LECTURE NO - 16

Derivatives

RATES OF CHANGE

Velocity can be viewed as *rate of change*-the rate of change of position with respect to time. Rates of change occur in other applications as well.

For example:

- A microbiologist might be interested in the rate at which the number of bacteria in a colony changes with time.
- An engineering might be interested in the rate at which the length of a metal rod changes with temperature.
- An economist might be interested in the rate at which production cost changes with the quantity of a product that is manufactured.
- A medical researcher might be interested in the rate at which the radius of an artery changes with the concentration of alcohol in the bloodstream.

Our next objective: "the rate of change of y with respect to x" when y is a function of x. In the case where y is a linear function of x, say $y = mx + b$, the slope m is the natural measure of the rate of change of y with respect to x. As in Figure 1, each 1-unit increase in x anywhere along the line produces an m-unit change in y. So we see that y changes at a constant rate with respect to x along the line and that m measures this rate of change.

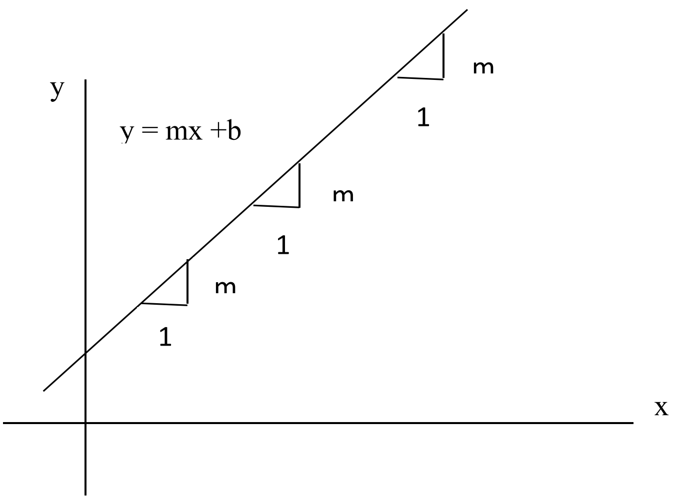


Figure 1: 1-unit increase in x always produces an m-unit change in y.

Example:

Find the rate of change of y with respect to x if

$$(a)\ y = 2x - 1 \qquad (b)\quad y = -5x + 1$$

Answer:

(a) The rate of change of equation (a) is 2, so each 1-unit increase in *x* produces a 2-unit increase in *y*.

(b) The rate of change of equation (b) is -5, so each 1-unit increase in *x* produces a 5-unit decrease in *y*.

Average rate of change:

If $y = f(x)$, then the average rate of change of *y* with respect to *x* over the interval $[x_0, x_1]$ is defined by

$$r_{ave} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Instantaneous rate of change:

If $y = f(x)$, then the instantaneous rate of change of *y* with respect to *x* over the interval $[x_0, x_1]$ is defined by

$$r_{inst} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Geometrically, the average rate of change of *y* with respect to *x* over the interval $[x_0, x_1]$ is the slope of the secant line through the points $P(x_0, f(x_0))$ and $P(x_1, f(x_1))$ and the instantaneous rate of change of *y* with respect to *x* at x_0 is the slope of the tangent line at the point $P(x_0, f(x_0))$.

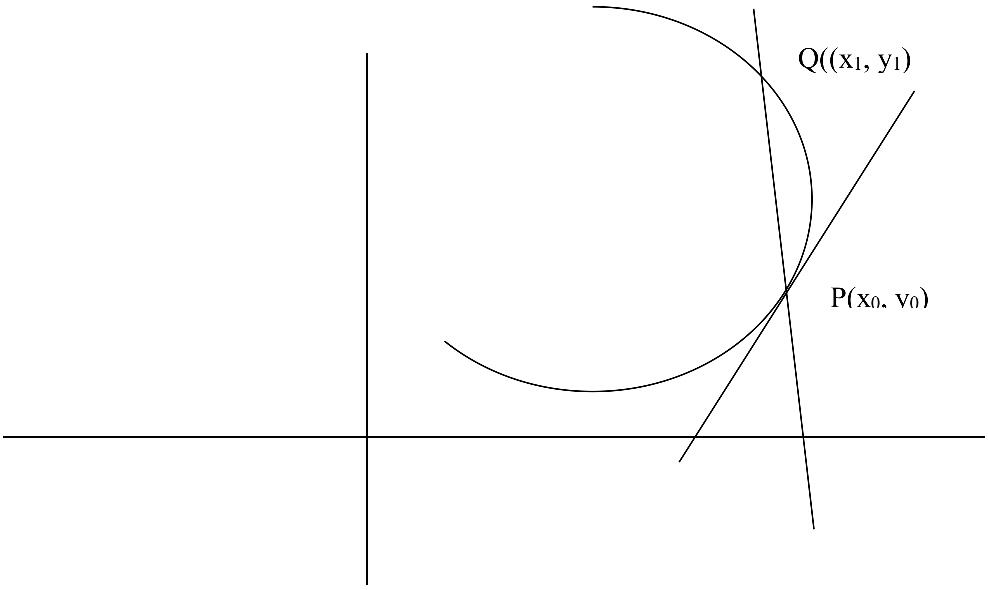


Figure:- 2

Example:

Let $y = x^2 + 1$.

(a) Find the average rate of change of y with respect to x over the interval [3,5].

(b) Find the instantaneous rate of change of y with respect to x when $x = -4$

Ans: (a) 8 and (b) -8

DEFINITION OF THE DERIVATIVE FUNCTION

The function $f^{'}$ defined by the formula

$$f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ is called the derivative of f with respect to x. The domain of $f^{'}$ consists of all x in the domain of f for which the limits exists.

Example:

(a) Find the derivative with respect to x of $f(x) = x^3 - x$

(b) Graph f and $f^{'}$ together, and discuss the relationship between the two graphs.

Solution: (a)

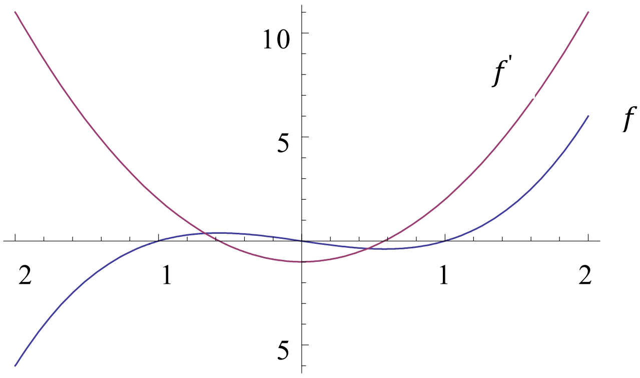
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{[(x+h)^3-(x+h)]-[x^3-x]}{h}$$

$$= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x-h-x^3+x}{h} = \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-h}{h}$$

$$= \lim_{h\to 0}[3x^2+3xh+h^2-1] \quad = 3x^2-1$$

Solution: (b)



Since $f'(x)$ can be interpreted as the slope of the tangent line to the graph of $y = f(x)$ at x, it follows that $f'(x)$ is positive where the tangent line has positive slope , is negative where the tangent line has negative slope, and is zero where the tangent line is horizontal. This is consistent with the graph of $f(x) = x^3 - x$ and $f'(x) = 3x^2 - 1$.

USING DERIVATIVES TO COMPUTE INSTANTANEOUS VELOCITY

Instantaneous velocity:

If $s = f(t)$ is the position function of a particle in rectilinear motion, then the instantaneous velocity at an arbitrary time t is given by

$$v_{inst} = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$$

i.e. the velocity function $v(t) = f'(t) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$ represents the instantaneous velocity of the particle at time t, more simply, the ***velocity function*** of the particle.

Example:

The position function for an object dropped from the Empire State Building from *1250 ft* above street level can be modeled by the position function $s = f(t) = 1250-16t^2$. Here, $f(t)$ is measured in feet above street level and t is measured in seconds after the object is released.

(a) Find the velocity function of the object.
(b) Find the time interval over which the velocity function is valid.
(c) What is the velocity of the object when it hits the ground?

Solution: (a)

The velocity function is

$$\begin{aligned}
v(t) &= \lim_{h\to 0}\frac{f(t+h)-f(t)}{h}\\
&= \lim_{h\to 0}\frac{[1250-16(t+h)^2]-[1250-16t^2]}{h}\\
&= \lim_{h\to 0}\frac{-16[t^2+2th+h^2-t^2]}{h}\\
&=-16\left(\lim_{h\to 0}\frac{2th+h^2}{h}\right)\\
&=-16.\lim_{h\to 0}(2t+h) = -32t
\end{aligned}$$

Where the units of velocity are feet per second.

Solution: (b)

The velocity function in part (a) is valid from the time the object is released ($t = 0$) until the time t_1 that it hits the ground, that is, when

$$1250 - 16t_1^2 = 0$$
$$\Rightarrow 1250 = 16t_1^2$$
$$\Rightarrow \sqrt{\frac{1250}{16}} \approx 8.8s$$

Solution: (c)

Putting the value of t_1 in v(t) = -32t then we get

$$v(t_1) = -32t_1 =-32\sqrt{\frac{1250}{16}} \approx -282.8ft/s.$$