x+h-x = \sec(y+k) - \sec y.$

when $h \to 0$ then $k \to 0$

$$\therefore f'(x) = \lim_{k \to 0} \frac{k}{\sec(y+k) - \sec y}$$

$$= \lim_{k \to 0} \frac{k}{\frac{1}{\cos(y+k)} - \frac{1}{\cos y}}$$

$$= \lim_{k \to 0} \frac{k}{\frac{\cos y - \cos(y+k)}{\cos(y+k)\cos y}}$$

$$= \lim_{k \to 0} \frac{k \cos(y+k) \cos y}{\cos y - \cos(y+k)}$$

$$= \lim_{k \to 0} \frac{k \cos(y+k) \cos y}{2 \sin \frac{y+y+k}{2} \sin \frac{y+k-y}{2}}$$

$$= \lim_{k\to 0} \frac{k}{2\sin(y+\frac{k}{2})\sin\frac{k}{2}} \cdot \lim_{k\to 0} \cos(y+k)\cos y$$

$$= \lim_{k\to 0} \frac{\frac{k}{2}}{\sin\frac{k}{2}} \cdot \lim_{k\to 0} \frac{1}{\sin(y+k/2)} \cdot \cos^2 y$$

$$= 1 \cdot \frac{\cos^2 y}{\sin y} = \frac{\cos^2 y}{\sin y}$$

$$= \frac{1}{\sec^2 y\sqrt{1-\cos^2 y}}$$

$$= \frac{1}{\sec^2 y\sqrt{1-\frac{1}{\sec^2 y}}}$$

$$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$$

$$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$$

$$= \frac{1}{x\,\frac{\sqrt{x^2-1}}{x}}$$

$$\therefore \quad f'(x) = \frac{1}{x\sqrt{x^2-1}} \quad \text{Ans.}$$

2. $x\ln x$

Let $f(x) = x\ln x$

$\therefore f(x+h) = (x+h)\ln(x+h)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)\ln(x+h) - x\ln x}{h}$$

$$= \lim_{h\to 0} \frac{x\ln(x+h) + h\ln(x+h) - x\ln x}{h}$$

$$= \lim_{h\to 0} \left[\frac{x\{\ln(x+h)-\ln x\}}{h} + \frac{h\ln(x+h)}{h}\right]$$

$$= \lim_{h\to 0} \frac{x\ln\left(\frac{x+h}{x}\right)}{h} + \lim_{h\to 0} \ln(x+h)$$

$$= x \lim_{h\to 0} \frac{1}{h} \ln\left(1+\frac{h}{x}\right) + \ln x$$

$$= x \lim_{h\to 0} \frac{1}{h} \left(\frac{h}{x} - \frac{1}{2}\cdot\frac{h^2}{x^2} + \frac{1}{3}\frac{h^3}{x^3} - \cdots\cdots\right) + \ln x$$

$$= x \lim_{h\to 0} \left(\frac{1}{x} - \frac{1}{2}\frac{h}{x^2} + \frac{1}{3}\frac{h^2}{x^3} - \cdots\right) + \ln x$$

$$= x\cdot\frac{1}{x} + \ln x$$

$$= 1 + \ln x$$

$\therefore f'(x) = 1 + \ln x$ Ans.

For $\ln(\cos x)$

$$f(x) = \ln \cos x$$

$$\therefore f(x+h) = \ln \cos(x+h).$$

$$\therefore f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\ln\cos(x+h) - \ln\cos x}{h}$$

let $\cos x = y$ $\therefore \cos(x+h) = y + K.$

$\therefore K = y + K - y = \cos(x+h) - \cos x.$

when $h \to 0$ then $K \to 0.$

$$\therefore f'(x) = \lim_{K\to 0} \frac{\ln(y+K) - \ln y}{K} \cdot \frac{K}{h}$$

$$= \lim_{K\to 0} \frac{\ln\left(\frac{y+K}{y}\right)}{K} \cdot \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{K\to 0} \frac{1}{K}\ln\left(1+\frac{K}{y}\right) \cdot \lim_{h\to 0} \frac{2\sin\frac{x+h+x}{2}\sin\frac{x-x-h}{2}}{h}$$

$$= \lim_{K\to 0} \frac{1}{K}\left(\frac{K}{y} - \frac{1}{2}\frac{K^2}{y^2} + \frac{1}{3}\frac{K^3}{y^3} - \cdots\right) \cdot \lim_{h\to 0} \frac{2\sin(x+\frac{h}{2})\sin(-\frac{h}{2})}{h}$$

$$= \lim_{K\to 0}\left(\frac{1}{y} - \frac{1}{2}\cdot\frac{K}{y^2} + \frac{1}{3}\frac{K^2}{y^3} - \cdots\right) \cdot \lim_{h\to 0} -\frac{\sin h/2}{h/2} \cdot \lim_{h\to 0} \sin\left(x+\frac{h}{2}\right)$$

$$= \frac{1}{y} \cdot \left\{ - \lim_{h\to 0} \frac{\sin h/2}{h/2} \right\} \cdot \sin x$$

$$= \frac{1}{y} \cdot \cdot -1 \cdot \sin x$$

$$= - \frac{\sin x}{y}$$

$$= - \frac{\sin x}{\cos x}$$

$$= -\tan x$$

$$\therefore \frac{d}{dx}(\ln \cos x) = -\tan x.$$

For $\cos(\ln x)$

$$f(x) = \cos(\ln x)$$

$$\therefore f(x+h) = \cos\{\ln(x+h)\}$$

$$\therefore f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos\{\ln(x+h)\} - \cos(\ln x)}{h}$$

let $y = \ln x \quad \therefore y+k = \ln(x+h).$

$\therefore k = y+k-y = \ln(x+h) - \ln x.$

when $h \to 0$ then $k \to 0$

$$\therefore f'(x) = \lim_{k\to 0} \frac{\cos(y+k) - \cos y}{k} \cdot \frac{k}{h}$$

$$= \lim_{k\to 0} \frac{2\sin\frac{y+k+y}{2} \sin\frac{y-y-k}{2}}{k} \cdot \lim_{k\to 0} \frac{\ln(x+h) - \ln x}{h}$$

$$= \lim_{k\to 0} \frac{2\sin(y+\frac{k}{2}) \sin(-\frac{k}{2})}{k} \cdot \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h}$$

$$= 2 \lim_{k\to 0} \frac{-\sin(k/2)}{\frac{k}{2}} \cdot \frac{1}{2} \cdot \lim_{k\to 0} \sin(y+\frac{k}{2}) \cdot \lim_{h\to 0} \frac{\ln(1+\frac{h}{x})}{h}$$

$$= -1 \cdot \lim_{k\to 0} \sin y \cdot \lim_{h\to 0} \frac{\frac{h}{x} - \frac{1}{2}\cdot\frac{h^2}{x^2} + \frac{1}{3}\cdot\frac{h^3}{h^3} - \cdots}{h}$$

$$= -\sin y \cdot \lim_{h\to 0} \frac{\cancel{h}\left(\frac{1}{x} - \frac{1}{2}\cdot\frac{h}{x^2} + \frac{1}{3}\cdot\frac{h^2}{x^3} - \cdots\right)}{\cancel{h}}$$

$$= -\sin y \cdot \frac{1}{x} = \frac{-\sin(\ln x)}{x}$$

$$\therefore \frac{d}{dx}\left(\cos(\ln x)\right) = -\frac{\sin(\ln x)}{x} \quad \underline{\underline{Ans.}}$$

For $\ln(\sin x)$

$$f(x) = \ln(\sin x)$$

$$\therefore f(x+h) = \ln \sin(x+h)$$

$$\therefore f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\ln \sin(x+h) - \ln \sin x}{h}$$

let $y = \sin x \quad \therefore y+k = \sin(x+h)$

$$\therefore k = y+k-y = \sin(x+h) - \sin x.$$

when $h \to 0$ then $k \to 0$

$$\therefore f'(x) = \lim_{k\to 0} \frac{\ln(y+k) - \ln y}{k} \cdot \frac{k}{h}$$

$$= \lim_{k\to 0} \frac{\ln\left(\frac{y+k}{y}\right)}{k} \cdot \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{k\to 0} \frac{\ln\left(1+\frac{k}{y}\right)}{k} \cdot \lim_{h\to 0} \frac{2\cos\frac{x+h+x}{2}\sin\frac{x+h-x}{2}}{h}$$

$$= \lim_{k\to 0} \frac{\frac{k}{y} - \frac{1}{2}\cdot\frac{k^2}{y^2} + \frac{1}{3}\cdot\frac{k^3}{y^3} - \cdots}{k} \quad \lim_{h\to 0} \frac{\cos\left(x+\frac{h}{2}\right)\sin h/2}{h/2}$$

$$= \lim_{k\to 0}\left(\frac{1}{y} - \frac{1}{2}\cdot\frac{k}{y^2} + \frac{1}{3}\frac{k^2}{y^3} - \cdots\right) \cdot \lim_{h\to 0}\frac{\sin h/2}{h/2} \cdot \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)$$

$$= \frac{1}{y} \cdot 1 \cdot \cos x = \frac{\cos x}{\sin x} = \cot x$$

$$\therefore \frac{d}{dx}\left(\ln \sin x\right) = \cot x \quad \underline{\underline{Ans.}}$$

For $x\tan^{-1}x$

$$f(x) = x\tan^{-1}x$$

$$\therefore f(x+h) = (x+h)\tan^{-1}(x+h)$$

$$\therefore f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)\tan^{-1}(x+h) - x\tan^{-1}x}{h}$$

let $y = \tan^{-1}x \quad \therefore y+k = \tan^{-1}(x+h)$

$\therefore \tan y = x \quad \Rightarrow \tan(y+k) = x+h.$

$\therefore k = y+k-y = \tan^{-1}(x+h) - \tan^{-1}x.$

$h = x+h-x = \tan(y+k) - \tan y.$

when $h\to 0$ then $k\to 0.$

$$\therefore f'(x) = \lim_{k\to 0}\frac{\tan(y+k)\,(y+k) - y\tan y}{k}\cdot\frac{k}{h}$$

$$= \lim_{k\to 0}\frac{y\tan(y+k) + k\tan(y+k) - y\tan y}{k}\cdot\lim_{k\to 0}\frac{k}{\tan(y+k)-\tan y}$$

$$= \lim_{k\to 0}\left\{\frac{y\tan(y+k) - y\tan y}{k} + \frac{k\tan(y+k)}{k}\right\}\cdot\lim_{k\to 0}\frac{k}{\frac{\sin(y+k)}{\cos(y+k)} - \frac{\sin y}{\cos y}}$$

$$= \left\{y\lim_{k\to 0}\frac{\tan(y+k)-\tan y}{k} + \tan y\right\}\cdot\lim_{k\to 0}\frac{k}{\frac{\sin(y+k)\cos y - \cos(y+k)\sin y}{\cos(y+k)\cos y}}$$

$$= \left\{y\lim_{k\to 0}\frac{\frac{\sin(y+k)}{\cos(y+k)} - \frac{\sin y}{\cos y}}{k} + \tan y\right\}\cdot\lim_{k\to 0}\frac{k\cos(y+k)\cos y}{\sin(y+k-y)}$$

$$= \left\{y\lim_{k\to 0}\frac{\sin(y+k)\cos y - \cos(y+k)\sin y}{k\cos(y+k)\cos y} + \tan y\right\}\cdot\lim_{k\to 0}\frac{k}{\sin k}\cdot\lim_{k\to 0}\cos(y+k)\cos y$$

$$= \left\{y\lim_{k\to 0}\frac{\sin(\cancel{y}+k-\cancel{y})}{k\cos(y+k)\cos y} + \tan y\right\}\cdot 1\cdot \cos y\cdot\cos y$$

$$= \left\{y\cdot\lim_{k\to 0}\frac{\sin k}{k}\cdot\lim_{k\to 0}\frac{1}{\cos(y+k)\cos y} + \tan y\right\}\cos^2 y$$

$$= \left(y \cdot 1 \cdot \frac{1}{\cos^2 y} + \tan y\right) \cos^2 y$$

$$= \left(\frac{y}{\cos^2 y} + \tan y\right) \cos^2 y$$

$$= y + \tan y \cos^2 y$$

$$= \tan^{-1} x + x \cdot \frac{1}{\sec^2 y}$$

$$= \frac{\tan^{-1} x}{} + \frac{x}{1+\tan^2 y} = \tan^{-1} x + \frac{x}{1+x^2}$$

$$\therefore \frac{d}{dx}(x \tan^{-1} x) = \tan^{-1} x + \frac{x}{1+x^2}$$

Ans

$$= \frac{\tan^{-1} x + x}{1 + \tan^2 y}$$

$$= \frac{x + \tan^{-1} x}{1 + x^2}$$

$$\therefore \frac{d}{dx}(x \tan^{-1} x) = \frac{x + \tan^{-1} x}{1+x^2}$$

For $x \sin x$

$$f(x) = x \sin x$$

$$\therefore f(x+h) = (x+h) \sin(x+h).$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)\sin(x+h) - x \sin x}{h}$$

$$= \lim_{h \to 0} \frac{x \sin(x+h) - x \sin x + h \sin(x+h)}{h}$$

$$= \lim_{h \to 0} \frac{x \sin(x+h) - x \sin x}{h} + \lim_{h \to 0} \frac{h \sin(x+h)}{h}$$

$$= x \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} + \sin x$$

$$= x \lim_{h \to 0} \frac{2 \cos \frac{x+h+x}{2} \sin \frac{x+h-x}{2}}{h} + \sin x$$

$$= x \lim_{h \to 0} \frac{\cos(x + h/2) \sin h/2}{h/2} + \sin x$$

$$= x \lim_{h\to 0} \frac{\sin h/2}{h/2} \cdot \lim_{h\to 0} \cos\left(x+\frac{h}{2}\right) + \sin x$$

$$= x \cdot 1 \cdot \cos x + \sin x$$

$$= x\cos x + \sin x$$

$$\therefore \frac{d}{dx}(x\sin x) = x\cos x + \sin x.$$

For $x^3 \ln x$

$$f(x) = x^3 \ln x$$

$$\therefore f(x+h) = (x+h)^3 \ln(x+h)$$

$$\therefore f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 \ln(x+h) - x^3 \ln x}{h}$$

$$= \lim_{h\to 0} \frac{(x^3+3x^2h+3xh^2+h^3)\ln(x+h) - x^3\ln x}{h}$$

$$= \lim_{h\to 0} \frac{x^3\ln(x+h) + h(3x^2+3xh+h^2)\ln(x+h) - x^3\ln x}{h}$$

$$= \lim_{h\to 0}\left[\frac{x^3\ln(x+h) - x^3\ln x}{h} + \frac{\cancel{h}(3x^2+3hx+h^2)\ln(x+h)}{\cancel{h}}\right]$$

$$= \lim_{h\to 0} \frac{x^3\{\ln(x+h) - \ln x\}}{h} + \lim_{h\to 0}(3x^2+3hx+h^2)\ln(x+h)$$

$$= x^3 \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} + 3x^2 \cdot \ln x$$

$$= x^3 \lim_{h\to 0} \frac{1}{h}\left(\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \cdots\right) + 3x^2\ln x$$

$$= x^3 \lim_{h\to 0}\left(\frac{1}{x} - \frac{h}{2x^2} + \frac{h^2}{3x^3} - \cdots\right) + 3x^2\ln x$$

$$= x^3 \cdot \frac{1}{x} + 3x^2\ln x$$

$$= x^2 + 3x^2\ln x = x^2(1+3\ln x) \quad \underline{\text{Ans.}}$$

Sol^n

Here $f(x) = e^{\sin x}$ $\therefore f(a) = e^{\sin a}$

$\therefore f(a+h) = e^{\sin(a+h)}$

$$\therefore f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{e^{\sin(a+h)} - e^{\sin a}}{h}$$

Let $\sin a = l$ and $\sin(a+h) = l + K$.

$\therefore K = \cancel{l} + K - \cancel{l} = \sin(a+h) - \sin a$

when $h \to 0$ then $K \to 0$.

$$\therefore f'(a) = \lim_{K\to 0} \frac{e^{l+K} - e^{l}}{K} \cdot \frac{K}{h}$$

$$= \lim_{K\to 0} \frac{e^{l} \cdot e^{K} - e^{l}}{K} \cdot \lim_{K\to 0} \frac{K}{h}$$

$$= e^{l} \lim_{K\to 0} \frac{e^{K} - 1}{K} \cdot \lim_{h\to 0} \frac{\sin(a+h) - \sin a}{h}$$

$$= e^{l} \cdot \lim_{K\to 0} \frac{\cancel{1} + \frac{K}{1!} + \frac{K^2}{2!} + \cdots - \cancel{1}}{K} \cdot \lim_{h\to 0} \frac{2\cos\frac{a+h+a}{2} \sin\frac{a+h-a}{2}}{h}$$

$$= e^{l} \cdot \lim_{K\to 0} \left(1 + \frac{K}{2!} + \cdots\right) \cdot \lim_{h\to 0} \frac{\sin h/2}{h/2} \cdot \lim_{h\to 0} \cos\left(a + \frac{h}{2}\right)$$

$= e^{l} \cdot 1 \cdot 1 \cdot$ ~~sin a~~ $\cos a$

$= e^{\sin a} \cos a$.

$\therefore f'(a) = e^{\sin a} \cos a$. Ans.