**Euler's method:** Let us consider the initial-value problem

$$\frac{dy}{dx} = f(x, y), \quad a \le x \le b, \quad y(a) = y_0. \tag{1}$$

In actuality, a continuous approximation to the solution $y(x)$ will not be obtained; instead, approximations to $y$ will be generated at various values, called mesh points, in the interval $[a, b]$. Once the approximate solution is obtained at the points, the approximate solutions at other points in the interval are found by interpolation.

We first make the stipulation that the mesh points are equally distributed throughout the interval $[a, b]$. This condition is ensured by choosing a positive integer $N$ and selecting the mesh points

$$x_i = a + ih, \quad \text{for each } i = 0, 1, 2, \ldots, N.$$

The common distance between the points $h = (b\text{-}a)/N$ is called the step size.

Suppose that $y(x)$, the unique solution to (1), has two continuous derivatives on $[a, b]$, so that for each $i = 0, 1, 2, \ldots, N-1$, Taylor's theorem gives

$$y(x_{i+1}) = y(x_i) + (x_{i+1} - x_i)\, y'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} y''(\xi_i),$$

for some number $\xi_i$ in $(x_i, x_{i+1})$. Since $h = x_{i+1} - x_i$, we have

$$y(x_{i+1}) = y(x_i) + h y'(x_i) + \frac{h^2}{2} y''(\xi_i),$$

and since $y(x)$ satisfies the differential equation (1),

$$y(x_{i+1}) = y(x_i) + h f(x_i, y(x_i)) + \frac{h^2}{2} y''(\xi_i).$$

If $h$ is very small, then we can neglect the second order or remainder term. Thus we have

$$y(x_{i+1}) = y(x_i) + h f(x_i, y(x_i)), \text{ for } i = 0, 1, 2, \ldots, N-1.$$

Using the notation $y(x_i) = y_i$ for $i = 0, 1, 2, \ldots, N$, we obtain

$$y_{i+1} = y_i + h f(x_i, y_i), \quad \text{for each } i = 0, 1, 2, \ldots, N-1.$$

This is known as Euler's method for obtaining an approximation to the well-posed initial-value problem.

**Problem:** Use Euler's method to approximate the solution $y(0.1)$ for the initial-value problem

$$\frac{dy}{dx} = \frac{y - x}{y + x}, \quad y(0) = 1.$$

**Solution:** Here we want the value at $x$ = 0.1 from $x$ = 0 in five steps. So we break up the interval 0 to 0.1 into five subintervals by introducing the points $x_1$, $x_2$, $x_3$, $x_4$, $x_5$. Then

$$h = \frac{0.1-0}{5} = 0.02.$$

We shall find the values of $y$ at $x$ = 0.02, 0.04, 0.06, 0.08 and 0.1 successively. Thus we have

$$x_0 = 0,\ y_0 = 1,\ h = 0.02,\ f(x,y) = \frac{y-x}{y+x}.$$

Using the Euler's formula, $y_{i+1} = y_i + hf(x_i, y_i)$, for each $i = 0, 1, 2, 4,$ we get

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.02\left(\frac{1-0}{1+0}\right) = 1 + 0.02 = 1.02$$

$$y_2 = y_1 + hf(x_1, y_1) = 1.02 + 0.02\left(\frac{1.02-0.02}{1.02+0.02}\right) = 1.02 + 0.02 \times 0.961538 = 1.0392308$$

$$y_3 = y_2 + hf(x_2, y_2) = 1.0392308 + 0.02\left(\frac{1.0392308-0.04}{1.0392308+0.04}\right)$$

$$= 1.0392308 + 0.02 \times 0.925873 = 1.05775$$

$$y_4 = y_3 + hf(x_3, y_3) = 1.05775 + 0.02\left(\frac{1.05775-0.06}{1.05775+0.06}\right) = 1.05775 + 0.02 \times 0.892641 = 1.0756$$

$$y_5 = y_4 + hf(x_4, y_4) = 1.0756 + 0.02\left(\frac{1.0756-0.08}{1.0756+0.08}\right) = 1.0756 + 0.02 \times 0.861544 = 1.09283.$$

Therefore, the value of $y(0.1) = 1.09283.$

**Problem:** Use Euler's method to approximate the solution $y(1.0)$ for the initial-value problem

$$\frac{dy}{dt} = y - t^2 + 1, \quad \text{for } 0 \le t \le 2 \quad \text{with} \quad y(0) = 0.5,$$

taking $h$ = 0.2.

**Solution:** Given the initial value problem

$$\frac{dy}{dt} = y - t^2 + 1, \quad \text{for } 0 \le t \le 2 \quad \text{with} \quad y(0) = 0.5.$$

Since $h$ = 0.2, we shall find the values of $y$ at $t$ = 0.2, 0.4, 0.6, 0.8 and 1.0 successively. Thus we have

$$t_0 = 0,\ y_0 = 1,\ h = 0.2,\ f(t,y) = y - t^2 + 1.$$

Using the Euler's formula, $y_{i+1} = y_i + hf(t_i, y_i)$, for each $i = 0, 1, 2, 4,$ we get

$$y_1 = y_0 + hf\left(t_0, y_0\right) = 0.5 + 0.2\left(0.5 - 0^2 + 1\right) = 0.5 + 0.3 = 0.8$$
$$y_2 = y_1 + hf\left(t_1, y_1\right) = 0.8 + 0.2\left(0.8 - 0.2^2 + 1\right) = 0.8 + 0.2 \times 1.76 = 1.152$$
$$y_3 = y_2 + hf\left(t_2, y_2\right) = 1.152 + 0.2\left(1.152 - 0.4^2 + 1\right)$$
$$= 1.152 + 0.2 \times 1.992 = 1.5504$$
$$y_4 = y_3 + hf\left(t_3, y_3\right) = 1.5504 + 0.2\left(1.5504 - 0.6^2 + 1\right) = 1.5504 + 0.2 \times 2.1904 = 1.98848$$
$$y_5 = y_4 + hf\left(t_4, y_4\right) = 1.98848 + 0.2\left(1.98848 - 0.8^2 + 1\right) = 1.98848 + 0.2 \times 2.34848 = 2.45818.$$

Therefore, the value of $y(1.0) = 2.45818$.

Modified Euler Method:

**Problem:** Use Modified Euler's method to approximate the solution $y(1.0)$ for the initial-value problem

$$\frac{dy}{dt} = y - t^2 + 1, \quad \text{for} \quad 0 \le t \le 2 \quad \text{with} \quad y(0) = 0.5,$$

taking $h = 0.2$.

**Solution:** Given the initial value problem

$$\frac{dy}{dt} = y - t^2 + 1, \quad \text{for} \quad 0 \le t \le 2 \quad \text{with} \quad y(0) = 0.5.$$

Since $h = 0.2$, we shall find the values of $y$ at $t = 0.2$, 0.4, 0.6, 0.8 and 1.0 successively. Thus we have

$$t_0 = 0,\ y_0 = 1,\ h = 0.2,\ f(t, y) = y - t^2 + 1.$$

Using the Modified Euler's Euler's formula, $y_{i+1}^{(n+1)} = y_i + \frac{h}{2}\left\{f\left(t_i, y_i\right) + f\left(t_{i+1}, y_{i+1}^{(n)}\right)\right\}$, for each $i = 0,\ 1,\ 2,\ 4,$ we get

$$y_1 = y_0 + hf\left(t_0, y_0\right) = 0.5 + 0.2\left(0.5 - 0^2 + 1\right) = 0.5 + 0.3 = 0.8$$
$$y_1^{(1)} = y_0 + \frac{h}{2}\left\{f\left(t_0, y_0\right) + f\left(t_1, y_1^{(0)}\right)\right\}$$
$$= 0.5 + \frac{0.2}{2}\left\{\left(0.5 - 0^2 + 1\right) + \left(0.8 - 0.2^2 + 1\right)\right\}$$
$$= 0.5 + 0.1\left(1.5 + 1.76\right) = 0.5 + 0.1 \times 3.26 = 0.826$$

$$y_2^{(0)} = y_1^{(1)} + hf\left(t_2, y_1^{(1)}\right) = 0.826 + 0.2\left(0.8 - 0.4^2 + 1\right) = 1.1832$$

$$y_2^{(1)} = y_1^{(1)} + \frac{h}{2}\left\{f\left(t_1, y_1^{(1)}\right) + f\left(t_2, y_2^{(0)}\right)\right\}$$

$$= 0.5 + \frac{0.2}{2}\left\{\left(0.826 - 0.2^2 + 1\right) + \left(1.1832 - 0.4^2 + 1\right)\right\}$$

$$= 1.20692$$

$$y_1 = y_0 + hf\left(t_0, y_0\right) = 0.5 + 0.2\left(0.5 - 0^2 + 1\right) = 0.5 + 0.3 = 0.8$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left\{f\left(t_0, y_0\right) + f\left(t_1, y_1^{(0)}\right)\right\}$$

$$= 0.5 + \frac{0.2}{2}\left\{\left(0.5 - 0^2 + 1\right) + \left(0.8 - 0.2^2 + 1\right)\right\}$$

$$= 0.5 + 0.1\left(1.5 + 1.76\right) = 0.5 + 0.1 \times 3.26 = 0.826$$

$$y_2 = y_1 + hf\left(t_1, y_1\right) = 0.8 + 0.2\left(0.8 - 0.2^2 + 1\right) = 0.8 + 0.2 \times 1.76 = 1.152$$

$$y_3 = y_2 + hf\left(t_2, y_2\right) = 1.152 + 0.2\left(1.152 - 0.4^2 + 1\right)$$

$$= 1.152 + 0.2 \times 1.992 = 1.5504$$

$$y_4 = y_3 + hf\left(t_3, y_3\right) = 1.5504 + 0.2\left(1.5504 - 0.6^2 + 1\right) = 1.5504 + 0.2 \times 2.1904 = 1.98848$$

$$y_5 = y_4 + hf\left(t_4, y_4\right) = 1.98848 + 0.2\left(1.98848 - 0.8^2 + 1\right) = 1.98848 + 0.2 \times 2.34848 = 2.45818.$$

Therefore, the value of $y(1.0) = 2.45818$.