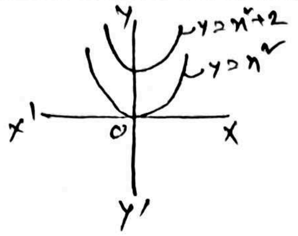
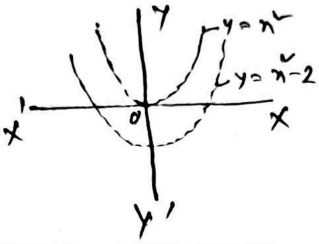
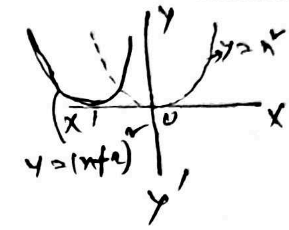
Translations:

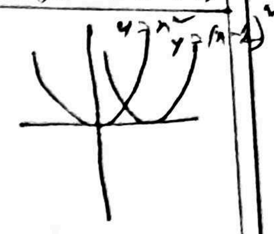
operation on $y = f(x)$	Add a positive constant c to $f(x)$	Subtract a positive constant c from $f(x)$	Add a positive constant c to x	Subtract a positive constant c from x
New equation	$y = f(x) + c$	$y = f(x) - c$	$y = f(x+c)$	$y = f(x-c)$
Geometric effect	Translates the graph of $y = f(x)$ up c units	Translates the graph of $y = f(x)$ down c units	Translates the graph of $y = f(x)$ left c units	Translates the graph of $y = f(x)$ right c units

Example





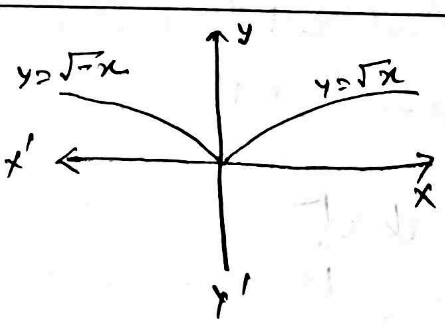


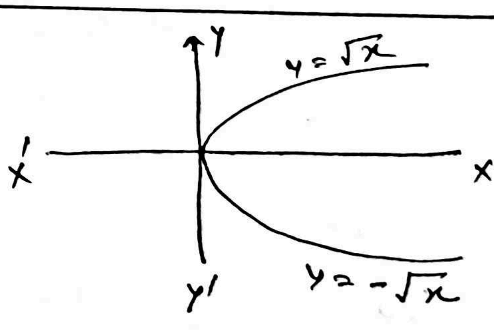


Reflections:

operation on $y = f(x)$	Replace x by $-x$	multiply $f(x)$ by -1
New equation	$y = f(-x)$	$y = -f(x)$
Geometric effect	Reflects the graph of $y = f(x)$ about y-axis	Reflects the graph of $y = f(x)$ about x-axis.

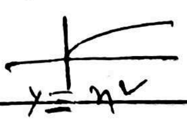
Example

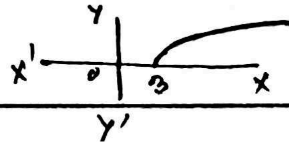




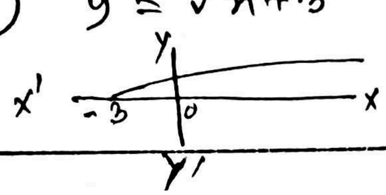
Ex: Given that the function $y = f(x) = \sqrt{x}$ i.e $y = \sqrt{x}$ write the new equation of $y = \sqrt{x}$ which is shifted right 3 units and left 3 units.

i) $y = \sqrt{x-3}$





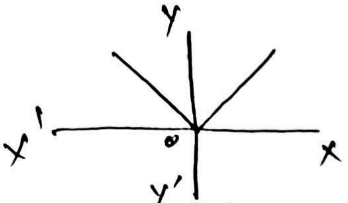
ii) $y = \sqrt{x+3}$



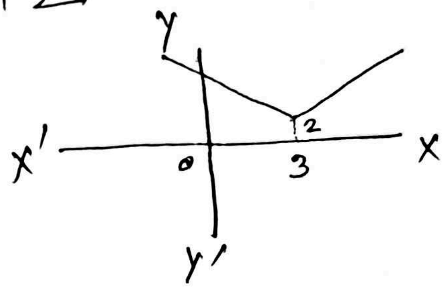
Ex: Given that the function $y = |x|$. Find the new equation of $y = |x|$ which is translated to right 3 units and up 2 units (together with). Also sketch the graph.

Solⁿ

$$y = |x-3| + 2$$



$y = |x|$



$y = |x-3| + 2$

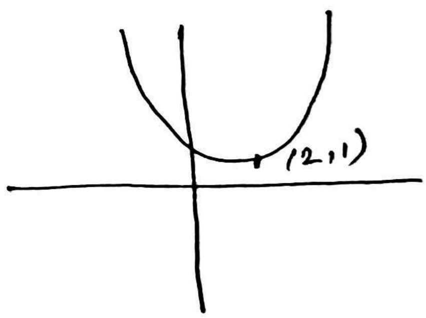
Ex: Sketch the graph of $y = x^2 - 4x + 5$. Also find, how much does the graph translate?

Solⁿ Given that the function

$$y = x^2 - 4x + 5$$

$$\Rightarrow y = x^2 - 4x + 4 + 1$$

$$\Rightarrow y = (x-2)^2 + 1$$



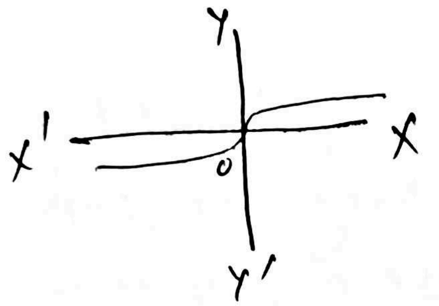
In this form we see that the graph can be obtained by translating the graph of $y = x^2$ right 2 units and up 1 unit.

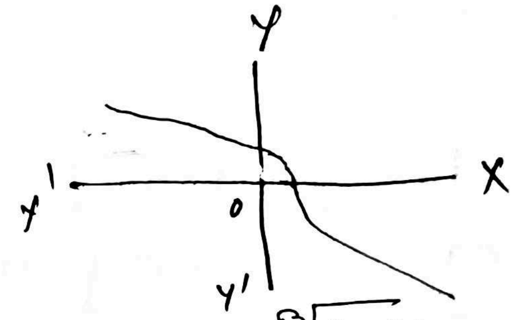
Ex: Given that the function $y = \sqrt[3]{x}$. Find the new equation which reflect about y-axis and translate right 2 units.

Sol^n: Given that the function $y = \sqrt[3]{x}$

First reflect the graph of $y = \sqrt[3]{x}$ about the y-axis to obtain the graph of $y = \sqrt[3]{-x}$ and then translate this graph right 2 units to obtain the graph of the equation $y = \sqrt[3]{-(x-2)}$

$= \sqrt[3]{2-x}$.



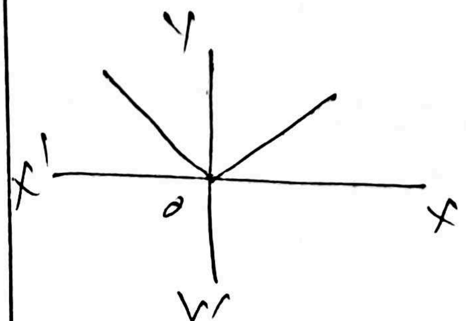


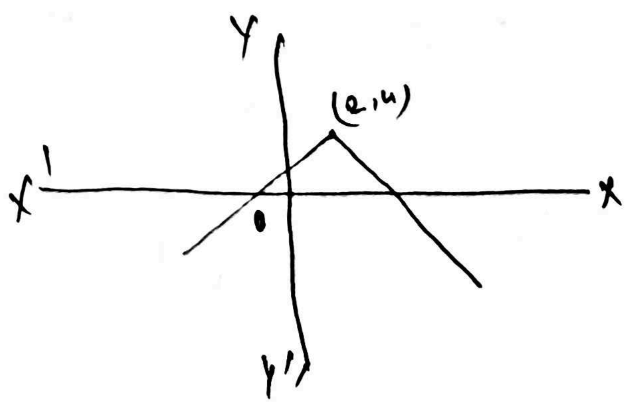
$y = \sqrt[3]{2-x}$

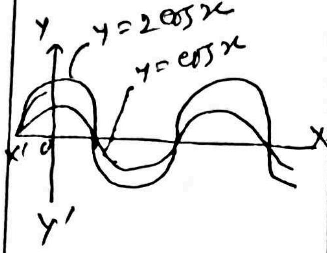
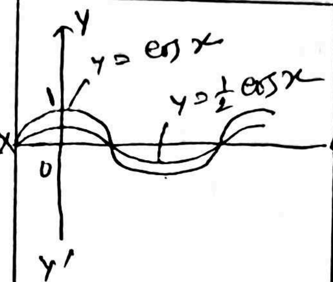
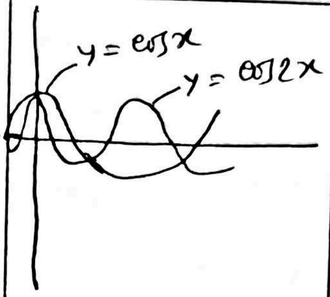
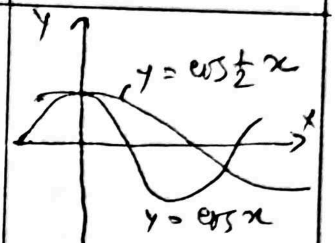
Ex: Given that the equation $y = |x|$. Find the equation which is reflect about the x-axis and translate to right by 2 units and up 4 units.

Sol^n: The required equation is

$$y = 4 - |x-2|$$





Operation on $y=f(x)$	multiply $f(x)$ by c $(c>1)$	multiply $f(x)$ by c $(0<c<1)$	multiply x by c $(c>1)$	multiply x by c $(0<c<1)$
New equation	$y=cf(x)$	$y=cf(x)$	$y=f(cx)$	$y=f(cx)$
Geometric effect	Stretches the graph of $y=f(x)$ vertically by a factor of c	Compresses the graph of $y=f(x)$ vertically by a factor of $\frac{1}{c}$.	Compresses the graph of $y=f(x)$ horizontally by a factor of c	Stretches the graph of $y=f(x)$ horizontally by a factor of $\frac{1}{c}$.
Example				

Ex-1: Sketch the graph of the equation by translating, reflecting, compressing and stretching the graph of $y=x^2$ appropriately and then use a graphing utility to confirm that your sketch is correct.

i) $y=-2(x+1)^2-3$

ii) $y=\frac{1}{2}(x-3)^2+2$

iii) $y=1+2x-x^2$

iv) $y=x^2+6x-10$

Ex-2: Sketch the graph of the equation by translating, reflecting, compressing and stretching the graph of (a) $y=\sqrt{x}$ (b) $y=\frac{1}{x}$ (c) $y=|x|$ (d) $y=\sqrt[3]{x}$ appropriately and then use a graphing utility to confirm that your sketch is correct

(a) (i) $y = 3 - \sqrt{x+1}$
(ii) $y = 1 + \sqrt{x-4}$
iii) $y = \frac{1}{2}\sqrt{x} + 1$
iv) $y = -\sqrt{3x}$

(c) (i) $y = |x+2| - 2$
(ii) $y = 1 - |x-3|$
(iii)

(b) (i) $y = \frac{1}{x-3}$
(ii) $y = \frac{1}{1-x}$
iii) $y = \frac{x-1}{x}$

(d) (i) $y = 1 - 2\sqrt[3]{x}$
(ii) $y = \sqrt[3]{x-2} - 3$
iii) $y = 2 + \sqrt[3]{x+1}$
iv) $y + \sqrt[3]{x-2} = 0$

Composition of functions:

Def^n: Given functions f and g, the composition of f with g denoted by $f \circ g$ is the function defined by $(f \circ g)(x) = f(g(x))$

The domain of $f \circ g$ is defined to consist of all x in the domain of g for which $g(x)$ is in the domain of f.

Example: Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$, Find
(a) $(f \circ g)(x)$ & (b) $(g \circ f)(x)$ Also domain and range.

Sol^n (a) $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = f(v) =$ $\quad v = \sqrt{x}$
$= v^2 + 3$
$= (\sqrt{x})^2 + 3 = x + 3$

$(f \circ g)(x)$ is defined for $(-\infty, \infty)$ but $g(x)$ is defined $[0, \infty)$, So ~~fog for~~ the domain of $(f \circ g)(x)$ is $[0, \infty)$ and range $(-\infty, \infty)$

(b) $(g\circ f)(x) = g(f(x)) = g(x^2+3) = g(v)$, $v = x^2+3$

$= \sqrt{v}$

$= \sqrt{x^2+3}$

$(g\circ f)(x)$ is defined for all values of x. Also $f(x)$ is defined for $(-\infty, \infty)$

So the domain of $(g\circ f)(x)$ is $(-\infty, \infty)$ or $\mathbb{R}$.

and range is $[0, \infty)$. Ans.

Compositions can also be defined for three or more functions: for example, $(f\circ g\circ h)(x)$ is computed as

$$(f\circ g\circ h)(x) = f(g(h(x)))$$

In other words, first find $h(x)$, then find $g(h(x))$ and then find $f(g(h(x)))$.

Example: Find $(f\circ g\circ h)(x)$ if

$f(x) = \sqrt{x}$, $g(x) = \frac{1}{x}$, $h(x) = x^3$.

Sol^n: $(f\circ g\circ h)(x) = f(g(h(x))) = f(g(x^3)) = f(g(v)) = f\left(\frac{1}{v}\right)$, $v = x^3$

$= f\left(\frac{1}{x^3}\right) = f(u)$, $u = \frac{1}{x^3}$

$= \sqrt{u}$

$= \sqrt{\frac{1}{x^3}}$

$= \sqrt{x^{-3}}$

$= x^{-3/2}$

$= \frac{1}{x^{3/2}}$

Expressing a function as a Composition

Many problems in mathematics are attacked by 'decomposing' functions into compositions of simpler functions.

For example, consider the function h given by

$$h(x) = (x+1)^2.$$

Solⁿ

To evaluate h(x) for a given value of x. We would first compute $x+1$ and then square the result. These two operations are performed by the functions

$$g(x) = x+1 \quad \text{and} \quad f(x) = x^2$$

we can express h in terms of f and g by writing

$$h(x) = (x+1)^2 = [g(x)]^2 = f(g(x)) = (f\circ g)(x)$$

$$\therefore h = f\circ g$$

Example: Express $h(x) = (x-4)^5$ as a composition of two functions.

Solⁿ To evaluate h(x) for a given value of x. We would first compute $x+1$ and then ~~square~~ raise the result to the fifth power. ~~These two~~ Therefore the inside function (first operation) is $g(x) = x-4$ and the outside function (2nd operation) is $f(x) = x^5$.

so $h(x) = f(g(x))$.

As a check, $f(g(x)) = [g(x)]^5 = (x-4)^5 = h(x)$

Example: Express $\sin(x^3)$ as a composition of two functions.

Sol^n $g(x) = x^3$ and $f(x) = \sin x$

Gives some more examples of decomposing functions into compositions.

Function	$g(x)$ inside	$f(x)$ outside	Composition
$(x^2+1)^{10}$	x^2+1	x^{10}	$(x^2+1)^{10} = f(g(x))$
$\sin^3 x$	$\sin x$	x^3	$\sin^3 x = f(g(x))$
$\tan(x^5)$	x^5	$\tan x$	$\tan(x^5) = f(g(x))$
$\sqrt{4-3x}$	$4-3x$	$\sqrt{x}$	$\sqrt{4-3x} = f(g(x))$
$8+\sqrt{x}$	$\sqrt{x}$	$8+x$	$8+\sqrt{x} = f(g(x))$
$\frac{1}{x+1}$	$x+1$	$\frac{1}{x}$	$\frac{1}{x+1} = f(g(x))$

Ex: Find formulas for $f \circ g$ and $g \circ f$ and state the domain of the compositions.

i) $f(x) = x^2$; $g(x) = \sqrt{1-x}$

ii) $f(x) = \sqrt{x-3}$; $g(x) = \sqrt{x^2+3}$

iii) $f(x) = \frac{1+x}{1-x}$; $g(x) = \frac{x}{1-x}$

iv) $f(x) = \frac{x}{1+x^2}$; $g(x) = \frac{1}{x}$

Inverse function:

Def^n: Let f be a function A into B and let $b \in B$. If $f: A \to B$ then

$$f^{-1}(b) = \{x : x \in A,\ f(x) = b\}$$

A procedure for finding the inverse of a function f.

Step-1: Write down the equation $y = f(x)$

" - 2: If possible, solve this equation for x as a function of y

" - 3: The resulting equation will be $x = f^{-1}(y)$, which provides a formula for f^{-1} with y as the independent variable.

" - 4: If y is acceptable as the independent variable for the inverse function, then you are done. but if you want to have x as the independent variable, then you need to interchange x and y in the equation $x = f^{-1}(y)$ to obtain $y = f^{-1}(x)$.

Ex: $f(x) = \sqrt{3x-2}$

Sol^n Let $y = \sqrt{3x-2}$

$\Rightarrow y^2 = 3x - 2$

$\Rightarrow y^2 + 2 = 3x$

$\Rightarrow x = \dfrac{y^2+2}{3}$

$\therefore f^{-1}(y) = \dfrac{y^2+2}{3}$

$\therefore f^{-1}(x) = \dfrac{x^2+2}{3}$ Ans.

Ex: Find formulas for $f \circ g \circ h$

i) $f(x) = x^2+1$, $g(x) = \frac{1}{x}$, $h(x) = x^3$

ii) $f(x) = \frac{1}{1+x}$, $g(x) = \sqrt[3]{x}$, $h(x) = \frac{1}{x^3}$

Ex: Express f as a composition of two functions; that is find g and h such that $f = g \circ h$.

i) $f(x) = \sqrt{x+2}$ ii) $f(x) = |x^2 - 3x + 5|$

iii) $f(x) = x^2+1$ iv) $f(x) = \frac{1}{x-3}$

v) $f(x) = \sin^2 x$ vi) $f(x) = \frac{3}{5+\cos x}$

vii) $f(x) = 3\sin(x^2)$ viii) $f(x) = 3\sin^2 x + 4\sin x$.

Ex: Express F as a composition of three functions; that is, find f, g and h such that $F = f \circ g \circ h$.

i) $F(x) = \{1 + \sin(x^2)\}^3$

ii) $F(x) = \sqrt{1 - \sqrt[3]{x}}$

iii) $F(x) = \frac{1}{1-x^2}$

iv) $F(x) = |5 + 2x|$