Differential Equations as Mathematical Models

It is often desirable to describe the behavior of some real-life system or phenomena, whether physical, sociological, or even economic, in mathematical terms. The mathematical description of a system of phenomena is called a *mathematical model* and is constructed with certain goals in mind. For example, we may wish to understand the mechanisms of a certain ecosystem by studying the growth of animal populations in that system, or we may wish to date fossils by analyzing the decay of a radioactive substance either in the fossil or in the stratum in which it was discovered.

We will apply our knowledge of differential equations in certain problems such as: (i) Population dynamics, (ii) Radioactive decay, (iii) Chemical reactions, (iv) Newton's law of cooling/warming, (v) Mixtures, (vi) Series circuits and (vii) Falling body problems.

Rate of Growth and Decay

The initial value problem

$$\frac{dx}{dt} = kx, \quad x(t_0) = x_0, \tag{1}$$

where k is a constant of proportionality, serves as a model for diverse phenomena involving either growth or decay. In biological applications the rate of growth of certain populations (bacteria, small animals) over short period of time is proportional to the population present at time t. In physics and chemistry (1) is seen in the form of a *first-order reaction*—that is, a reaction whose rate, or velocity, dx/dt is directly proportional to the amount x of a substance that is unconverted or remaining at time t.

Example: A certain radioactive substance has a half time of 38 hours. Find how long it takes for 90% of the radioactivity to be dissipated.

Solution: Let R be the amount of the substance at time t hours and R_0 be the amount of substance at time $t = 0$. Since the rate of dissipation of radioactive substance is proportional to the amount of substance at that time, we have

$$\frac{dR}{dt} = kR \tag{2}$$

where k is a proportionality constant.

Rewriting the equation (2), we get

$$\frac{dR}{R} = kdt$$

Integrating,

$\ln R = kt + \ln A,$ where $\ln A$ is an integrating constant.

$R = e^{kt+\ln A} = e^{kt}.e^{lnA} = Ae^{kt}$

$R = e^{kt+C} = e^{kt}.e^{C} = Ae^{kt}$ [Let $e^{C} = A$]

$$\therefore \quad R = Ae^{kt} \tag{3}$$

At time $t = 0$, $R = R_0$.

Thus from (3), we have $A = R_0$.

Thus

$$R = R_0 e^{kt} \tag{4}$$

When $t = 38$, $R = R_0/2$.

Therefore,

$$\frac{R_0}{2} = R_0 e^{38k}$$

$$\Rightarrow \frac{1}{2} = e^{38k}$$

$$\Rightarrow 38k = \ln\left(\frac{1}{2}\right)$$

$$\therefore \; k = \frac{1}{38}\ln\left(\frac{1}{2}\right)$$

Now the amount of the remaining substance is (100 − 90) = 10 percent of its original amount. Hence, when $R = 10\%$ of $R_0 = \dfrac{R_0}{10}$ then (4) gives

$$\frac{R_0}{10} = R_0 e^{kt}$$

$$\Rightarrow \frac{1}{10} = e^{kt}$$

$$\Rightarrow kt = \ln\left(\frac{1}{10}\right)$$

$$\Rightarrow t = \frac{1}{k}\ln\left(\frac{1}{10}\right)$$

$$\Rightarrow t = 38\times\frac{\ln\left(\frac{1}{10}\right)}{\ln\left(\frac{1}{2}\right)} = 126.233 \approx 126 \text{ hours.}$$

So it takes 126 hours for 90% of the radioactivity to be dissipated.

Exercise 1: The population of a community is known to increase at a rate proportional to the number of people present at time t. If the population has doubled in 50 years, how long will it take to triple? to quadruple?

Exercise 2: The rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. Half of the original number of radioactive nuclei have undergone disintegration in a period of 1500 years.

(i) What percentage of the original radioactive nuclei will remain after 4500 years?

(ii) In how many years will only one-tenth of the original number remain?

Answers: 1. 79 years, 100 years; 2. (i) One-eighth or 12.5% of the original number, (ii) 4985 years.

Newton's Law of Cooling/Warming

Newton's law of cooling states that the rate of change of the temperature of a cooling body is proportional to the difference between the temperature of the body and the constant temperature of the medium surrounding the body. The mathematical formulation is given by the linear first-order differential equation

$$\frac{dT}{dt} = k\left(T - T_m\right), \tag{5}$$

where k is a constant of proportionality, $T(t)$ is the temperature of the body for $t > 0$, and T_m is the ambient temperature—that is, the temperature of the medium surrounding the body.

Example: A body of temperature 80° F is placed in a room of constant temperature 50° F at time $t = 0$; and at the end of 5 minutes, the body has cooled to a temperature of 70° F. Determine the temperature of the body as a function of time for $t > 0$.

1. What is the temperature of the body at the end of 10 minutes?

2. When will the temperature of the body be 60° F?

3. After how many minutes will the temperature of the body be within 1° F of the constant 50° F temperature of the room?

Solution: Let *x* bet the Fahrenheit temperature of the body at time *t*. By Newton's law of cooling, we at once have the differential equation

$$\frac{dx}{dt} = k(x-50), \tag{6}$$

where *k* is the constant of proportionality.

The initial temperature of 80° F gives the initial condition

$$x(0) = 80 \tag{7}$$

and the 70° F temperature at the end of 5 minutes gives the additional condition

$$x(5) = 70. \tag{8}$$

Rewriting the equation (6), we have

$$\frac{dx}{x-50} = kdt$$

Integrating, we find

$\ln(x-50) = kt + \ln C$, where ln *C* is an integrating constant.

Thus we get

$$x = 50 + Ce^{kt}. \tag{9}$$

Equation (7) gives

$$80 = 50 + Ce^{0} \quad \Rightarrow C = 30.$$

Now applying (8), we get

$$70 = 50 + 30e^{5k}$$
$$\Rightarrow 30e^{5k} = 20$$
$$\Rightarrow e^{5k} = \frac{2}{3}$$
$$\Rightarrow e^{k} = \left(\frac{2}{3}\right)^{\frac{1}{5}}.$$

Thus equation (9) yields

$$x = 50 + 30\left(\frac{2}{3}\right)^{\frac{t}{5}}. \tag{10}$$

(1) At the end of t = 10 minutes,

$$x = 50 + 30\left(\frac{2}{3}\right)^{2} \approx 63.33° \text{ F.}$$

(2) When x = 60, (10) gives

$$60 = 50 + 30\left(\frac{2}{3}\right)^{\frac{t}{5}}$$

$$\Rightarrow 30\left(\frac{2}{3}\right)^{\frac{t}{5}} = 10$$

$$\Rightarrow \left(\frac{2}{3}\right)^{\frac{t}{5}} = \frac{1}{3}$$

$$\Rightarrow \frac{t}{5}\ln\left(\frac{2}{3}\right) = \ln\left(\frac{1}{3}\right)$$

$$\Rightarrow t = 5\frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{2}{3}\right)}$$

$$\therefore t \approx 13.55 \text{ (minutes)}.$$

(3) Now we have to find out after how many minutes the temperature of the body will be within 1° F of the constant 50° F temperature of the room. Thus we seek the time when the temperature is x = 51.

From (10), we have

$$51 = 50 + 30\left(\frac{2}{3}\right)^{\frac{t}{5}} \quad \Rightarrow t = 5\frac{\ln\left(\frac{1}{30}\right)}{\ln\left(\frac{2}{3}\right)} \approx 41.94 \text{ (minutes)}.$$

Exercise 1: When a cake is removed from an oven, its temperature is measure at 300° F. Three minutes later its temperature is 200° F. How long will it take for the cake to cool off to a room temperature of 70° F?

Exercise 2: Suppose that you turn off the heat in your home at night 2 hours before you go to bed, call this time $t = 0$. If the temperature T at time $t = 0$ is 66° F and at the time you go to bed ($t - 2$) has dropped to 60° F, what temperature can you expect in the morning, say 8 hours later ($t - 10$)? Of course, this process of cooling off will depend on the outside temperature T_m, which we assume to be constant at 32° F.

Exercise 3: A thermometer is removed from where the air temperature is 70° F to the outside, where the temperature is 10° F. After $\frac{1}{2}$ minute the thermometer reads 50° F. What is the reading at $t = 1$ minute? How long will it take for the thermometer to reach 15° F?

Answers: 1. 1 hr 30 minutes, approx.; 3. 36.67° F, $3.06'$ approx. (3 hours 6 minutes).

Mixture Problem

Letting x denote the amount of S present at time t, the derivative dx/dt denotes the rate of change of x with respect to t. If IN denotes the rate at which S enters the mixture and OUT the rate at which it leaves, we have at once the equation

$$\frac{dx}{dt} = IN - OUT, \tag{11}$$

from which to determine the amount of x of S at time t.

Example: A tank initially contains 50 gal of pure water. Starting at time $t = 0$ a brine containing 2 lb of dissolved salt per gallon flows into the tank at the rate of 3 gal/min. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate.

1. How much salt is in the tank at any time $t > 0$?

2. How much salt is present at the end of 25 min?

3. How much salt is present after a long time?

Solution: Let x denote the amount of salt in the tank at time t. We know

$$\frac{dx}{dt} = IN - OUT. \tag{12}$$

The brine flows in at the rate of 3 gal/min, and each gallon contains 2 lb of salt. Thus

IN = (2 lb/gal)(3 gal/min) = 6 lb/min.

Since the rate of outflow equals the rate of inflow, the tank contains 50 gal of the mixture at any time *t*. Thus 50 gal contains *x* lb of salt at any time *t*, and so the concentration of salt at time *t* is $\frac{x}{50}$ lb/gal. Thus, since the mixture flows out at the rate of 3 gal/min, we have

$$OUT = \left(\frac{x}{50}\text{ lb/gal}\right)(3\text{ gal/min}) = \frac{3x}{50}\text{ lb/min.}$$

Thus, using equation (12) we have the differential equation for *x* as a function of *t* as follows:

$$\frac{dx}{dt} = 6 - \frac{3x}{50}. \qquad (13)$$

Since initially there was no salt in the tank, we also have the initial condition

$$x(0) = 0. \qquad (14)$$

Equation (13) gives

$$\frac{dx}{dt} = \frac{300-3x}{50}$$
$$\Rightarrow \frac{dx}{dt} = \frac{3(100-x)}{50}$$
$$\therefore \frac{dx}{100-x} = \frac{3}{50}dt$$

Integrating, we get

$\ln(100-x) = \frac{3}{50}t + \ln C$, where $\ln C$ is an integrating constant.

$$\therefore \; x = 100 + Ce^{-\frac{3t}{50}}.$$

Using the condition (14), we find

$C = -100$.

Thus we obtain

$$x = 100\left(1 - e^{-\frac{3t}{50}}\right). \qquad (15)$$

At the end of 25 min, $t = 25$ and equation (15) gives

$$x(25) = 100\left(1 - e^{-\frac{3\times 25}{50}}\right) = 100(1 - 0.22313) = 77.687 \approx 78 \text{ (lb)}.$$

So there will be 78 lb salt in the tank at the end of 25 min.

When $t \to \infty$, we observe that $x \to 100$. Therefore after a long time there will be 100 lb salt in the tank.

Exercise 1: A large tank is partially filled with 100 gallons of fluid in which 10 pounds of salt is dissolved. Brine containing 1/2 pound of dissolved salt per gallon is pumped into the tank at the rate of 6 gal/min. The well-mixed solution is then pumped out at a slower rate of 4 gal/min. Find the number of pounds of salt in the tank after 30 minutes.

Answer: 1. 64.38 pounds.

Series Circuits: For a series circuit containing only a resistor and an inductor, Kirchhoff's second law states that the sum of the voltage drop across the inductor ($L(di/dt)$) and the voltage drop across the resistor (iR) is the same as the impressed voltage ($E(t)$) on the circuit (see Fig. 1).

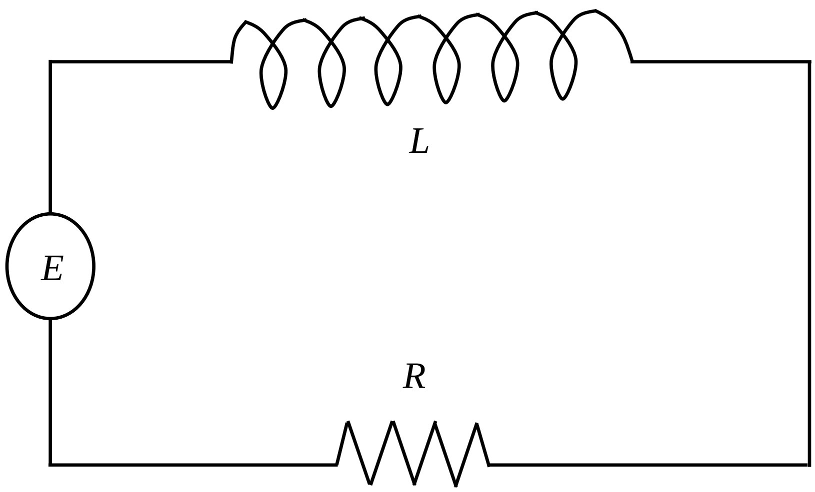


Fig. 1: *LR* series circuit

Thus we obtain the linear differential equation for the current $i(t)$,

$$L\frac{di}{dt} + Ri = E(t), \tag{16}$$

where L and R are constants known as the inductance and the resistance, respectively.

The voltage drop across a capacitor with capacitance C is given by $q(t)/C$, where q is the charge on the capacitor. Hence, for series circuit shown in Fig. 2, Kirchhoff's second law gives

$$Ri + \frac{1}{C}q = E(t). \tag{17}$$

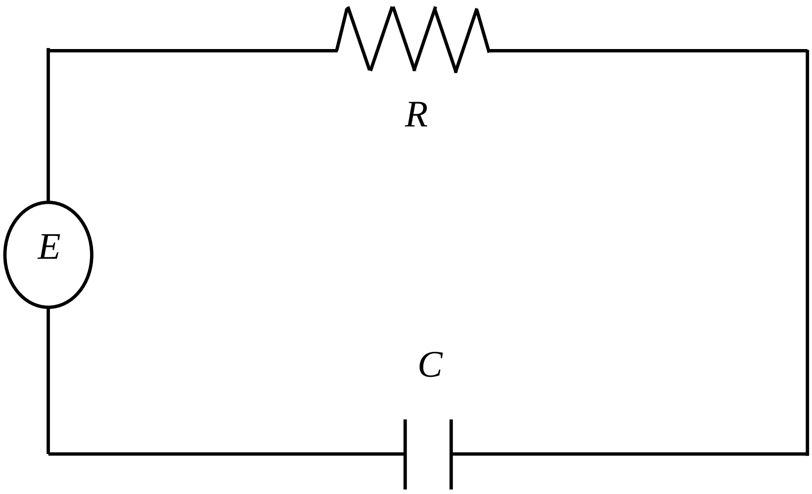


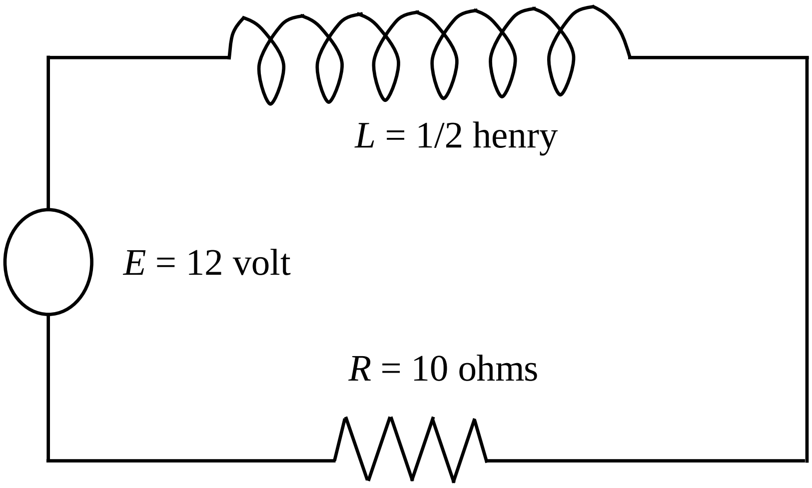
Fig. 2: *RC* series circuit

But current I and charge q are related by $i = dq/dt$, so (17) becomes the linear differential equation

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{18}$$

Example: A 12-volt battery is connected to a series circuit in which the inductance is $\frac{1}{2}$ henry and the resistance is 10 ohms. Determine the current I if the initial current is zero. What will happen after a long time?

Solution: Here E = 12 volt, L = 1/2 henry and R = 10 ohms.



By Kirchhoff's second law we obtain the linear differential equation for the current $i(t)$,

$$L\frac{di}{dt}+Ri=E(t)$$
$$\Rightarrow \frac{1}{2}\frac{di}{dt}+10i=12$$
$$\Rightarrow \frac{di}{dt}+20i=24$$
$$\Rightarrow \frac{di}{dt}=-20\left(i-\frac{6}{5}\right)$$
$$\Rightarrow \frac{di}{i-\frac{6}{5}}=-20dt$$

Integrating,

$$\ln\left(i-\frac{6}{5}\right)=-20t+\ln A$$

$$\therefore\ i=\frac{6}{5}+Ae^{-20t} \tag{19}$$

Since the initial current is zero, that is, $i(0) = 0$.

Therefore, equation (19) gives

$$0=\frac{6}{5}+Ae^{0} \quad \Rightarrow A=-\frac{6}{5}.$$

Hence the current in the series is

$$i = \frac{6}{5} - \frac{6}{5}e^{-20t}$$
$$= \frac{6}{5}\left(1 - e^{-20t}\right).$$

After a long time, i.e., as $t \to \infty$, the second term in the above expression $e^{-20t} \to 0$. Thus we have

$$i = \frac{6}{5}$$
$$= \frac{12}{10}$$
$$= \frac{E}{R}$$

$$\therefore\ E = iR$$

So, after a long time the current in the circuit is simply governed by Ohm's law ($E = iR$).

Exercise 1: A 30-volt electromotive force is applied to an *LR* series circuit in which the inductance is 0.1 henry and the resistance is 50 ohms. Find the current $i(t)$ if $i(0) = 0$. Determine the current as $t \to \infty$.

Exercise 2: A 100-volt electromotive force is applied to an *RC* series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4} farad. Find the charge $q(t)$ on the capacitor if $q(0) = 0$. Find the current $i(t)$.

Answers: 1. $i(t) = \frac{3}{5}\left(1 - e^{-500t}\right),\ i \to \frac{3}{5}$ as $t \to \infty$; 2. $q(t) = \frac{1}{100}\left(1 - e^{-50t}\right);\ i(t) = \frac{1}{2}e^{-50t}$.

Falling Body Problems

We consider that a body is falling through air toward the earth. In such a circumstance the body encounters air resistance as it falls. The amount of air resistance depends upon the velocity of the body, but no general law exactly expressing this dependence is known. In some instances the law $R = kv$ appears to be quite satisfactory, while in others $R = kv^2$ appears to be more exact.

Note: The acceleration due to gravity, g = 980 cm/sec^2 in cgs system and 32 ft/sec^2 in the British system.

If m be the mass of a body and w denotes it weight then we have $w = mg$.

Also $v = dx/dt$, $a = dv/dt = d^2x/dt^2$, $dv/dt = (dv/dx)(dx/dt) = v(dv/dx)$.

By Newton's second law of motion ($F = ma$), we have

$$m\frac{dv}{dt} = F, \tag{20}$$

$$m\frac{d^2x}{dt^2} = F, \tag{21}$$

$$mv\frac{dv}{dx} = F, \tag{22}$$

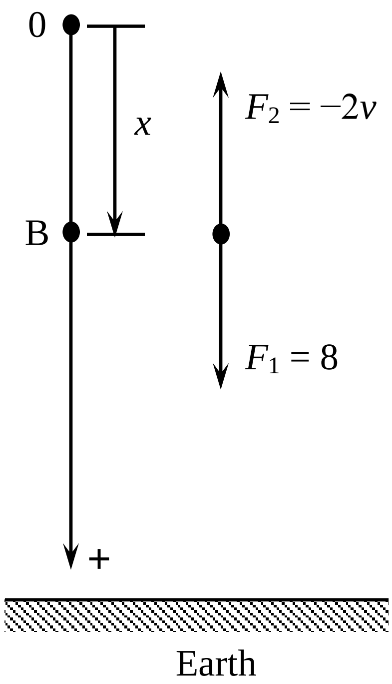
where F is the resultant force acting on the body.

Example: A body weighting 8 lb falls from rest toward the earth from a great height. As it falls, air resistance acts upon it, and we shall assume that this resistance (in pounds) is numerically equal to $2v$, where v is the velocity (in feet per second). Find the velocity and distance at time t seconds. Hence determine the velocity and distance at time 1 second.

Solution: We choose the positive x axis vertically downward along the path of the body B and the origin at the point from which the body fell. The forces acting on the body are:

1. F_1, its weight, 8 lb, which acts downward and hence is positive.

2. F_2, the air resistance, numerically equal to $2v$, which acts upward and hence is the negative quantity $-2v$.



By Newton's second law, we have

$$m\frac{dv}{dt} = F_1 + F_2. \tag{23}$$

Here $m = \frac{w}{g} = \frac{8}{32} = \frac{1}{4}$.

Thus (23) reduces to

$$\frac{1}{4}\frac{dv}{dt} = 8 - 2v. \tag{24}$$

Since the body was initially at rest, we have the initial condition

$$v(0) = 0. \tag{25}$$

Rewriting the equation (24), we get

$$\frac{dv}{v-4} = -8dt.$$

Integrating,

$\ln(v-4) = -8t + \ln c_1$, where $\ln c_1$ is an integrating constant.

$$\therefore\ v = 4 + c_1 e^{-8t}. \tag{26}$$

Using (25), we find

$$0 = 4 + c_1 e^0 \ \Rightarrow c_1 = -4.$$

Thus the velocity at time t is given by

$$v = 4\left(1 - e^{-8t}\right). \tag{27}$$

Now to determine the distance fallen at time t, we write (27) in the form

$$\frac{dx}{dt} = 4\left(1 - e^{-8t}\right). \tag{28}$$

Here it is note that $x(0) = 0$.

Integrating the equation (28), we obtain

$$x = 4\left(t + \frac{1}{8}e^{-8t}\right) + c_2. \tag{29}$$

Since $x(0) = 0$, we find

$$0 = 4\left(0 + \frac{1}{8}\right) + c_2 \ \Rightarrow c_2 = -\frac{1}{2}.$$

Hence the distance fallen at time t is given by

$$x = 4\left(t + \frac{1}{8}e^{-8t}\right) - \frac{1}{2}$$

That is, $x = 4\left(t + \frac{1}{8}e^{-8t} - \frac{1}{8}\right).$

$$\tag{30}$$

At time $t = 1$, equation (27) gives

$$v = 4\left(1 - e^{-8}\right) \approx 4 \text{ (ft/sec)}$$

and equation (30) yields

$$x = 4\left(1 + \frac{1}{8}e^{-8} - \frac{1}{8}\right) \approx 3.5 \ (\text{ft}).$$