LECTURE NO-24

Related Rates

In this section, we will study related rates problems. In such problems one tries to find the rate at which some quantity is changing by relating it to other quantities whose rates of change are known.

A strategy for solving related rates problems

Step-1:

Draw a figure and label the quantities that vary. Give a definition for each label.

Step-2:

Identify the rates of change that are known and the rate of change that is to found.

Step-3:

Find an equation that relates the variables whose rate of change were identified in step 2. To do this, it will often be helpful to draw an appropriately labeled figure that illustrates the relationship.

Step-4:

Differentiate both sides of the equation obtained in step 3 with respect to the time to produce a relationship between the known rates of change and the unknown rate of change.

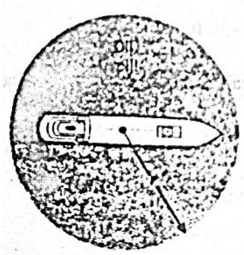
Step-5:

After completing step 4, substitute all known values for the rates of change and the variables, and then solve for the unknown rate of change.

Problem:

Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 ft/s. How fast is the area of the spill increasing when the radius of the spill is 60 ft.?

Solution:



Let t = number of seconds elapsed from the time of the spill.

r = radius of the spill in feet after t seconds.

A = Area of the spill in square feet after t seconds.

We have to find $\left.\frac{dA}{dt}\right|_{r=60}$. Given that $\frac{dr}{dt} = 2\ ft/sec$

We know that, the area of a circle is A

$$A = \pi r^2$$

Differentiating w. r. to t, we have

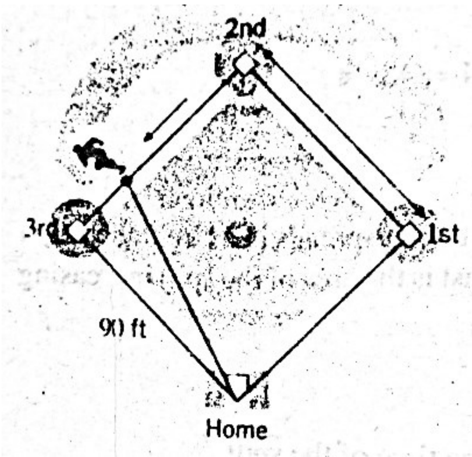
$$\frac{dA}{dt} = 2\pi r.\frac{dr}{dt}$$

Where, r = 60 ft, then $\frac{dr}{dt} = 2\ ft/sec$

$$\therefore \frac{dA}{dt} = 2\pi.60.2 = 240\pi \cong 754\ ft^2/sec$$

Problem: A baseball diamond is a square whose sides are 90 ft. long. Suppose that a player running from second base to third base has a speed of 30 ft./sec at the instant when he is 20 ft. from third base. At what rate is the player's distance from home plate changing at that instant?

Solution:



The rate at which the distance from third base is changing is $\frac{dx}{dt}$, and the rate at which the distance from home plate is changing is $\frac{dy}{dt}$.

Let t = number of seconds since the player left second base.

x = distance in feet from the player to third base.

y = distance in feet from the player to home plate.

We have to find $\left.\frac{dy}{dt}\right|_{x=20}$. Given that $\left.\frac{dx}{dt}\right|_{x=20} = -30\ \ ft/sec$

[The quantity $\left.\frac{dx}{dt}\right|_{x=20}$ is negative because x is decreasing with respect to t]

From the figure, we have

$$x^2 + 90^2 = y^2$$

Differentiating with respect to t, then we have

$$2x.\frac{dx}{dt} = 2y.\frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{x}{y}.\frac{dx}{dt}$$

$$= \frac{20}{10\sqrt{85}}\ (-30)$$

$$\therefore \frac{dy}{dt} = -\frac{60}{\sqrt{85}} \cong -6.51\, ft/sec.$$

$$y = \sqrt{x^2 + 90^2}$$

$$= \sqrt{20^2 + 90^2}$$

$$= \sqrt{8500}$$

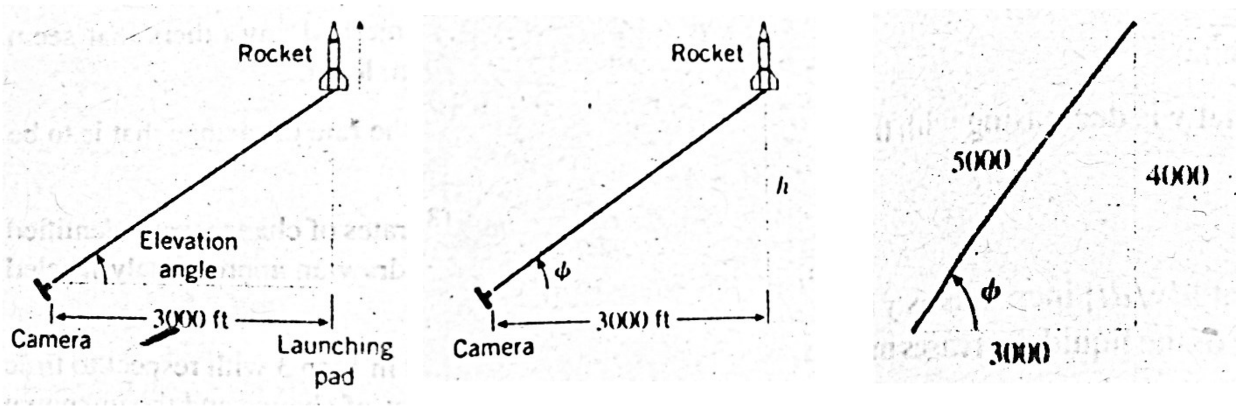
$$= 10\sqrt{85}$$

The negative sign tells us that y is decreasing.

(Ans.)

Problem: A camera mounted at a point 3000 ft. from the base of a rocket launching pad. Let us assume that the rocket rises vertically and the camera is to take a series of photographs of the rocket. The elevation angle of the camera will have to vary at just the right rate to keep the rocket in sight.

If the rocket is rising vertically at 880 ft. /s when it is 4000 ft. up. How fast must the camera elevation angle change at that instant to keep the rocket in sight?



Let t be the time, φ be the elevation angle and h be the height from the foot after t seconds.

We have to find $\left.\frac{d\varphi}{dt}\right|_{x=4000\,ft}$. Given that $\left.\frac{dh}{dt}\right|_{x=4000} = 880\, ft./sec$

From the above figure, we have

$$\tan\varphi = \frac{h}{3000}$$

$$\Rightarrow \varphi = \tan^{-1}\frac{h}{3000}$$

$$\Rightarrow \frac{d\varphi}{dt} = \frac{1}{1+\left(\frac{h}{3000}\right)^2}.\frac{1}{3000}.\frac{dh}{dt}$$

$$\Rightarrow \frac{d\varphi}{dt}\bigg|_{x=4000} = \frac{3000^2}{3000^2+4000^2} \cdot \frac{1}{3000} . 880$$

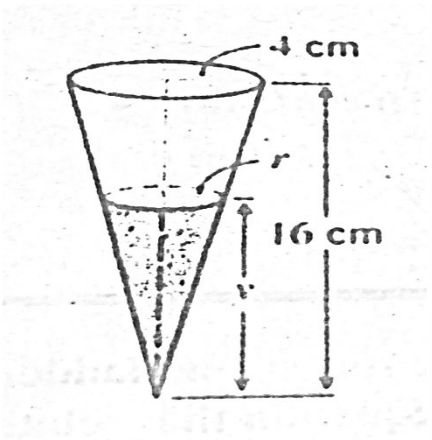
$$= \frac{66}{625} \approx .11 rad/sec \qquad \text{Ans.}$$

Problem: Suppose that liquid is to be cleared of sediment by pouring it through a conical filter that 16 cm high and has a radius of 4 cm at the top. Suppose also that the liquid is forced out of the cane at a constant rate of$2cm^3/min$**.**

(a) Find a formula that expresses the rate of change to the depth of the liquid in terms the depth.

(b) At what rate is the depth of the liquid changing at the instant when the level is 8 cm deep?

Solution:



Let

t = time elapsed from the initial observation (min)

v = volume of liquid in the cone at time t (cm^3)

y = depth of the liquid in the cone at time t (cm)

r = radius of the liquid surface at time t (cm)

Given that $\frac{dv}{dt} = -2$ (-ve sign because v decreases as t increases)

By formula, the volume of a cone

$$v = \frac{1}{3}\pi r^2 y \quad \ldots\ldots\ldots\ldots..(1)$$

From figure (ii) using similar triangle

$$\frac{r}{y} = \frac{4}{16} = \frac{1}{4}$$

$$\therefore r = \frac{y}{4}$$

$$\therefore v = \frac{1}{3}\pi\frac{y^2}{16}y = \frac{\pi}{48}y^3$$

$$\therefore \frac{dv}{dt} = \frac{\pi}{48}3y^2\frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{16}{\pi y^2}.\frac{dv}{dt} = \frac{16}{\pi y^2}.(-2) = -\frac{32}{\pi y^2}.$$

$$\therefore \frac{dy}{dt} = -\frac{32}{\pi y^2}$$

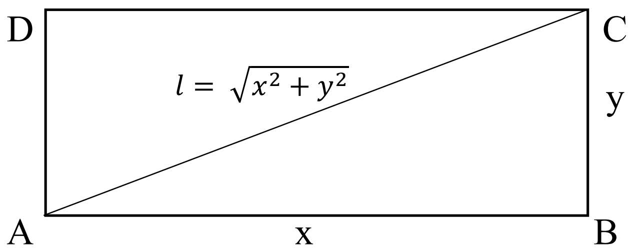
Which express $\frac{dy}{dt}$ in terms of y

(b) When the depth = 8 cm, then we have

$$\left.\frac{dy}{dt}\right|_{y=8} = --\frac{32}{\pi 8^2} = \frac{1}{2\pi} \approx -0.16\ cm/min$$

Problem: Let l be the length of a diagonal of a rectangle whose sides have lengths x and y and assume that x and y vary with time. If x increases at a constant rate of $\frac{1}{2}$ft/s and y decreases at a constant rate of $\frac{1}{4}$ ft./s. How fast is the size of the diagonal changing when x = 3 ft. and y = 4 ft. Is the diagonal increasing or decreasing at that instant?

Solution:



From the theorem of Pythagoras we have

$$l^2 = x^2 + y^2 \quad \text{----------------(i)}$$

Given that $\frac{dx}{dt} = \frac{1}{2}$ft./sec, $\frac{dy}{dt} = \frac{1}{4}$ ft./sec

Now differentiating (i) w.r.to t, we have

$$2l\frac{dl}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\Rightarrow l\frac{dl}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

Now putting x = 3 and y = 4.

$$\therefore \frac{dl}{dt} = \frac{1}{l}\left(3.\frac{1}{2} + 4.\frac{-1}{4}\right)$$

$$= \frac{1}{\sqrt{9+16}}\left(\frac{3}{2} - 1\right)$$

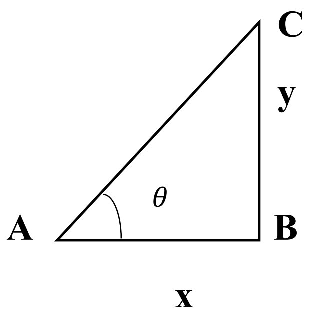
$$=\frac{1}{5}.\frac{1}{2}=\frac{1}{10}$$

At the instant, the diagonal is increasing.

Problem: Let θ be an acute angle in a right triangle and let x and y respectively be the lengths of the sides adjacent and opposite θ. Suppose also that x and y vary with time.

At a certain instant, $x = 2$ units and is increasing at 1 units/sec while y = 2 units and is decreasing at $\frac{1}{4}$ $unit/sec$. How fast is θ changing at that instant? Is θ increasing or decreasing at that instant?

Solution:



Given that $\frac{dx}{dt} = 1$unit/sec and $\frac{dy}{dt} = -\frac{1}{4}$ unit/sec.

From the figure, we get $\theta = \tan^{-1}\frac{y}{x}$ -------------------------- (i)

Differentiating (i) w.r.to t then we have

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\frac{dy}{dt} - \frac{y}{x^2}\frac{dx}{dt}\right)$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\frac{dy}{dt} - \frac{y}{x^2}\frac{dx}{dt}\right)$$

Putting x = 2, y = 2, $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = -\frac{1}{4}$.

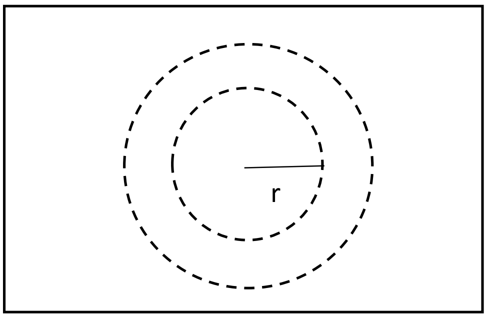
$$\frac{d\theta}{dt} = \frac{4}{4+4}\left(\frac{1}{2}\frac{-1}{4} - \frac{2}{4}.1\right)$$

$$= -\frac{5}{16} \text{ rad/sec}$$

(-ve) sign tells us that θ is decreasing at that instant.

Problem: A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 3 ft./sec. How rapidly is the area enclosed by the ripple increasing at the end of 10 s?

Solution:



Given that $\frac{dr}{dt} = 3$ft/sec

From the formula, we have

A(Area of a circle) = πr^2

$\therefore \quad \frac{dA}{dt} = \pi . 2r . \frac{dr}{dt}$

$= 2\pi r . 3 = 6\pi r$

After 1 sec the radius increases 3 ft.

After 10 seconds the radius increases $3\times 10 = 30$ ft.

$\therefore \quad \left.\frac{dA}{dt}\right|_{t=10} = 6\pi \times 3 = 18\pi ft^2/sec$ Ans.

Problem: Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 6 mi^2/hr. How past is the radius of the spill increasing when the area is 9 $mi.^2$?

Solution: We know that**,** the area of the circle A is

$$A = \pi r^2$$

Differentiating w.r.to t, we have

$$\frac{dA}{dt} = \pi . 2r . \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{2\pi r}\frac{dA}{dt}$$

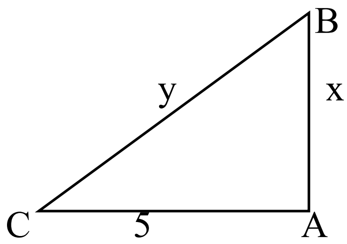
$$= \frac{1}{2\pi\sqrt{\frac{A}{\pi}}}\frac{dA}{dt}$$

$$= \frac{1}{2\pi\frac{3}{\sqrt{\pi}}} 6$$

$$= \frac{1}{\sqrt{\pi}} \quad \text{Ans.}$$

Problem: A rocket, rising vertically, is tracked by a radar. How fast is the rocket rising 4 mile high and distance from the radar station is increasing at rate of 200 mi/h?

Solution:



Let AB = x mile height

BC = y mile hypotenuse

$\therefore \; y^2 = x^2 + 5^2$ -----(i)

Here, $\frac{dy}{dt} = 2000$ mile/hr

Differentiating (i) w.r.to t, we get

$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$

$\Rightarrow y\frac{dy}{dt} = x\frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt}$

When, x = 4 then $y^2 = 4^2 + 5^2$

$y = \sqrt{41}$

$\therefore \frac{dx}{dt} = \frac{\sqrt{41}}{4}.2000$

$= 3201.56$ mile/hr. Ans.

Problem: If the area of a circle increases at a uniform rate, then show that the rate of increase of the circumference will vary inversely as the radius.

Problem: A 13 ft. ladder is leaning against a wall. If the top of the ladder slips down the wall at a rate of 2ft/sec. How fast will the foot be moving away from the wall when the top is 5 ft. above the ground? Ans. 5/6 ft./sec