Lecture No - 23

Parametric Equation

Sometimes x and y are expressed in terms of third variable usually called a parameter.

In such cases we can find $\frac{dy}{dx}$ without eliminating the parameter. The process of differentiation in such cases is shown below.

Let $x = f_1(t)$ and $y = f_2(t)$

$\therefore \frac{dx}{dt} = f_1'(t)$ $\therefore \frac{dy}{dt} = f_2'(t)$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{f_2'(t)}{f_1'(t)}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{f_2'(t)}{f_1'(t)}$$

Example:

01. $x = a\cos t$ and $y = b\sin t$. Find $\frac{dy}{dx}$

Solution

Given that $x = a\cos t$ and $y = b\sin t$

$\therefore \frac{dx}{dt} = \frac{d}{dx}(a\cos t) = -a\sin t$ $\therefore \frac{dy}{dt} = \frac{d}{dx}(b\sin t) = a\cos t$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{bsint}{-acost} = -\frac{b}{a}\tan t$

02. Find $\frac{dy}{dx}$ if $x = acos^3t$, $y = a\sin^3 t$

Solution

Given that $x = acos^3t$ and $y = asin^3t$

$\therefore \frac{dx}{dt} = a.3\,cos^2t\,(-sint)$

$\frac{dy}{dt} = a.3\,sin^2t\,(cost)$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3a\,sin^2t\,cost}{-3a\,cos^2t\,sint} = -\frac{asint}{acost} = -\tan t$

(Ans.)

03. $\tan y = \frac{2t}{1-t^2}$, $\sin x = \frac{2t}{1+t^2}$

Solution

Given that $\tan y = \frac{2t}{1-t^2}$ $\sin x = \frac{2t}{1+t^2}$

$=> y = \tan^{-1}\frac{2t}{1-t^2}$, $=> x = \sin^{-1}\frac{2t}{1+t^2}$

$=> y = 2\tan^{-1} t$ $=> x = 2\tan^{-1} t$

$\therefore \frac{dy}{dt} = 2.\frac{1}{1+t^2}$ $=> \frac{dx}{dt} = 2.\frac{1}{1+t^2}$

$=> \frac{dy}{dt} = \frac{2}{1+t^2}$ $=> \frac{dx}{dt} = \frac{2}{1+t^2}$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2}{1+t^2}}{\frac{2}{1+t^2}} = 1$ (Ans.)

04. $= \frac{3at^2}{1+t^3}$, $x = \frac{3at}{1+t^3}$

Solution

Given that $y = \frac{3at^2}{1+t^3}$ $x = \frac{3at}{1+t^3}$

$\therefore \frac{dy}{dt} = \frac{(1+t^3)6at - 3at^2.3t^2}{(1+t^3)^2}$ $\therefore \frac{dx}{dt} = \frac{(1+t^3)3a - 3at.3t^2}{(1+t^3)^2}$

$=> \frac{dy}{dt} = \frac{6at + 6at^4 - 9at^4}{(1+t^3)^2}$ $=> \frac{dx}{dt} = \frac{(1+t^3)3a - 3at.3t^2}{(1+t^3)^2}$

$=> \frac{dy}{dt} = \frac{6at - 3at^4}{(1+t^3)^2}$ $=> \frac{dx}{dt} = \frac{3a + 3at^3 - 9at^3}{(1+t^3)^2}$

$=> \frac{dy}{dt} = \frac{3at(2-t^3)}{(1+t^3)^2}$ $=> \frac{dx}{dt} = \frac{3a - 6at^3}{(1+t^3)^2}$

$=> \frac{dx}{dt} = \frac{3a(1-2t^3)}{(1+t^3)^2}$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{3at(2-t^3)}{(1+t^3)^2}}{\frac{3a(1-2t^3)}{(1+t^3)^2}} = \frac{3at(2-t^3)}{3a(1-2t^3)} = \frac{2-t^3}{1-2t^3}$ Ans.

05. $x = a(\theta + \sin\theta)\,,\, y = a(1 - \cos\theta)$

Solution

Given that $x = a(\theta + \sin\theta)\,,\, y = a(1 - \cos\theta)$

$x = a(\theta + \sin\theta)$ $\qquad$ $y = a(1 - \cos\theta)$

$\therefore \frac{dx}{d\theta} = a(1 + cos\theta)$ $\quad$ $\therefore \frac{dy}{d\theta} = asin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{asin\theta}{a(1+cos\theta)} = \frac{\sin 2.\frac{\theta}{2}}{1 + cos2.\frac{\theta}{2}} = \frac{2sin\frac{\theta}{2}cos\frac{\theta}{2}}{2cos^2\frac{\theta}{2}}$$

$$= \frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}} = tan\frac{\theta}{2} \;\; Ans.$$

Exercise

Find $\frac{dy}{dx}$.

01. $x = a(\cos t + t\sin t)\,,\, y = a(sint - tcost)$.

Ans. $tant$

02. $x = sin^2t\,,\, y = tant$.

03. $x = logt + sint\,,\, y = e^t + cost$.

04. $x = acos^3(e^t), y = asin^3(e^t)$

Differentiating *f(x)* with respect to *g(x)*

Let $y = f(x)$ $\qquad$ and $z = g(x)$

$\therefore \frac{dy}{dx} = f'(x)$ $\qquad$ $\therefore \frac{dz}{dx} = g'(x)$

$$\therefore \frac{dy}{dz} = \frac{f'(x)}{g'(x)}$$

Example :

01. Differentiate $\tan^{-1}\frac{2x}{1-x^2}$ with respect to $\sin^{-1}\frac{2x}{1+x^2}$.

02. Differentiate $x^{\sin^{-1}x}$ with respect to $\sin^{-1}x$.

03. Differentiate $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$ with respect to $\sec^{-1}\frac{1}{2x^2-1}$.

04. Differentiate $e^{\sin^{-1}x}$ with respect to $\cos 3x$.

05. Differentiate $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\tan^{-1}\frac{2x}{1-x^2}$

06. Differentiate $(sinx)^x$ with respect to $x^{\sin x}$.

Solution

01. Let $y = \tan^{-1}\frac{2x}{1-x^2}$ $\quad$ and $\quad$ $z = \sin^{-1}\frac{2x}{1+x^2}$

$=> y = 2\tan^{-1}x$ $\qquad$ $=> z = 2\tan^{-1}x$

$\therefore \frac{dy}{dx} = \frac{2}{1+x^2}$ $\qquad$ $\therefore \frac{dz}{dx} = \frac{2}{1+x^2}$

$$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}} = 1$$

Solution

02. Let $y = x^{\sin^{-1}x}$ $\qquad$ and $z = \sin^{-1}x$

Taking ln on both sides $\qquad$ $\therefore \frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}}$

$lny = lnx^{\sin^{-1}x}$

$=> lny = \sin^{-1}x \; lnx$

$\therefore \frac{d}{dx}(lny) = \frac{d}{dx}(\sin^{-1}x \; lnx)$

$=> \frac{1}{y}\frac{dy}{dx} = \sin^{-1}x\frac{d}{dx}(lnx) + lnx\frac{d}{dx}(\sin^{-1}x)$

$=> \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\sin^{-1}x + lnx\frac{1}{\sqrt{1-x^2}}$

$\therefore \frac{dy}{dx} = y(\frac{\sin^{-1}x}{x} + \frac{lnx}{\sqrt{1-x^2}})$

$$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{y\left(\frac{\sin^{-1}x}{x} + \frac{lnx}{\sqrt{1-x^2}}\right)}{\frac{1}{\sqrt{1-x^2}}}$$

$$=> \frac{dy}{dz} = \frac{y(\sqrt{1-x^2}\sin^{-1}x + xlnx)}{x}.$$

Solution

03. Let $y = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$ and $z = \sec^{-1}\frac{1}{2x^2-1}$

Putting $x = \sin\theta$ | Putting $x = \cos\theta$

$\therefore \theta = \sin^{-1}x$ | $\therefore \theta = \cos^{-1}x$

$\therefore y = \tan^{-1}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$ | $\therefore z = \sec^{-1}\frac{1}{2\cos^2\theta-1}$

$=> y = \tan^{-1}\frac{\sin\theta}{\sqrt{\cos^2\theta}}$ | $=> z = \sec^{-1}\frac{1}{\cos 2\theta}$

$=> y = \tan^{-1}\frac{\sin\theta}{\cos\theta}$ | $=> z = \sec^{-1}\sec 2\theta$

$=> y = \tan^{-1}\tan\theta$ | $=> z = 2\theta$

$=> y = \theta$ | $=> z = 2\cos^{-1}x$

$=> y = \sin^{-1}x$ | $=> \frac{dz}{dx} = \frac{2}{\sqrt{1-x^2}}$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}} = \frac{1}{2}$$

Solution

04. Let $y = e^{\sin^{-1}x}$ and $z = \cos 3x$

$\therefore \frac{dy}{dx} = e^{\sin^{-1}x}.\frac{1}{\sqrt{1-x^2}}$ | $\therefore \frac{dz}{dx} = -3\sin 3x$

$=> \frac{dy}{dx} = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

$$\therefore \frac{dy}{dz} = \frac{\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}}{-3\sin 3x} = \frac{e^{\sin^{-1}x}}{-3\sqrt{1-x^2}\sin 3x}$$ Ans.

Solution

05. Let $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ and $z = \tan^{-1}\frac{2x}{1-x^2}$

Putting $x = \tan\theta \therefore \theta = \tan^{-1}x$ $=> z = 2\tan^{-1}x$

$\therefore y = \tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}$ | $\therefore \frac{dz}{dx} = \frac{2}{1+x^2}$

$= \tan^{-1}\frac{\sec\theta-1}{\tan\theta}$

$= \tan^{-1}\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}$ | $\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}} = \frac{1}{4}$ Ans.

$= \tan^{-1}\frac{1-\cos\theta}{\sin\theta}$

$= \tan^{-1}\frac{1-\cos 2\frac{\theta}{2}}{\sin 2\frac{\theta}{2}}$

$= \tan^{-1}\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$= \tan^{-1}\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$

$= \tan^{-1}\tan\frac{\theta}{2}$

$= \frac{\theta}{2}$

$= \frac{1}{2}\tan^{-1}x$

$\therefore \frac{dy}{dx} = \frac{1}{2(1+x^2)}$

Solution

06. Let $y = (\sin x)^x$ *and* $z = x^{\sin x}$

Taking ln on both sides | Taking ln on both sides

$=> \ln y = \ln(\sin x)^x$ | $=> \ln z = \sin x \ln x$

$=> \ln y = x\ln(\sin x)$ | $\therefore \frac{1}{z}\frac{dz}{dx} = \frac{\sin x}{x} + \cos x \ln x$

$\therefore \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{\sin x}\cos x + \ln(\sin x) \times 1$ | $=> \frac{dz}{dx} = z(\frac{\sin x}{x} + \cos x \ln x)$

$=> \frac{dy}{dx} = y\{x\cot x + \ln(\sin x)\}$ | $\therefore \frac{dz}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \ln x\right)$

$=> \frac{dy}{dx} = (\sin x)^x\{x\cot x + \ln(\sin x)\}$

$$\therefore \frac{dy}{dz} = \frac{(\sin x)^x\{x\cot x + \ln(\sin x)\}}{x^{\sin x}\left(\frac{\sin x}{x} + \cos x \ln x\right)}$$ Ans.

Inverse function

RULE: When differentiating the inverse function, first set the value of x of that function and then simplify. Usually the value of x will be $\sin n\theta$ *or* $\cos n\theta$ *or* $\tan n\theta$ *or* $\sec n\theta$ or $\cot n\theta$, where n = 1, 2, 3, 4, 5, ………… if can. If can't then differentiate with respect to x directly (without putting the value of x).

Examples (Putting the value of x)

1. $y = \sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$

2. $y = \sin^2\left(\cot^{-1}\sqrt{\frac{1\cancel{+}x}{1+x}}\right)$ or $y = \sin^2\left(\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)$

3. $y = \sin^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right)$

4. $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

5. $y = \sin^{-1}\left(\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right)$

6. $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

9. $y = \cos\left\{2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$

Solution-1 Given that $y = \sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$

putting $x = \cos\theta \quad \therefore \theta = \cos^{-1}x$

$$y = \sin\left\{2\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\}$$

$$\Rightarrow y = \sin\left\{2\tan^{-1}\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\theta/2}}\right\}$$

$$\Rightarrow y = \sin\left(2\tan^{-1}\tan\theta/2\right)$$

$\Rightarrow y = \sin(2 \cdot \theta/2)$

$\Rightarrow y = \sin\theta$

$\Rightarrow y = \sqrt{1-\cos^2\theta}$

$\Rightarrow y = \sqrt{1-x^2}$

$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}(1-x^2)$

$= \frac{1}{2\sqrt{1-x^2}}(-2x) = \frac{-x}{\sqrt{1-x^2}}$

$\therefore \frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}$ <u>Ans.</u>

<u>Solution-2</u> $y = \sin^2\left(\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)$ putting $x = \cos\theta \therefore \theta = \cos^{-1}x$

$\Rightarrow y = \sin^2\left(\cot^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)$

$\Rightarrow y = \sin^2\left(\cot^{-1}\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\theta/2}}\right)$

$\Rightarrow y = \sin^2\left(\cot^{-1}\tan\theta/2\right)$

$\Rightarrow y = \sin^2\left\{\cot^{-1}\cot\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}$

$\Rightarrow y = \sin^2\left(\frac{\pi}{2}-\theta/2\right)$

$\Rightarrow y = \left\{\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}^2$

$\Rightarrow y = \cos^2\theta/2$

$\Rightarrow y = \frac{1}{2}\cdot 2\cos^2\theta/2$

$\Rightarrow y = \frac{1}{2}(1+\cos 2\cdot\theta/2)$

$\Rightarrow y = \frac{1}{2}(1+\cos\theta)$

$\Rightarrow y = \frac{1}{2}(1+x)$

$\therefore \frac{dy}{dx} = \frac{1}{2}(0+1) \quad \therefore \frac{dy}{dx} = \frac{1}{2}$ <u>Ans.</u>

Solution-3: Given that $y = \sin^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right)$

putting $x = \cos\theta \quad \therefore \theta = \cos^{-1}x$

$$\therefore y = \sin^{-1}\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{2}\right)$$

$$\Rightarrow y = \sin^{-1}\left(\frac{\sqrt{2\cos^2\theta/2}+\sqrt{2\sin^2\theta/2}}{2}\right)$$

$$\Rightarrow y = \sin^{-1}\left(\frac{\sqrt{2}\cos\theta/2+\sqrt{2}\sin\theta/2}{2}\right)$$

$$\Rightarrow y = \sin^{-1}\left(\frac{1}{\sqrt{2}}\cos\theta/2+\frac{1}{\sqrt{2}}\sin\theta/2\right)$$

$$\Rightarrow y = \sin^{-1}\left(\sin\pi/4\cos\theta/2+\cos\pi/4\sin\theta/2\right)$$

$$\Rightarrow y = \sin^{-1}\sin\left(\frac{\pi}{4}+\theta/2\right)$$

$$\Rightarrow y = \frac{\pi}{4}+\frac{\theta}{2}$$

$$\Rightarrow y = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\cdot\frac{-1}{\sqrt{1-x^2}} = \frac{-1}{2\sqrt{1-x^2}} \quad \underline{\underline{\text{Ans.}}}$$

Solution-4: Given that $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

putting $x = \tan\theta \quad \therefore \theta = \tan^{-1}x$

$$\therefore y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{\cancel{2}\sin^{\cancel{2}}\theta/2}{\cancel{2}\cancel{\sin\theta/2}\cos\theta/2}\right)$$

$\Rightarrow y = \tan^{-1}\tan \theta/2$

$\Rightarrow y = \dfrac{\theta}{2}$

$\Rightarrow y = \dfrac{1}{2}\tan^{-1}x$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{2}\cdot\dfrac{1}{1+x^2} = \dfrac{1}{2(1+x^2)}$ Ans

Solution-5: $y = \sin^{-1}\left(\dfrac{x+\sqrt{1-x^2}}{\sqrt{2}}\right)$

putting $x = \sin\theta \quad \therefore \theta = \sin^{-1}x$

$\therefore y = \sin^{-1}\left(\dfrac{\sin\theta + \sqrt{1-\sin^2\theta}}{\sqrt{2}}\right)$

$\Rightarrow y = \sin^{-1}\left(\dfrac{\sin\theta + \cos\theta}{\sqrt{2}}\right)$

$\Rightarrow y = \sin^{-1}\left(\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right)$

$\Rightarrow y = \sin^{-1}\left(\sin\theta\cos\pi/4 + \cos\theta\sin\pi/4\right)$

$\Rightarrow y = \sin^{-1}\sin(\theta + \pi/4)$

$\Rightarrow y = \theta + \pi/4$

$\Rightarrow y = \sin^{-1}x + \dfrac{\pi}{4}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}} + 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$ Ans.

Solution-6: Given that, $y = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

putting $x = \tan\theta \quad \therefore \theta = \tan^{-1}x$

$\therefore y = \sin^{-1}\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$\Rightarrow y = \sin^{-1}\cos 2\theta = \sin^{-1}\sin\left(\frac{\pi}{2} + 2\theta\right)$

$\Rightarrow y = \frac{\pi}{2} + 2\theta = \frac{\pi}{2} + 2\tan^{-1}x$

$\Rightarrow y = \frac{\pi}{2} + 2\tan^{-1}x$

$\therefore \frac{dy}{dx} = 0 + 2 \cdot \frac{1}{1+x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$ <u>Ans.</u>

Solution-17: Given that $y = \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan x/2\right)$

Solution-2: $y = \cos\left\{2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$

putting $x = \cos\theta$ $\therefore \theta = \cos^{-1}x$

$\therefore y = \cos\left\{2\cot^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\}$

$\Rightarrow y = \cos\left\{2\cot^{-1}\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\theta/2}}\right\}$

$\Rightarrow y = \cos\left(2\cot^{-1}\tan\theta/2\right)$

$\Rightarrow y = \cos\left\{2\cot^{-1}\cot\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right\}$

$\Rightarrow y = \cos\left\{2\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right\}$

$\Rightarrow y = \cos\left(\pi - \frac{\theta}{2}\right)$

$\Rightarrow y = -\cos\frac{\theta}{2}$ $\Rightarrow y = -\cos\theta$

$\Rightarrow y = -x$

$\Rightarrow y = -\sqrt{\cos^2\theta/2}$ $\therefore \frac{dy}{dx} = -1$ Ans.

$\Rightarrow y = -\sqrt{\frac{1}{2}\cdot 2\cos^2\theta/2}$

$\Rightarrow y = -\frac{1}{\sqrt{2}}\sqrt{1+\cos\theta}$

$\Rightarrow y = -\frac{1}{\sqrt{2}}\sqrt{1+x}$

$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{1+x}} \cdot 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{2}\sqrt{1+x}}$ <u>Ans.</u>

Examples (Without putting the value of x)

1. $y = \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

2. $y = \cos^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$

3. $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

4. $y = \tan^{-1}\left(\frac{1+\tan x}{1-\tan x}\right)$

5. $y = \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$

6. $y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

7. $y = \frac{x\sin^{-1}x}{\sqrt{1-x^2}}$

8. $y = \tan^{-1}\left(\frac{x\sin\alpha}{1-x\cos\alpha}\right)$

Solution-1: Given that $y = \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1+x^2}}\right)^2}} \cdot \frac{d}{dx}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{\sqrt{1-\frac{1}{1+x^2}}} \cdot \frac{d}{dx}\left((1+x^2)^{-\frac{1}{2}}\right)$$

$$= \frac{1}{\sqrt{\frac{\cancel{1}+x^2-\cancel{1}}{1+x^2}}} \cdot \frac{-1}{2}(1+x^2)^{-\frac{1}{2}-1} \cdot 2x$$

$$= \frac{\sqrt{1+x^2}}{\cancel{x}} \cdot \frac{-\cancel{x}}{(1+x^2)^{3/2}}$$

$$= \frac{\sqrt{1+x^2}}{(1+x^2)^{3/2}} = (1+x^2)^{\frac{1}{2}-\frac{3}{2}} = (1+x^2)^{-1} = \frac{1}{1+x^2} \text{ Ans}$$

Solution-2: $y = \cos^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{a+b\cos x}{b+a\cos x}\right)^2}} \cdot \frac{d}{dx}\left(\frac{a+b\cos x}{b+a\cos x}\right)$$

$$= -\frac{b+a\cos x}{\sqrt{(b+a\cos x)^2-(a+b\cos x)^2}} \cdot \frac{(b+a\cos x)(-b\sin x)-(a+b\cos x)(-a\sin x)}{(b+a\cos x)^2}$$

$$= -\frac{-\sin x\,(b^2 + \cancel{ab\cos x} - a^2 - \cancel{ab\cos x})}{\sqrt{b^2 + \cancel{2ab\cos x} + a^2\cos^2 x - a^2 - \cancel{2ab\cos x} - b^2\cos^2 x}\;(b + a\cos x)}$$

$$= \frac{\sin x\,(b^2 - a^2)}{\sqrt{(b^2 - a^2) - (b^2 - a^2)\cos^2 x}\;(b + a\cos x)}$$

$$= \frac{(b^2 - a^2)\sin x}{\sqrt{(b^2 - a^2)(1 - \cos^2 x)}\;(b + a\cos x)}$$

$$= \frac{(b^2 - a^2)\cancel{\sin x}}{\sqrt{b^2 - a^2}\cdot\cancel{\sin x}\;(b + a\cos x)}$$

$$= \frac{\sqrt{b^2 - a^2}}{b + a\cos x}$$ Ans.

Solution-3

Given that, $y = \tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right)$

$$\Rightarrow y = \tan^{-1}\left(\frac{\cos^2 x/2 - \sin^2 x/2}{\cos^2 x/2 + 2\sin x/2\cos x/2 + \sin^2 x/2}\right) \quad \left[\cos 2\cdot x/2 = \cos^2 x/2 - \sin^2 x/2\right]$$

$$\Rightarrow y = \tan^{-1}\frac{(\cos x/2 + \sin x/2)(\cos x/2 - \sin x/2)}{(\cos x/2 + \sin x/2)^2}$$

$$\Rightarrow y = \tan^{-1}\frac{\cos x/2 - \sin x/2}{\cos x/2 + \sin x/2}$$

$$= \tan^{-1}\frac{\cos x/2\left(1 - \frac{\sin x/2}{\cos x/2}\right)}{\cos x/2\left(1 + \frac{\sin x/2}{\cos x/2}\right)}$$

$$= \tan^{-1}\frac{1 - \tan x/2}{1 + \tan x/2}$$

$$= \tan^{-1}\frac{\tan \pi/4 - \tan x/2}{1 + \tan \pi/4\,\tan x/2}$$

$$= \tan^{-1}\tan(\pi/4 - x/2)$$

$\Rightarrow y = \frac{\pi}{4} - \frac{x}{2}$

$\therefore \frac{dy}{dx} = -\frac{1}{2} \cdot \frac{d}{dx}(x)$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}$ Ans.

Solution-4:

Given that $y = \tan^{-1}\left(\frac{1+\tan x}{1-\tan x}\right)$

$\Rightarrow y = \tan^{-1} \frac{\tan \pi/4 + \tan x}{1 - \tan \pi/4 \tan x}$

$\Rightarrow y = \tan^{-1} \tan(\pi/4 + x)$

$\Rightarrow y = \frac{\pi}{4} + x$

$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} + x\right)$

$\Rightarrow \frac{dy}{dx} = 1$ Ans.

Solution-5:

Given that $y = \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan x/2\right)$

$\therefore \frac{dy}{dx} = \frac{1}{1 + \frac{a-b}{a+b}\tan^2 x/2} \cdot \frac{d}{dx}\left(\sqrt{\frac{a-b}{a+b}} \tan x/2\right)$

$= \frac{1}{1 + \frac{(a-b)\sin^2 x/2}{(a+b)\cos^2 x/2}} \cdot \sqrt{\frac{a-b}{a+b}} \cdot \sec^2 x/2 \cdot \frac{1}{2}$

$= \frac{(a+b)\cos^2 x/2}{(a+b)\cos^2 x/2 + (a-b)\sin^2 x/2} \cdot \sqrt{\frac{a-b}{a+b}} \cdot \sec^2 x/2 \cdot \frac{1}{2}$

$= \frac{(a+b)\cos^2 x/2}{a\cos^2 x/2 + b\cos^2 x/2 + a\sin^2 x/2 - b\sin^2 x/2} \cdot \frac{\sqrt{a-b}}{\sqrt{a+b}} \cdot \frac{1}{\cos^2 x/2} \cdot \frac{1}{2}$

$= \frac{\sqrt{a+b} \cdot \sqrt{a-b}}{a(\cos^2 x/2 + \sin^2 x/2) + b(\cos^2 x/2 - \sin^2 x/2)} \cdot \frac{1}{2}$

$= \frac{\sqrt{a^2-b^2}}{a + b\cos 2 \cdot x/2} \cdot \frac{1}{2} = \frac{\sqrt{a^2-b^2}}{2(a+b\cos x)}$ Ans

Solution-6 Given that

$$y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

$$\Rightarrow y = \tan^{-1}\sqrt{\frac{2\sin^2 x/2}{2\cos^2 x/2}}$$

$$\Rightarrow y = \tan^{-1}\tan x/2$$

$$\Rightarrow y = \frac{x}{2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \quad \underline{\underline{\text{Ans.}}}$$

Ex- $y = \tan^{-1}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

Sol^n Given that $y = \tan^{-1}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

$\Rightarrow y$ putting $x = \sin\theta$. $\therefore \theta = \sin^{-1}x$

$$\therefore \sqrt{1+x} = \sqrt{1+\sin\theta} = \sqrt{\sin^2\theta/2 + 2\sin\theta/2\cos\theta/2 + \cos^2\theta/2}$$

$$= \sin\theta/2 + \cos\theta/2$$

$$\therefore \sqrt{1-x} = \sqrt{1-\sin\theta} = \cos\theta/2 - \sin\theta/2$$

$$\therefore y = \tan^{-1}\frac{\sin\theta/2 + \cancel{\cos\theta/2} - \cancel{\cos\theta/2} + \sin\theta/2}{\cancel{\sin\theta/2} + \cos\theta/2 + \cos\theta/2 - \cancel{\sin\theta/2}}$$

$$\Rightarrow y = \tan^{-1}\frac{2\sin\theta/2}{2\cos\theta/2}$$

$$\Rightarrow y = \tan^{-1}\tan\theta/2$$

$$\Rightarrow y = \frac{\theta}{2} = \frac{1}{2}\sin^{-1}x$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}} \quad \underline{\underline{\text{Ans.}}}$$

Ex. Diff. w.r.to x. $y = \sin^{-1}\left(\sqrt{\frac{x^2}{1+x^2}}\right)$

Solution—

Given that $y = \sin^{-1}\sqrt{\frac{x^2}{1+x^2}}$

putting $x = \cancel{\sin\theta}\ \tan\theta \quad \therefore \theta = \tan^{-1}x$

$\therefore y = \sin^{-1}\sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}}$

$\Rightarrow y = \sin^{-1}\sqrt{\frac{\tan^2\theta}{\sec^2\theta}}$

$\Rightarrow y = \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)$

$\Rightarrow y = \sin^{-1}\left(\frac{\sin\theta}{\cos\theta} \times \cos\theta\right)$

$\Rightarrow y = \sin^{-1}\sin\theta$

$\Rightarrow y = \theta$

$\Rightarrow y = \tan^{-1}x$

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}x)$

$= \frac{1}{1+x^2}$ Ans.

Ex. Differentiate $\tan^{-1}\frac{x-\sqrt{a^2-x^2}}{x+\sqrt{a^2-x^2}}$ w.r.to x.

Solution: Given that the function $y = \tan^{-1}\frac{x-\sqrt{a^2-x^2}}{x+\sqrt{a^2-x^2}}$

putting $x = a\sin\theta \quad \therefore \theta = \sin^{-1}\frac{x}{a}$.

$\therefore y = \tan^{-1}\frac{a\sin\theta - \sqrt{a^2-a^2\sin^2\theta}}{a\sin\theta + \sqrt{a^2-a^2\sin^2\theta}}$

$= \tan^{-1}\frac{a\sin\theta - a\cos\theta}{a\sin\theta + a\cos\theta}$

$= \tan^{-1}\frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}$

$= \tan^{-1}\frac{\cos\theta\left(\frac{\sin\theta}{\cos\theta}-1\right)}{\cos\theta\left(\frac{\sin\theta}{\cos\theta}+1\right)}$

$= \tan^{-1}\frac{\tan\theta-1}{\tan\theta+1}$

$= \tan^{-1}\frac{\tan\theta-\tan\pi/4}{1+\tan\theta\tan\pi/4}$

$= \tan^{-1}\tan(\theta-\pi/4)$

$= \theta - \pi/4 = \sin^{-1}\frac{x}{a} - \frac{\pi}{4}$

$\Rightarrow y = \sin^{-1}\frac{x}{a} - \frac{\pi}{4}$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$

$= \frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}} \cdot \frac{1}{a}$

$= \frac{a}{\sqrt{a^2-x^2}} \cdot \frac{1}{a}$

$= \frac{1}{\sqrt{a^2-x^2}}$ Ans.