Area between $y = f(x)$ and $y = g(x)$

AREA FORMULA:

The area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

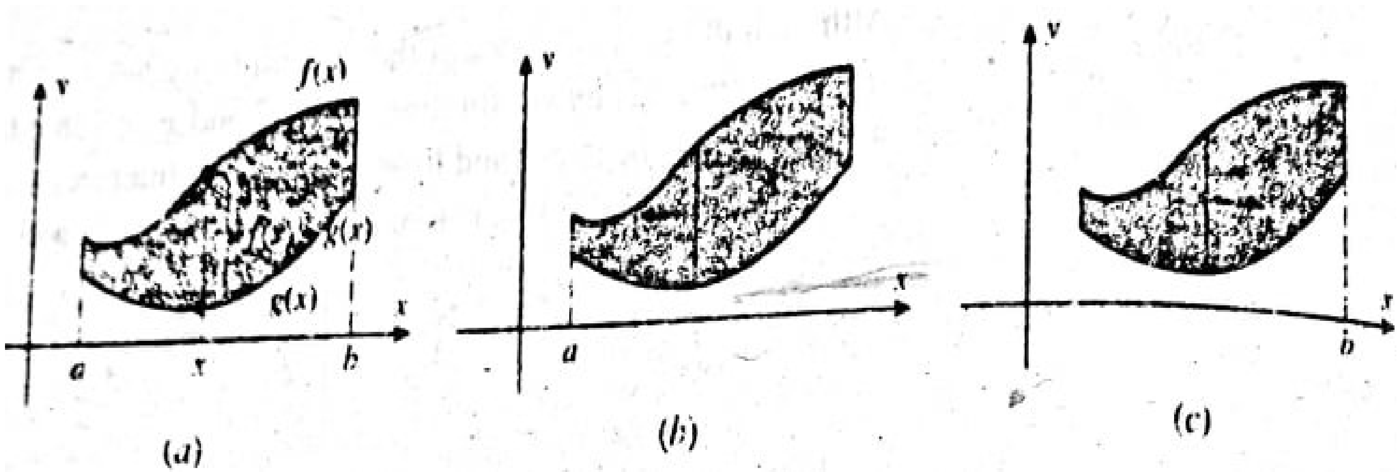
$$A = \int_a^b [f(x) - g(x)]dx$$

Findings the Limits of Integrations for the area between two curves:

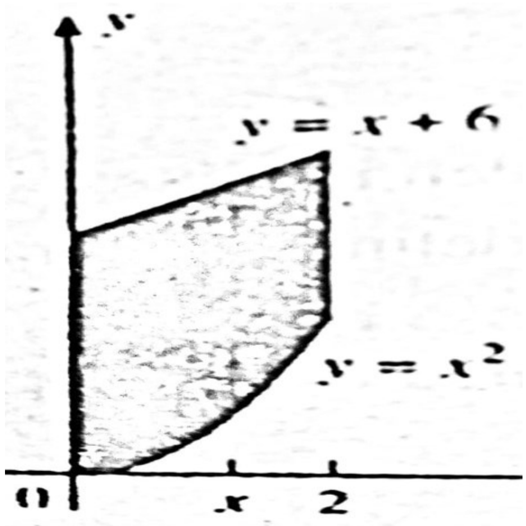
Step1: Sketch the region and then draw a vertical line segment through the region at an arbitrary point x on the x-axis, corresponding the top and bottom boundaries (Fig. a)

Step 2: The y-coordinate of the top endpoint of the line segment sketched in step1 will be *f(x),* the bottom one *g(x)*, and the length of the line segment will be *f(x)-g(x)*. This is the integrand in (1).

Step 3: To determine the limits of integration, imagine moving the line segment left and then right. The left most position at which the line segment intersects the region is $x = a$ and the right most is $x = b$ (Fig. b, c)



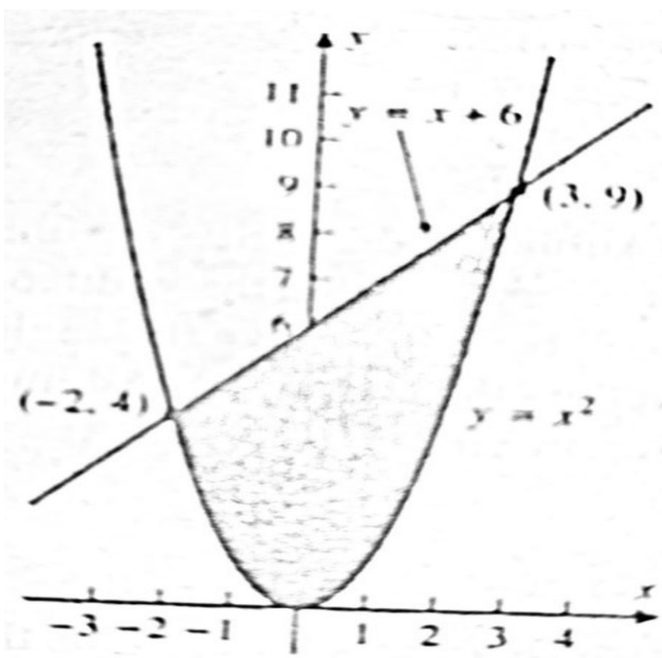
Example: Find the area of the region that is enclosed by the curves $y = x + 6, y = x^2, x = 0, x = 2$



$$A = \int_0^2 [(x + 6) - x^2]dx = 34/3$$

Example: Find the area of the region that is enclosed by the curves $y = x + 6, y = x^2$

Solution:



Solving $y = x+6$ and $y=x^2$, We have $x = -2$ and $x = 3$

$$y = 4 \text{ and } y = 9$$

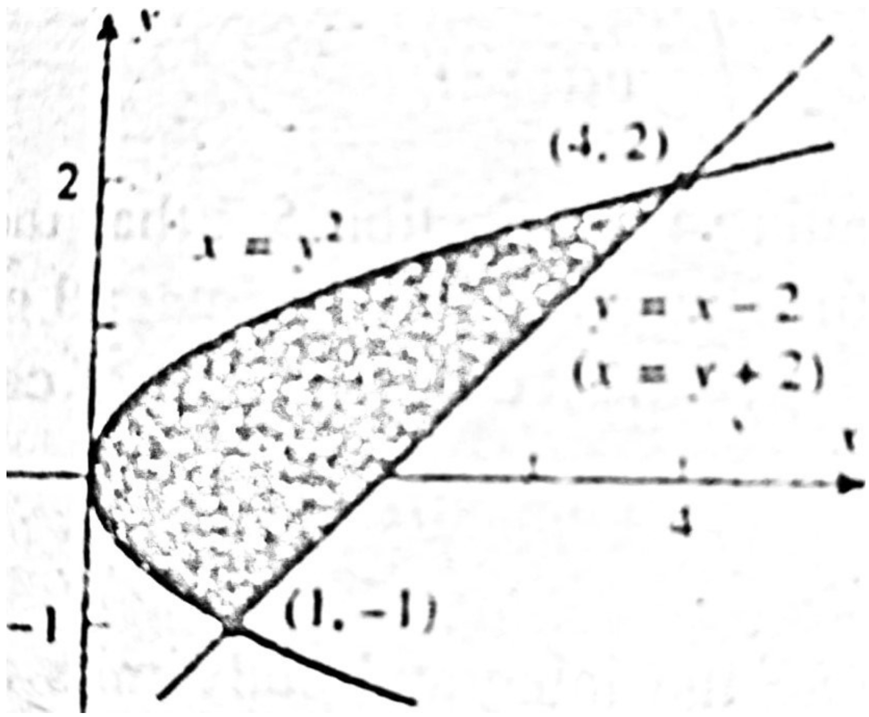
$$A = \int_{-2}^{3} [(x+6) - x^2]dx = 125/6$$

AREA FORMULA:

The area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$ and above by y $= d$ is

$$A = \int_{c}^{d} [w(y) - v(y)]dy$$

Example: Find the area of the region enclosed by $x = y^2$ and $y = x - 2$



Given equation

$$x = y^2 \qquad \text{(i)}$$

$$y = x - 2 \qquad \text{(ii)}$$

Solving (i) and (ii) then we get $y = -1$ and $y = 2$

$$A = \int_{-1}^{2} [(y+2) - y^2] = \frac{9}{2} \text{ Ans.}$$

Volumes by cylindrical shells about the y-axis:

Let *f* be continuous and nonnegative on [a, b] and let R be the region that is bounded above by *y = f(x)*, below by the x-axis, and on the sides by the lines x = a and x = b. Then the volume V of the solid of revolution that is generated by revolving the region R about the y-axis is given by

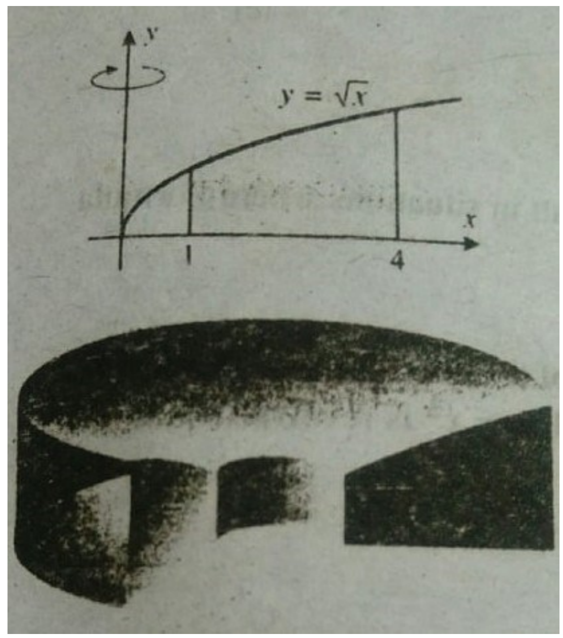
$$V = \int_a^b 2\pi x f(x) dx$$

Example:

Find the volume of the solid generated when the region enclosed between $y = \sqrt{x}, x = 1, x = 4,$ and x-axis is revolved about the y-axis.

Solution:

Since $f(x) = \sqrt{x}, a = 1 \; and \; b = 4.$



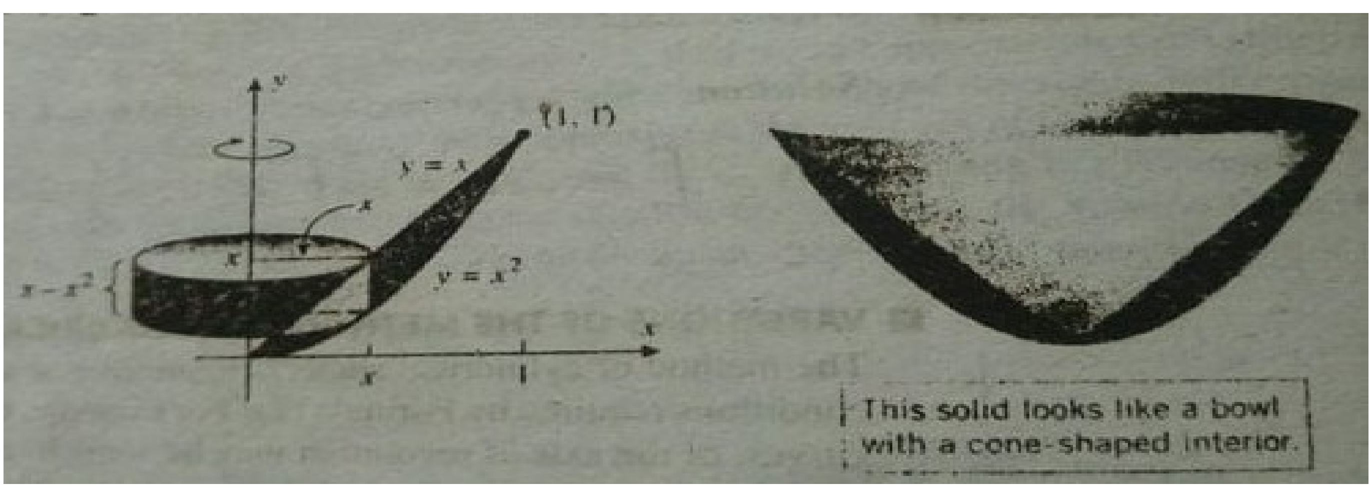
By using formula

$$V = \int_a^b 2\pi x f(x) dx$$

$$V = \int_1^4 2\pi x\sqrt{x}\, dx$$

$$= 2\pi \int_1^4 x^{2/3} dx = \frac{124\pi}{5}$$

Example: Find the volume of the solid generated when the region R in the first quadrant enclosed between y = x and y = x^2 is revolved about the *y*-axis.



This solid looks like a bowl with a cone-shaped interior.

Given equation

$$y = x \text{ ....(1)}$$

$$y = x^2 \text{ ....(2)}$$

Solving equation (1) and (2) x = 0 and x = 1

$$V = \int_0^1 2\pi x(x - x^2)dx = \frac{\pi}{6}$$

Average value of a function

Definition:

If *f* is continuous on *[a, b],* then the average value (or mean value) of *f* on *[a, b]* is defined to be

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example:

Find the average value of the function $f(x) = \sqrt{x}$ over the interval [1, 4].

Solution:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx_{=} = \frac{1}{4-1}\int_1^4 \sqrt{x}dx = \frac{14}{9} \approx 1.6$$

Find the average value of the function over the given interval

$$(i) f(x) = \sqrt[3]{x}; [-1, 8]$$

$$(ii) f(x) = \frac{x}{(5x^2+1)^2}; [0, 2]$$

Theorem (Mean Value Theorem for Integrals)

If *f* is a continuous function on the closed, bounded interval [a, b], then there is at least one number *c* in *(a, b)* for which

$$f(c) = \frac{1}{b-a}\int_a^b f(t)dt$$

Proof:

Consider the function F so that

$$F(x) = \int_a^x f(t)\,dt$$

for every value of x in [a, b].

The First Fundamental theorem of calculus tells us that F is continuous on [a, b], is differentiable on (a, b), and $F'(x) = f(x)$. These are exactly the conditions needed to apply the Mean Value Theorem for Derivatives to F on [a, b]. That is, there is at least one point c in (a, b) for which

$$F'(c) = \frac{F(b) - F(a)}{b - a}$$

Now, from the definition of F,

$$F(a) = \int_a^a f(t)dt = 0$$

and

$$F'(c) = f(c)$$

Thus,

$$f(c) = \frac{F(b)}{b-a} = \frac{1}{b-a}\int_a^b f(t)dt \quad \text{(Proved)}$$