The Sandwich theorem:

For all values of x in the interval $0 < |x-a| < \delta$,
$f(x) \le g(x) \le h(x)$ and

$$\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x).$$

then $\displaystyle\lim_{x \to a} g(x) = l.$

It is also known as Squeezing or Pinching theorem.

Summary: If $f(x) \le g(x) \le h(x)$ and

$\displaystyle\lim_{x \to a} f(x) = l$ $\quad\quad$ $\displaystyle\lim_{x \to a} h(x) = l$ then

$$\lim_{x \to a} g(x) = l.$$

Ex: we know that

$$-1 \le \sin x \le 1$$

$$\Rightarrow -\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

$$\Rightarrow \lim_{x \to \infty} \frac{-1}{x} \le \lim_{x \to \infty} \frac{\sin x}{x} \le \lim_{x \to \infty} \frac{1}{x}$$

$$\Rightarrow 0 \le \lim_{x \to \infty} \frac{\sin x}{x} \le 0$$

$$\Rightarrow \lim_{x \to \infty} \frac{\sin x}{x} = 0. \quad \text{Ans.}$$

Ex: If $3x \le f(x) \le x^3+2$ is defined for $[0,2]$ then find $\lim_{x\to 1} f(x)$.

Sol^n: Given that

$$3x \le f(x) \le x^3+2$$

$$\Rightarrow \lim_{x\to 1} 3x \le \lim_{x\to 1} f(x) \le \lim_{x\to 1} (x^3+2)$$

$$\Rightarrow 3 \le \lim_{x\to 1} f(x) \le \lim_{x\to 1} 3$$

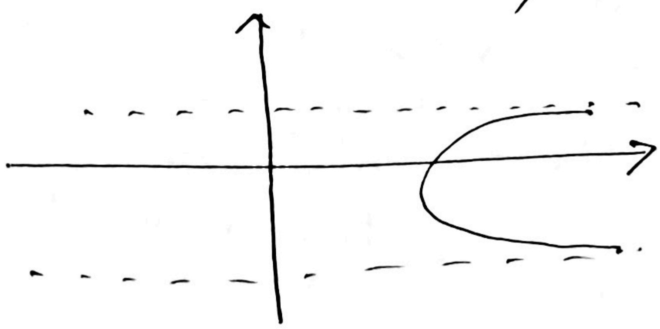
$$\Rightarrow \lim_{x\to 1} f(x) = 3.$$

Ex: If $4 \le f(x) \le x^3+6x-3$ then

$$\lim_{x\to 1} f(x) = ?$$

The horizontal Asymptotes:

A horizontal line that tells you how the function will behave at the very edges of a graph.



Horizontal Asymptotes Rules:

To find horizontal asymptotes compare the degree of the numerator "M" to the degree of the denominator "N"

- If $M < N$, then $y = 0$ is horizontal asymptote
- If $M > N$, then no horizontal asymptote.
- If $M = N$, then divide leading co-efficients.

Given $f(x) = \dfrac{ax^n + \cdots}{bx^m + \cdots}$

if $n = m$ then the line $y = \frac{a}{b}$ is the horizontal asymptote.

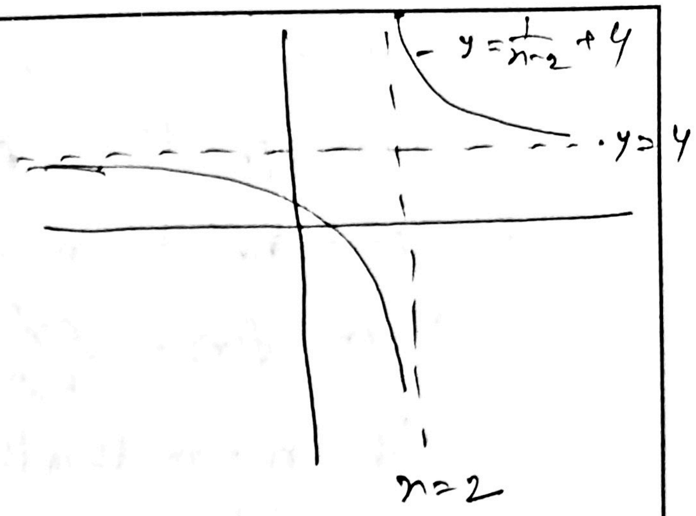
Ex:

$f(x) = \dfrac{8x+1}{4x^2 - 3x}$. Since degree of numerator is less than the degree of the denominator, A horizontal asymptote occurs at $y = 0$.

$f(x) = \dfrac{9x}{5x-2}$, Since degrees of the numerator and denominator are equal, divide the co-efficients of the highest degree terms.
A horizontal asymptote occurs at $y = \frac{9}{5}$.

$f(x) = \dfrac{x^2 + 3x - 6}{4x + 5}$. Since the degree of numerator is larger than the degree of the denominator.
No horizontal asymptote.

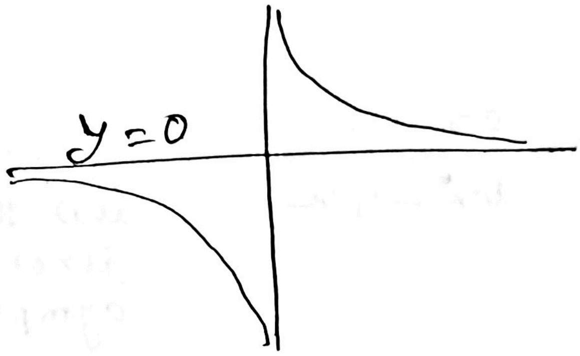
Ex: $y = \frac{1}{x-2} + 4$



Here $x = 2$, the vertical asymptote.

$y = 4$, the horizontal asymptote.

Ex: $y = \frac{4x^2}{x^3}$



Ex: $y = \frac{2x-4}{x-1}$

$\therefore y = \frac{2(x-1) - 2}{x-1}$

$= 2 - \frac{2}{x-1}$

when x is large, then the term $\frac{2}{x-1}$ approaches 0. So the horizontal asymptote is $y = 2$.