Zia

WAVES AND OSCILLATIONS

Differential equation of a Simple Harmonic Motion (SHM):

Defination of SHM: If the acceleration of a body is directly proportional to its displacement from a certain point and is always directed towards this point, then the motion of the body is called simple harmonic motion.

Thus, in case of simple harmonic oscillation, the relationship between acceleration a and displacement x is

$$a \propto -x$$

$$\text{or, } a = -k'x$$

Since the force is proportional to the acceleration so in case of simple harmonic motion we can say force is ~~always~~ also proportional to the displacement. i.e

$$F \propto -x$$

$$\text{or, } F = -kx$$

Hence, the constant k is called the force constant.

Differential equation of Simple Harmonic Motion (SHM):

From the defination of simple harmonic motion we know that acceleration is proportional to displacement but in opposite direction.
If F be force acting on a particle and x be its displacement then for simple harmonic motion;

$$F \propto -x$$

$$\text{or, } F = -kx \quad \text{---}(1); \quad k = \text{force constant}$$

Again from Newton's second law of motion we know that if m be the mass and a be the acceleration then $F = ma$

$$\therefore ma = -kx \quad \text{---}(2)$$

but we know, acceleration

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$$a = \frac{d^2x}{dt^2}$$

From eqn (2) $\Rightarrow$

$$m\frac{d^2x}{dt^2} = -kx$$

$$\text{or, } \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

or, $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$

If we write $\frac{k}{m} = \omega^2$ then the eqn becomes

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{— (3)}$$

Equation (3) is known as the differential equation of SHM.

* Eqn (3) is a 2nd order differential equation and its solution is, $x = A\sin(\omega t + \delta)$.

* Time period: The time taken for a complete oscillation by a particle executing simple harmonic ~~oscillation~~ motion is called time period T.

* Frequency: The number of oscillations performed by an oscillator in one second is called its frequency, f.

$$f = \frac{1}{T} \text{ or, } T = \frac{1}{f}$$

Total energy in Simple Harmonic Oscillation

Suppose a particle executing simple harmonic oscillation has amplitude A, angular frequency ω and phase constant δ. If the displacement of the particle in time t is x then from the eqn of SHM we know

$$x = A\sin(\omega t + \delta) \quad \text{--- (1)}$$

Potential energy (P.E):

We know that the force acting on a particle executing simple harmonic oscillation towards its equilibrium position is $F = -kx$. Now to displace the particle from $x=0$ to $x=x$ position, the work done by the force would be the potential energy, U of the particle at position x.

$$\therefore \; U = \int_0^x F' dx$$

$$= \int_0^x kx\,dx$$

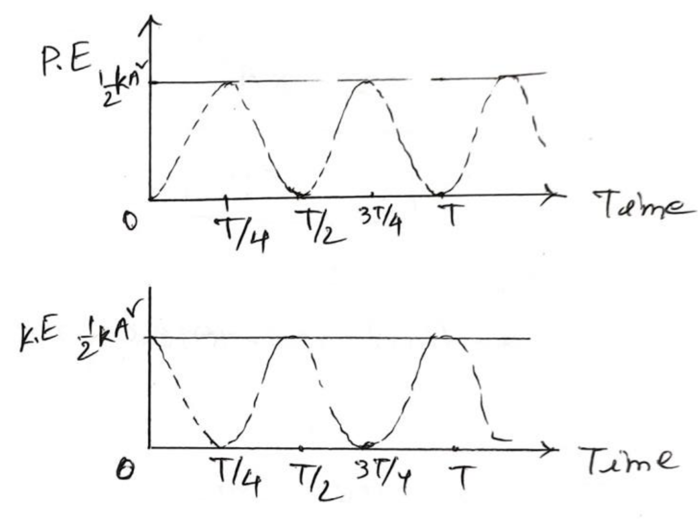
$$= k\left[\frac{x^2}{2}\right]_0^x$$

$$\text{or, } U = \frac{1}{2}kx^2 \quad \text{--- (2)}$$

Since $x = A\sin(\omega t+\delta)$

$$\therefore \quad U = \frac{1}{2}KA^2\sin^2(\omega t+\delta) \quad \text{— (3)}$$

Since the maximum value of $\sin^2(\omega t+\delta)$ is 1, then the maximum value of potential energy is $\frac{1}{2}KA^2$.



Kinetic energy, K :

At any instant, the kinetic energy of the particle is $K = \frac{1}{2}mv^2$

We know, velocity $v = \frac{dx}{dt} = \frac{d}{dt}\{A\sin(\omega t+\delta)\}$

$$= \omega A\cos(\omega t+\delta)$$

Hence, $K = \frac{1}{2}mv^2$

$$K = \frac{1}{2}m\omega^2A^2\cos^2(\omega t+\delta) \quad \text{— (4)}$$

Since the maximum value of $\cos^2(\omega t+\delta)$ is 1, from eqn (4) we see that the maximum K.E of the particle is $\frac{1}{2}KA^2$.

Total mechanical energy E is the sum of kinetic and potential energies. Using eqn (3) and (4) we get

$$E = K + U$$
$$= \frac{1}{2} KA^2 \{\cos^2(\omega t+\delta) + \sin^2(\omega t+\delta)\}$$
$$= \frac{1}{2} KA^2$$
$$\therefore \boxed{E = \frac{1}{2} KA^2}$$

Since k and A are constant quantities, we see that the total energy is constant.

Average kinetic energy:

We get average kinetic energy ~~by~~ dividing the multiplication of ~~of~~ K.E and time intervel by total time.
For a simple harmonic motion time pereiod is T, then for a cycle -

$$K_{ave} = \frac{\int_0^T K dt}{\int_0^T dt}$$
$$= \frac{\frac{1}{2} KA^2 \int_0^T \cos^2(\omega t+\delta) dt}{[t]_0^T}$$

$$K = \frac{1}{2} m v^2$$
$$= \frac{1}{2} m \left[\frac{dx}{dt}\right]^2$$
$$= \frac{1}{2} m A^2 \omega^2 \cos^2(\omega t+\delta)$$
$$K = \frac{1}{2} KA^2 \cos^2(\omega t+\delta)$$

$$\text{or, } K = \frac{\frac{1}{2}KA^2}{T}\int_0^T \frac{1}{2}\times 2\cos^2(\omega t+\delta)dt$$

$$= \frac{KA^2}{4T}\int_0^T \{1+\cos 2(\omega t+\delta)\}dt$$

$$= \frac{KA^2}{4T}\left\{[T]_0^T + \left[\frac{\sin 2(\omega t+\delta)}{2\omega}\right]_0^T\right\}$$

$$= \frac{KA^2}{4T}\left\{(T-0) - \frac{1}{2\omega}\left(\sin 2(\omega\times\frac{1}{T}+0) - \sin 2\delta\right)\right.$$

$$= \frac{1}{4}\cdot\frac{KA^2}{\cancel{T}}(T+0)$$

$$= \frac{1}{4}\times\frac{KA^2}{\cancel{T}}\times\cancel{T}$$

$$= \frac{1}{4}KA^2$$

$$= \frac{1}{2}\times\frac{1}{2}KA^2$$

$$E_{ave} = \frac{1}{2}\times E$$

Thus the average K.E is the half of total energy.

* Same as: Average Potential energy

$$E_p = \frac{\int_0^T E_p\,dt}{\int_0^T dt}$$

$$E_p = \frac{1}{2}kx^2$$
$$= \frac{1}{2}k\{A\sin(\omega t+\delta)\}^2$$

$$= \frac{1}{2}\times\frac{1}{2}KA^2$$

$$= \frac{1}{2}\times E$$

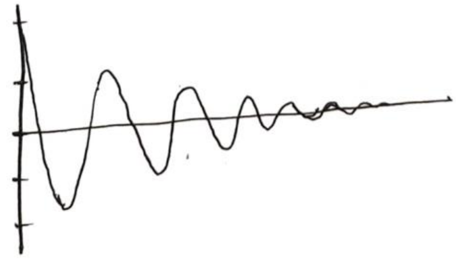
Lissajou's figure:

When a particle is affected by two waves acting parpendicularly, then the resultant motion of that particle is a curve. These curved lines are called Lissajou's figure. Size of these curved lines depend on time period phase difference and the amplitude of two original amplitude.

Damped harmonic motion:

Oscillation of a simple pendulum or a spring stops after a time interval. This causes because a frictional force acts against the motion of oscillation. This frictional force is called damping force.

Thus, "When a damping force acts on the motion of an oscillation, then the motion is called the damped harmonic motion."



Forced vibration:

If the frequency of an applied periodic frequency on a body is different from its natural frequency the body will first oscillate irregularly then it oscillates with the frequency of the applied frequency of the applied frequency. Oscillation of this type is called forced vibration.

Resonance:

If the natural frequency of a body is equal to the periodic frequency applied on it, the body starts to vibrate the with the maximum amplitude. Then the vibration of this type is called resonance.

* We know that the equation of a progressive wave is -

$$y = a\sin\frac{2\pi}{\lambda}(vt - x)$$

Hence, $y =$ Displacement of the particle of the wave.

$x =$ " of the wave.

Differential equation of a progressive wave:

We know that the ~~d~~ equation of a progressive wave is -

$$y = a\sin \frac{2\pi}{\lambda}(vt - x) \quad \text{--- (1)}$$

Here,

y = Displacement of the particle from source at a distance x for time t.

a = Amplitude of the particle

v = velocity of the wave

λ = wave length.

Differentiating eqn (1) w.r.t t, we get -

$$\frac{dy}{dt} = \frac{2\pi v}{\lambda} \cdot a\cos \frac{2\pi}{\lambda}(vt - x)$$

Again differentiating $\frac{dy}{dt}$ w.r.t t we get

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{2\pi a v}{\lambda} \frac{d}{dt} \cos \frac{2\pi}{\lambda}(vt - x)$$

or, $$\frac{d^2 y}{dt^2} = -\frac{4\pi^2 v^2}{\lambda^2} a\sin \frac{2\pi}{\lambda}(vt - x)$$

or, $$\frac{d^2 y}{dt^2} = -\frac{4\pi^2 v^2}{\lambda^2} y \quad \text{--- (2)}$$

Again, differentiating eqn (1) w.r.t x we get -

$$\frac{dy}{dx} = -\frac{2\pi}{\lambda} a\cos \frac{2\pi}{\lambda}(vt - x)$$

Again, Differentiating w.r.t x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{4\pi^2}{\lambda^2} a \sin\frac{2\pi}{\lambda}(vt-x)$$

$$\text{or, } \frac{d^2y}{dx^2} = -\frac{4\pi^2}{\lambda^2} \cdot y \quad \text{—} (3)$$

From eq^n (2) & (3) we get

$$\frac{d^2y}{dx^2} = \frac{1}{v^2}\left(-\frac{4\pi^2 v^2}{\lambda^2} y\right)$$

$$\text{or, } \frac{d^2y}{dx^2} = \frac{1}{v^2} \cdot \frac{d^2y}{dt^2}$$

$$\therefore \frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dx^2} \quad \text{—} (4)$$

Eq^n (4) is called the differential eq^n of a progressive wave.

Principle of seperposition:

When two waves are incident on a particle simultaneously, the resultant displacement of the particle from the mean position is the vector sum of the displacement produced by the individual wave.

If y_1 and y_2 be the displacements of

the two waves respectively, then the resulta
displacement -

$$\vec{Y} = \vec{Y_1} + \vec{Y_2}$$

As a result the above vector equation can be written as -

$$y = y_1 \pm y_2$$

What is intensity? Derive mathematical express for intensity.

<u>Intensity:</u> Intensity is the amount of energy flowing per second per unit area perpendicular to the direction of propagation of wave.

If the incident sound energy on the surface of a sphere of radius r drawn around the point ~~P~~ source P, then the intensity at any point on the surface of area A of the sphere is

$$I = \frac{P}{A} = \frac{P}{4\pi r^2}$$

Intensity is measured by $Js^{-1}m^{-1}$ or Wm^{-2}.

Mathematical expression for intensity:

Let a wave of amplitude a and frequency f propagating through a medium with a velocity v. Along the path of propagation of this wave of E amount of energy is flowing through the area A around a point in the medium in time t then the energy flowing perpendicularly in the unit time through ~~the~~ unit area that is the intensity of the wave, I will be,

$$I = \frac{E}{At}$$

$$\text{or, } I = \frac{EL}{ALt} \quad [L = \text{Length of a portion of the medium}]$$

$$= \frac{EL}{Vt} \quad [V = AL = \text{Volume of the portion}]$$

$$= \frac{Ev}{V} \quad \left[v = \frac{L}{t}\right]$$

$$\therefore I = \frac{Ev}{V}$$

In case of simple harmonic motion the total energy is equal to the maximum P.E or maximum K.E.

$$\therefore E = \frac{1}{2} m v_{max}^2 \qquad \text{Hence, } v_{max} = \omega a$$

$$= \frac{1}{2} m (\omega a)^2$$

$$so,\ I = \frac{1}{2}\frac{m(\omega a)^2 v}{V}$$

$$= \frac{1}{2}\rho\omega^2 a^2 v \quad [\because \rho = \frac{m}{V}]$$

$$= \frac{1}{2}\rho(2\pi f)^2 a^2 v$$

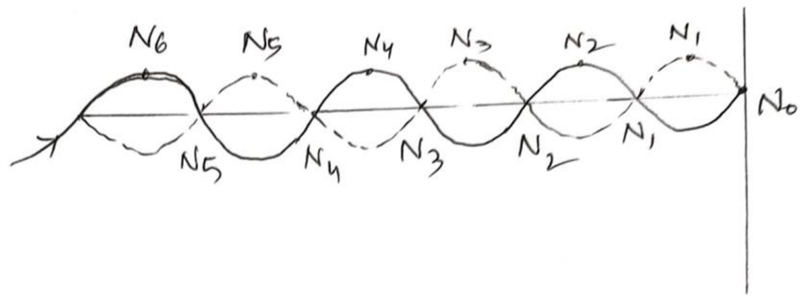
$$= \frac{1}{2}\rho \times 4\pi^2 f^2 a^2 v$$

$$\therefore\ I = 2\pi^2 \rho v a^2 f^2$$

This is the equation for intensity of sound wave.

What is stationary wave? Derive the equation of stationary wave discussing the conditions for production of nodes and antinodes.

<u>Stationary wave:</u> The resultant wave produced in a limited portion of a medium by superposition of two progressive waves having the same wavelength and amplitude travelling in opposite direction is called a stationary or standing wave.



Let, two progressive waves having of same amplitude a and wavelength λ moving with same velocity v along X axis from opposite direction.

Along positive x direction, $y_1 = a \sin \frac{2\pi}{\lambda}(vt - x)$

Along negativ x direction, $y_2 = a \sin \frac{2\pi}{\lambda}(vt + x)$

So, the resultant displacement of the particle is

$$y = y_1 + y_2$$
$$= a\left[\sin \frac{2\pi}{\lambda}(vt - x) + \sin \frac{2\pi}{\lambda}(vt + x)\right]$$
$$= 2a \sin\left(\frac{2\pi}{\lambda} vt\right) \cos \frac{2\pi x}{\lambda}$$
$$= 2a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi}{\lambda} vt$$
$$y = A \sin \frac{2\pi}{\lambda} vt \quad \text{--- (1)}$$

Eqn (1) is the eqn of a stationary wave whose amplitude, $A = 2a \cos \frac{2\pi x}{\lambda}$

Antinodes:

The points where the resultant amplitude is maximum i.e $A = \pm 2a$, antinodes are formed. That is antinodes are formed at the points where $\cos \frac{2\pi}{\lambda} x = \pm 1$. So the antinodes are formed at the points were —

$$\frac{2\pi x}{\lambda} = 0, \pi, 2\pi \ldots n\pi \quad (n = 0, 1, 2 \ldots)$$

$$\text{or,} \quad x = 0, \frac{\lambda}{2}, 2\frac{\lambda}{2}, \ldots \frac{n\lambda}{2}$$

$$\text{or,} \quad x = 0, \frac{2\lambda}{4}, \frac{4\lambda}{4}, \frac{6\lambda}{4}, \ldots \frac{2n\lambda}{4} \, (n = 0, 1, 2, \ldots)$$

the points on the stationary waves which are at the distance of even multiple of $\frac{\lambda}{4}$, antinode are formed.

Nodes: The points where there is no vibration i.e the amplitude $A = 0$, nodes are formed. That is, nodes are formed at the points where $\cos \frac{2\pi}{\lambda} x = 0$. So nodes are formed at the points where -

$$\frac{2\pi x}{\lambda} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots (2n+1)\frac{\pi}{2} \, (n = 0, 1, \ldots$$

$$\text{or,} \quad x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots (2n+1)\frac{\lambda}{4}$$

So the points on the stationary waves which are at distance of odd multiple of $\frac{\lambda}{4}$, nodes are formed.