(***Manufacturing Costs***) A furniture manufacturer makes chairs and tables, each of which must go through an assembly process and a finishing process. The times required for these processes are given (in hours) by the matrix

$$A = \begin{array}{cc} \text{Assembly process} & \text{Finishing process} \end{array}$$
$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \begin{array}{l} \text{Chair} \\ \text{Table} \end{array}.$$

The manufacturer has a plant in Salt Lake City and another in Chicago. The hourly rates for each of the processes are given (in dollars) by the matrix

$$\begin{array}{cc} \text{Salt Lake City} & \text{Chicago} \end{array}$$
$$B = \begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix} \begin{array}{l} \text{Assembly process} \\ \text{Finishing process} \end{array}.$$

What do the entries in the matrix product AB tell the manufacturer?

Find the total amount to manufacture the chairs and tables in each plant.

Solution: Given that

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix}.$$

Therefore,

$$AB = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix}$$
$$= \begin{bmatrix} 2\times9+2\times10 & 2\times10+2\times12 \\ 3\times9+4\times10 & 3\times10+4\times12 \end{bmatrix}$$
$$= \begin{bmatrix} 38 & 44 \\ 67 & 78 \end{bmatrix}$$

(*Medicine*) A diet research project includes adults and children of both sexes. The composition of the participants in the project is given by the matrix

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \text{Adults} & \text{Children} \\ \left[\begin{array}{c} 80 \\ 100 \end{array}\right. & \left.\begin{array}{c} 120 \\ 200 \end{array}\right] \end{array}\begin{array}{l} \\ \text{Male} \\ \text{Female} \end{array}.$$

The number of daily grams of protein, fat, and carbohydrate consumed by each child and adult is given by the matrix

$$B = \begin{array}{ccc} \text{Protein} & \text{Fat} & \text{Carbo-hydrate} \\ \left[\begin{array}{c} 20 \\ 10 \end{array}\right. & \begin{array}{c} 20 \\ 20 \end{array} & \left.\begin{array}{c} 20 \\ 30 \end{array}\right] \end{array}\begin{array}{l} \\ \text{Adult} \\ \text{Child} \end{array}.$$

(a) How many grams of protein are consumed daily by the males in the project?

(b) How many grams of fat are consumed daily by the females in the project?

Solution: Given that

$$A = \begin{bmatrix} 80 & 120 \\ 100 & 200 \end{bmatrix} \text{ and } B = \begin{bmatrix} 20 & 20 & 20 \\ 10 & 20 & 30 \end{bmatrix}.$$

(a) The number of daily grams of protein by the males is 80×20+120×10=1600+1200=2800

(b) The number of daily grams of fat by the females is 100×20+200×20=2000+4000=6000

Page 34 (Example 2.9) Schaum's Outline series

Given that,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

and $f(x) = 2x^2 - 3x + 5$

$\therefore f(A) = 2A^2 - 3A + 5I$, where I is an identity matrix.

Now,

$$A^2 = AA = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1\times1+2\times3 & 1\times2+2\times(-4) \\ 3\times1+(-4)\times3 & 3\times2+(-4)\times(-4) \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}$$

$$\therefore f(A)=2\begin{bmatrix}7 & -6\\ -9 & 22\end{bmatrix}-3\begin{bmatrix}1 & 2\\ 3 & -4\end{bmatrix}+5\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}14 & -12\\ -18 & 44\end{bmatrix}-\begin{bmatrix}3 & 6\\ 9 & -12\end{bmatrix}+\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}$$

$$=\begin{bmatrix}14-3+5 & -12-6+0\\ -18-9+0 & 44+12+5\end{bmatrix}$$

$$=\begin{bmatrix}16 & -18\\ -27 & 61\end{bmatrix}$$

Hermitian Matrix: The conjugate transpose of a complex matrix is called the Hermitian matrix. Symbolically, if A is a complex matrix then $A^H=\left(\bar{A}\right)^T$

Here

$$A=\begin{bmatrix}3-5i & 2+4i\\ 6+7i & 1+8i\end{bmatrix}$$

$$\bar{A}=\begin{bmatrix}3+5i & 2-4i\\ 6-7i & 1-8i\end{bmatrix}$$

Therefore, $A^H=\left(\bar{A}\right)^T=\begin{bmatrix}3+5i & 2-4i\\ 6-7i & 1-8i\end{bmatrix}^T=\begin{bmatrix}3+5i & 6-7i\\ 2-4i & 1-8i\end{bmatrix}$

Here

$$A=\begin{bmatrix}2-3i & 5+8i\\ -4 & 3-7i\\ -6-i & 5i\end{bmatrix}$$

$$\bar{A}=\begin{bmatrix}2+3i & 5-8i\\ -4 & 3+7i\\ -6+i & -5i\end{bmatrix}$$

Therefore, $A^H=\left(\bar{A}\right)^T=\begin{bmatrix}2+3i & 5-8i\\ -4 & 3+7i\\ -6+i & -5i\end{bmatrix}^T=\begin{bmatrix}2+3i & -4 & -6+i\\ 5-8i & 3+7i & -5i\end{bmatrix}$

Adjoint of a matrix: Let $A=[a_{ij}]$ be an $n\times n$ matrix over a field K and let A_{ij} denote the cofactor of a_{ij}. The classical adjoint of A, denoted by adj A, is the transpose of the matrix of cofactors of A. Namely,

$$\text{adj } A=\left[A_{ij}\right]^T$$

Inverse of a matrix: $A^{-1} = \dfrac{1}{D}\text{adj } A = \dfrac{1}{|A|}\text{adj } A$

Rule of signs: $A = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Problem: Find the inverse of the matrix $A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$.

Solution: Given the matrix

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$$

Determinant of A, $|A| = 2\times(-20+2) - 3\times(0-2) - 4\times(0+4) = -36+6-16 = -46 \neq 0$

The cofactors of A are

$$A_{11} = \begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} = -20+2 = -18,\quad A_{12} = -\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = -(0-2) = 2,\quad A_{13} = \begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} = 0+4 = 4$$

$$A_{21} = -\begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix} = -(15-4) = -11,\quad A_{22} = \begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} = 10+4 = 14,\quad A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -(-2-3) = 5$$

$$A_{31} = \begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix} = 6-16 = -10,\quad A_{32} = -\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} = -(4-0) = -4,\quad A_{33} = \begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix} = -8+0 = -8$$

So,

$$\text{adj } A = \begin{bmatrix} -18 & 2 & 4 \\ -11 & 14 & 5 \\ -10 & -4 & -8 \end{bmatrix}^T = \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{|A|}\text{adj } A = -\frac{1}{46}\begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix} = \begin{bmatrix} \frac{9}{23} & \frac{11}{46} & \frac{5}{23} \\ -\frac{1}{23} & -\frac{7}{23} & \frac{2}{23} \\ -\frac{2}{23} & -\frac{5}{46} & \frac{4}{23} \end{bmatrix}$$

Problem: Find the inverse of the matrix $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

Solution: Given the matrix

$$A=\begin{bmatrix}2 & -1 & -1\\ 1 & -2 & 1\\ 1 & -1 & 2\end{bmatrix}$$

Determinant of A, $|A|=2\times(-4+1)+1\times(2-1)-1\times(-1+2)=-6+1-1=-6\neq 0$

The cofactors of A are

$$A_{11}=\begin{vmatrix}-2 & 1\\ -1 & 2\end{vmatrix}=-4+1=-3,\ A_{12}=-\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix}=-(2-1)=-1,\ A_{13}=\begin{vmatrix}1 & -2\\ 1 & -1\end{vmatrix}=-1+2=1$$

$$A_{21}=-\begin{vmatrix}-1 & -1\\ -1 & 2\end{vmatrix}=-(-2-1)=3,\ A_{22}=\begin{vmatrix}2 & -1\\ 1 & 2\end{vmatrix}=4+1=5,\ A_{23}=-\begin{vmatrix}2 & -1\\ 1 & -1\end{vmatrix}=-(-2+1)=1$$

$$A_{31}=\begin{vmatrix}-1 & -1\\ -2 & 1\end{vmatrix}=-1-2=-3,\ A_{32}=-\begin{vmatrix}2 & -1\\ 1 & 1\end{vmatrix}=-(2+1)=-3,\ A_{33}=\begin{vmatrix}2 & -1\\ 1 & -2\end{vmatrix}=-4+1=-3$$

So,

$$\operatorname{adj} A=\begin{bmatrix}-3 & -1 & 1\\ 3 & 5 & 1\\ -3 & -3 & -3\end{bmatrix}^T=\begin{bmatrix}-3 & 3 & -3\\ -1 & 5 & -3\\ 1 & 1 & -3\end{bmatrix}$$

Therefore,

$$A^{-1}=\frac{1}{|A|}\operatorname{adj} A=-\frac{1}{6}\begin{bmatrix}-3 & 3 & -3\\ -1 & 5 & -3\\ 1 & 1 & -3\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ \frac{1}{6} & -\frac{5}{6} & \frac{1}{2}\\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2}\end{bmatrix}$$

$x+2y=3$

$2x+4y=6$

$2(3-2y)+4y=6\Rightarrow 6-4y+4y=6\Rightarrow 0=0$

$\Rightarrow x+2y=3$

$x=1,$

$1+2y=3$

$\therefore y=1$

Variables=2

Equation=1

Number of Variables-Number of Equations=2-1=1

Therefore, number of free variable is 1.

$x+2y=3$

$2x+4y=2$

$\Rightarrow x+2y=3$
$1=3$

$x+2y=3$
$4y=2$
$\Rightarrow y=1/2$

System of linear equations

$AX = B$

Multiplying both sides by A^{-1} then we have
$A^{-1}AX = A^{-1}B$
$\Rightarrow IX = A^{-1}B$
$\therefore X = A^{-1}B$

$$A=\begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

Cramer's Rule

Theorem 2.3.1 Cramer's Rule

Let A be an $n \times n$ nonsingular matrix, and let $\mathbf{b} \in \mathbb{R}^n$. Let A_i be the matrix obtained by replacing the ith column of A by $\mathbf{b}$. If $\mathbf{x}$ is the unique solution of $A\mathbf{x} = \mathbf{b}$, then

$$x_i = \frac{\det(A_i)}{\det(A)} \quad \textit{for} \quad i = 1, 2, \ldots, n$$

Proof Since

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)}(\operatorname{adj} A)\mathbf{b}$$

it follows that

$$\begin{aligned} x_i &= \frac{b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni}}{\det(A)} \\ &= \frac{\det(A_i)}{\det(A)} \end{aligned}$$

■

EXAMPLE 3 Use Cramer's rule to solve

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 3x_3 &= 9 \end{aligned}$$

Solution

$$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -4 \quad \det(A_1) = \begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix} = -4$$

$$\det(A_2) = \begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix} = -4 \quad \det(A_3) = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix} = -8$$

Therefore,

$$x_1 = \frac{-4}{-4} = 1, \qquad x_2 = \frac{-4}{-4} = 1, \qquad x_3 = \frac{-8}{-4} = 2 \qquad \blacksquare$$

Book-Abdur Rahman Example 10 (page 25):
Solution: Given the system of equations

$$\left.\begin{array}{l} x+y-z=1 \\ 2x+3y+\lambda z=3 \\ x+\lambda y+3z=2 \end{array}\right\} \xrightarrow[L_3 \to -L_1+L_3]{L_2 \to -2L_1+L_2} \left.\begin{array}{l} x+y-z=1 \\ y+(\lambda+2)z=1 \\ (\lambda-1)y+4z=1 \end{array}\right\}$$

$$\xrightarrow{L_3 \to -(\lambda-1)L_2+L_3} \left.\begin{array}{l} x+y-z=1 \\ y+(\lambda+2)z=1 \\ \{-(\lambda-1)(\lambda+2)+4\}z=-\lambda+1+1 \end{array}\right\}$$

$$\longrightarrow \left.\begin{array}{l} x+y-z=1 \\ y+(\lambda+2)z=1 \\ (\lambda+3)(2-\lambda)z=2-\lambda \end{array}\right\}$$

(i) The above system has a unique solution if $(\lambda+3)(2-\lambda)\neq 0$,

=> $\lambda+3\neq 0$ and $2-\lambda\neq 0$ => $\lambda\neq -3$ and $\lambda\neq 2$

(ii) The above system has more than one solution if $2-\lambda=0$ => $\lambda=2$

(iii) The above system has no solution if $\lambda+3=0$ => $\lambda=-3$

Exercise 30 (Page 36): Given the system of equations

$$\left.\begin{array}{l} x-3z=-3 \\ 2x+\lambda y-z=-2 \\ x+2y+\lambda z=1 \end{array}\right\} \xrightarrow{L_1 \leftrightarrow L_3}$$

$$\left.\begin{array}{l} x+2y+\lambda z=1 \\ 2x+\lambda y-z=-2 \\ x-3z=-3 \end{array}\right\} \xrightarrow[L_3 \to -L_1+L_3]{L_2 \to -2L_1+L_2} \left.\begin{array}{l} x+2y+\lambda z=1 \\ (\lambda-4)y-(2\lambda+1)z=-4 \\ -2y-(\lambda+3)z=-4 \end{array}\right\}$$

$$\xrightarrow[\sim]{L_3 \to 2L_2+(\lambda-4)L_3} \left.\begin{aligned} & x+2y+\lambda z=1 \\ & (\lambda-4)y-(2\lambda+1)z=0 \\ & \{-(4\lambda+2)-(\lambda-4)(\lambda+3)\}z=-4(\lambda-4)-8 \end{aligned}\right\}$$

$$\sim \left.\begin{aligned} & x+2y+\lambda z=1 \\ & (\lambda+4)y+(2\lambda-1)z=0 \\ & -(\lambda^2-\lambda-12+4\lambda+2)z=-4(\lambda-2) \end{aligned}\right\}$$

Or, $\left.\begin{aligned} & x+2y+\lambda z=1 \\ & (\lambda+4)y+(2\lambda-1)z=0 \\ & (\lambda^2+3\lambda-10)z=4(\lambda-2) \end{aligned}\right\}$

Or, $\left.\begin{aligned} & x+2y+\lambda z=1 \\ & (\lambda+4)y+(2\lambda-1)z=0 \\ & (\lambda+5)(\lambda-2)z=4(\lambda-2) \end{aligned}\right\}$

(i) The above system has a unique solution if $(\lambda+5)(\lambda-2)\neq 0$,

=> $\lambda+5\neq 0$ and $\lambda-2\neq 0$ => $\lambda\neq -5$ and $\lambda\neq 2$

(ii) The above system has more than one solution if $\lambda-2=0$ => $\lambda=2$

(iii) The above system has no solution if $\lambda+5=0$ => $\lambda=-5$

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 6 & 2 \\ 2 & 10 & -4 \end{bmatrix} \underset{R_3 \to (-1)R_1+R_3}{\overset{R_2 \to -(3/2)R_1+R_2}{\sim}} \begin{bmatrix} 2 & -1 & 4 \\ 0 & \frac{15}{2} & -4 \\ 0 & 11 & -8 \end{bmatrix} \overset{R_3 \to (-22/15)R_2+R_3}{\sim} \begin{bmatrix} 2 & -1 & 4 \\ 0 & \frac{15}{2} & -4 \\ 0 & 0 & -\frac{32}{15} \end{bmatrix}$$

Since this matrix is in echelon form and has no zero row, hence the given vectors are linearly independent.

{(1,0),(0,1)}

(3,4)=3(1,0)+4(0,1)=(3,0)+(0,4)=(3+0,0+4)=(3,4)

() For S is a subspace,

(i) S is nonempty

(ii) $u,v\in S$, $\alpha u+\beta v\in S$

Let,

$v=\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3$, where $\alpha_1,\alpha_2,\alpha_3\in F$

$R_1 \leftrightarrow R_3$

$L_3 \rightarrow L_3/2$

Solution: We form the following matrix using the given vectors:

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 6 & 2 \\ 2 & 10 & -4 \end{bmatrix} \xrightarrow[L_3 \rightarrow L_1 - L_3]{L_2 \rightarrow 3L_1 - 2L_2} \begin{bmatrix} 2 & -1 & 4 \\ 0 & -15 & 8 \\ 0 & -11 & 8 \end{bmatrix} \xrightarrow[L_3 \rightarrow L_3/(-11)]{L_2 \rightarrow L_2/(-15)} \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & -\frac{8}{15} \\ 0 & 1 & -\frac{8}{11} \end{bmatrix}$$

$$\xrightarrow{L_3 \rightarrow L_2 - L_3} \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & -\frac{8}{15} \\ 0 & 0 & \frac{32}{165} \end{bmatrix}$$

The matrix is in echelon form and it has no zero row. Hence the given vectors are linearly independent.

We set
(i) $y=1$, $z=0$.
Then $x-3y+z=0 \Rightarrow x-3.1+0=0 \Rightarrow x=3$

(ii) $y=0$, $z=1$.
Then $x-3y+z=0 \Rightarrow x-3.0+1=0 \Rightarrow x=-1$
Thus we obtain the solutions $(3, 1, 0)$ and $(-1, 0, 1)$.
Hence the set $\{(3, 1, 0), (-1, 0, 1)\}$ is a basis of the solution space.

x=1, y=0, z=0
x=0, y=1, z=0
x=0, y=0, z=1

(1,2,1)=1(1,0,1)-2(0,-1,0)

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 5 & 15 & 10 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -12 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

u=(u1, u2, u3)
v=(v1, v2, v3)
Inner product, (u, v)=u1*v1+u2*v2+u3*v3
|| u ||=Sqrt(u, u)=Sqrt(u1*u1+u2*u2+u3*u3)
=Sqrt(u1^2+ u2^2+ u3^2)

$$\frac{\frac{1}{3}}{\frac{\sqrt{6}}{3}} = \frac{1}{3} \times \frac{3}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

$$2x + y = 1$$
$$x - y + 3z = 2$$
$$2y - 4z = 6$$

Or,

$$x - y + 3z = 2$$
$$3y - 6z = -3$$
$$2y - 4z = 6$$

Or,

$$x - y + 3z = 2$$
$$y - 2z = -1$$
$$y - 2z = 3$$

$$u_1 = \left(\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}\right)$$
$$\|u_1\| = \sqrt{(u_1, u_1)} = \sqrt{\left(\frac{i}{\sqrt{3}}\right)^2 + \left(\frac{i}{\sqrt{3}}\right)^2 + \left(\frac{i}{\sqrt{3}}\right)^2}$$
$$= \sqrt{-\frac{1}{3} - \frac{1}{3} - \frac{1}{3}} = \sqrt{-1} = \sqrt{i^2} = i$$

$(fog)(x)=f(g(x))$

Lu Factorization

3.39. Find the LU factorization of (a) $A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & -4 & 7 \\ -1 & -2 & 1 \end{bmatrix}$, (b) $B = \begin{bmatrix} 1 & 4 & -3 \\ 2 & 8 & 1 \\ -5 & -9 & 7 \end{bmatrix}$.

(a) Reduce A to triangular form by the following operations:

"Replace R_2 by $-2R_1 + R_2$," "Replace R_3 by $R_1 + R_3$," and then "Replace R_3 by $\frac{5}{2}R_2 + R_3$"

These operations yield the following, where the triangular form is U:

$$A \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & -5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix} = U \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{5}{2} & 1 \end{bmatrix}$$

The entries $2, -1, -\frac{5}{2}$ in L are the negatives of the multipliers $-2, 1, \frac{5}{2}$ in the above row operations. (As a check, multiply L and U to verify $A = LU$.)

$$A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & -4 & 7 \\ -1 & -2 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_1 + R_3]{R_2 \to -2R_1 + R_2} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & -5 & 6 \end{bmatrix} \xrightarrow{R_3 \to (5/2)R_2 + R_3} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{5}{2} & 1 \end{bmatrix}$$

(b) Reduce B to triangular form by first applying the operations "Replace R_2 by $-2R_1 + R_2$" and "Replace R_3 by $5R_1 + R_3$." These operations yield

$$B \sim \begin{bmatrix} 1 & 4 & -3 \\ 0 & 0 & 7 \\ 0 & 11 & -8 \end{bmatrix}.$$

Observe that the second diagonal entry is 0. Thus, B cannot be brought into triangular form without row interchange operations. Accordingly, B is not LU-factorable. (There does exist a PLU factorization of such a matrix B, where P is a permutation matrix, but such a factorization lies beyond the scope of this text.)

3.41. Find the LU factorization of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & -10 & 2 \end{bmatrix}$.

Reduce A to triangular form by the following operations:

(1) "Replace R_2 by $-2R_1 + R_2$," (2) "Replace R_3 by $3R_1 + R_3$," (3) "Replace R_3 by $-4R_2 + R_3$"

These operations yield the following, where the triangular form is U:

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix}$$

The entries $2, -3, 4$ in L are the negatives of the multipliers $-2, 3, -4$ in the above row operations. (As a check, multiply L and U to verify $A = LU$.)

Lu Factorization

3.69. Find the LU factorization of each of the following matrices:

(a) $\begin{bmatrix} 1 & -1 & -1 \\ 3 & -4 & -2 \\ 2 & -3 & -2 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & 1 \\ 3 & 4 & 2 \end{bmatrix}$, (c) $\begin{bmatrix} 2 & 3 & 6 \\ 4 & 7 & 9 \\ 3 & 5 & 4 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 7 & 10 \end{bmatrix}$

Least Squares Problem Given a linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns, find a vector $\boldsymbol{x}$, if possible, that minimizes $\|A\mathbf{x} - \mathbf{b}\|$ with respect to the Euclidean inner product on R^m. Such a vector is called a ***least squares solution*** of $A\mathbf{x} = \mathbf{b}$.

For any linear system $A\mathbf{x} = \mathbf{b}$, *the associated normal system*

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

is consistent, and all solutions of the normal system are least squares solutions of $A\mathbf{x} = \mathbf{b}$. Moreover, if W is the column space of A, and $\boldsymbol{x}$ is any least squares solution of $A\mathbf{x} = \mathbf{b}$, then the orthogonal projection of $\boldsymbol{b}$ on W is

$$\text{proj}_W \mathbf{b} = A\mathbf{x}$$

Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned} x_1 - x_2 &= 4 \\ 3x_1 + 2x_2 &= 1 \\ -2x_1 + 4x_2 &= 3 \end{aligned}$$

and find the orthogonal projection of $\boldsymbol{b}$ on the column space of A.

Solution

Here

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

Observe that A has linearly independent column vectors, so we know in advance that there is a unique least squares solution. We have

$$A^T A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

so the normal system $\boxed{A^T A\mathbf{x} = A^T \mathbf{b}}$ in this case is

$$\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

Solving this system yields the least squares solution

$$x_1 = \frac{17}{95}, \quad x_2 = \frac{143}{285}$$

From Formula 5, the orthogonal projection of $\boldsymbol{b}$ on the column space of A is

$$A\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \frac{17}{95} \\ \frac{143}{285} \end{bmatrix} = \begin{bmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{bmatrix}$$

Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$, and find the orthogonal projection of $\boldsymbol{b}$ onto the column space of A.

(a)
$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$$

(b)
$$A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

(c)
$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}$$

Solution:

The associated normal system is $A^T A\mathbf{x} = A^T\mathbf{b}$, or

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$$

or

$$\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

This system has solution $x_1 = 5$, $x_2 = 1/2$, which is the least squares solution of $A\mathbf{x} = \mathbf{b}$.

The orthogonal projection of $\mathbf{b}$ on the column space of A is $A\mathbf{x}$, or

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 11/2 \\ -9/2 \\ -4 \end{bmatrix}$$

(c) The associated normal system is

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}$$

or

$$\begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ -9 \end{bmatrix}$$

This system has solution $x_1 = 12$, $x_2 = -3$, $x_3 = 9$, which is the least squares solution of $A\mathbf{x} = \mathbf{b}$.

The orthogonal projection of $\mathbf{b}$ on the column space of A is $A\mathbf{x}$, or

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 12 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 9 \\ 0 \end{bmatrix}$$

which can be written as (3, 3, 9, 0).

$$\langle u-\langle u,v\rangle tv, u-\langle u,v\rangle tv\rangle = \langle u,u\rangle - \langle u,v\rangle t\langle v,u\rangle - \langle u,v\rangle t\langle u,v\rangle + \langle u,v\rangle t\langle u,v\rangle t\langle v,v\rangle$$

$$= \langle u,u\rangle - \langle u,v\rangle t\overline{\langle u,v\rangle} - \langle u,v\rangle t\langle u,v\rangle + \langle u,v\rangle t\langle u,v\rangle t\langle v,v\rangle$$

$$\|u\|^2 - 2t\left|\langle u,v\rangle^2\right| + \left|\langle u,v\rangle^2\right| t^2 \|v\|^2 \geq 0$$

Set

$$t = \frac{1}{\|v\|^2}$$

Then

$$\|u\|^2 - 2.\frac{1}{\|v\|^2}.\left|\langle u,v\rangle^2\right| + \left|\langle u,v\rangle^2\right|.\frac{1}{\|v\|^4}.\|v\|^2 \geq 0$$

$$\Rightarrow \|u\|^2 - 2.\frac{1}{\|v\|^2}.\left|\langle u,v\rangle^2\right| + \left|\langle u,v\rangle^2\right|.\frac{1}{\|v\|^2} \ge 0$$

$$\Rightarrow \|u\|^2 - \frac{1}{\|v\|^2}\left|\langle u,v\rangle^2\right| \ge 0$$

$$\Rightarrow \|u\|^2 \ge \frac{1}{\|v\|^2}\left|\langle u,v\rangle^2\right|$$

$$\Rightarrow \|u\|^2\|v\|^2 \ge \left|\langle u,v\rangle^2\right|$$

u=(1,1,2), v=(2,3,3)

$$\|u\|^2 = \langle u,u\rangle = 1^2 + 1^2 + 2^2 = 1+1+4 = 6$$

$$\|v\|^2 = \langle v,v\rangle = 2^2 + 3^2 + 3^2 = 4+9+9 = 22$$

$$\left|\langle u,v\rangle^2\right| = |1.2+1.3+2.3| = 2+3+6 = 11$$

$$\left|\langle u,v\rangle^2\right| = |1.2+1.3+2.3| = 2+3+6 = 11 \qquad (1)$$

Now, $\|u\|^2\|v\|^2 = 6.22 = 132 \qquad (2)$

From (1) and (2), we have

$$\|u\|^2\|v\|^2 > \left|\langle u,v\rangle^2\right|$$

An $n \times n$ matrix U is said to be unitary if its column vectors form an orthonormal set in $\mathbb{C}^n$. Thus, U is unitary if and only if $U^H U = I$. If U is unitary, then, since the column vectors are orthonormal, U must have rank n. It follows that

$U^{-1} = IU^{-1} = U^H U U^{-1} = U^H$

A real unitary matrix is an orthogonal matrix.

Let

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}$$

Find a unitary matrix U that diagonalizes A.

Solution

The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 0$, with corresponding eigenvectors $\mathbf{x}_1 = (1-i, 1)^T$ and $\mathbf{x}_2 = (-1, 1+i)^T$. Let

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{x}_1\|}\mathbf{x}_1 = \frac{1}{\sqrt{3}}(1-i, 1)^T$$

and

$$\mathbf{u}_2 = \frac{1}{\|\mathbf{x}_2\|}\mathbf{x}_2 = \frac{1}{\sqrt{3}}(-1, 1+i)^T$$

Thus

$$U = \frac{1}{\sqrt{3}}\begin{bmatrix} 1-i & -1 \\ 1 & 1+i \end{bmatrix}$$

and

$$\begin{aligned} U^H A U &= \frac{1}{3}\begin{bmatrix} 1+i & 1 \\ -1 & 1-i \end{bmatrix}\begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}\begin{bmatrix} 1-i & -1 \\ 1 & 1+i \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

■

The characteristic equation of the matrix A is

$$\begin{vmatrix} \lambda-2 & -(1-i) \\ -(1+i) & \lambda-1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-2)(\lambda-1)-(1-i)(1+i)=0$$

$$\Rightarrow \lambda^2-3\lambda+2-1+i^2=0$$

$$\Rightarrow \lambda^2-3\lambda=0$$

$$\Rightarrow \lambda(\lambda-3)=0$$

$$\therefore \lambda=0,3$$

(i) When $\lambda=0$, we seek a non-zero vector $\boldsymbol{v}_1=(x, y)^T$ such that

$$\begin{pmatrix} -2 & -(1-i) \\ -(1+i) & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\text{or,}\quad \begin{aligned} -2x-(1-i)y&=0 \\ -(1+i)x-y&=0 \end{aligned}$$

$$\text{or,}\quad \begin{aligned} 2x+(1-i)y&=0 \\ (1+i)(1-i)x+(1-i)y&=0 \end{aligned}$$

$$\text{or,}\quad \begin{aligned} 2x+(1-i)y&=0 \\ 2x+(1-i)y&=0 \end{aligned}$$

or, $2x+(1-i)y=0$

Here are 2 variables and 1 equation. So, there is 2−1=1 free variable. Let x is the free variable and we set x=−1.

Then

$-2+(1-i)y=0$

$\therefore y=\dfrac{2}{1-i}=\dfrac{2(1+i)}{(1-i)(1+i)}=\dfrac{2(1+i)}{(1-i^2)}=\dfrac{2(1+i)}{2}=1+i$

So, $\boldsymbol{v_1}=(-1, 1+i)^T$ is an eigenvector corresponding to the eigenvalue λ=0.

(ii) When λ=3, we seek a non-zero vector $\boldsymbol{v_2}=(x, y)^T$ such that

$$\begin{pmatrix} 3-2 & -(1-i) \\ -(1+i) & 3-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0$$

or, $\begin{pmatrix} 1 & -(1-i) \\ -(1+i) & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0$

or, $x-(1-i)y=0$

$\quad -(1+i)x+2y=0$

or, $x-(1-i)y=0$

$\quad (1+i)x-2y=0$

or, $x-(1-i)y=0$

$\quad (1+i)(1-i)x-2(1-i)y=0$

or, $x-(1-i)y=0$

$\quad 2x-2(1-i)y=0$

or, $x-(1-i)y=0$

$\quad x-(1-i)y=0$

or, $x-(1-i)y=0$

Here are 2 variables and 1 equation. So, there is 2−1=1 free variable. Let y is the free variable and we set y=1.

Then

$x-(1-i)=0$

$\therefore x=1-i$

So, $\boldsymbol{v_2}=(1-i, 1)^T$ is an eigenvector corresponding to the eigenvalue λ=3.

Now, we let

$$\boldsymbol{u}_1=\frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}=\frac{1}{\sqrt{|-1|^2+|1+i|^2}}\begin{pmatrix} -1 \\ 1+i \end{pmatrix}=\frac{1}{\sqrt{1+\left(\sqrt{1^2+1^2}\right)^2}}\begin{pmatrix} -1 \\ 1+i \end{pmatrix}==\frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1+i \end{pmatrix}$$

Spectral Theorem

If A is Hermitian, then there exists a unitary matrix U that diagonalizes A.

By Theorem 6.4.3, there is a unitary matrix U such that $U^H AU = T$, where T is upper triangular. Furthermore,

$$T^H = (U^H AU)^H = U^H A^H U = U^H AU = T$$

Therefore, T is Hermitian and consequently must be diagonal. ■

Given

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}$$

find an orthogonal matrix U that diagonalizes A.

Solution

The characteristic polynomial

$$p(\lambda) = -\lambda^3 + 3\lambda^2 + 9\lambda + 5 = (1+\lambda)^2(5-\lambda)$$

has roots $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$. Computing eigenvectors in the usual way, we see that $\mathbf{x}_1 = (1, 0, 1)^T$ and $\mathbf{x}_2 = (-2, 1, 0)^T$ form a basis for the eigenspace $N(A + I)$. We can apply the Gram–Schmidt process to obtain an orthonormal basis for the eigenspace corresponding to $\lambda_1 = \lambda_2 = -1$:

$$\begin{aligned}
\mathbf{u}_1 &= \frac{1}{\|\mathbf{x}_1\|}\mathbf{x}_1 = \frac{1}{\sqrt{2}}(1, 0, 1)^T \\
\mathbf{p} &= \left(\mathbf{x}_2^T \mathbf{u}_1\right)\mathbf{u}_1 = -\sqrt{2}\mathbf{u}_1 = (-1, 0, 1)^T \\
\mathbf{x}_2 - \mathbf{p} &= (-1, 1, 1)^T \\
\mathbf{u}_2 &= \frac{1}{\|\mathbf{x}_2 - \mathbf{p}\|}(\mathbf{x}_2 - \mathbf{p}) = \frac{1}{\sqrt{3}}(-1, 1, 1)^T
\end{aligned}$$

The eigenspace corresponding to $\lambda_3 = 5$ is spanned by $\mathbf{x}_3 = (-1, -2, 1)^T$. Since $\mathbf{x}_3$ must be orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$ (Theorem 6.4.1), we need only normalize

$$\mathbf{u}_3 = \frac{1}{\|\mathbf{x}_3\|}\mathbf{x}_3 = \frac{1}{\sqrt{6}}(-1, -2, 1)^T$$

Thus, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set and

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

diagonalizes A. ■

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$$A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Characteristic equation is

$$\left|\sigma I - A^T A\right| = 0$$

$$\Rightarrow \begin{vmatrix} \sigma - 2 & 2 \\ 2 & \sigma - 2 \end{vmatrix} = 0$$

$$\Rightarrow (\sigma - 2)^2 - 4 = 0$$

$$\Rightarrow (\sigma - 2)^2 = 4$$

$$\Rightarrow \sigma - 2 = \pm 2$$

$$\Rightarrow \sigma = 2 \pm 2$$

$$\therefore \sigma = 4, 0$$

When σ=4: We seek a nonzero vector **x** such that

$$\begin{pmatrix} 4-2 & 2 \\ 2 & 4-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow 2x_1 + 2x_2 = 0$$

$$\therefore x_1 + x_2 = 0$$

Let x_1 is a free variable and set x_1=1.

Then

$$1 + x_2 = 0$$

$$\therefore x_2 = -1$$

So, $\boldsymbol{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to σ=4.

When σ=0: We seek a nonzero vector **x** such that

$$\begin{pmatrix} 0-2 & 2 \\ 2 & 0-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow 2x_1 - 2x_2 = 0$$

$$\therefore x_1 - x_2 = 0$$

Let x_1 is a free variable and set x_1=1.

Then

$$1 - x_2 = 0$$

$$\therefore x_2 = 1$$

So, $\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\sigma=0$.