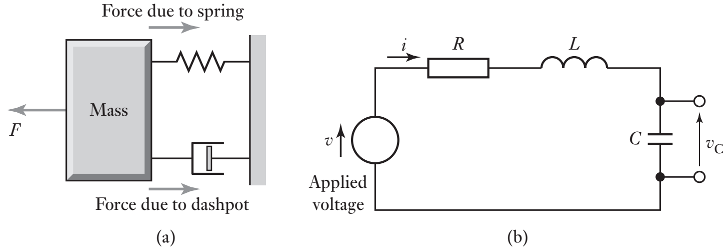

Figure 17.13 Analogous systems.

current as being analogous to the force, then the potential difference is analogous to the velocity and the dashpot constant c to the reciprocal of the resistance, i.e. $(1/R)$. These analogies between current and force, potential difference and velocity, hold for the other building blocks with the spring being analogous to inductance and mass to capacitance.

The mechanical system in Figure 17.1(a) and the electrical system in Figure 17.1(b) have input/output relationships described by similar differential equations:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F \quad \text{and} \quad RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2} + v_C = v$$

The analogy between current and force is the one most often used. However, another set of analogies can be drawn between potential difference and force.

17.4 Fluid system building blocks

In fluid flow systems there are three basic building blocks which can be considered to be the equivalent of electrical resistance, capacitance and inductance. Fluid systems can be considered to fall into two categories: hydraulic, where the fluid is a liquid and is deemed to be incompressible; and pneumatic, where it is a gas which can be compressed and consequently shows a density change.

Hydraulic resistance is the resistance to flow which occurs as a result of a liquid flowing through valves or changes in a pipe diameter (Figure 17.14(a)). The relationship between the volume rate of flow of liquid q through the resistance element and the resulting pressure difference $(p_1 - p_2)$ is

$$p_1 - p_2 = Rq$$

where R is a constant called the hydraulic resistance. The bigger the resistance, the bigger the pressure difference for a given rate of flow. This equation, like that for the electrical resistance and Ohm's law, assumes a linear relationship. Such hydraulic linear resistances occur with orderly flow through capillary tubes and porous plugs, but non-linear resistances occur with flow through sharp-edged orifices or if flow is turbulent.

Hydraulic capacitance is the term used to describe energy storage with a liquid where it is stored in the form of potential energy. A height of liquid in a container (Figure 17.14(b)), i.e. a so-called pressure head, is one form of such a storage. For such a capacitance, the rate of change of volume V in the

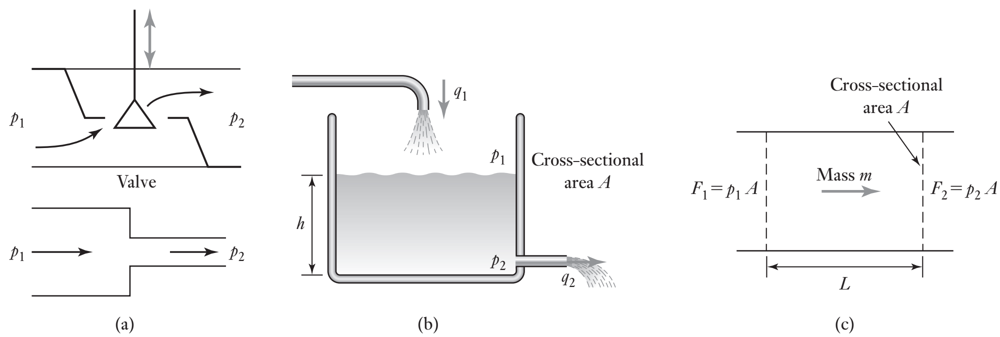

Figure 17.14 Hydraulic examples: (a) resistance, (b) capacitance, (c) inertance.

container, i.e. $\mathrm{d}V/\mathrm{d}t$, is equal to the difference between the volumetric rate at which liquid enters the container q_1 and the rate at which it leaves, q_2,

$$q_1 - q_2 = \frac{\mathrm{d}V}{\mathrm{d}t}$$

But $V = Ah$, where A is the cross-sectional area of the container and h the height of liquid in it. Hence

$$q_1 - q_2 = \frac{\mathrm{d}(Ah)}{\mathrm{d}t} = A\frac{\mathrm{d}h}{\mathrm{d}t}$$

But the pressure difference between the input and output is p, where $p = h\rho g$ with ρ being the liquid density and g the acceleration due to gravity. Thus, if the liquid is assumed to be incompressible, i.e. its density does not change with pressure,

$$q_1 - q_2 = A\frac{\mathrm{d}(p/\rho g)}{\mathrm{d}t} = \frac{A}{\rho g}\frac{\mathrm{d}p}{\mathrm{d}t}$$

The hydraulic capacitance C is defined as being

$$C = \frac{A}{\rho g}$$

Thus

$$q_1 - q_2 = C\frac{\mathrm{d}p}{\mathrm{d}t}$$

Integration of this equation gives

$$p = \frac{1}{C}\int (q_1 - q_2)\,\mathrm{d}t$$

Hydraulic inertance is the equivalent of inductance in electrical systems or a spring in mechanical systems. To accelerate a fluid and so increase its velocity, a force is required. Consider a block of liquid of mass m (Figure 17.14(c)). The net force acting on the liquid is

$$F_1 - F_2 = p_1 A - p_2 A = (p_1 - p_2)A$$

where $(p_1 - p_2)$ is the pressure difference and A the cross-sectional area. This net force causes the mass to accelerate with an acceleration a, and so

$$(p_1 - p_2)A = ma$$

But a is the rate of change of velocity $\mathrm{d}v/\mathrm{d}t$, hence

$$(p_1 - p_2)A = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

But the mass of liquid concerned has a volume of AL, where L is the length of the block of liquid or the distance between the points in the liquid where the pressures p_1 and p_2 are measured. If the liquid has a density ρ then $m = AL\rho$ and so

$$(p_1 - p_2)A = AL\rho\frac{\mathrm{d}v}{\mathrm{d}t}$$

But the volume rate of flow $q = Av$, hence

$$(p_1 - p_2)A = L\rho\frac{\mathrm{d}q}{\mathrm{d}t}$$

$$p_1 - p_2 = I\frac{\mathrm{d}q}{\mathrm{d}t}$$

where the hydraulic inertance I is defined as

$$I = \frac{L\rho}{A}$$

With pneumatic systems the three basic building blocks are, as with hydraulic systems, resistance, capacitance and inertance. However, gases differ from liquids in being compressible, i.e. a change in pressure causes a change in volume and hence density. **Pneumatic resistance** R is defined in terms of the mass rate of flow $\mathrm{d}m/\mathrm{d}t$ (note that this is often written as an m with a dot above it to indicate that the symbol refers to the mass rate of flow and not just the mass) and the pressure difference $(p_1 - p_2)$ as

$$p_1 - p_2 = R\frac{\mathrm{d}m}{\mathrm{d}t} = R\dot{m}$$

Pneumatic capacitance C is due to the compressibility of the gas, and is comparable with the way in which the compression of a spring stores energy. If there is a mass rate of flow $\mathrm{d}m_1/\mathrm{d}t$ entering a container of volume V and a mass rate of flow of $\mathrm{d}m_2/\mathrm{d}t$ leaving it, then the rate at which the mass in the container is changing is $\mathrm{d}m_1/\mathrm{d}t - \mathrm{d}m_2/\mathrm{d}t$. If the gas in the container has a density ρ then the rate of change of mass in the container is

$$\text{rate of change of mass in container} = \frac{\mathrm{d}(\rho V)}{\mathrm{d}t}$$

But, because a gas can be compressed, both ρ and V can vary with time. Hence

$$\text{rate of change of mass in container} = \rho\frac{\mathrm{d}V}{\mathrm{d}t} + V\frac{\mathrm{d}\rho}{\mathrm{d}t}$$

Since $(\mathrm{d}V/\mathrm{d}t) = (\mathrm{d}v/\mathrm{d}p)(\mathrm{d}p/\mathrm{d}t)$ and, for an ideal gas, $pV = mRT$, with consequently $p = (m/V)RT = \rho RT$ and $\mathrm{d}\rho/\mathrm{d}t = (1/RT)(\mathrm{d}p/\mathrm{d}t)$, then

$$\text{rate of change of mass in container} = \rho\frac{\mathrm{d}V}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{V}{RT}\frac{\mathrm{d}p}{\mathrm{d}t}$$

where R is the gas constant and T the temperature, assumed to be constant, on the Kelvin scale. Thus

$$\frac{\mathrm{d}m_1}{\mathrm{d}t} - \frac{\mathrm{d}m_2}{\mathrm{d}t} = \left(\rho\frac{\mathrm{d}V}{\mathrm{d}p} + \frac{V}{RT}\right)\frac{\mathrm{d}p}{\mathrm{d}t}$$

The pneumatic capacitance due to the change in volume of the container C_1 is defined as

$$C_1 = \rho\frac{\mathrm{d}V}{\mathrm{d}p}$$

and the pneumatic capacitance due to the compressibility of the gas C_2 as

$$C_2 = \frac{V}{RT}$$

Hence

$$\frac{\mathrm{d}m_1}{\mathrm{d}t} - \frac{\mathrm{d}m_2}{\mathrm{d}t} = (C_1 + C_2)\frac{\mathrm{d}p}{\mathrm{d}t}$$

or

$$p_1 - p_2 = \frac{1}{C_1 + C_2}\int(\dot{m}_1 - \dot{m}_2)\mathrm{d}t$$

Pneumatic inertance is due to the pressure drop necessary to accelerate a block of gas. According to Newton's second law, the net force is $ma = \mathrm{d}(mv)/\mathrm{d}t$. Since the force is provided by the pressure difference $(p_1 - p_2)$, then if A is the cross-sectional area of the block of gas being accelerated

$$(p_1 - p_2)A = \frac{\mathrm{d}(mv)}{\mathrm{d}t}$$

But m, the mass of the gas being accelerated, equals ρLA with ρ the gas density and L the length of the block of gas being accelerated, and the volume rate of flow, $q = Av$ where v is the velocity. Thus

$$mv = \rho LA\frac{q}{A} = \rho Lq$$

and so

$$(p_1 - p_2)A = L\frac{\mathrm{d}(\rho q)}{\mathrm{d}t}$$

But $\dot{m} = \rho q$ and so

$$p_1 - p_2 = \frac{L}{A}\frac{\mathrm{d}\dot{m}}{\mathrm{d}t}$$

$$p_1 - p_2 = I\frac{\mathrm{d}\dot{m}}{\mathrm{d}t}$$

with the pneumatic inertance I being $I = L/A$.

Table 17.3 shows the basic characteristics of the fluid building blocks, both hydraulic and pneumatic.

For hydraulics the volumetric rate of flow and for pneumatics the mass rate of flow are analogous to the electric current in an electrical system. For both hydraulics and pneumatics the pressure difference is analogous to the potential difference in electrical systems. Compare Table 17.3 with Table 17.2. Hydraulic and pneumatic inertance and capacitance are both energy storage elements; hydraulic and pneumatic resistance are both energy dissipaters.

17.4.1 Building up a model for a fluid system

Figure 17.15 shows a simple hydraulic system, a liquid entering and leaving a container. Such a system can be considered to consist of a capacitor, the liquid in the container, with a resistor, the valve.

Table 17.3 Hydraulic and pneumatic building blocks.

Building block	Describing equation	Energy stored or power dissipated
Hydraulic		
Inertance	$q = \dfrac{1}{L}\int (p_1 - p_2)\,\mathrm{d}t$ $p = L\dfrac{\mathrm{d}q}{\mathrm{d}t}$	$E = \dfrac{1}{2}Iq^2$
Capacitance	$q = C\dfrac{\mathrm{d}(p_1 - p_2)}{\mathrm{d}t}$	$E = \dfrac{1}{2}C(p_1 - p_2)^2$
Resistance	$q = \dfrac{p_1 - p_2}{R}$	$P = \dfrac{1}{R}(p_1 - p_2)^2$
Pneumatic		
Inertance	$\dot{m} = \dfrac{1}{L}\int (p_1 - p_2)\,\mathrm{d}t$	$E = \dfrac{1}{2}I\dot{m}^2$
Capacitance	$\dot{m} = C\dfrac{\mathrm{d}(p_1 - p_2)}{\mathrm{d}t}$	$E = \dfrac{1}{2}C(p_1 - p_2)^2$
Resistance	$\dot{m} = \dfrac{p_1 - p_2}{R}$	$P = \dfrac{1}{R}(p_1 - p_2)^2$

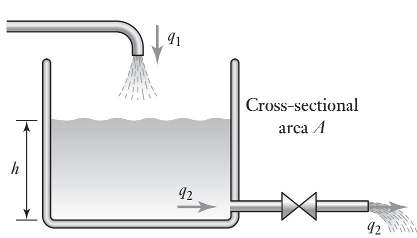

Figure 17.15 A fluid system.

Inertance can be neglected since flow rates change only very slowly. For the capacitor we can write

$$q_1 - q_2 = C\frac{\mathrm{d}p}{\mathrm{d}t}$$

The rate at which liquid leaves the container q_2 equals the rate at which it leaves the valve. Thus for the resistor

$$p_1 - p_2 = Rq_2$$

The pressure difference $(p_1 - p_2)$ is the pressure due to the height of liquid in the container and is thus $h\rho g$. Thus $q_2 = h\rho g/R$ and so substituting for q_2 in the first equation gives

$$q_1 - \frac{h\rho g}{R} = C\frac{\mathrm{d}(h\rho g)}{\mathrm{d}t}$$

and, since $C = A/\rho g$,

$$q_1 = A\frac{\mathrm{d}h}{\mathrm{d}t} + \frac{\rho g h}{R}$$

This equation describes how the height of liquid in the container depends on the rate of input of liquid into the container.

A bellows is an example of a simple pneumatic system (Figure 17.16). Resistance is provided by a constriction which restricts the rate of flow of gas into the bellows and capacitance is provided by the bellows itself. Inertance can be neglected since the flow rate changes only slowly.

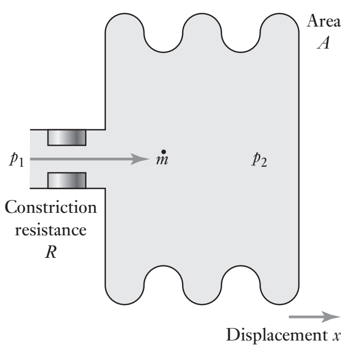

Figure 17.16 A pneumatic system.

The mass flow rate into the bellows is given by

$$p_1 - p_2 = R\dot{m}$$

where p_1 is the pressure prior to the constriction and p_2 the pressure after the constriction, i.e. the pressure in the bellows. All the gas that flows into the bellows remains in the bellows, there being no exit from the bellows. The capacitance of the bellows is given by

$$\dot{m}_1 - \dot{m}_2 = (C_1 + C_2)\frac{\mathrm{d}p_2}{\mathrm{d}t}$$

The mass flow rate entering the bellows is given by the equation for the resistance and the mass leaving the bellows is zero. Thus

$$\frac{p_1 - p_2}{R} = (C_1 + C_2)\frac{\mathrm{d}p_2}{\mathrm{d}t}$$

Hence

$$p_1 = R(C_1 + C_2)\frac{\mathrm{d}p_2}{\mathrm{d}t} + p_2$$

This equation describes how the pressure in the bellows p_2 varies with time when there is an input of a pressure p_1.

The bellows expands or contracts as a result of pressure changes inside it. Bellows are just a form of spring and so we can write $F = kx$ for the relationship between the force F causing an expansion or contraction and the resulting displacement x, where k is the spring constant for the bellows. But