# SECOND ORDER CIRCUITS

*order of a circuit* = *order of the differential equations required to describe the circuit* = *the number of independent energy storage elements*

# SECOND ORDER CIRCUITS

- $2^{nd}$-order circuits have 2 independent energy storage elements (inductors and/or capacitors)
- Analysis of a $2^{nd}$-order circuit yields a $2^{nd}$-order differential equation (DE)
- A 2nd-order differential equation has the form:

$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_o x(t) = f(t)$$

- Solution of a $2^{nd}$-order differential equation requires two initial conditions: x(0) and x'(0)
- All higher order circuits ($3^{rd}$, $4^{th}$, etc) have the same types of responses as seen in $1^{st}$-order and $2^{nd}$-order circuits

•Since 2nd-order circuits have two energy-storage types, the circuits can have the following forms:

1) Two capacitors
2) Two inductors
3) One capacitor and one inductor

A) Series RLC circuit B) Parallel RLC circuit C) Others

# SOLUTION TO A 2ND ORDER DIFFERENTIAL EQUATION

$$\text{Complete response} = \underset{\text{stored energy}}{\text{natural response}} + \underset{\text{independent source}}{\text{forced response}}$$

**The general solution to a differential equation has two parts:**

$x(t) = x_h + x_p$ = **homogeneous solution + particular solution**

**or**

$x(t) = x_n + x_f$ = **natural solution + forced solution**

# NATURAL RESPONSE

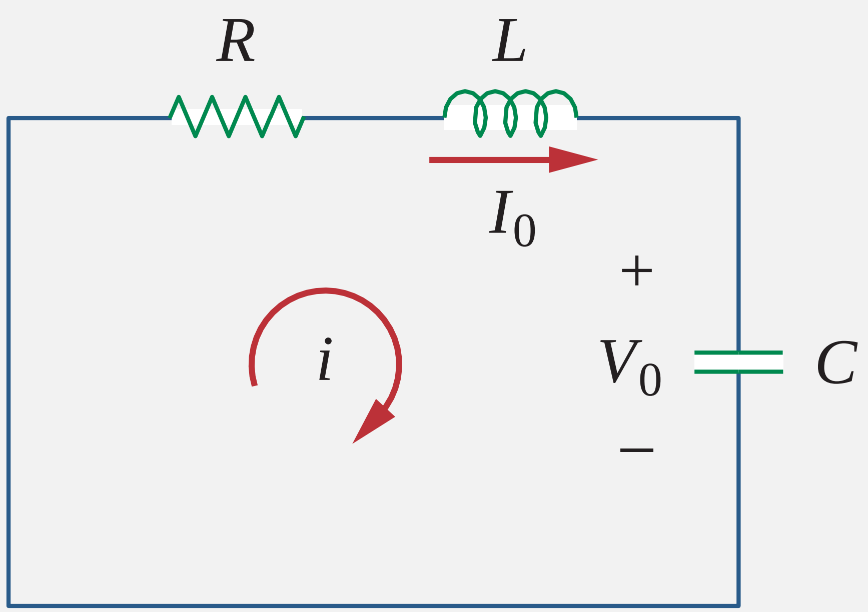
$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_o x(t) = 0$$

- Substituting the general form of the solution $Ae^{st}$ yields the ***characteristic*** equation:
  - $s^2 + a_1 s + a_o = 0$

- Finding the roots of this quadratic (called the ***characteristic roots*** or ***natural frequencies***) yields:

$$\boxed{s_1\,,\ s_2 = \frac{-a_1 \pm \sqrt{(a_1)^2 - 4a_o}}{2}}$$

# THE SOURCE--FREE SERIES RLC CIRCUITS




- KVL around the loop:

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0$$

- Differentiate to avoid integration

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

- Solution of this differential equation is what we are looking for.

# SERIES RLC - NATURAL FREQUENCIES

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

A more compact way of expressing the roots is

$$\boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}}$$

$$\boxed{\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}}$$

$\alpha$ = damping coefficient

$w_o$ = resonant frequency

# NATURAL RESPONSE OF SERIES RLC

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- constants are determined by the initial conditions
- three possible outcomes:

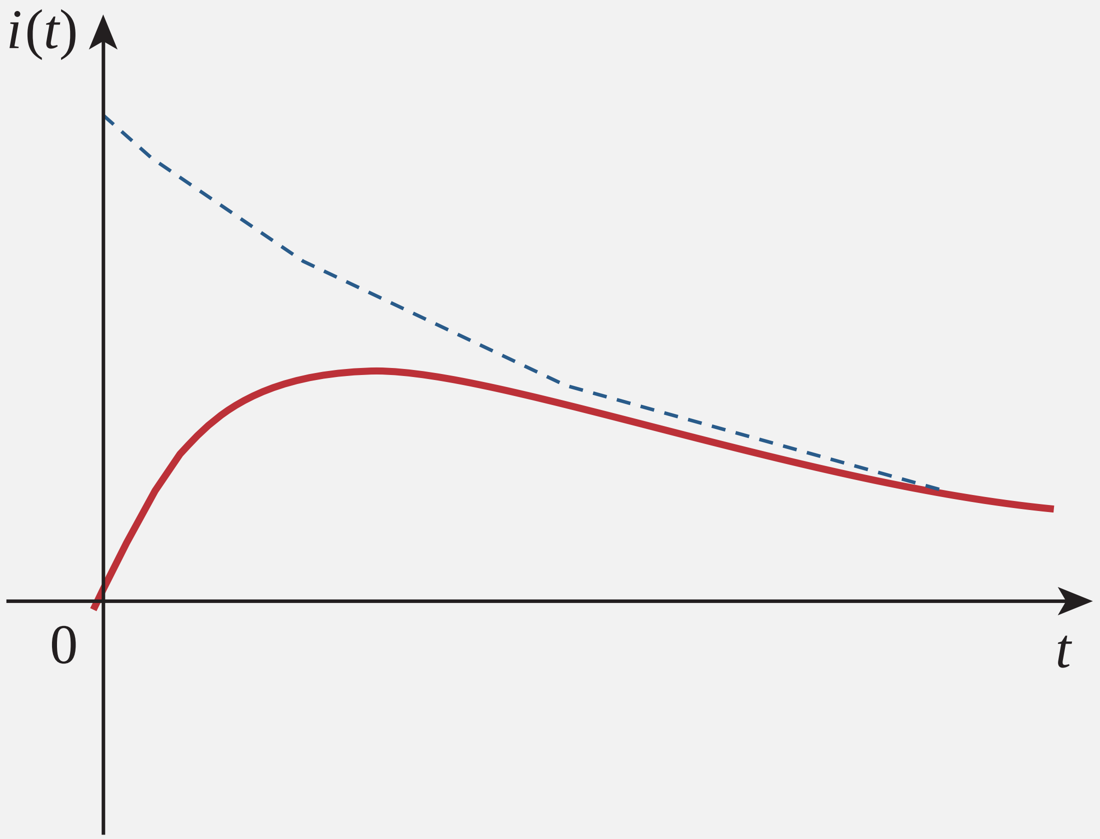
1. If $a > \omega_0$, we have the *overdamped* case.
2. If $a = \omega_0$, we have the *critically damped* case.
3. If $a < \omega_0$, we have the *underdamped* case.

# OVER-DAMPED RESPONSE

$(A > \omega_0)$

- roots are real and distinct

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

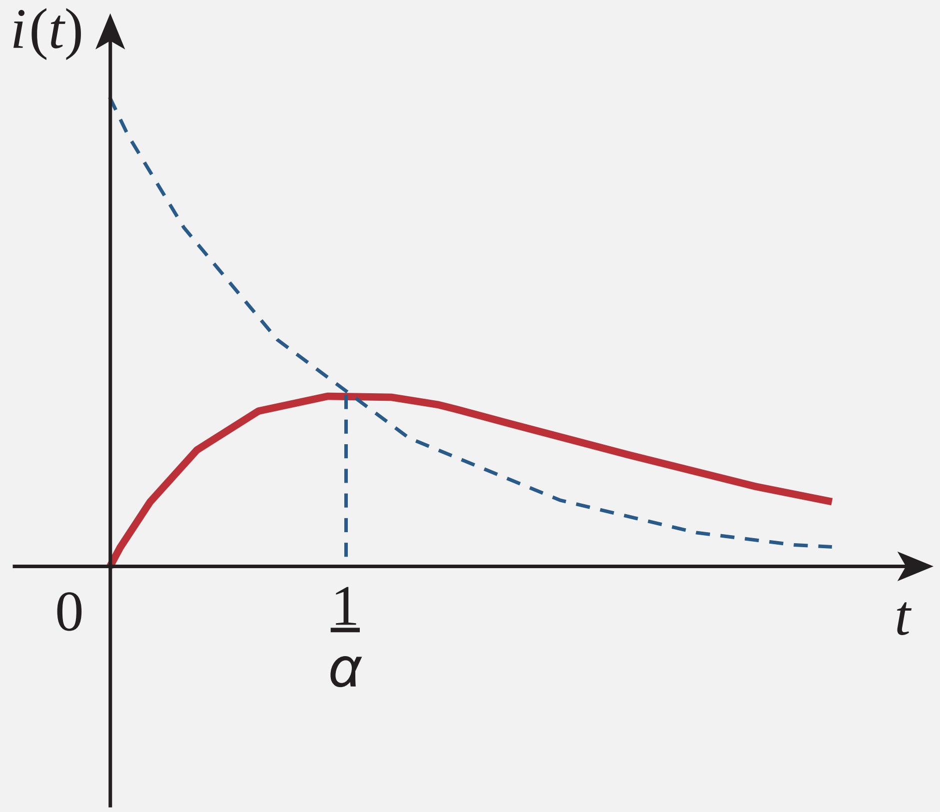


# CRITICALLY DAMPED RESPONSE

## $(A = \omega_0)$

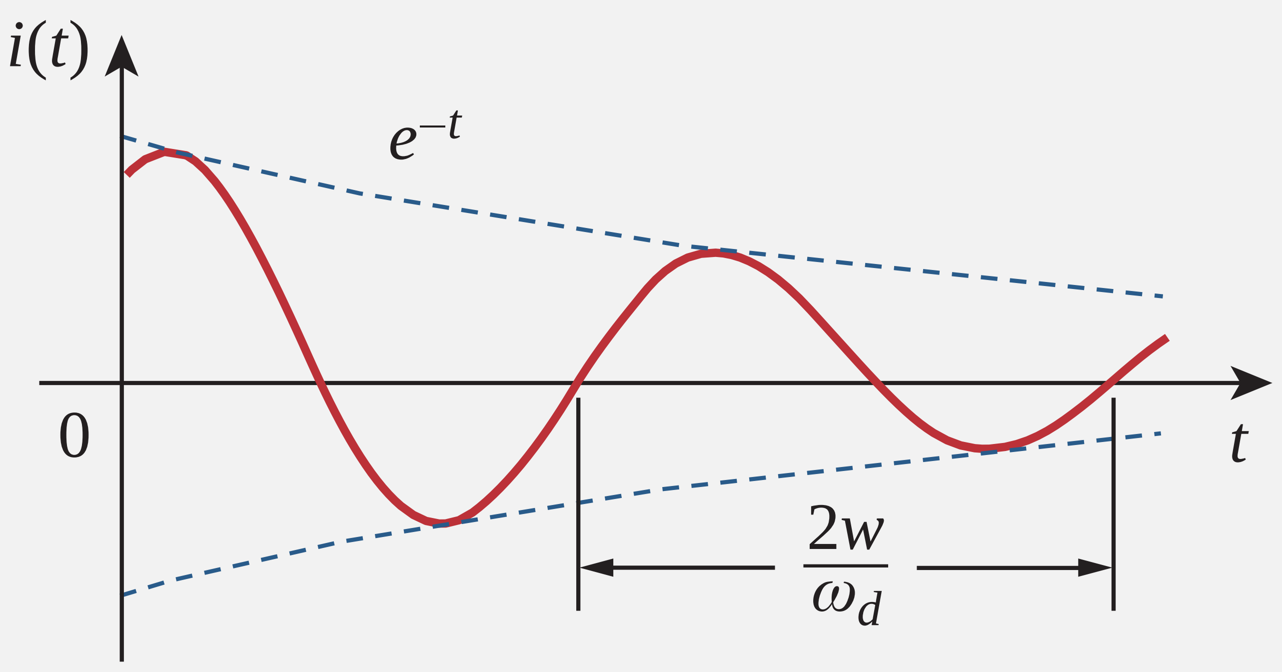
- roots are repeated

$$i(t) = (A_2 + A_1 t)e^{-at}$$



# UNDER DAMPED RESPONSE

**$(A < \omega_0)$**




- roots are complex

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$
$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

$$i(t) = e^{-at}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

# EXAMPLE

**Example 8.3**

In Fig. 8.8, $R = 40\ \Omega$, $L = 4$ H, and $C = 1/4$ F. Calculate the characteristic roots of the circuit. Is the natural response overdamped, underdamped, or critically damped?

# EXAMPLE

**Solution:**

We first calculate

$$\alpha = \frac{R}{2L} = \frac{40}{2(4)} = 5, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times \frac{1}{4}}} = 1$$

The roots are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -5 \pm \sqrt{25 - 1}$$

or

$$s_1 = -0.101, \qquad s_2 = -9.899$$

Since $\alpha > \omega_0$, we conclude that the response is overdamped. This is also evident from the fact that the roots are real and negative.

# EXAMPLE

**Example 8.4**

Find $i(t)$ in the circuit of Fig. 8.10. Assume that the circuit has reached steady state at $t = 0^-$.

# EXAMPLE

**Solution:**

For $t < 0$, the switch is closed. The capacitor acts like an open circuit while the inductor acts like a shunted circuit. The equivalent circuit is shown in Fig. 8.11(a). Thus, at $t = 0$,

$$i(0) = \frac{10}{4 + 6} = 1 \text{ A}, \qquad v(0) = 6i(0) = 6 \text{ V}$$

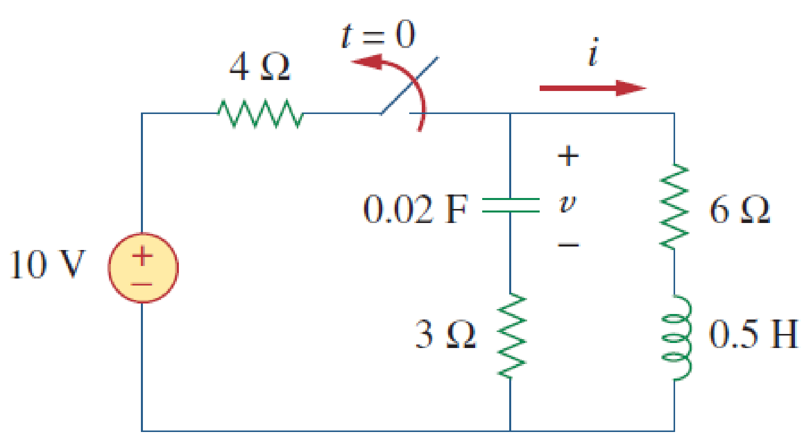
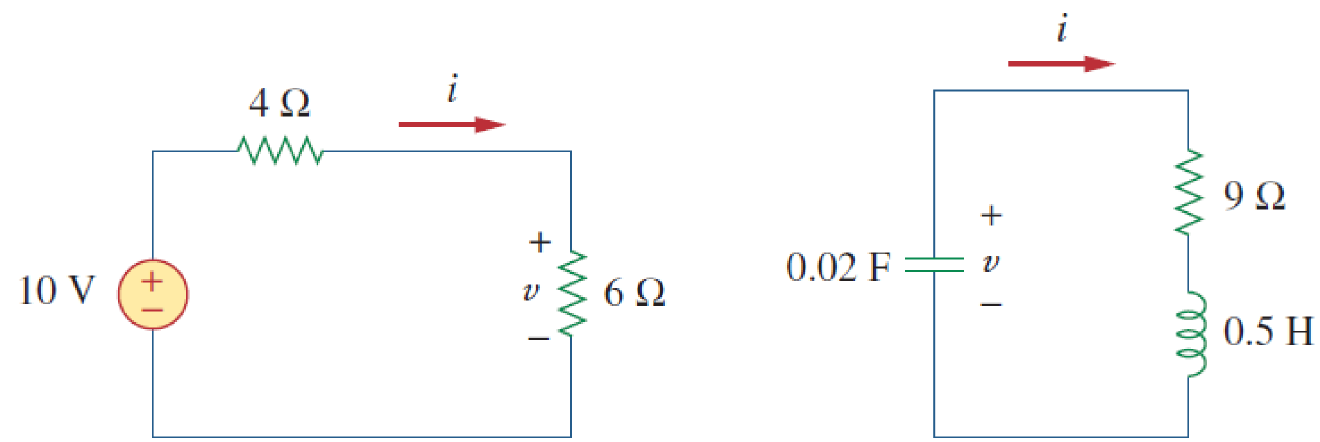



**Figure 8.10**
For Example 8.4.




(a) (b)

**Figure 8.11**
The circuit in Fig. 8.10: (a) for $t < 0$, (b) for $t > 0$.

# EXAMPLE

For $t > 0$, the switch is opened and the voltage source is disconnected. The equivalent circuit is shown in Fig. 8.11(b), which is a source-free series *RLC* circuit. Notice that the 3-Ω and 6-Ω resistors, which are in series in Fig. 8.10 when the switch is opened, have been combined to give $R = 9\ \Omega$ in Fig. 8.11(b). The roots are calculated as follows:

$$\alpha = \frac{R}{2L} = \frac{9}{2(\frac{1}{2})} = 9, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times \frac{1}{50}}} = 10$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -9 \pm \sqrt{81 - 100}$$

or

$$s_{1,2} = -9 \pm j4.359$$

Hence, the response is underdamped ($\alpha < \omega$); that is,

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) \qquad \textbf{(8.4.1)}$$

# EXAMPLE

We now obtain $A_1$ and $A_2$ using the initial conditions. At $t = 0$,

$$i(0) = 1 = A_1 \tag{8.4.2}$$

From Eq. (8.5),

$$\left.\frac{di}{dt}\right|_{t=0} = -\frac{1}{L}[Ri(0) + v(0)] = -2[9(1) - 6] = -6 \text{ A/s} \tag{8.4.3}$$

Note that $v(0) = V_0 = -6$ V is used, because the polarity of $v$ in Fig. 8.11(b) is opposite that in Fig. 8.8. Taking the derivative of $i(t)$ in Eq. (8.4.1),

$$\begin{aligned}\frac{di}{dt} &= -9e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)\\ &\quad + e^{-9t}(4.359)(-A_1 \sin 4.359t + A_2 \cos 4.359t)\end{aligned}$$

Imposing the condition in Eq. (8.4.3) at $t = 0$ gives

$$-6 = -9(A_1 + 0) + 4.359(-0 + A_2)$$

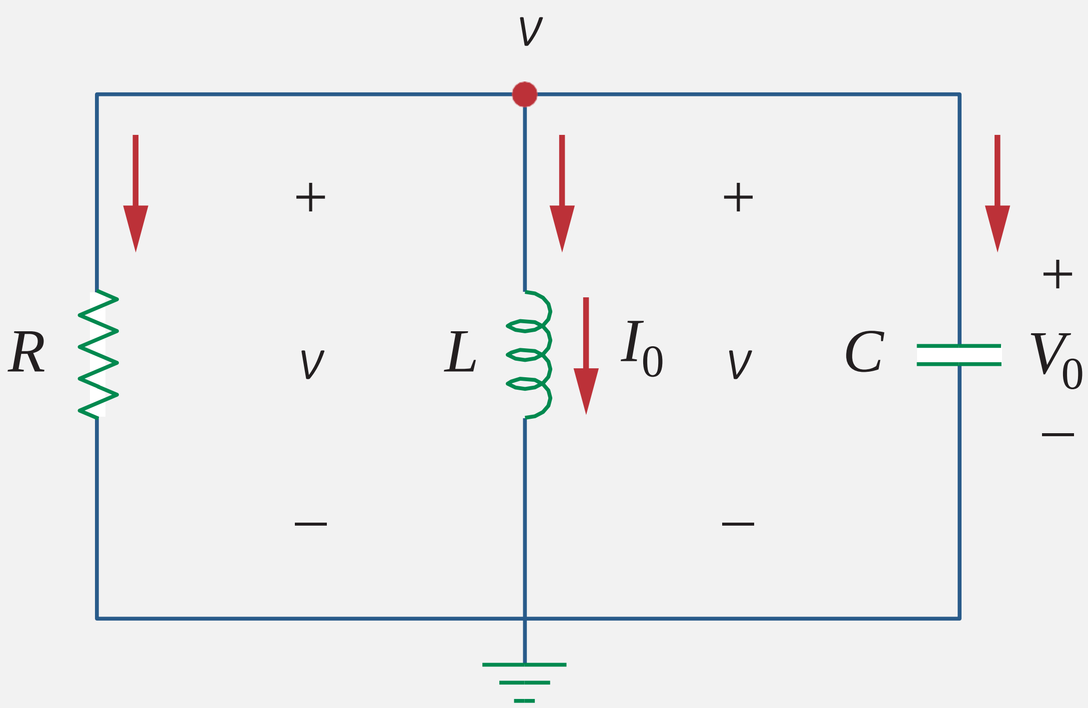
But $A_1 = 1$ from Eq. (8.4.2). Then

$$-6 = -9 + 4.359A_2 \qquad \Rightarrow \qquad A_2 = 0.6882$$

Substituting the values of $A_1$ and $A_2$ in Eq. (8.4.1) yields the complete solution as

$$i(t) = e^{-9t}(\cos 4.359t + 0.6882 \sin 4.359t) \text{ A}$$

# THE SOURCE--FREE PARALLEL RLC CIRCUITS




- KCL at the top node

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v\,dt + C\frac{dv}{dt} = 0$$

- differentiate to avoid integration

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

# CHARACTERISTIC EQUATION AND THE ROOTS

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$s_{1,2} = \frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$\boxed{s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}}$$

where

$$\boxed{\alpha = \frac{1}{2RC}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}}$$

# OVER-DAMPED RESPONSE

## $(A > \omega_0)$

- roots are real and distinct

$$\boxed{v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

# CRITICALLY DAMPED RESPONSE

$$(A = \omega_0)$$

- roots are repeated

$$v(t) = (A_1 + A_2 t)e^{-a\,t}$$

# UNDER DAMPED RESPONSE

## $(\alpha < \omega_0)$

- roots are complex

$$s_1 = -\alpha + \sqrt{(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$

$$s_2 = -\alpha - \sqrt{(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

$$\omega_d = \sqrt{(\omega_0^2 - \alpha^2)}$$

$$\boxed{v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)}$$

# EXAMPLE

**Example 8.5**

In the parallel circuit of Fig. 8.13, find $v(t)$ for $t > 0$, assuming $v(0) = 5$ V, $i(0) = 0$, $L = 1$ H, and $C = 10$ mF. Consider these cases: $R = 1.923\ \Omega$, $R = 5\ \Omega$, and $R = 6.25\ \Omega$.

# PROCEDURE:

**1. Find the characteristic equation and the natural response**

A) Determine if the circuit is a series RLC or parallel RLC (for $t > 0$ with independent sources killed). If the circuit is not series RLC or parallel RLC determine the describing equation of capacitor voltage or inductor current.

B) Obtain the characteristic equation. Use the standard formulas for $\alpha$ and $w_o$ for a series RLC circuit or a parallel RLC circuit. Use these values of $\alpha$ and $w_o$ in the characteristic equation as: $s^2 + 2\alpha s + w_o^2$.

C) Find the roots of the characteristic equation (characteristic roots).

D) Determine the form of the natural response based on the type of characteristic roots:

**2. Find the forced response** ***- Analyze the circuit at $t = \infty$ to find $x_f = x(\infty)$.***

**3. Find the initial conditions, x(0) and x'(0).**

**4. Find**

A) Find the total response, $x(t) = x_n + x_f$ .

B) Use the two initial conditions to solve for the two unknowns in the total response