R-C and R-L Circuits

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First-Order Circuits

- Introduction
- The Source Free RC Circuit
- The Source Free RL Circuit
- Step Response of an RC Circuit
- Step Response of an RL Circuit

First-Order Circuits: Introduction

- A first-order circuit can only contain one energy storage element (a capacitor or an inductor)
- The circuit will also contain resistance.
- So there are two types of first--order circuits:
  - RC circuit
  - RL circuit
- A first-order circuit is characterized by a first-order differential equation.

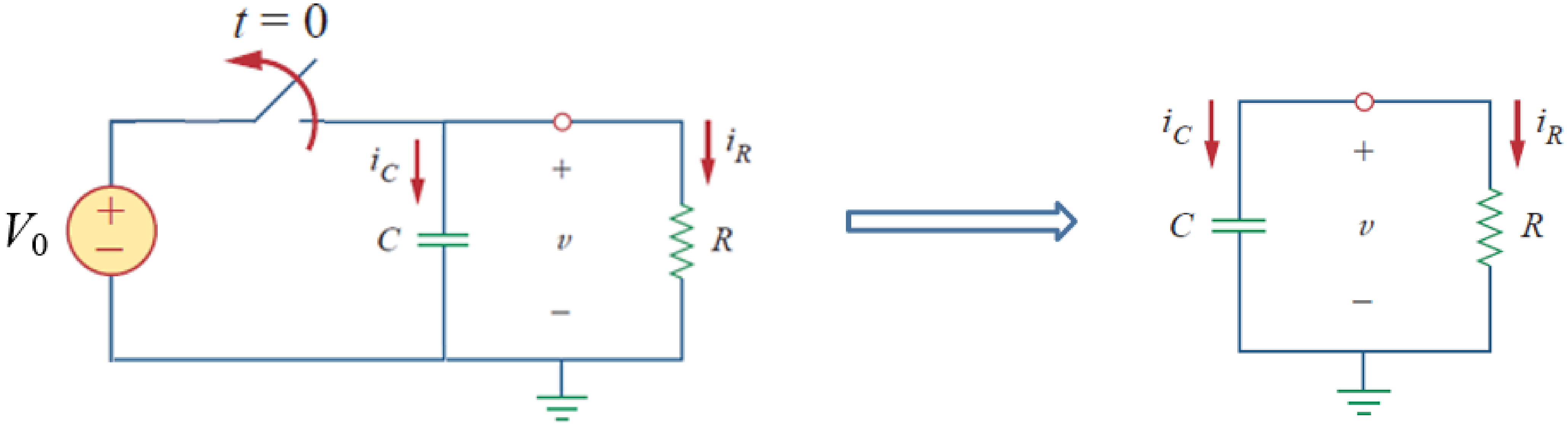
First-Order Circuits: The Source-Free Circuits

- A source-free circuit is one where all independent sources have been disconnected from the circuit after some switch action
- The voltages and currents in the circuit typically will have some transient response due to initial conditions (initial capacitor voltages and initial inductor currents)
- We will begin by analyzing source-free circuits as they are the simplest type. Later we will analyze circuits that also contain sources after the initial switch action

First-Order Circuits: The Source-Free RC Circuits

- A source free RC circuit occurs when its dc source is suddenly disconnected.
- The energy already stored in the capacitor is released to the resistors.

First-Order Circuits: The Source-Free RC Circuits



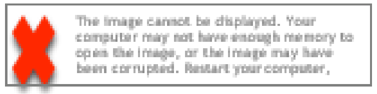
•Since the capacitor is **initially charged**, we can assume that at time t=0, **the initial voltage** is

$$v(0) = V_0$$

- the energy **stored:**

$$w(0) = \frac{1}{2} C V_0^2$$

- Applying **KCL** at the top node

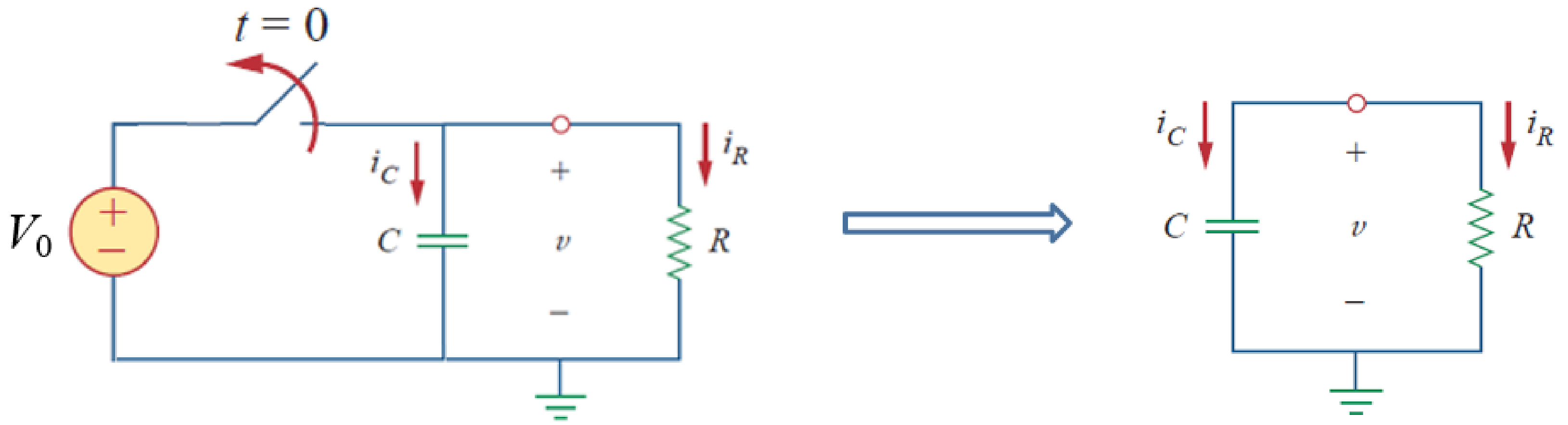


, $i_C = C\, dv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

First-Order Circuits: The Source-Free RC Circuits



$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

- This is a ***first--order differential equation***, since only the ***first derivative*** of v is involved.
- Rearranging the terms:

$$\int \frac{dv}{v} = -\frac{1}{RC} \int dt$$

- Integrating both sides:

$$\ln v = -\frac{t}{RC} + \ln A$$

- $\ln A$ is the integration constant. Thus

$$\ln \frac{v}{A} = -\frac{t}{RC}$$

- Taking powers of e produces:

$$v(t) = Ae^{-t/RC}$$

- From the ***initial conditions:*** $v(0)=A=V_0$

$$\boxed{v(t) = V_0 e^{-t/RC}}$$

- The **natural response** of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

First-Order Circuits: The Source-Free RC Circuits

General form of the Differential Equations (**DE**) and the response for a **1st--order** source--free circuit:

- **In general, a first--order D.E. has the form:**

$$\frac{dx}{dt} + \frac{1}{\tau}x(t) = 0 \qquad for \quad t \geq 0$$

- **Solving this DE (as we did with the RC circuit) yields:**

$$x(t) = x(0)e^{-\frac{t}{\tau}} \qquad for \quad t \geq 0$$

- **here τ= (Greek letter "Tau")** = **time constant**(in seconds)

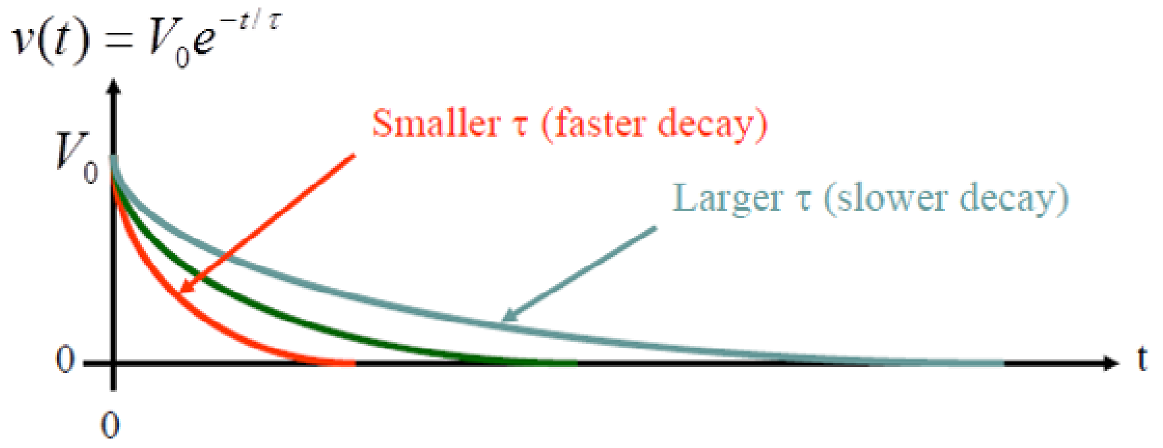
First-Order Circuits: The Source-Free RC Circuits

- Notes concerning τ:

1) For the Source--Free RC circuit the DE is: $\frac{dv}{dt} + \frac{1}{RC} v(t) = 0 \quad for \quad t \geq 0$

  - So, for an RC circuit: $\boxed{\tau = RC}$

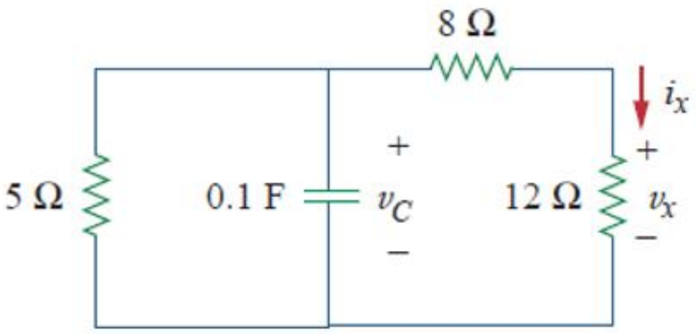
2) τ is related to the rate of exponential decay in a circuit as shown below.



3) It is typically easier to sketch a response in terms of multiples of τ than to be concerning with scaling of the graph.

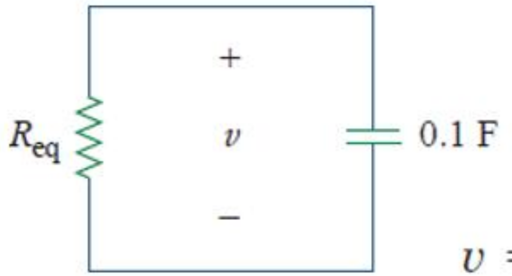
First-Order Circuits: The Source-Free RC Circuits

Ex. 7.1: In Fig. 7.5, let $v_C(0) = 15\ \text{V}$. Find v_C, v_x and i_x for $t>0$.



Solution

- Equivalent Circuit for the above circuit can be generated:



$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$

$$\tau = R_{eq}C = 4(0.1) = 0.4\ \text{s}$$

$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4}\ \text{V}, \qquad v_C = v = 15e^{-2.5t}\ \text{V}$$

we can use voltage division to get v_x

$$v_x = \frac{12}{12 + 8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}\ \text{V}$$

Finally,

$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t}\ \text{A}$$

First-Order Circuits: The Source-Free RC Circuits

Equivalent Resistance seen by a Capacitor

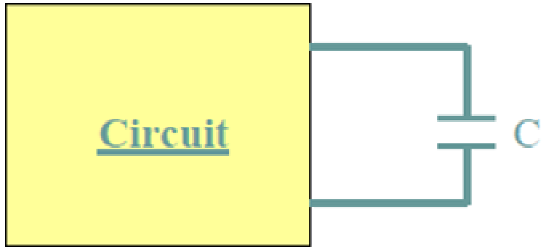
- For the RC circuit in the previous example, it was determined that $\tau = RC$. But what value of R should be used in circuits with multiple resistors?
- In general, a first--order RC circuit has the following time constant:

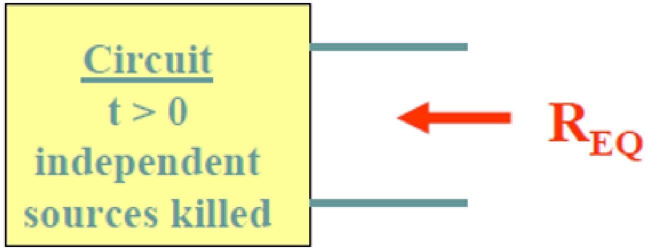
$$\boxed{\tau = R_{EQ}C}$$

- where R_{EQ} is the Thevenin resistance seen by the capacitor.

- More specifically,

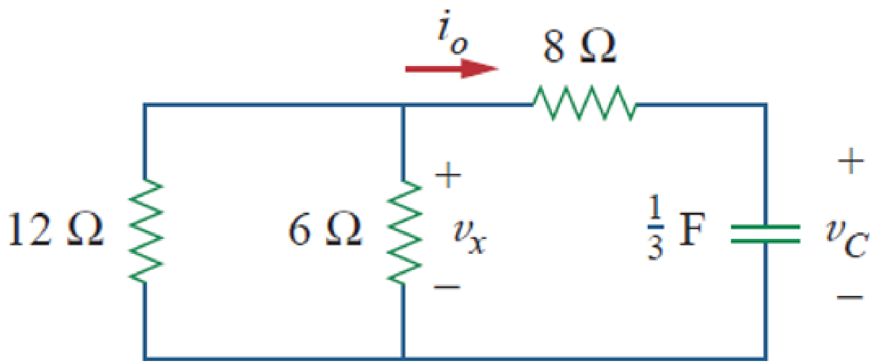
  $R_{EQ} = R$ (seen from the terminals of the capacitor for t>0 with independent sources killed.)





First-Order Circuits: The Source-Free RC Circuits

Ex. : Refer to the circuit below. Let $v_C(0) = 45$ V. Determine v_C, v_x and i_o for $t \geq 0$.



Solution

- Consider R_{eq} seen from the capacitor.

$$R_{eq} = \frac{12 \times 6}{18} + 8 = 12\ \Omega$$

- Time constant τ **:** $\tau = R_{eq}C = 12 \times \frac{1}{3} = 4\,s$

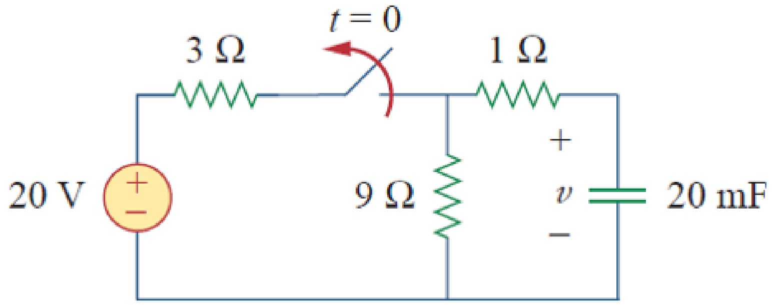
- Then: $v_C(t) = v_C(0)e^{\frac{-t}{4}} = 45e^{-0.25t}$ V

$$v_x(t) = \frac{4}{4+8} v_C(t) = \frac{1}{3} 45e^{-0.25t} = 15e^{-0.25t} \text{ V}$$

$$i_o(t) = \frac{v_x(t) - v_C(t)}{8} = \frac{15e^{-0.25t} - 45e^{-0.25t}}{8} = -3.75e^{-0.25t} \text{ V}$$

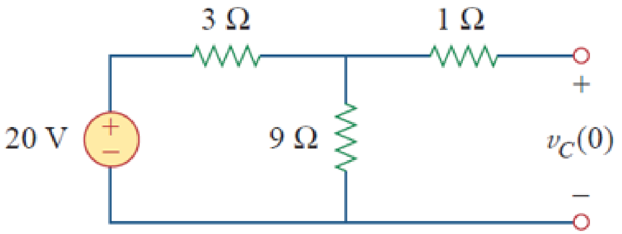
First-Order Circuits: The Source-Free RC Circuits

Ex. 7.2: The switch in the circuit below has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.



Solution

- For $t<0$ the switch is closed; the capacitor is an **open circuit to dc**, as represented in

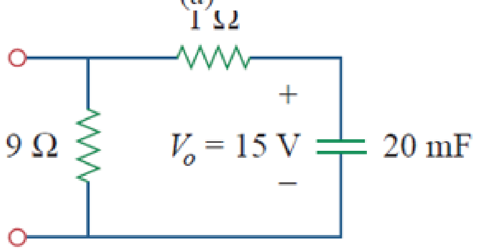


(a)

$$v_C(t) = \frac{9}{9+3}(20) = 15 \text{ V}, \qquad t < 0$$

$$v_C(0) = V_0 = 15 \text{ V}$$

- For $t>0$ the switch is opened, and we have the *RC* circuit shown in Fig. (b).



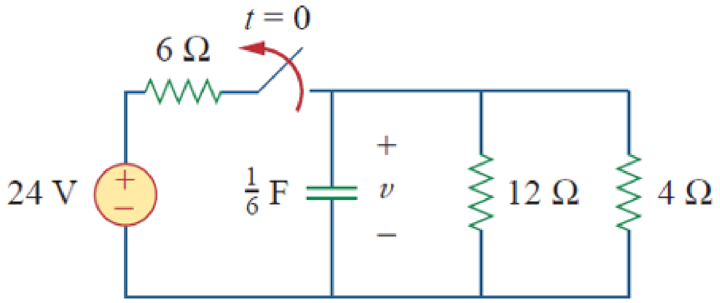
(b)

$$R_{eq} = 1 + 9 = 10\ \Omega$$

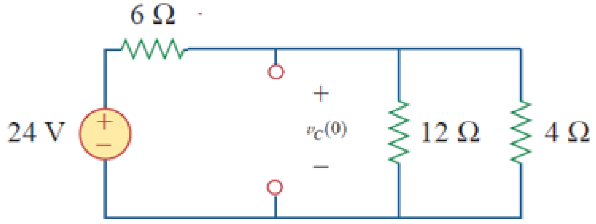
- Time constant τ : $\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2$ s
- Then: $v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2} = 15e^{-5t}$ V
- The **initial energy** stored in the capacitor: $w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25$ J

First-Order Circuits: The Source-Free RC Circuits

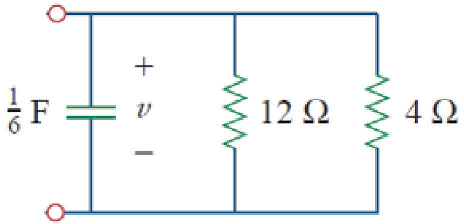
Ex. : If the switch in Fig. below opens at $t = 0$, find $v(t)$ for $t \geq 0$ *and* $w_C(0)$.



Solution



(a)



(b)

- For $t<0$ the switch is closed; the capacitor is an open circuit to dc as shown in Fig. (a).

$$v_C(t) = \frac{3}{3+6} 24 = 8 \text{ V} \quad for \quad t < 0$$

$$v_C(0) = V_0 = 8 \text{ V}$$

- For $t>0$ the switch is opened, and we have the *RC* circuit shown in Fig. (b).

$$R_{eq} = \frac{12 \times 4}{16} = 3 \, \Omega$$

- Time constant $\boldsymbol{\tau}$ **:** $\quad \tau = R_{eq} C = 3 \times \frac{1}{6} = 0.5\, s$

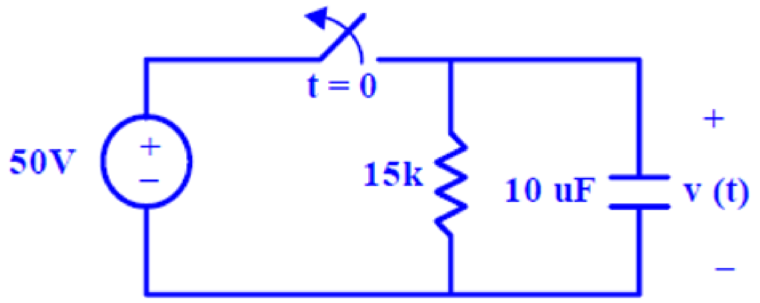
- Then: $\quad v(t) = v_C(0) e^{\frac{-t}{0.5}} = 8e^{-2t} \text{ V}$

- The **initial energy** stored in the capacitor:

First-Order Circuits: The Source-Free RC Circuits

Ex. : The switch in the circuit shown had been closed for a long time and then opened at time t = 0.

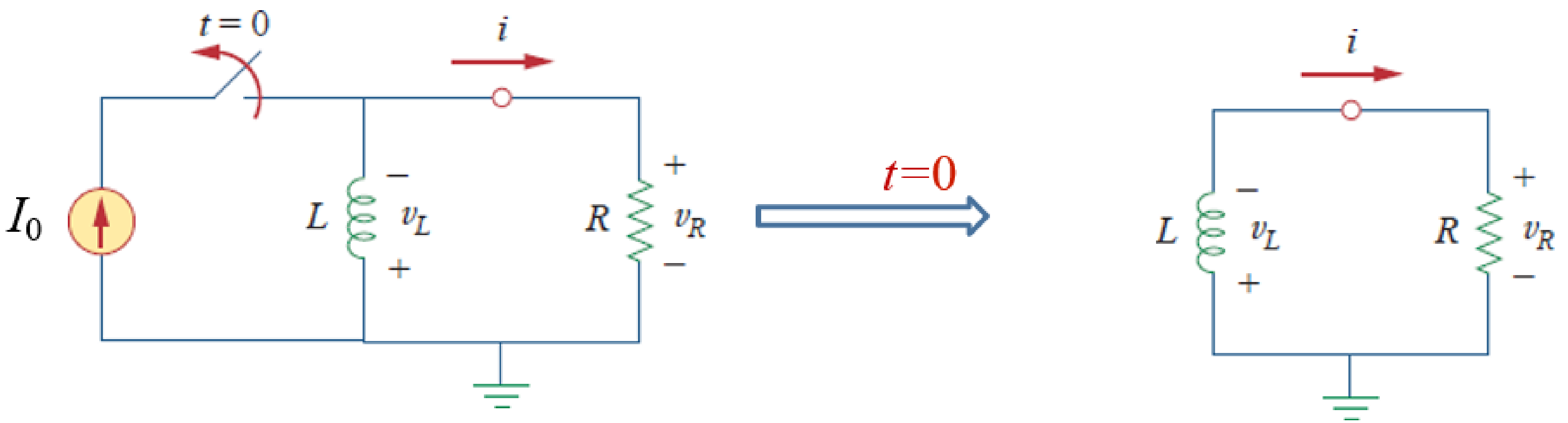
a) Determine an expression for $v(t)$.
b) Graph $v(t)$ versus t.
c) How long will it take for the capacitor to completely discharge?
d) Determine the capacitor voltage at time t=100ms.
e) Determine the time at which the capacitor voltage is 10V.



First-Order Circuits: The Source-Free RL Circuits

- A **source--free RL circuit** occurs when its dc source is suddenly disconnected.
- The **energy already stored** in the inductor is released to the resistors.

First-Order Circuits: The Source-Free RL Circuits



At time, t=0 , the intuctor has **the initial current:**

$$i(0) = I_0$$

Then the energy **stored:**

$$w(0) = \frac{1}{2} L I_0^2$$

- We can apply **KVL** around the loop above :

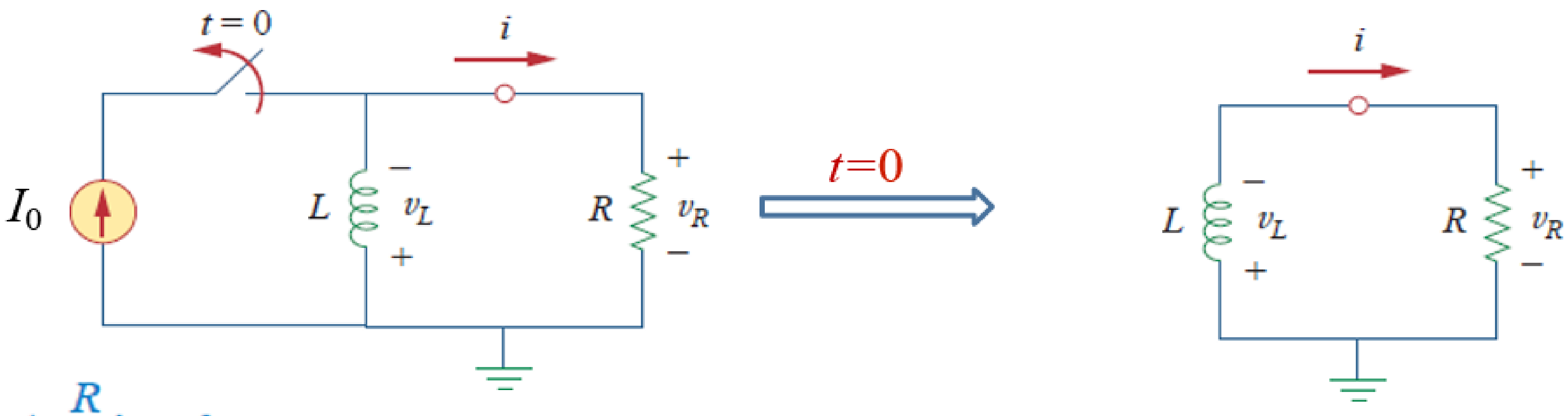
$$v_L + v_R = 0$$

- By definition, $v_L = L\, di/dt$ and $v_R = Ri$. Thus,

$$L\frac{di}{dt} + Ri = 0$$

$$\frac{di}{dt} + \frac{R}{L} i = 0$$

First-Order Circuits: The Source-Free RL Circuits



$$\frac{di}{dt} + \frac{R}{L} i = 0$$

- This is a ***first-order differential equation***, since only the ***first derivative*** of i is involved.
- Rearranging the terms and integrating:

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt$$

- Then:

$$\ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

$$\ln \frac{i(t)}{I_0} = -\frac{Rt}{L}$$

- Taking powers of e produces:

$$i(t) = I_0 e^{-Rt/L}$$

- ***Time constant for RL circuit becomes:*** $\boxed{\tau = \frac{L}{R}}$

$$\boxed{i(t) = I_0 e^{-t/\tau}}$$

The **natural response** of the *RL* circuit is an exponential decay of the initial current.

First-Order Circuits: The Source-Free RL Circuits

General form of the Differential Equations (**DE**) and the response for a **1st--order** source--free circuit:

In general, a first--order D.E. has the form:

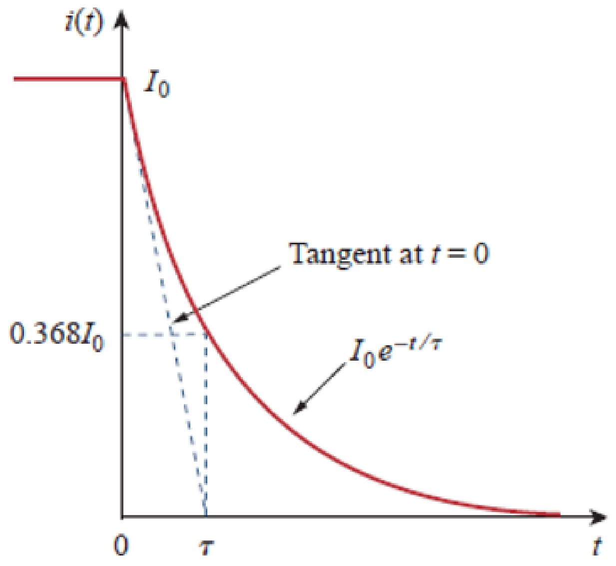
$$\frac{dx}{dt} + \frac{1}{\tau}x(t) = 0 \qquad for \quad t \geq 0$$

Solving this DE (as we did with the RL circuit) yields:

$$x(t) = x(0)e^{-\frac{t}{\tau}} \qquad for \quad t \geq 0$$

Then: $$i(t) = i(0)e^{-\frac{t}{\tau}} = I_0 e^{-\frac{t}{\tau}} \qquad for \quad t \geq 0$$

Where: $$\tau = \frac{L}{R}$$



First-Order Circuits: The Source-Free RL Circuits

For the RL circuit , it was determined that $\tau = L/R$. As with the RC circuit, the value of R should actually be the equivalent (*or Thevenin*) resistance seen by the inductor.

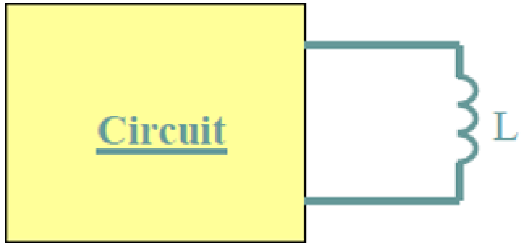
In general, a first--order RL circuit has the following time constant:

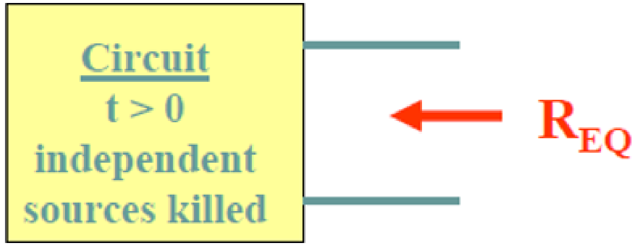
$$\tau = \frac{L}{R_{EQ}}$$

where R_{EQ} is the Thevenin resistance seen by the inductor.

More specifically,

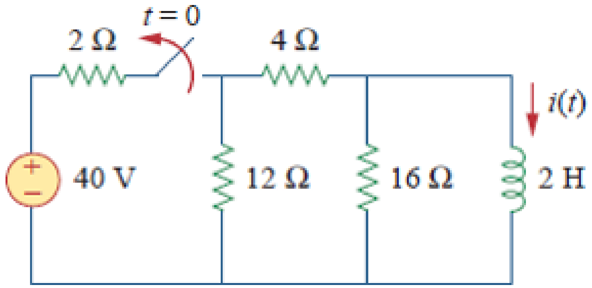
$R_{EQ} = R$ (seen from the terminals of the capacitor for t>0 with independent sources killed.)





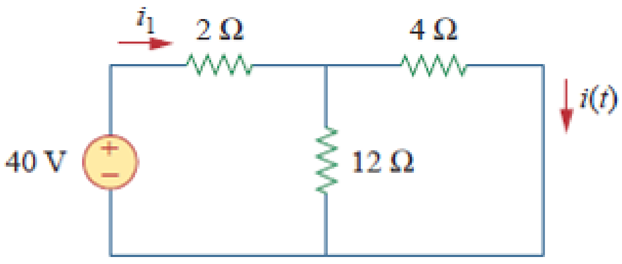
First-Order Circuits: The Source-Free RL Circuits

Ex. 7.4: The switch in the circuit below has been closed for a long time. At $t=0$ the switch is opened. Calculate $i(t)$ for $t>0$.



Solution

- When $t<0$ the switch is closed, and the inductor acts as a short circuit to dc,



(a)

$$\frac{4 \times 12}{4 + 12} = 3\ \Omega \quad \Rightarrow \quad i_1 = \frac{40}{2 + 3} = 8\ \text{A}$$

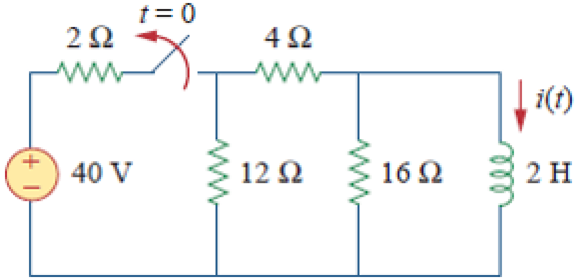
- Using current division: $$i(t) = \frac{12}{12 + 4} i_1 = 6\ \text{A}, \qquad t < 0$$

- Current through an inductor cannot change instantaneously,

$$i(0) = i(0^-) = 6\ \text{A}$$

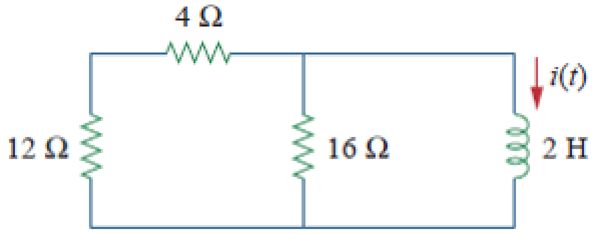
First-Order Circuits: The Source-Free RL Circuits

Ex. 7.4: The switch in the circuit below has been closed for a long time. At $t=0$ the switch is opened. Calculate $i(t)$ for $t>0$.



Solution

When $t>0$ the switch is open and the voltage source is disconnected. We now have the source-free RL circuit in Fig. (b).



(b)

$$R_{eq} = (12 + 4) \parallel 16 = 8\ \Omega$$

- The time constant is :

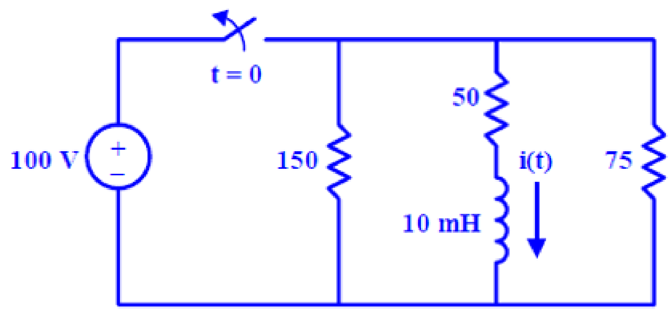
$$\tau = \frac{L}{R_{eq}} = \frac{2}{8} = \frac{1}{4}\ \text{s}$$

- Thus,

$$\boxed{i(t) = i(0)e^{-t/\tau} = 6e^{-4t}\ \text{A}}$$

First-Order Circuits: The Source-Free RL Circuits

Ex. : Determine an expression for $i(t)$. Sketch $i(t)$ versus t.



First-Order Circuits: The Source-Free RL Circuits

Step Response (DC forcing functions)

- Consider circuits having DC forcing functions for t > 0 (i.e., circuits that have independent DC sources for t > 0).

- The general solution to a differential equation has two parts:
- $x(t) = x_h + x_p$ = homogeneous solution + particular solution
- or
- $x(t) = x_n + x_f$ = natural solution + forced solution

$$\boxed{\text{Complete response} = \underbrace{\text{natural response}}_{\text{stored energy}} + \underbrace{\text{forced response}}_{\text{independent source}}}$$

- x_n is due to the initial conditions in the circuit
- and x_f is due to the forcing functions (independent voltage and current sources for t > 0).

- x_f in general take on the "form" of the forcing functions,
- So DC sources imply that the forced response function will be a constant(DC),
- Sinusoidal sources imply that the forced response will be sinusoidal, etc.

First-Order Circuits: The Source-Free RL Circuits

Step Response (DC forcing functions)

- Since we are only considering DC forcing functions in this chapter, we assume that : $x_f = B$ (constant).
- Recall that a 1st-order source--free circuit had the form $Ae^{-t/\tau}$. Note that there was a natural response only since there were no forcing functions (sources) for t > 0. So the natural response was

$$x_n = Ae^{-t/\tau} \quad for \quad t > 0$$

- The complete response for 1st--order circuit with DC forcing functions therefore will have the form: $x(t) = x_f + x_n$

$$x(t) = B + Ae^{-t/\tau}$$

- The “Shortcut Method”: An easy way to find the constants B and A is to evaluate $x(t)$ at 2 points. Two convenient points at $t = 0$ and $t = \infty$ since the circuit is under dc conditions at these two points. This approach is sometimes called the “shortcut method.”

First-Order Circuits: The Source-Free RL Circuits

- **Step Response (DC forcing functions)**

- **The “Shortcut Method” :**

  So, $x(0) = B + Ae^{0} = B + A$

  And $x(\infty) = B + Ae^{-\infty} = B$

- **Complete response yields the following expression:**

$$\boxed{x(t) = x(\infty) + [x(0) - x(\infty)]e^{-t/\tau}}$$

- **The Shortcut Method-- Procedure: The shortcut method will be the key method used to analyze 1st--order circuit with DC forcing functions:**

1. **Analyze the circuit at** $t = 0-$**: Find** $x(0-) = x(0+)$
2. **Analyze the circuit at** $t = \infty$**: Find** $x(\infty)$
3. **Find** $\tau = R_{EQ}C$ **or** $\tau = L/R_{EQ}$
4. **Assume that** $x(t)$ **has the form** $x(t) = x(\infty) + [x(0) - x(\infty)]\ e^{-t/\tau}$ **using** $x(0)$ **and** $x(\infty)$

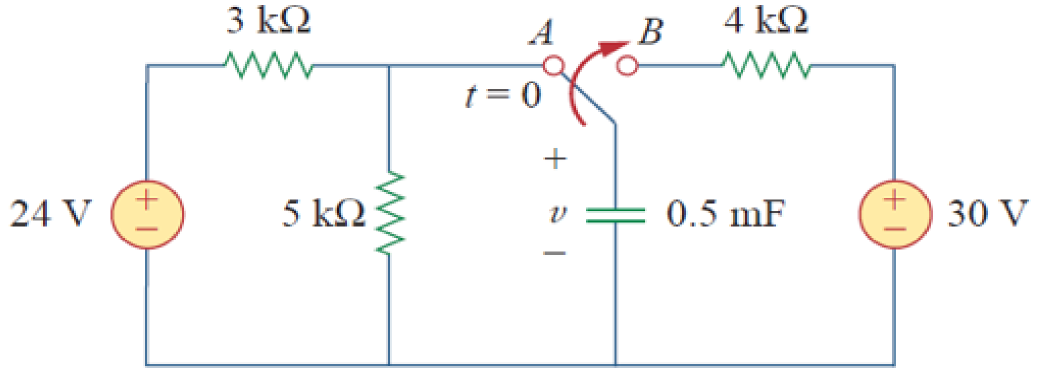
First-Order Circuits: The Source-Free RL Circuits

Step Response (DC forcing functions)

- Notes: The “shortcut method” also works for source--free circuits, but $x(\infty)$ = B=0 since the circuit is dead at t = ∞. If variables other than vC or iL are needed, it is generally easiest to solve for vC or iL first and then use the result to find the desired variable.

First-Order Circuits: The Source-Free RL Circuits

Ex. 7.10: The switch in Fig. Below has been in position A for a long time. At $t=0$ the switch moves to B. Determine $v(t)$ for $t>0$ and calculate its value at $t=1$ s and 4 s.



Solution

- Voltage across the capacitor just before $t=0$. Capacitor is open circuit under dc conditions:

$$v(0^-) = \frac{5}{5+3}(24) = 15 \text{ V}$$

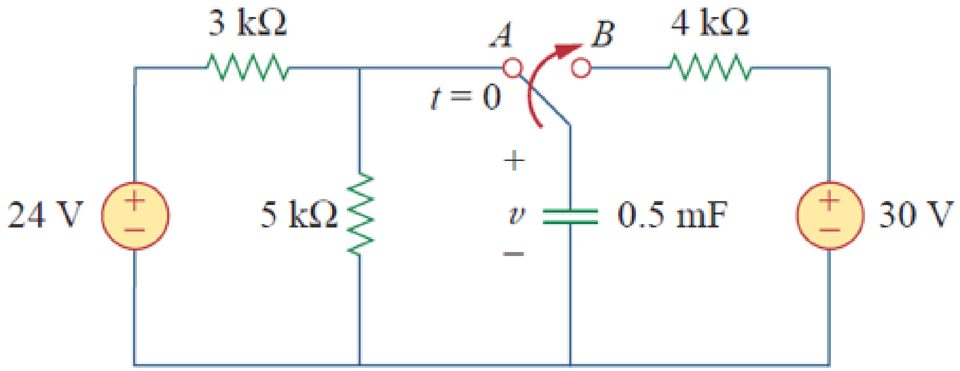
- Capacitor voltage cannot change instantaneously: $v(0) = v(0^-) = v(0^+) = 15 \text{ V}$

- For t>0 (switch to B). Thevenin Resistance connected to the capacitor: $R_{Th} = 4 \text{ k}\Omega$

- Time constant: $\tau = R_{Th}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ s}$

First-Order Circuits: The Source-Free RL Circuits

Ex. 7.10: The switch in Fig. below has been in position A for a long time. At $t=0$ the switch moves to B. Determine $v(t)$ for $t>0$ and calculate its value at t 1 s and 4 s.



Solution

- Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) = 30$ V. Thus,

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$
$$= 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t})\text{ V}$$

At $t = 1$,

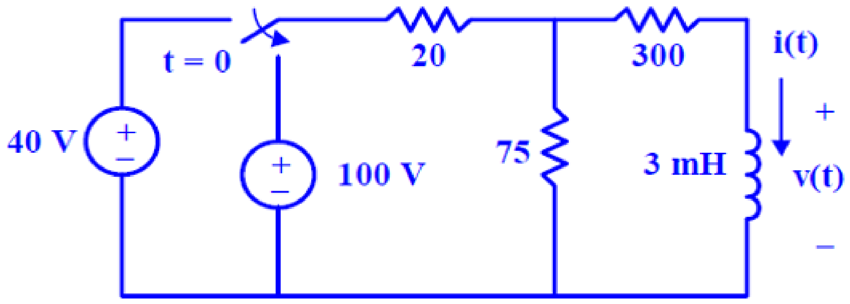
$$v(1) = 30 - 15e^{-0.5} = 20.9\text{ V}$$

At $t = 4$,

$$v(4) = 30 - 15e^{-2} = 27.97\text{ V}$$

First-Order Circuits: The Source-Free RL Circuits

Ex. 1 : Find $v(t)$ and $i(t)$ for $t \geq 0$.



Ex. 2: Find $v(t)$ and $i(t)$ for $t \geq 0$.