Rules of Inference

ARGUMENTS

DEFINITION 1 An *argument* in propositional logic is a sequence of propositions. All but the final proposition in the argument are called *premises* and the final proposition is called the *conclusion*. An argument is *valid* if the truth of all its premises implies that the conclusion is true.

An *argument* is an assertion that a given set of propositions P_1, P_2,... ,P_n, called *premises*, (has a consequence) another proposition Q, called the *conclusion*. Such an argument is denoted by,

$P_1, P_2, \ldots, P \dashv Q$

An argument $P_1, P_2, \ldots, P_n \dashv Q$ is said to be *valid* if Q is true whenever all the premises $P_1, P_2, \ldots, P_n$ are true.

An argument which is not valid is called *fallacy*.

Valid Arguments in Propositional Logic

Consider the following argument involving propositions (which, by definition, is a sequence of propositions):

"If you have a current password, then you can log onto the network."

"You have a current password."

Therefore,
"You can log onto the network."

We would like to determine whether this is a valid argument. That is, we would like to determine whether the conclusion "You can log onto the network" must be true when the premises "If you have a current password, then you can log onto the network" and "You have a current password" are both true.

Before we discuss the validity of this particular argument, we will look at its form. Use p to represent "You have a current password" and q to represent "You can log onto the network." Then, the argument has the form

$$\begin{array}{l} p \rightarrow q \\ \underline{P \qquad\quad} \\ \therefore q \end{array}$$

where ∴ is the symbol that denotes "therefore."

The argument we obtain by substituting these values of p and q into the argument form is

"If you have access to the network, then you can change your grade."
"You have access to the network."

∴ "You can change your grade."

We know that when p and q are propositional variables, the statement $((p \rightarrow q) \land p) \rightarrow q$ is a tautology (shown in previous lecture). In particular, when both $p \rightarrow q$ and p are true, we know that q must also be true. We say this form of argument is **valid** because whenever all its premises (all statements in the argument other than the final one, the conclusion) are true, the conclusion must also be true.

Rules of Inference for Propositional Logic

✓We can always use a truth table to show that an argument form is valid. We do this by showing that whenever the premises are true, the conclusion must also be true.

✓However, this can be a tedious approach. For example, when an argument form involves 10 different propositional variables, to use a truth table to show this argument form is valid requires $2^{10} = 1024$ different rows.

✓Fortunately, we do not have to resort to truth tables. Instead, we can first establish the validity of some relatively simple argument forms, called **rules of inference**.

TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition

Rule of Inference	*Tautology*	*Name*
$\begin{array}{rl} & p \land q \\ \hline \therefore & p \end{array}$	$(p \land q) \rightarrow p$	Simplification
$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \land q \end{array}$	$((p) \land (q)) \rightarrow (p \land q)$	Conjunction
$\begin{array}{rl} & p \lor q \\ & \neg p \lor r \\ \hline \therefore & q \lor r \end{array}$	$((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$	Resolution

The tautology $(p \wedge (p \rightarrow q)) \rightarrow q$ is the basis of the rule of inference called **modus ponens**. (Modus ponens is Latin for *mode that affirms*.) This tautology leads to the following valid argument form, which we have already seen in our initial discussion about arguments (where, as before, the symbol $\therefore$ denotes "therefore"):

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Example-1

Show that the following argument is a fallacy: $p \rightarrow q,\ \neg p \dashv\ \neg q$.

Ans.

Construct the truth table for $[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$ as in Fig. below. Since the proposition $[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$ is not a tautology, the argument is a fallacy. Equivalently, the argument is a fallacy since in the third line of the truth table $p \rightarrow q$ and $\neg p$ are true but $\neg q$ is false.

p	q	$p \rightarrow q$	$\neg p$	$(p \rightarrow q) \wedge \neg p$	$\neg q$	$[(p \rightarrow q) \wedge \neg p] \rightarrow \neg p$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

Example-2

Prove the following argument is valid: $p \to \neg q$, $r \to q$, $r \dashv \neg p$.

Construct the truth table of the premises and conclusions as in Fig. below. Now, $p \to \neg q$, $r \to q$, and r are true simultaneously only in the fifth row of the table, where $\neg p$ is also true. Hence the argument is valid

	p	q	r	$p \to \neg q$	$r \to q$	$\neg q$
1	T	T	T	F	T	F
2	T	T	F	F	T	F
3	T	F	T	T	F	F
4	T	F	F	T	T	F
5	F	T	T	T	T	T
6	F	T	F	T	T	T
7	F	F	T	T	F	T
8	F	F	F	T	T	T

EXAMPLE 3 Consider the following argument:

S_1 : If a man is a bachelor, he is unhappy.

S_2 : If a man is unhappy, he dies young.

S : Bachelors die young

Here the statement S below the line denotes the conclusion of the argument, and the statements S_1 and S_2 above the line denote the premises. We claim that the argument $S_1, S_2 \dashv S$ is valid. For the argument is of the form

$p \rightarrow q, q \rightarrow r \dashv p \rightarrow r$

where p is "He is a bachelor," q is "He is unhappy" and r is "He dies young;" this argument (Law of Syllogism) is valid.

$\begin{array}{rl} & p \rightarrow q \\ & q \rightarrow r \\ \hline \therefore & p \rightarrow r \end{array}$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism

EXAMPLE 4

Determine the validity of the following argument:
If 7 is less than 4, then 7 is not a prime number.
7 is not less than 4.

∴ 7 is a prime number.
First translate the argument into symbolic form. Let *p* be "7 is less than 4" and *q* be "7 is a prime number." Then the argument is of the form
$p \rightarrow \neg q, \neg q \dashv q$
Now, we construct a truth table as shown below. The above argument is shown to be a fallacy since, in the fourth line of the truth table, the premises $p \rightarrow \neg q$ and $\neg p$ are true, but the conclusion q is false.

p	q	$\neg q$	$p \rightarrow \neg q$	$\neg p$
T	T	F	F	F
T	F	T	T	F
F	T	F	T	T
F	F	T	T	T

EXAMPLE 5

Test the validity of the following argument:

If two sides of a triangle are equal, then the opposite angles are equal.

Two sides of a triangle are not equal.

$\therefore$ The opposite angles are not equal.

First translate the argument into the symbolic form $p \to q, \neg p \dashv \neg q$, where p is "Two sides of a triangle are equal" and q is "The opposite angles are equal." By previous exampe-1, this argument is a fallacy.

EXAMPLE 7 State which rule of inference is the basis of the following argument: "It is below freezing now. Therefore, it is either below freezing or raining now."

Solution: Let p be the proposition "It is below freezing now" and q the proposition "It is raining now." Then this argument is of the form

$$\frac{p}{\therefore p \vee q}$$

This is an argument that uses the addition rule.

Rule		
$\dfrac{p}{\therefore\ p \vee q}$	$p \rightarrow (p \vee q)$	Addition

Example-6

Determine whether the argument given here is valid and determine whether its conclusion must be true because of the validity of the argument.

"If $\sqrt{2} > \frac{3}{2}$, then $(\sqrt{2})^2 > (\frac{3}{2})^2$. We know that $\sqrt{2} > \frac{3}{2}$. Consequently, $(\sqrt{2})^2 = 2 > (\frac{3}{2})^2 = \frac{9}{4}$."

Solution: Let p be the proposition "$\sqrt{2} > \frac{3}{2}$" and q the proposition "$2 > (\frac{3}{2})^2$." The premises of the argument are $p \to q$ and p, and q is its conclusion. This argument is valid because it is constructed by using modus ponens, a valid argument form. However, one of its premises, $\sqrt{2} > \frac{3}{2}$, is false. Consequently, we cannot conclude that the conclusion is true. Furthermore, note that the conclusion of this argument is false, because $2 < \frac{9}{4}$.

EXAMPLE 8 State which rule of inference is the basis of the following argument: "It is below freezing and raining now. Therefore, it is below freezing now."

Solution: Let p be the proposition "It is below freezing now," and let q be the proposition "It is raining now." This argument is of the form

$$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$

This argument uses the simplification rule.

Rule of Inference	*Tautology*	*Name*
$\begin{array}{l} \quad p \wedge q \\ \hline \therefore \quad p \end{array}$	$(p \wedge q) \rightarrow p$	Simplification

EXAMPLE 9 State which rule of inference is used in the argument:

If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Solution: Let p be the proposition "It is raining today," let q be the proposition "We will not have a barbecue today," and let r be the proposition "We will have a barbecue tomorrow." Then this argument is of the form

$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$$

Hence, this argument is a hypothetical syllogism.

EXAMPLE 10 Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Solution: Let p be the proposition "It is sunny this afternoon," q the proposition "It is colder than yesterday," r the proposition "We will go swimming," s the proposition "We will take a canoe trip," and t the proposition "We will be home by sunset." Then the premises become $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$. The conclusion is simply t. We need to give a valid argument with premises $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$ and conclusion t. We construct an argument to show that our premises lead to the desired conclusion as follows.

Step	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Rule of Inference	*Tautology*	*Name*
$\begin{array}{l} \quad p \\ \quad p \rightarrow q \\ \hline \therefore q \end{array}$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens
$\begin{array}{l} \quad \neg q \\ \quad p \rightarrow q \\ \hline \therefore \neg p \end{array}$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus tollens
$\begin{array}{l} \quad p \wedge q \\ \hline \therefore p \end{array}$	$(p \wedge q) \rightarrow p$	Simplification

Note that we could have used a truth table to show that whenever each of the four hypotheses is true, the conclusion is also true. However, because we are working with five propositional variables, p, q, r, s, and t , such a truth table would have 32 rows.

EXAMPLE 11 Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

Solution: Let p be the proposition "You send me an e-mail message," q the proposition "I will finish writing the program," r the proposition "I will go to sleep early," and s the proposition "I Will wake up feeling refreshed." Then the premises are $p \to q, \neg p \to r$, and $r \to s$. The desired conclusion is $\neg q \to s$.We need to give a valid argument with premises $p \to q$, $\neg p \to r$, and $r \to s$ and conclusion $\neg q \to s$. This argument form shows that the premises lead to the desired conclusion.

Step	**Reason**
1. $p \rightarrow q$	Premise
2. $\neg q \rightarrow \neg p$	Contrapositive of (1)
3. $\neg p \rightarrow r$	Premise
4. $\neg q \rightarrow r$	Hypothetical syllogism using (2) and (3)
5. $r \rightarrow s$	Premise
6. $\neg q \rightarrow s$	Hypothetical syllogism using (4) and (5)

EXAMPLE 12 Use resolution to show that the hypotheses "Jasmine is skiing or it is not snowing" and "It is snowing or Bart is playing hockey" imply that "Jasmine is skiing or Bart is playing hockey."

Solution: Let p be the proposition "It is snowing," q the proposition "Jasmine is skiing," and r the proposition "Bart is playing hockey."We can represent the hypotheses as $\neg p \lor q$ and $p \lor r$, respectively. Using resolution, the proposition $q \lor r$, "Jasmine is skiing or Bart is playing hockey," follows.

$p \lor q$ $\neg p \lor r$ $\therefore\ q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Rules of Inference for Quantified Statements

We have discussed rules of inference for propositions. We will nowdescribe some important rules of inference for statements involving quantifiers. These rules of inference are used extensively in mathematical arguments, often without being explicitly mentioned.

TABLE 2 **Rules of Inference for Quantified Statements.**

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existential generalization

EXAMPLE 13 Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Solution: Let $D(x)$ denote "x is in this discrete mathematics class," and let $C(x)$ denote "x has taken a course in computer science." Then the premises are $\forall x(D(x) \rightarrow C(x))$ and D(Marla). The conclusion is C(Marla). The following steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\forall x(D(x) \rightarrow C(x))$	Premise
2. $D(\text{Marla}) \rightarrow C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3)

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation

EXAMPLE 14: Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Solution: Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam." The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$. These steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

Methods of Proving Theorems

Direct Proofs

A direct proof shows that a conditional statement $p \rightarrow q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs.

EXAMPLE 1 Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

By the definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer. We want to show that n^2 is also odd. We can square both sides of the equation $n = 2k + 1$ to obtain a new equation that expresses n^2. When we do this, we find that $n2 = (2k + 1)^2 = 4k2 + 4k + 1 = 2(2k^2 + 2k) + 1$. By the definition of an odd integer, we can conclude that n^2 is an odd integer (it is one more than twice an integer). Consequently, we have proved that if n is an odd integer, then n^2 is an odd integer.

Proof by contraposition

Proofs by contraposition make use of the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$. This means that the conditional statement $p \rightarrow q$ can be proved by showing that its contrapositive, $\neg q \rightarrow \neg p$, is true.

EXAMPLE 2 Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Solution: The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $3n + 2$ is odd, then n is odd" is false; namely, assume that n is even. Then, by the definition of an even integer, $n = 2k$ for some integer k. Substituting $2k$ for n, we find that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. This tells us that $3n + 2$ is even (because it is a multiple of 2), and therefore not odd. This is the negation of the premise of the theorem. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded; we have proved the theorem "If $3n + 2$ is odd, then n is odd."

Proofs by Contradiction

Suppose we want to prove that a statement p is true. we can conclude that $\neg p$ is false, which means that p is true.

Example-3

Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution: To start a proof by contradiction, we suppose that $\neg p$ is true. Note that $\neg p$ is the statement "It is not the case that $\sqrt{2}$ is irrational," which says that $\sqrt{2}$ is rational. If $\sqrt{2}$ is rational, there exist integers a and b with $\sqrt{2} = a/b$, where $b \neq 0$ and a and b have no common factors (so that the fraction a/b is in lowest terms.) (Here, we are using the fact that every rational number can be written in lowest terms.)

$$2 = \frac{a^2}{b^2}.$$

Hence,

$$2b^2 = a^2.$$

By the definition of an even integer it follows that a^2 is even. We next use the fact that if a^2 is even, a must also be even, which follows by Exercise 16. Furthermore, because a is even, by the definition of an even integer, $a = 2c$ for some integer c. Thus,

$$2b^2 = 4c^2.$$

Dividing both sides of this equation by 2 gives

$$b^2 = 2c^2.$$

By the definition of even, this means that b^2 is even. Again using the fact that if the square of an integer is even, then the integer itself must be even, we conclude that b must be even as well.

Note that the statement that $\sqrt{2} = a/b$, where a and b have no common factors, means, in particular, that 2 does not divide both a and b. Because our assumption of $\neg p$ leads to the contradiction that 2 divides both a and b and 2 does not divide both a and b, $\neg p$ must be false. That is, the statement p, "$\sqrt{2}$ is irrational," is true. We have proved that $\sqrt{2}$ is irrational.

EXHAUSTIVEPROOF Some theorems can be proved by examining a relatively small number of examples. Such proofs are called **exhaustive proofs**, or **proofs by exhaustion** because these proofs proceed by exhausting all possibilities. An exhaustive proof is a special type of proof by cases where each case involves checking a single example. We now provide some illustrations of exhaustive proofs.

EXAMPLE 1 Prove that $(n + 1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.

Solution: We use a proof by exhaustion. We only need verify the inequality $(n + 1)^3 \geq 3^n$ when $n = 1, 2, 3$, and 4. For $n = 1$, we have $(n + 1)^3 = 2^3 = 8$ and $3^n = 3^1 = 3$; for $n = 2$, we have $(n + 1)^3 = 3^3 = 27$ and $3^n = 3^2 = 9$; for $n = 3$, we have $(n + 1)^3 = 4^3 = 64$ and $3^n = 3^3 = 27$; and for $n = 4$, we have $(n + 1)^3 = 5^3 = 125$ and $3^n = 3^4 = 81$. In each of these four cases, we see that $(n + 1)^3 \geq 3^n$. We have used the method of exhaustion to prove that $(n + 1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.