ANTIDERIVATIVES

A function F is called an anti-derivatives of a function f on a given interval I if $F'(x) = f(x)$ for all x in the interval.

$$\begin{aligned}
&F'(x) = f(x) \\
&\Rightarrow \frac{d}{dx}(F(x)) = f(x) \\
&\Rightarrow \int \frac{d}{dx}(F(x))dx = \int f(x)dx + c \\
&\Rightarrow F(x) = \int f(x)dx + c
\end{aligned}$$

DEFINITION (Area under a curve):

If the function f is continuous on [a, b] and if $f(x) \geq 0$ for all x in [a, b], then the area under the curve $y = f(x)$ over the interval [a, b] is defined by

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*) \Delta x$$

RIEMANN SUMS AND THE DEFINITE INTEGRAL

Partition

A partition of the interval [a, b] is a collection of points $a = x_0 < x_1 < ..... < x_{n-1} < x_n = b$ that divides [a, b] into n subintervals of lengths

$$\Delta x_1 = x_1 - x_0, \Delta x_2 = x_2 - x_1, \Delta x_3 = x_3 - x_2, ... \Delta x_n = x_n - x_{n-1}$$

The partition is said to be ***regular*** provided the subintervals all have the same length

$$\Delta x_k = \Delta x = \frac{b-a}{n}$$

Otherwise the partition is called **general** partition.

The magnitude $\max \Delta x_k$ is called the ***mesh*** size of the partition.

We must replace the constant length Δx by the variable length Δx_k, When this is done the sum $\sum_{k=1}^{n} f(x_k^*) \Delta x$ is replaced by $\sum_{k=1}^{n} f(x_k^*) \Delta x_k$.

The area A between the graph of f and the interval $\max \Delta x_k$ satisfy the equation

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

The limit that appears in this expression is one of the fundamental concepts of integral calculus.

Integrable

A function *f* is said to be integrable on a finite closed interval [a, b] if the limit

$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and does not depend on the choice of partition or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol $\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$, which is called the definite integral of *f* from a to b. The numbers a and b are called the ***lower limit of integration*** and the ***upper limit of integration***, respectively, and *f(x)* is called the **integrand.**

Theorem:

If a function *f* is continuous on an interval [a, b], then *f* is integrable on [a, b], and the net signed area A between the graph of *f* and the interval [a, b] is

$$A = \int_a^b f(x)dx$$

PROPERTIES OF THE DEFINITE INTEGRAL

a. If a is in the domain of *f*, we define $\int_a^a f(x)dx = 0$

b. If *f* is integrable on [a, b], then we define

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

c. If *f* and *g* are integrable on [a, b] and if c is a constant, then *cf*, f+g, f-g are integrable on [a, b] and

$$(i)\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$(ii)\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

d. If *f* is integrable on a closed interval containing the three points a, c and b, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

THE FUNDAMENTAL THEOREM OF CALCULUS

Statement: If *f* is continuous on *[a, b]* and *F* is any antiderivative of *f* on [a, b]

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof:

PROOF. Let $x_1, x_2, \ldots, x_{n-1}$ be any points in $[a, b]$ such that

$$a < x_1 < x_2 < \cdots < x_{n-1} < b$$

These values divide $[a, b]$ into n subintervals

$$[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b] \tag{3}$$

whose lengths, as usual, we denote by

$$\Delta x_1, \Delta x_2, \ldots, \Delta x_n$$

By hypothesis, $F'(x) = f(x)$ for all x in $[a, b]$, so F satisfies the hypotheses of the Mean-Value Theorem (4.7.2) on each subinterval in (3). Hence, we can find points $x_1^*, x_2^*, \ldots, x_n^*$ in the respective subintervals in (3) such that

$$\begin{aligned}
F(x_1) - F(a) &= F'(x_1^*)(x_1 - a) = f(x_1^*)\Delta x_1 \\
F(x_2) - F(x_1) &= F'(x_2^*)(x_2 - x_1) = f(x_2^*)\Delta x_2 \\
F(x_3) - F(x_2) &= F'(x_3^*)(x_3 - x_2) = f(x_3^*)\Delta x_3 \\
&\vdots \\
F(b) - F(x_{n-1}) &= F'(x_n^*)(b - x_{n-1}) = f(x_n^*)\Delta x_n
\end{aligned}$$

Adding the preceding equations yields

$$F(b) - F(a) = \sum_{k=1}^{n} f(x_k^*)\Delta x_k \tag{4}$$

Let us now increase n in such a way that $\max \Delta x_k \to 0$. Since f is assumed to be continuous, the right side of (4) approaches $\int_a^b f(x)\,dx$ by Theorem 5.5.2 and Definition 5.5.1. However, the left side of (4) is independent of n; that is, the left side of (4) remains constant as n increases. Thus,

$$F(b) - F(a) = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx \qquad \blacksquare$$

Mean value theorem

If *f(x)* is

(i) continuous in [a, b]

(ii) $f'(x)$ exists in (a, b)

then there exists at least one value of x say c between a and b s.t.

$$f(b)-f(a)=(b-a)f'(c)$$

Substitution method:

$\int f(g(x))g'(x)dx$

Let $g(x) = u$, $du = g'(x)dx$

$\therefore \int f(u)du$

Example:

(i) $\int \cos^3 x \sin x dx$

(ii) $\int \frac{3xdx}{\sqrt{4x^2+5}}dx$

(iii) $\int x^2\sqrt{1+x}dx$

$(iv)\int \frac{6}{(1-2x)^3}dx$

$(v)\int \frac{x^2+1}{\sqrt{x^3+3x}}dx$

$(vi)\int (a+bx)^n dx$

$(vii)\int \sqrt[n]{a+bx}dx$

$$(viii)\int \frac{x^2+1}{x^4+x^2+1}dx$$

Solution: $I = \int \frac{x^2+1}{x^4+x^2+1}\,dx$ putting $x - \frac{1}{x} = z$

$\therefore \left(1+\frac{1}{x^2}\right)dx = dz.$

$$= \int \frac{x^2\left(1+\frac{1}{x^2}\right)dx}{x^2\left(x^2+1+\frac{1}{x^2}\right)}$$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx$$

$$= \int \frac{1+\frac{1}{x^2}}{x^2 - 2\cdot x\cdot\frac{1}{x} + \left(\frac{1}{x}\right)^2 + 3}\,dx$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+3}$$

$$\therefore I = \int \frac{dz}{z^2+3}$$

$$= \int \frac{dz}{z^2+(\sqrt{3})^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) + C \text{ Ans.}$$

Ex. $\int \sqrt{\frac{a+x}{a-x}}\,dx.$

Ans.

Sol^n: $I = \int \sqrt{\frac{a+x}{a-x}}\,dx.$

putting $x = a\cos 2\theta. \quad \therefore \theta = \frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right)$

$\therefore dx = -a\sin 2\theta \cdot 2\,d\theta$

$$\therefore I = \int \sqrt{\frac{a+a\cos 2\theta}{a-a\cos 2\theta}}\,(-2a\sin 2\theta)\,d\theta$$

$$= -\int \sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}}\cdot 2a\sin 2\theta$$

$$I = -\int \frac{\cos\theta}{\cancel{\sin\theta}}\cdot 2a\cdot 2\cancel{\sin\theta}\cos\theta\,d\theta$$

$$= -4a\int \cos^2\theta\,d\theta$$

$$= -2a\int 2\cos^2\theta\,d\theta$$

$$= -2a\int (1+\cos 2\theta)\,d\theta$$

$$= -2a\left[\theta + \frac{\sin 2\theta}{2}\right]$$

$$= -2a\left[\frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\sin 2\cdot\frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right)\right]$$

$$= -2a\left[\frac{1}{2}\cos^{-1}\frac{x}{a} + \frac{1}{2}\sin\cos^{-1}\left(\frac{x}{a}\right)\right]$$

$$= -a\left[\cos^{-1}\frac{x}{a} + \sin\cos^{-1}\left(\frac{x}{a}\right)\right] + c \text{ Ans.}$$

Ex. $\displaystyle\int \frac{dx}{\{(x+a)^2-a^2\}^{3/2}}$

Sol^n $\displaystyle I = \int \frac{dx}{\{(x+a)^2-a^2\}^{3/2}}$

putting $x+a = a\sec\theta$

$\therefore dx = a\sec\theta\tan\theta\, d\theta$

$$\therefore I = \int \frac{a\sec\theta\tan\theta\, d\theta}{(a^2\sec^2\theta - a^2)^{3/2}}$$

$$= \int \frac{a\sec\theta\tan\theta\, d\theta}{a^3\cdot\tan^3\theta}$$

$$= \frac{1}{a^2}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \frac{1}{a^2}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{a^2}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{a^2}\int \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta}\, d\theta$$

$$= \frac{1}{a^2}\int \operatorname{cosec}\theta\cot\theta\, d\theta$$

$$= \frac{1}{a^2}(-\operatorname{cosec}\theta)$$

$$= -\frac{1}{a^2}\operatorname{cosec}\theta$$

$$= -\frac{1}{a^2}\,\frac{1}{\sin\theta}$$

$$= -\frac{1}{a^2}\,\frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\cos\theta}$$

$$= -\frac{1}{a^2}\,\cot\theta\cdot\sec\theta$$

$$= -\frac{1}{a^2}\,\frac{\sec\theta}{\tan\theta} = \frac{-1}{a^2}\,\frac{\frac{x+a}{a}}{\sqrt{\sec^2\theta-1}}$$

$$= -\frac{x+a}{a^3\sqrt{\left(\frac{x+a}{a}\right)^2-1}}$$

$$= -\frac{x+a}{a^3\sqrt{\frac{(x+a)^2-a^2}{a^2}}}$$

$$= -\frac{x+a}{a^3\cdot\frac{1}{a}\sqrt{x^2+2ax}}$$

$$= -\frac{x+a}{a^2\sqrt{x^2+2ax}} + c$$

Ans.

Ex. $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$

Solution: $I = \int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$

putting $x-\alpha = z^2 \quad \therefore x = z^2+\alpha$.

$\therefore dx = 2z\,dz$.

$$\therefore I = \int \frac{2z\,dz}{\sqrt{z^2.(z^2+\alpha-\beta)}}$$

$$= 2\int \frac{dz}{\sqrt{z^2+(\sqrt{\alpha-\beta})^2}}$$

$$= 2\ln\left|z+\sqrt{z^2+\alpha-\beta}\right|$$

$$= 2\ln\left|\sqrt{x-\alpha}+\sqrt{x-\alpha+\alpha-\beta}\right|$$

$$= 2\ln\left|\sqrt{x-\alpha}+\sqrt{x-\beta}\right| + c$$ Ans.

Ex $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

Sol^n $I = \int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

putting $x-\alpha = z^2 \quad \therefore x = \alpha+z^2$

$\therefore dx = 2z\,dz$

$$\therefore I = \int \frac{2z\,dz}{\sqrt{z^2(\beta-z^2-\alpha)}}$$

$$= \int \frac{2\cancel{z}\,dz}{\cancel{z}\sqrt{\beta-\alpha-z^2}}$$

$$= 2\int \frac{dz}{\sqrt{(\sqrt{\beta-\alpha})^2-z^2}}$$

$$= 2.\sin^{-1}\left(\frac{z}{\sqrt{\beta-\alpha}}\right)$$

$$= 2\sin^{-1}\left(\frac{\sqrt{x-\alpha}}{\sqrt{\beta-\alpha}}\right) = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + c$$ Ans.

Ex. $\int \frac{dx}{\sqrt{(x-1)(2-x)}}$

Ex. $\int \sqrt{\frac{x+1}{x-1}}\, dx$

Sol^n $\int \sqrt{\frac{x+1}{x-1}}\, dx$

$= \int \frac{\sqrt{x+1}\,\sqrt{x+1}}{\sqrt{x-1}\,\sqrt{x+1}}\, dx$

$= \int \frac{x+1}{\sqrt{x^2-1}}\, dx$

$= \int \frac{x\,dx}{\sqrt{x^2-1}}\, dx + \int \frac{dx}{\sqrt{x^2-1}}$

$= I_1 + I_2$

where $I_1 = \int \frac{x\,dx}{\sqrt{x^2-1}}$

putting $x^2 - 1 = z^2$

$\therefore 2x\,dx = 2z\,dz$

$\Rightarrow x\,dx = z\,dz$

$\therefore I_1 = \int \frac{z\,dz}{z}$

$= \int dz = z = \sqrt{x^2-1}$

and $I_2 = \int \frac{dx}{\sqrt{x^2-1}}$

$= \cosh^{-1} x$

$\therefore I = I_1 + I_2$

$= \sqrt{x^2-1} + \cosh^{-1} x + c$

i) $\int_0^3 \frac{dx}{(x+2)\sqrt{x+1}}$ Put $x+1=z^2$

ii) $\int_0^a \sqrt{\frac{a+x}{a-x}}\,dx$ put $x = a\cos\theta$

iii) $\int_{1/2}^1 \frac{dx}{x\sqrt{1-x^2}}$ put $x = \sin\theta$

iv) $\int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ put $x = \sin\theta$