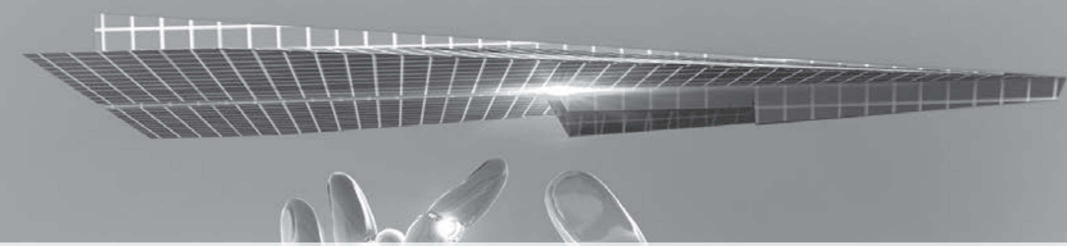


Chapter seventeen Basic system models

Objectives

The objectives of this chapter are that, after studying it, the reader should be able to:

- **Explain the importance of models in predicting the behaviour of systems.**
- **Devise models from basic building blocks for mechanical, electrical, fluid and thermal systems.**
- **Recognise analogies between mechanical, electrical, fluid and thermal systems.**

17.1 Mathematical models

Consider the following situation. A microprocessor switches on a motor. How will the rotation of the motor shaft vary with time? The speed will not immediately assume the full-speed value but will only attain that speed after some time. Consider another situation. A hydraulic system is used to open a valve which allows water into a tank to restore the water level to that required. How will the water level vary with time? The water level will not immediately assume the required level but will only attain that level after some time.

In order to understand the behaviour of systems, **mathematical models** are needed. These are simplified representations of certain aspects of a real system. Such a model is created using equations to describe the relationship between the input and output of a system and can then be used to enable predictions to be made of the behaviour of a system under specific conditions, e.g. the outputs for a given set of inputs, or the outputs if a particular parameter is changed. In devising a mathematical model of a system it is necessary to make assumptions and simplifications and a balance has to be chosen between simplicity of the model and the need for it to represent the actual real-world behaviour. For example, we might form a mathematical model for a spring by assuming that the extension x is proportional to the applied force F, i.e. $F = kx$. This simplified model might not accurately predict the behaviour of a real spring where the extension might not be precisely proportional to the force and where we cannot apply this model regardless of the size of the force, since large forces will permanently deform the spring and might even break it and this is not predicted by the simple model.

The basis for any mathematical model is provided by the fundamental physical laws that govern the behaviour of the system. In this chapter a range of systems will be considered, including mechanical, electrical, thermal and fluid examples.

Like a child building houses, cars, cranes, etc., from a number of basic building blocks, systems can be made up from a range of building blocks. Each building block is considered to have a single property or function. Thus, to take a simple example, an electrical circuit system may be made up from building blocks which represent the behaviour of resistors, capacitors and inductors. The resistor building block is assumed to have purely the property of resistance, the capacitor purely that of capacitance and the inductor purely that of inductance. By combining these building blocks in different ways, a variety of electrical circuit systems can be built up and the overall input/output relationships obtained for the system by combining in an appropriate way the relationships for the building blocks. Thus a mathematical model for the system can be obtained. A system built up in this way is called a **lumped parameter** system. This is because each parameter, i.e. property or function, is considered independently.

There are similarities in the behaviour of building blocks used in mechanical, electrical, thermal and fluid systems. This chapter is about the basic building blocks and their combination to produce mathematical models for physical, real, systems. Chapter 18 looks at more complex models. It needs to be emphasised that such models are only aids in system design. Real systems often exhibit non-linear characteristics and can depart from the ideal models developed in these chapters. This matter is touched on in Chapter 18.

17.2 Mechanical system building blocks

The models used to represent mechanical systems have the basic building blocks of springs, dashpots and masses. **Springs** represent the stiffness of a system, **dashpots** the forces opposing motion, i.e. frictional or damping effects, and **masses** the inertia or resistance to acceleration (Figure 17.1). The mechanical system does not have to be really made up of springs, dashpots and masses but have the properties of stiffness, damping and inertia. All these building blocks can be considered to have a force as an input and a displacement as an output.

The stiffness of a spring is described by the relationship between the forces F used to extend or compress a spring and the resulting extension or compression x (Figure 17.1(a)). In the case of a spring where the extension or compression is proportional to the applied forces, i.e. a linear spring,

$$F = kx$$

where k is a constant. The bigger the value of k, the greater the forces have to be to stretch or compress the spring and so the greater the stiffness. The

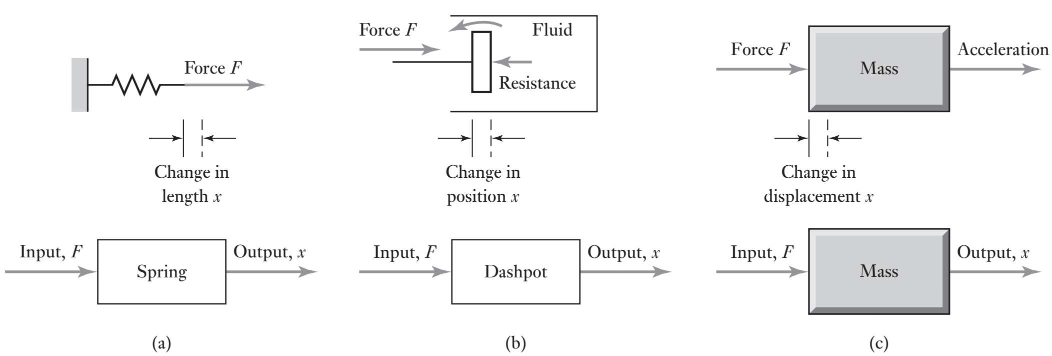


Figure 17.1 Mechanical systems: (a) spring, (b) dashpot, (c) mass.

object applying the force to stretch the spring is also acted on by a force, the force being that exerted by the stretched spring (Newton's third law). This force will be in the opposite direction and equal in size to the force used to stretch the spring, i.e. kx.

The dashpot building block represents the types of forces experienced when we endeavour to push an object through a fluid or move an object against frictional forces. The faster the object is pushed, the greater the opposing forces become. The dashpot which is used pictorially to represent these damping forces which slow down moving objects consists of a piston moving in a closed cylinder (Figure 17.1(b)). Movement of the piston requires the fluid on one side of the piston to flow through or past the piston. This flow produces a resistive force. In the ideal case, the damping or resistive force F is proportional to the velocity v of the piston. Thus

$$F = cv$$

wherer c is a constant. The larger the value of c, the greater the damping force at a particular velocity. Since velocity is the rate of change of displacement x of the piston, i.e. $v = \mathrm{d}x/\mathrm{d}t$, then

$$F = c\frac{\mathrm{d}x}{\mathrm{d}t}$$

Thus the relationship between the displacement x of the piston, i.e. the output, and the force as the input is a relationship depending on the rate of change of the output.

The mass building block (Figure 17.1(c)) exhibits the property that the bigger the mass, the greater the force required to give it a specific acceleration. The relationship between the force F and the acceleration a is (Newton's second law) $F = ma$, where the constant of proportionality between the force and the acceleration is the constant called the mass m. Acceleration is the rate of change of velocity, i.e. $\mathrm{d}v/\mathrm{d}t$, and velocity v is the rate of change of displacement x, i.e. $v = \mathrm{d}x/\mathrm{d}t$. Thus

$$F = ma = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}(\mathrm{d}x/\mathrm{d}t)}{\mathrm{d}t} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

Energy is needed to stretch the spring, accelerate the mass and move the piston in the dashpot. However, in the case of the spring and the mass we can get the energy back, but with the dashpot we cannot. The spring when stretched stores energy, the energy being released when the spring springs back to its original length. The energy stored when there is an extension x is $\frac{1}{2}kx^2$. Since $F = kx$, this can be written as

$$E = \frac{1}{2}\frac{F^2}{k}$$

There is also energy stored in the mass when it is moving with a velocity v, the energy being referred to as kinetic energy, and released when it stops moving:

$$E = \frac{1}{2}mv^2$$

However, there is no energy stored in the dashpot. It does not return to its original position when there is no force input. The dashpot dissipates energy rather than storing it, the power P dissipated depending on the velocity v and being given by

$$P = cv^2$$

17.2.1 Rotational systems

The spring, dashpot and mass are the basic building blocks for mechanical systems where forces and straight line displacements are involved without any rotation. If there is rotation then the equivalent three building blocks are a **torsional spring,** a **rotary damper** and the **moment of inertia,** i.e. the inertia of a rotating mass. With such building blocks the inputs are torque and the outputs angle rotated. With a torsional spring the angle θ rotated is proportional to the torque T. Hence

$$T = k\theta$$

With the rotary damper a disc is rotated in a fluid and the resistive torque T is proportional to the angular velocity ω, and since angular velocity is the rate at which angle changes, i.e. $\mathrm{d}\theta/\mathrm{d}t$,

$$T = c\omega = c\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

The moment of inertia building block has the property that the greater the moment of inertia I, the greater the torque needed to produce an angular acceleration α:

$$T = I\alpha$$

Thus, since angular acceleration is the rate of change of angular velocity, i.e. $\mathrm{d}\omega/\mathrm{d}t$, and angular velocity is the rate of change of angular displacement, then

$$T = I\frac{\mathrm{d}\omega}{\mathrm{d}t} = I\frac{\mathrm{d}(\mathrm{d}\theta/\mathrm{d}t)}{\mathrm{d}t} = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

The torsional spring and the rotating mass store energy; the rotary damper just dissipates energy. The energy stored by a torsional spring when twisted through an angle θ is $\frac{1}{2}k\theta^2$ and since $T = k\theta$ this can be written as

$$E = \frac{1}{2}\frac{T^2}{k}$$

The energy stored by a mass rotating with an angular velocity ω is the kinetic energy E, where

$$E = \frac{1}{2}I\omega^2$$

The power P dissipated by the rotatory damper when rotating with an angular velocity ω is

$$P = c\omega^2$$

Table 17.1 summarises the equations defining the characteristics of the mechanical building blocks when there is, in the case of straight line displacements (termed translational), a force input F and a displacement x output and, in the case of rotation, a torque T and angular displacement θ.

Table 17.1 Mechanical building blocks.

Building block	Describing equation	Energy stored or power dissipated
Translational		
Spring	$F = kx$	$E = \frac{1}{2}\frac{F^2}{k}$
Dashpot	$F = c\frac{dx}{dt} = cv$	$P = cv^2$
Mass	$F = m\frac{d^2x}{dt^2} = m\frac{dv}{dt}$	$E = \frac{1}{2}mv^2$
Rotational		
Spring	$T = k\theta$	$E = \frac{1}{2}\frac{T^2}{k}$
Rotational damper	$T = c\frac{d\theta}{dt} = c\omega$	$P = c\omega^2$
Moment of inertia	$T = I\frac{d^2\theta}{dt^2} = I\frac{d\omega}{dt}$	$E = \frac{1}{2}I\omega^2$

17.2.2 Building up a mechanical system

Many systems can be considered to be essentially a mass, a spring and dashpot combined in the way shown in Figure 17.2(a) and having an input of a force F and an output of displacement x (Figure 17.2(b)). To evaluate the relationship between the force and displacement for the system, the procedure to be adopted is to consider just one mass, and just the forces acting on that body. A diagram of the mass and just the forces acting on it is called a **free-body diagram** (Figure 17.2(c)).

When several forces act concurrently on a body, their single equivalent resultant can be found by vector addition. If the forces are all acting along the same line or parallel lines, this means that the resultant or net force acting on the block is the algebraic sum. Thus for the mass in Figure 17.2(c), if we consider just the forces acting on that block then the net force applied to the

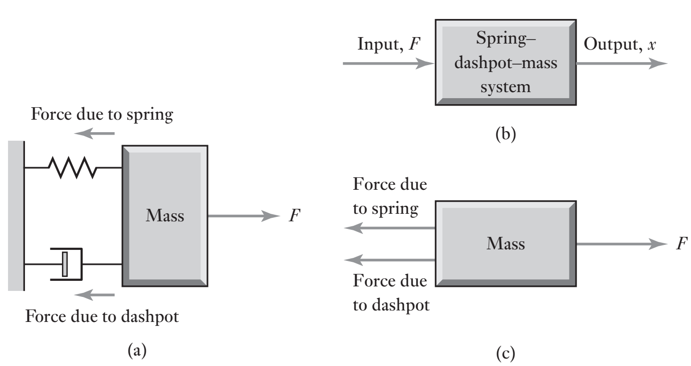


Figure 17.2 (a) Spring–dashpot–mass, (b) system, (c) free-body diagram.

mass is the applied force F minus the force resulting from the stretching or compressing of the spring and minus the force from the damper. Thus

$$\text{net force applied to mass } m = F - kx - cv$$

where v is the velocity with which the piston in the dashpot, and hence the mass, is moving. This net force is the force applied to the mass to cause it to accelerate. Thus

$$\text{net force applied to mass} = ma$$

Hence

$$F - kx - c\frac{\mathrm{d}x}{\mathrm{d}t} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

or, when rearranged,

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F$$

This equation, called a **differential equation,** describes the relationship between the input of force F to the system and the output of displacement x. Because of the $\mathrm{d}^2x/\mathrm{d}t^2$ term, it is a second-order differential equation; a first-order differential equation would only have $\mathrm{d}x/\mathrm{d}t$.

There are many systems which can be built up from suitable combinations of the spring, dashpot and mass building blocks. Figure 17.3 illustrates some.

Figure 17.3(a) shows the model for a machine mounted on the ground and could be used as a basis for studying the effects of ground disturbances on the displacements of a machine bed. Figure 17.3(b) shows a model for the wheel and its suspension for a car or truck and can be used for the study

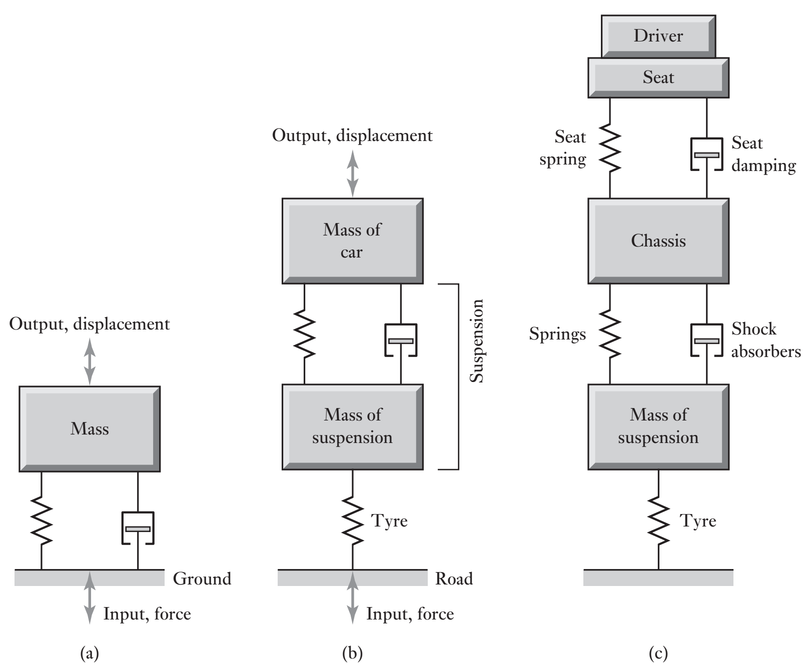


Figure 17.3 Model for (a) a machine mounted on the ground, (b) the chassis of a car as a result of a wheel moving along a road, (c) the driver of a car as it is driven along a road.

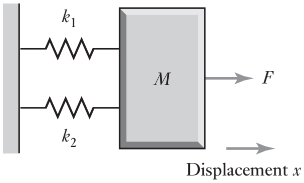


Figure 17.4 Example.

of the behaviour that could be expected of the vehicle when driven over a rough road and hence as a basis for the design of the vehicle suspension. Figure 17.3(c) shows how this model can be used as part of a larger model to predict how the driver might feel when driving along a road. The procedure to be adopted for the analysis of such models is just the same as outlined above for the simple spring–dashpot–mass model. A free-body diagram is drawn for each mass in the system, such diagrams showing each mass independently and just the forces acting on it. Then for each mass the resultant of the forces acting on it is equated to the product of the mass and the acceleration of the mass.

To illustrate the above, consider the derivation of the differential equation describing the relationship between the input of the force F and the output of displacement x for the system shown in Figure 17.4.

The net force applied to the mass is F minus the resisting forces exerted by each of the springs. Since these are k_1x and k_2x, then

$$\text{net force} = F - k_1x - k_2x$$

Since the net force causes the mass to accelerate, then

$$\text{net force} = m\frac{d^2x}{dt^2}$$

Hence

$$m\frac{d^2x}{dt^2} + (k_1 + k_2)x = F$$

The procedure for obtaining the differential equation relating the inputs and outputs for a mechanical system consisting of a number of components can be summarised as follows:

1 Isolate the various components in the system and draw free-body diagrams for each.
2 Hence, with the forces identified for a component, write the modelling equation for it.
3 Combine the equations for the various system components to obtain the system differential equation.

As an illustration, consider the derivation of the differential equation describing the motion of the mass m_1 in Figure 17.5(a) when a force F is applied. Consider the free-body diagrams (Figure 17.5(b)). For mass m_2 these are the force F and the force exerted by the upper spring. The force exerted by the upper spring is due to its being stretched by $(x_2 - x_1)$ and so is $k(x_3 - x_2)$. Thus the net force acting on the mass is

$$\text{net force} = F - k_2(x_3 - x_2)$$

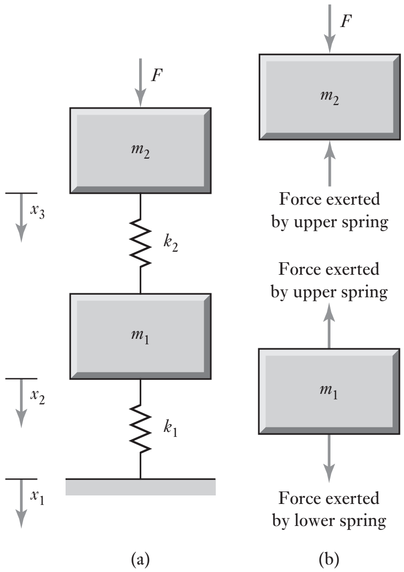
This force will cause the mass to accelerate and so

$$F - k_2(x_3 - x_2) = m_2\frac{d^2x_3}{dt}$$

For the free-body diagram for mass , the force exerted by the upper spring is $k_2(x_3 - x_2)$ and that by the lower spring is $k_1(x_1 - x_2)$. Thus the net force acting on the mass is

$$\text{net force} = k_1(x_2 - x_1) - k_2(x_3 - x_2)$$

Figure 17.5 Mass–spring system.



This force will cause the mass to accelerate and so

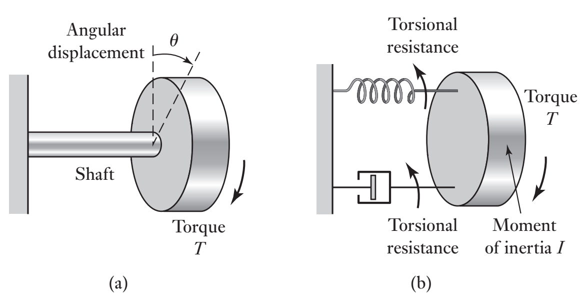
$$k_1(x_2 - x_1) - k_2(x_3 - x_2) = m_1 \frac{d^2 x_2}{dt}$$

We thus have two simultaneous second-order differential equations to describe the behaviours of the system.

Similar models can be constructed for rotating systems. To evaluate the relationship between the torque and angular displacement for the system, the procedure to be adopted is to consider just one rotational mass block, and just the torques acting on that body. When several torques act on a body simultaneously, their single equivalent resultant can be found by addition in which the direction of the torques is taken into account. Thus a system involving a torque being used to rotate a mass on the end of a shaft (Figure 17.6(a)) can be considered to be represented by the rotational building blocks shown in Figure 17.6(b). This is a comparable situation with that analysed above (Figure 17.2) for linear displacements and yields a similar equation

$$I \frac{d^2\theta}{dt^2} + c \frac{d\theta}{dt} + k\theta = T$$

Figure 17.6 Rotating a mass on the end of a shaft: (a) physical situation, (b) building block model.



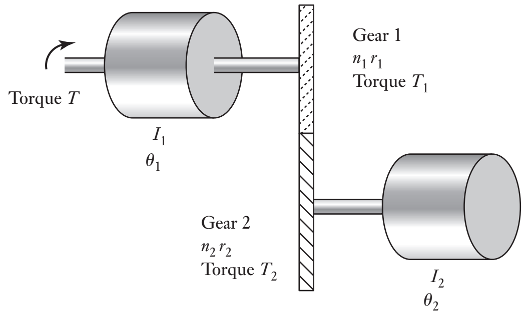


Figure 17.7 A two-gear train system.

Motors operating through gear trains to rotate loads are a feature of many control systems. Figure 17.7 shows a simple model of such a system. It consists of a mass of moment of inertia I_1 connected to gear 1 having n_1 teeth and a radius r_1 and a mass of moment of inertia I_2 connected to a gear 2 with n_2 teeth and a radius r_2. We will assume that the gears have negligible moments of inertia and also that rotational damping can be ignored.

If gear 1 is rotated through an angle θ_1 then gear 2 will rotate through an angle θ_2 where

$$r_1\theta_1 = r_2\theta_2$$

The ratio of the gear teeth numbers is equal to the ratio n of the gear radii:

$$\frac{r_1}{r_2} = \frac{n_1}{n_2} = n$$

If a torque T is applied to the system and torque T_1 is applied to gear 1 then the net torque is $T - T_1$ and so

$$T - T_1 = I_1\frac{d^2\theta_1}{dt^2}$$

If the torque T_2 occurs at gear 2 then

$$T_2 = I_2\frac{d^2\theta_2}{dt^2}$$

We will assume that the power transmitted by gear 1 is equal to that transmitted by gear 2 and so, as the power transmitted is the product of the torque and angular velocity, we have

$$T_1\frac{d\theta_1}{dt} = T_2\frac{d\theta_2}{dt}$$

Since $r_2\theta_1 = r_2\theta_2$ it follows that

$$r_1\frac{d\theta_1}{dt} = r_2\frac{d\theta_2}{dt^2}$$

and so

$$\frac{T_1}{T_2} = \frac{r_1}{r_2} = n$$

Thus we can write

$$T - T_1 = T - nT_2 = T - n\left(I_2\frac{\mathrm{d}^2\theta_2}{\mathrm{d}t^2}\right)$$

and so

$$T - n\left(I_2\frac{\mathrm{d}^2\theta_2}{\mathrm{d}t^2}\right) = I_1\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2}$$

Since $\theta_2 = n\theta_1$, $\mathrm{d}\theta_2/\mathrm{d}t = n\mathrm{d}\theta_1/\mathrm{d}t$ and $\mathrm{d}^2\theta_2/\mathrm{d}t^2 = n\mathrm{d}^2\theta_1/\mathrm{d}t$ then

$$T - n^2\left(I_2\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2}\right) = I_1\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2}$$

$$(I_1 + n^2I_2)\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} = T$$

Without the gear train we would have had simply

$$I_1\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} = T$$

Thus the moment of inertia of the load is reflected back to the other side of the gear train as an additional moment of inertia term n^2I_2.

17.3 Electrical system building blocks

The basic building blocks of electrical systems are inductors, capacitors and resistors (Figure 17.8).

For an **inductor** the potential difference v across it at any instant depends on the rate of change of current ($\mathrm{d}i/\mathrm{d}t$) through it:

$$v = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

where L is the inductance. The direction of the potential difference is in the opposite direction to the potential difference used to drive the current through the inductor, hence the term back e.m.f. The equation can be rearranged to give

$$i = \frac{1}{L}\int v\,\mathrm{d}t$$

For a **capacitor**, the potential difference across it depends on the charge q on the capacitor plates at the instant concerned:

$$v = \frac{q}{C}$$

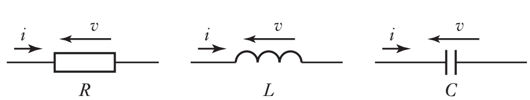


Figure 17.8 Electrical building blocks.