Propositions

Mala Rani Barman
Lecturer
Dept. of CSE
Sheikh Hasina University

Propositions

A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

EXAMPLE 1 All the following declarative sentences are propositions.

1. Washington, D.C., is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. $1 + 1 = 2$.
4. $2 + 2 = 3$.

Propositions 1 and 3 are true, whereas 2 and 4 are false.

Some sentences that are not propositions are given in Example 2.

EXAMPLE 2 Consider the following sentences.
1. What time is it?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables.

We use letters to denote **propositional variables.** The **truth value** of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition. The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic**.

DEFINITION 1 Let p be a proposition. The *negation of p*, denoted by $\neg p$ (also denoted by $\overline{p}$), is the statement
"It is not the case that p."
The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

EXAMPLE 3 Find the negation of the proposition
"Michael's PC runs Linux"
and express this in simple English.

Solution: The negation is
"It is not the case that Michael's PC runs Linux."
This negation can be more simply expressed as
"Michael's PC does not run Linux."

Table 1 displays the **truth table** for the negation of a proposition p.

TABLE 1 The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

DEFINITION 2 Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

EXAMPLE 4 Find the conjunction of the propositions p and q where p is the proposition "Rebecca's PC has more than 16 GB free hard disk space" and q is the proposition "The processor in Rebecca's PC runs faster than 1 GHz."

Solution*:* The conjunction of these propositions, $p \wedge q$, is the proposition "Rebecca's PC has more than 16 GB free hard disk space, and the processor in Rebecca's PC runs faster than 1 GHz." This conjunction can be expressed more simply as "Rebecca's PC has more than 16 GB free hard disk space, and its processor runs faster than 1 GHz."

DEFINITION 3 Let p and q be propositions. The *disjunction* of p and q, denoted by $p \lor q$, is the proposition "p or q." The disjunction $p \lor q$ is false when both p and q are false and is true otherwise.

EXAMPLE 5 What is the disjunction of the propositions p and q where p and q are the same propositions as in Example 5?

Solution: The disjunction of p and q, $p \lor q$, is the proposition "Rebecca's PC has at least 16 GB free hard disk space, or the processor in Rebecca's PC runs faster than 1 GHz."

TABLE 2 **The Truth Table for the Conjunction of Two Propositions.**

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

TABLE 3 **The Truth Table for the Disjunction of Two Propositions.**

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

DEFINITION 4 Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

DEFINITION 5 Let p and q be propositions. The *conditional statement* $p \rightarrow q$ is the proposition "if p, then q." The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

TABLE 5 The Truth Table for the Conditional Statement $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

DEFINITION 6 Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

EXAMPLE 6 Construct the truth table of the compound proposition $(p \lor \neg q) \rightarrow (p \land q)$.

TABLE 7 The Truth Table of $(p \lor \neg q) \rightarrow (p \land q)$.

p	q	$\neg q$	$p \lor \neg q$	$p \land q$	$(p \lor \neg q) \rightarrow (p \land q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Translating English Sentences

There are many reasons to translate English sentences into expressions involving propositional variables and logical connectives. In particular, English (and every other human language) is often ambiguous. Translating sentences into compound statements (and other types of logical expressions, which we will introduce later in this chapter) removes the ambiguity.

EXAMPLE 7 How can this English sentence be translated into a logical expression?
"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Solution: Let a, c, and f represent "You can access the Internet from campus," "You are a computer science major," and "You are a freshman," respectively. Noting that "only if" is one way a conditional statement can be represented as, this sentence can be represented as
$a \rightarrow (c \vee \neg f)$.

EXAMPLE 8 How can this English sentence be translated into a logical expression?
"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Solution: Let q, r, and s represent "You can ride the roller coaster," "You are under 4 feet tall," and "You are older than 16 years old," respectively. Then the sentence can be translated to
$(r \wedge \neg s) \rightarrow \neg q$.
Of course, there are other ways to represent the original sentence as a logical expression, but the one we have used should meet our needs.

Propositional Equivalences

DEFINITION 1 A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

EXAMPLE 9 We can construct examples of tautologies and contradictions using just one propositional variable. Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$, shown in Table 1. Because $p \vee \neg p$ is always true, it is a tautology. Because $p \wedge \neg p$ is always false, it is a contradiction.

Logical Equivalences

Compound propositions that have the same truth values in all possible cases are called **logically equivalent**. We can also define this notion as follows.

DEFINITION 2 The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

EXAMPLE 10 Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution: The truth tables for these compound propositions are displayed in Table 3. Because the truth values of the compound propositions $\neg(p \vee q)$ and $\neg p \wedge \neg q$ agree for all possible combinations of the truth values of p and q, it follows that $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology and that these compound propositions are logically equivalent.

TABLE 3 **Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.**

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Show that $p \rightarrow q$ and $\neg p \lor q$ are logically equivalent.

Solution: We construct the truth table for these compound propositions in Table 4. Because the truth values of $\neg p \lor q$ and $p \rightarrow q$ agree, they are logically equivalent.

TABLE 4 **Truth Tables for $\neg p \lor q$ and $p \rightarrow q$.**

p	q	$\neg p$	$\neg p \lor q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

EXAMPLE 11 Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the *distributive law* of disjunction over conjunction.

TABLE 5 A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Table 6 contains some important equivalences. In these equivalences, **T** denotes the compound proposition that is always true and **F** denotes the compound proposition that is always false.

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$$p \to q \equiv \neg p \vee q$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \vee q \equiv \neg p \to q$$

$$p \wedge q \equiv \neg(p \to \neg q)$$

$$\neg(p \to q) \equiv p \wedge \neg q$$

$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$

$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

De Morgan's laws:

$\neg(p_1 \vee p_2 \vee \cdot\cdot\cdot \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \cdot\cdot\cdot \wedge \neg p_n)$

And

$\neg(p1 \wedge p2 \wedge \cdot\cdot\cdot \wedge pn) \equiv (\neg p1 \vee \neg p2 \vee \cdot\cdot\cdot \vee \neg pn)$.

EXAMPLE 12 Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Ans.

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ by De Morgan law

$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$ by De Morgan law

$\equiv \neg p \wedge (p \vee \neg q)$ by the double negation law

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ by the distributive law

$\equiv \mathbf{F} \vee (\neg p \wedge \neg q)$ because $\neg p \wedge p \equiv \mathbf{F}$

$\equiv (\neg p \wedge \neg q) \vee \mathrm{F}$ by the commutative law for disjunction

$\equiv \neg p \wedge \neg q$ by the identity law for **F**

EXAMPLE 13 Show that $(p \land q) \rightarrow (p \lor q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to **T**. (*Note:* This could also be done using a truth table.)

$(p \land q) \rightarrow (p \lor q) \equiv \neg(p \land q) \lor (p \lor q)$ by Example 3

$\equiv (\neg p \lor \neg q) \lor (p \lor q)$ by De Morgan law

$\equiv (\neg p \lor p) \lor (\neg q \lor q)$

by the associative and commutative laws for disjunction

$\equiv \mathbf{T} \lor \mathbf{T}$

$\equiv \mathbf{T}$

Example-14

Let p be "It is cold" and let q be "It is raining". Give a simple verbal sentence which describes each of the following statements: (a) $\neg p$; (b) $p \wedge q$; (c) $p \vee q$; (d) $q \vee \neg p$. In each case, translate $\wedge$, $\vee$, and ~ to read "and," "or," and "It is false that" or "not," respectively, and then simplify the English sentence.

(a) It is not cold. (c) It is cold or it is raining.

(b) It is cold and raining. (d) It is raining or it is not cold.

Example-15

Use the laws show that $\neg(p \wedge q) \vee (\neg p \wedge q) \equiv \neg p$.

Statement	**Reason**
(1) $\neg(p \vee q) \vee (\neg p \wedge q) \equiv (\neg p \wedge \neg q) \vee (\neg p \wedge q)$	DeMorgan's law
(2) $\equiv \neg p \wedge (\neg q \vee q)$	Distributive law
(3) $\equiv \neg p \wedge T$	Complement law
(4) $\equiv \neg p$	Identity law

Example-16

Show that the propositions $\neg(p \land q)$ and $\neg p \lor \neg q$ are logically equivalent.

p	q	$p \land q$	$\neg(p \land q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(a) $\neg(p \land q)$

p	q	$\neg p$	$\neg q$	$\neg p \lor \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(b) $\neg p \lor \neg q$