Network Theorem

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Course Outline

- Superposition Theorem
- Thevenin's Theorem
- Norton's Theorem
- Maximum Power Transfer Theorem
- Reciprocity and Millman's Theorem

Reference Books

- **Introductory Circuit Analysis (11th Edition)** 11th Edition by Robert L. Boylestad

Superposition Theorem

- The superposition theorem extends the use of Ohm's Law to circuits with multiple sources.

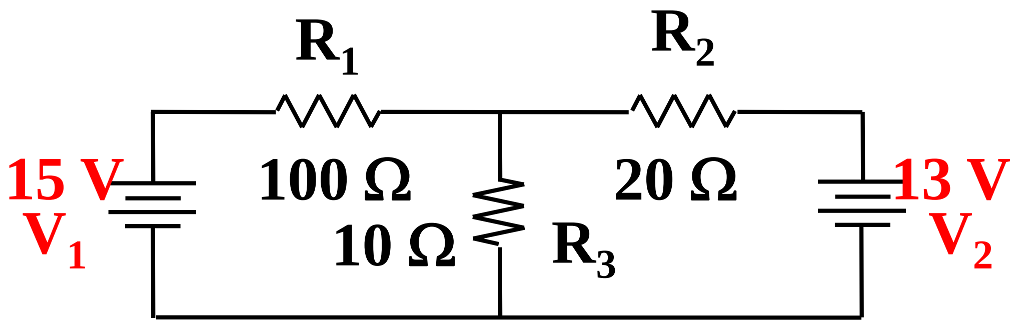
- In order to apply the superposition theorem to a network, certain conditions must be met:
  1. All the components must be **linear**, meaning that the current is proportional to the applied voltage.
  2. All the components must be **bilateral**, meaning that the current is the same amount for opposite polarities of the source voltage.
  3. **Passive components** may be used. These are components such as resistors, capacitors, and inductors, that do not amplify or rectify.
  4. **Active components** may not be used. Active components include transistors, semiconductor diodes, and electron tubes. Such components are never bilateral and seldom linear.

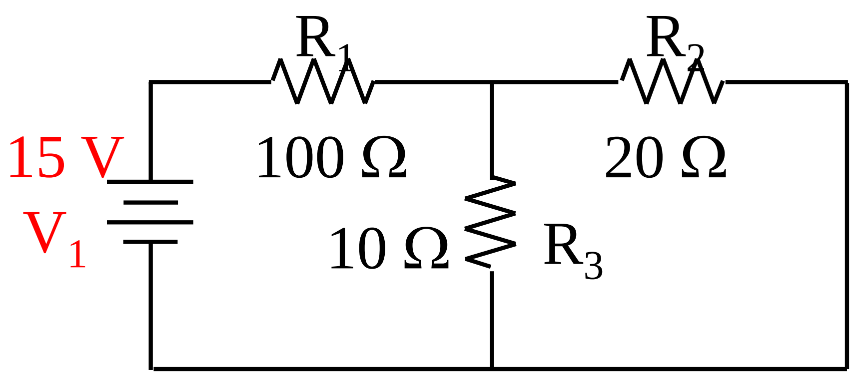
Superposition Theorem

- In a linear, bilateral network that has more than one source, the current or voltage in any part of the network can be found by adding algebraically the effect of each source separately.

- This analysis is done by:
  - shorting each voltage source in turn.
  - opening each current source in turn.

Superposition Theorem



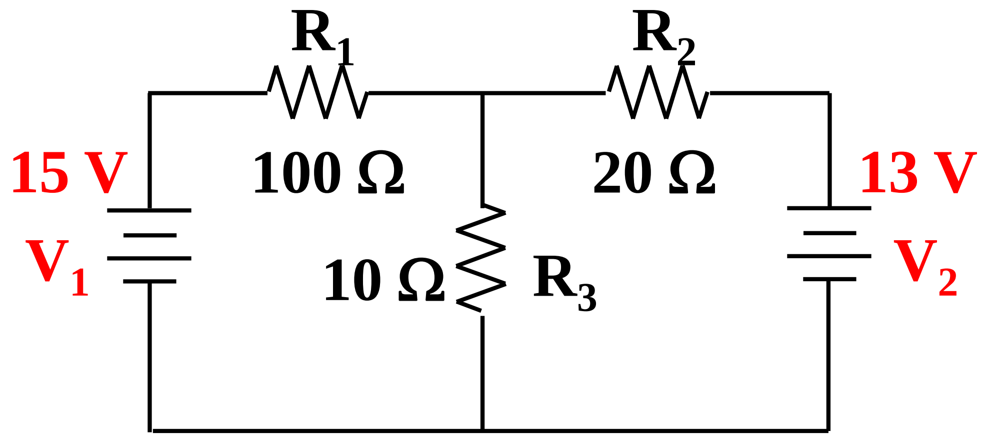


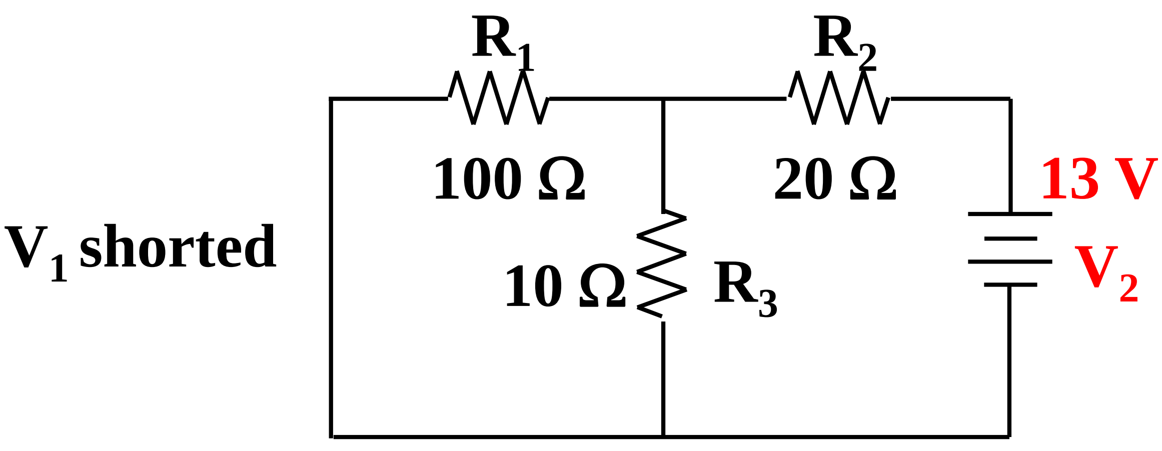
V_2 shorted

$R_{EQ} = 106.7\ \Omega$, $I_T = 0.141$ A and $I_{R_3} = 0.094$ A

Superposition Theorem

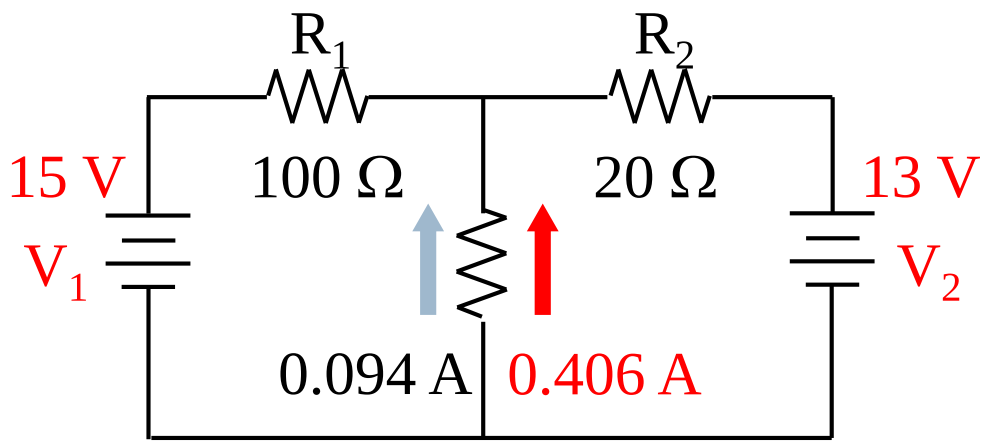
Superposition Theorem (Applied)





R_{EQ} = 29.09 Ω, I_T = 0.447 A and I_{R_3} = 0.406 A

Superposition Theorem (Applied)



With V_2 shorted

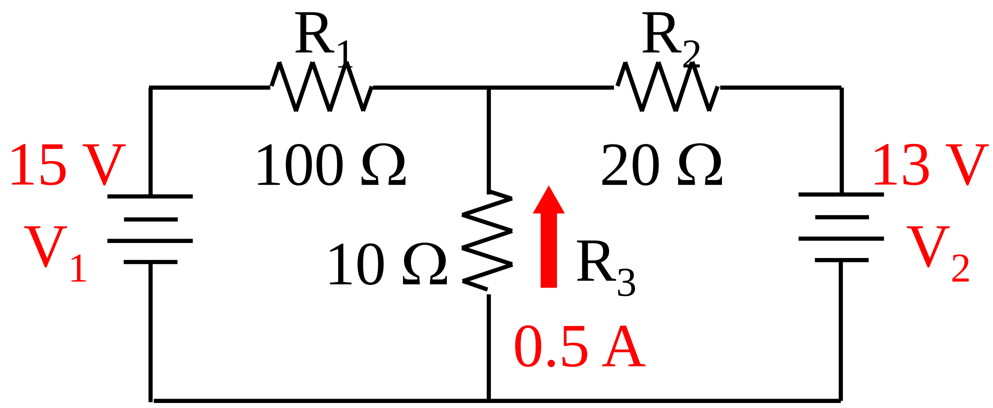
$R_{EQ} = 106.7\ \Omega$, $I_T = 0.141$ A and $I_{R_3} = 0.094$ A

With V_1 shorted

$R_{EQ} = 29.09\ \Omega$, $I_T = 0.447$ A and $I_{R_3} = 0.406$ A

Adding the currents gives $I_{R_3} = 0.5$ A

Superposition Theorem (Check)

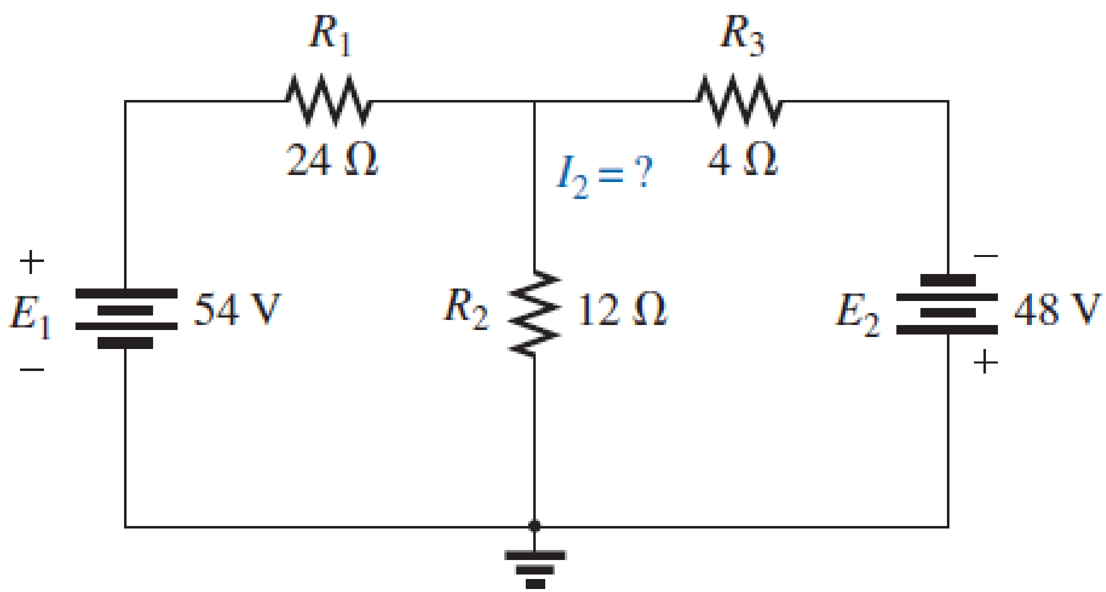


With 0.5 A flowing in R_3, the voltage across R_3 must be 5 V (Ohm's Law). The voltage across R_1 must therefore be 10 volts (KVL) and the voltage across R_2 must be 8 volts (KVL). Solving for the currents in R_1 and R_2 will verify that the solution agrees with KCL.

$$I_{R_1} = 0.1 \text{ A and } I_{R_2} = 0.4 \text{ A}$$

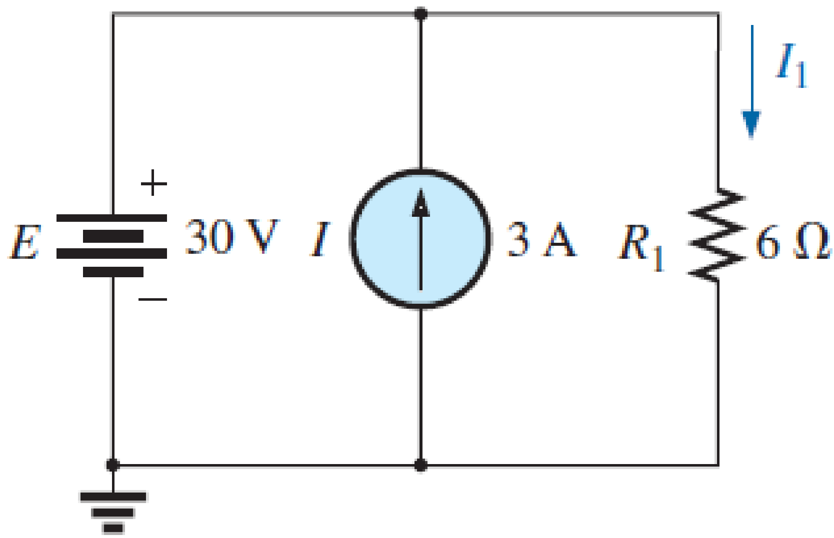
$$I_{R_3} = 0.1 \text{ A} + 0.4 \text{ A} = 0.5 \text{ A}$$

Superposition Theorem



Superposition Theorem

Example 3



Superposition Theorem

Example 4

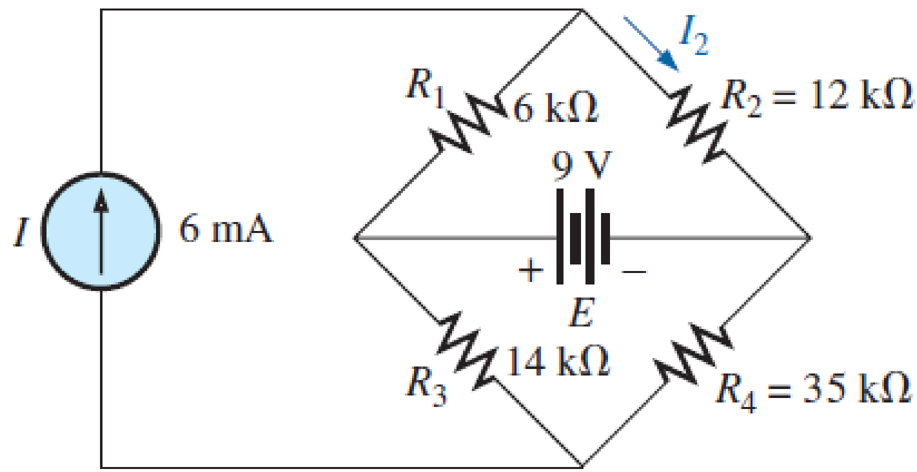


FIG. 15
Example 4.

Superposition Theorem

Example 5

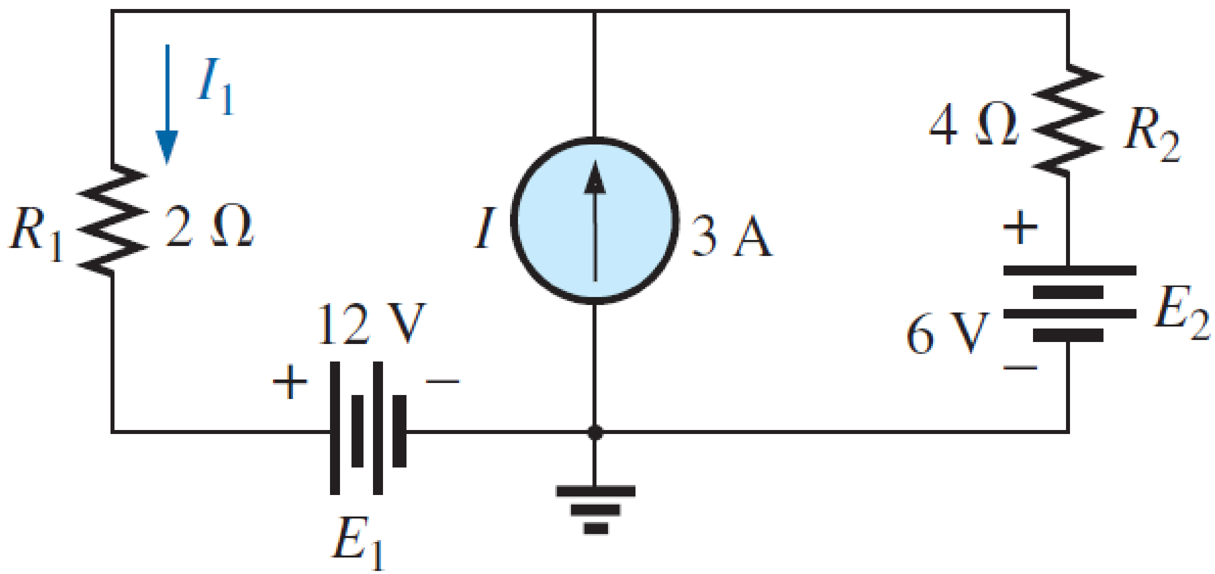


FIG. 18

Example 5.

Thevenin's Theorem

- Thevenin's theorem simplifies the process of solving for the unknown values of voltage and current in a network by reducing the network to an equivalent series circuit connected to any pair of network terminals.

- Any network with two open terminals can be replaced by a **single voltage source (V_{TH})** and a **series resistance (R_{TH})** connected to the open terminals. A component can be removed to produce the open terminals.

Thevenin's Theorem

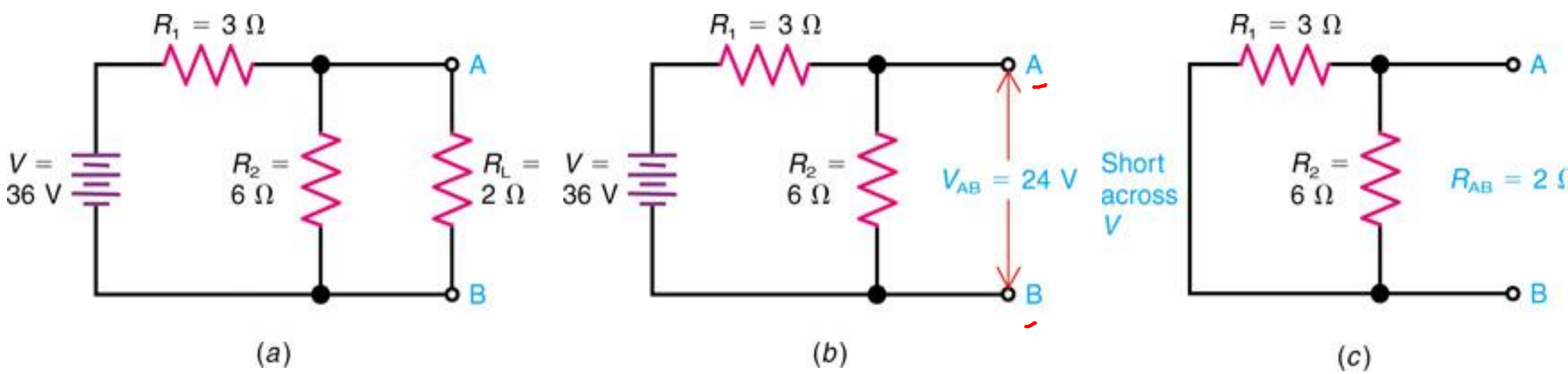


Fig. 10-3: Application of Thevenin's theorem. (*a*) Actual circuit with terminals A and B across R_L. (*b*) Disconnect R_L to find that V_{AB} is 24V. (*c*) Short-circuit V to find that R_{AB} is 2Ω.

Thevenin's Theorem

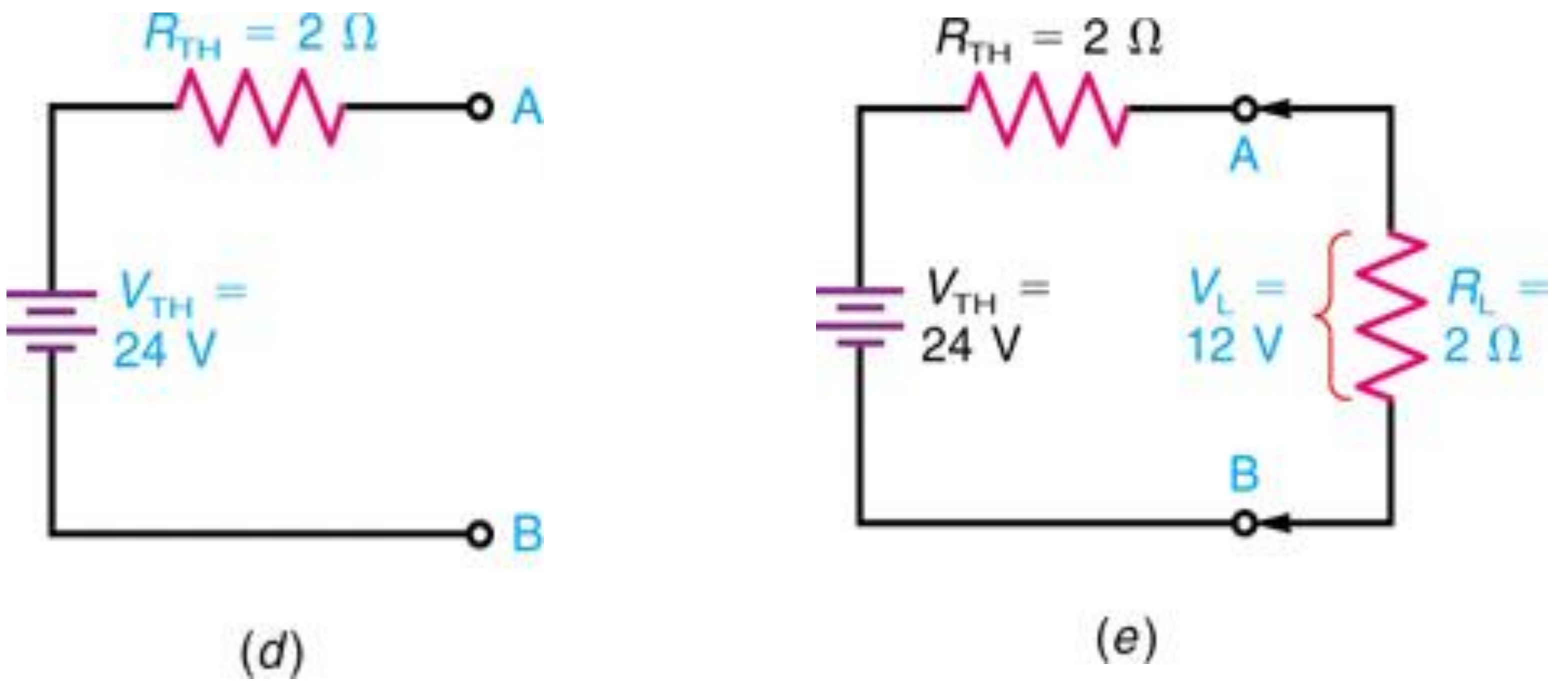
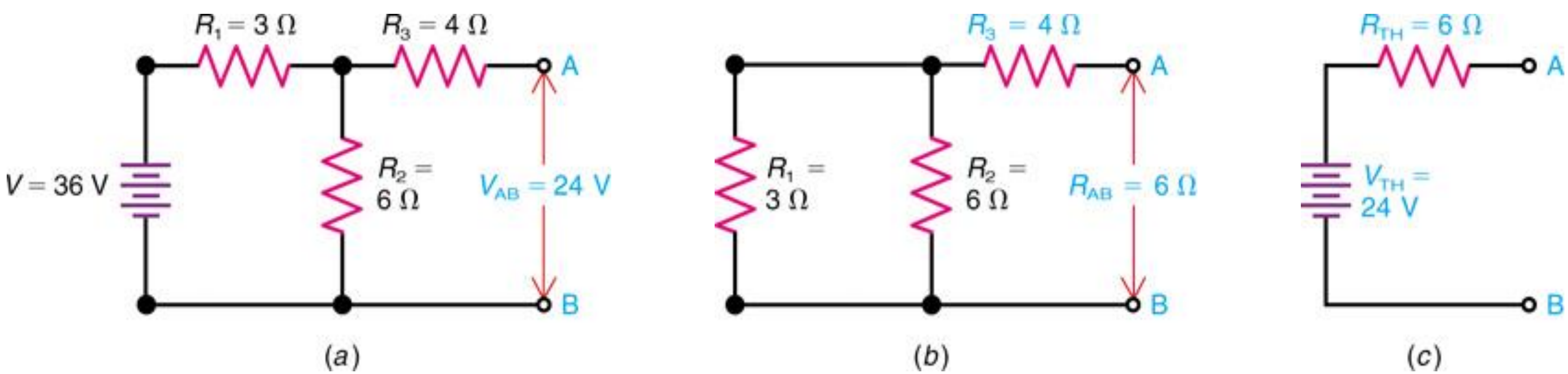


Fig. 10-3 (*d*) Thevenin equivalent circuit. (*e*) Reconnect R_L at terminals A and B to find that V_L is 12V.

Thevenin's Theorem

- Determining Thevenin Resistance and Voltage
  - R_{TH} is determined by shorting the voltage source and calculating the circuit's total resistance as seen from open terminals A and B.
  - V_{TH} is determined by calculating the voltage between open terminals A and B.

Thevenin's Theorem



Note that **R_3** does not change the value of **V_{AB}** produced by the source V, but R_3 <u>does</u> increase the value of **R_{TH}**.

Fig. 10-4: Thevenizing the circuit of Fig. 10-3 but with a 4-Ω R_3 in series with the A terminal. (*a*) V_{AB} is still 24V. (*b*) Now the R_{AB} is 2 + 4 = 6 Ω. (*c*) Thevenin equivalent circuit.

Thevenizing a Circuit with Two Voltage Sources

- The circuit in Figure 10-5 can b solved by Kirchhoff's laws, but **Thevenin's theorem** can be used to find the current I_3 through the middle resistance R_3.
  - Mark the terminals A and B across R_3.
  - Disconnect R_3.
  - To calculate V_{TH}, find V_{AB} across the open terminals

Thevenizing a Circuit with Two Voltage Sources

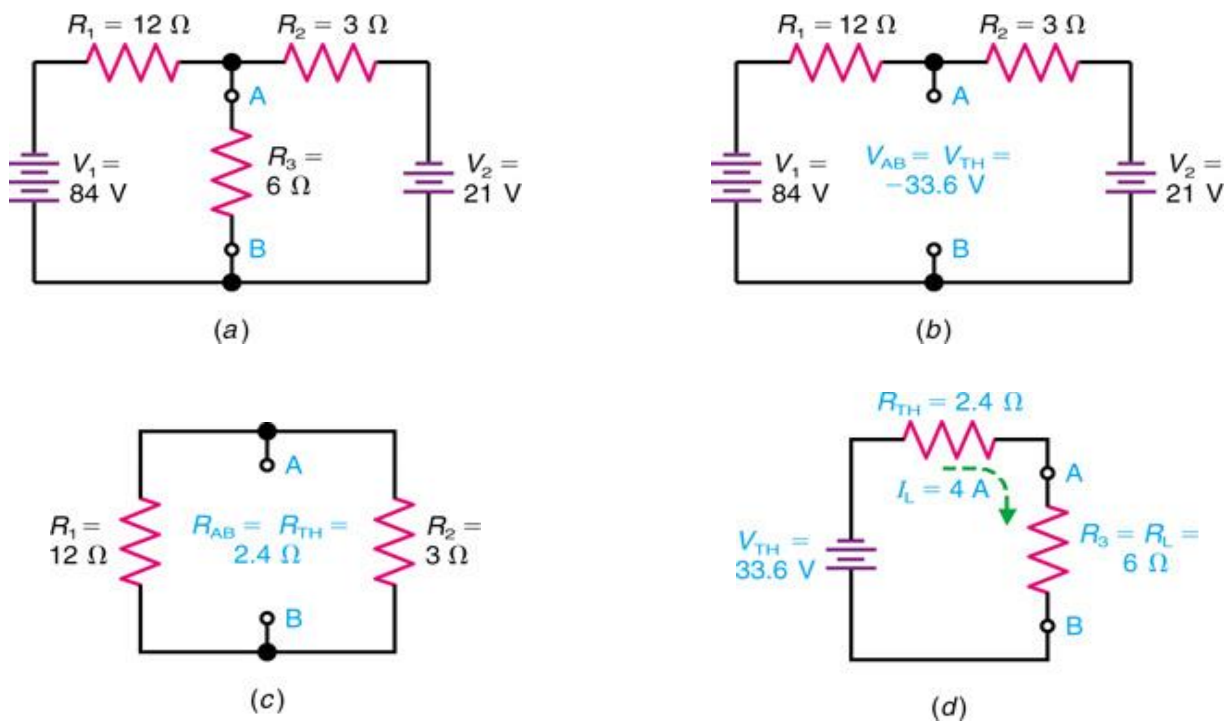


Fig. 10-5: Thevenizing a circuit with two voltage sources V_1 and V_2. (*a*) Original circuit with terminals A and B across the middle resistor R_3. (*b*) Disconnect R_3 to find that V_{AB} is −33.6V. (*c*) Short-circuit V_1 and V_2 to find that R_{AB} is 2.4 Ω. (*d*) Thevenin equivalent with R_L reconnected to terminals A and B.

Thevenizing a Bridge Circuit

- A Wheatstone Bridge Can Be Thevenized.
  - Problem: Find the voltage drop across R_L.
  - The bridge is unbalanced and Thevenin's theorem is a good choice.
  - R_L will be removed in this procedure making **A** and **B** the Thevenin terminals.

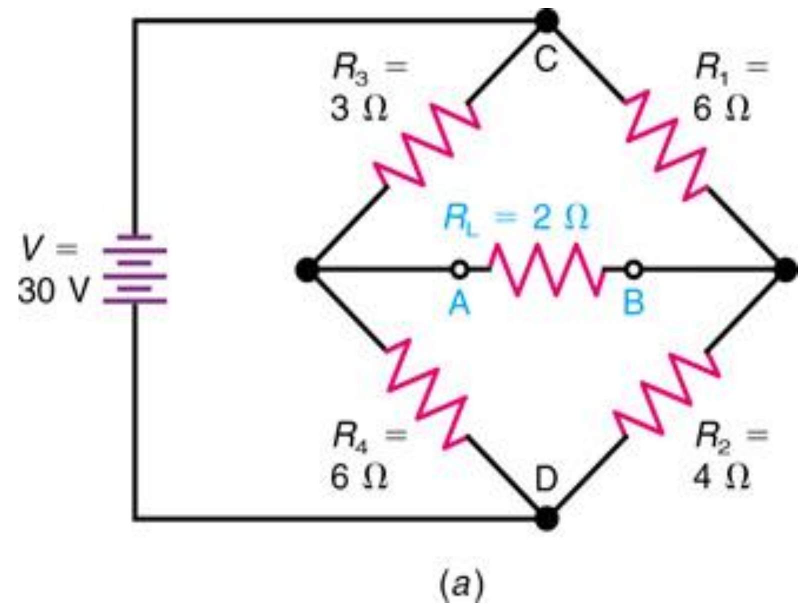
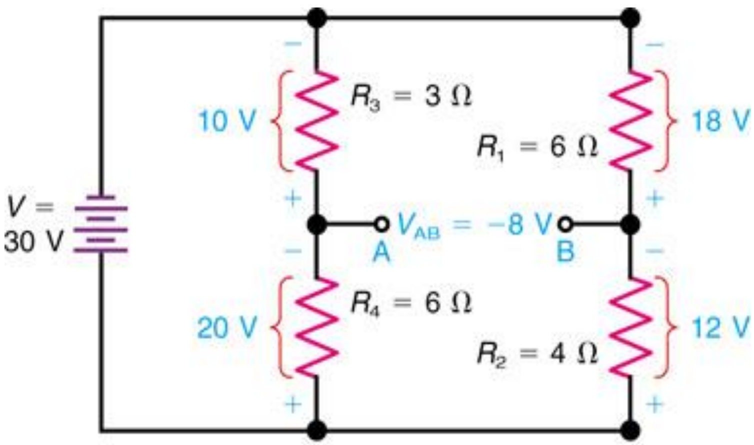


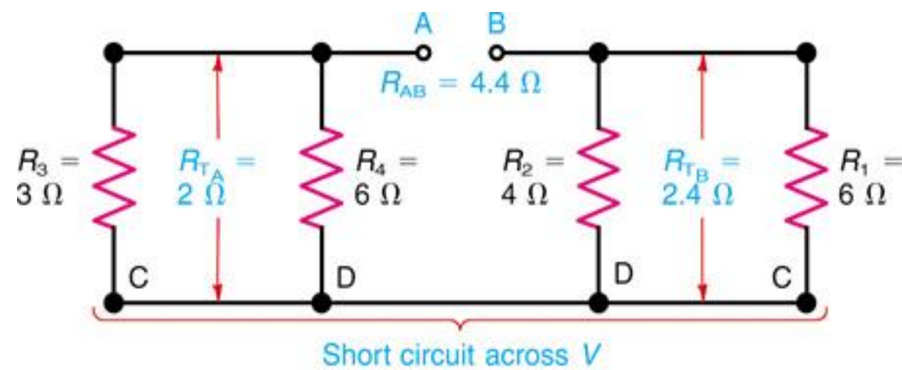
Fig. 10-6: Thevenizing a bridge circuit. (*a*) Original circuit with terminals A and B across middle resistor R_L.

Thevenizing a Bridge Circuit



(b)

V_{AB} = −20 −(−12) = −8V



(c)

$R_{AB} = R_{TA} + R_{TB}$ = 2 + 2.4 = 4.4 Ω

Fig. 10-6(*b*) Disconnect R_L to find V_{AB} of −8 V. (*c*) With source V short-circuited, R_{AB} is 2 + 2.4 = 4.4 Ω.

Thevenizing a Bridge Circuit

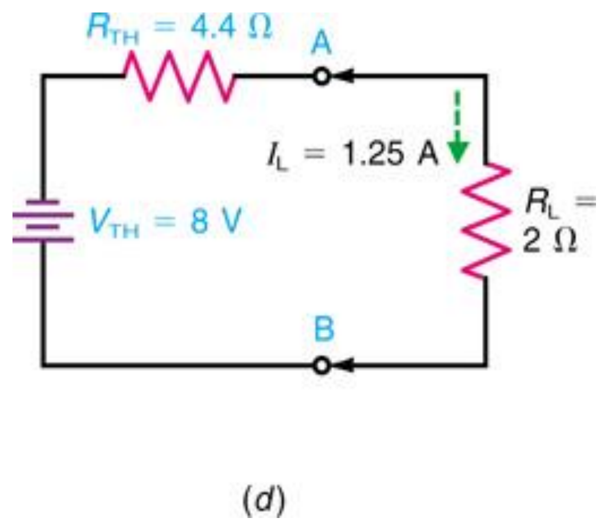


Fig. 10-6(*d*) Thevenin equivalent with R_L reconnected to terminals A and B.

Norton Theorem

9.4 NORTON'S THEOREM

- In Section 8.3, we learned that every voltage source with a series internal resistance has a current source equivalent.
- The current source equivalent can be determined by Norton's theorem. It can also be found through the conversions of Section 8.3.
- The theorem states the following:
  - Any two-terminal linear bilateral dc network can be replaced by an equivalent circuit consisting of a current source and a parallel resistor.

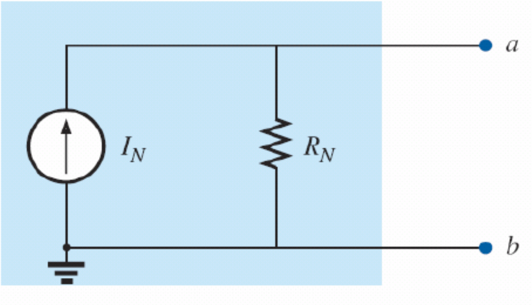


FIG. 9.65 *Norton equivalent circuit.*

Norton Theorem

9.4 NORTON'S THEOREM
Norton's Theorem Procedure

Preliminary:

1. Remove that portion of the network across which the Norton equivalent circuit is found.
2. Mark the terminals of the remaining two-terminal network.
3. RN:

Calculate RN by first setting all sources to zero (voltage sources are replaced with short circuits and current sources with open circuits) and then finding the resultant resistance between the two marked terminals. (If the internal resistance of the voltage and/or current sources is included in the original network, it must remain when the sources are set to zero.)

Norton Theorem

9.4 NORTON'S THEOREM
Norton's Theorem Procedure

Since RN = RTh, the procedure and value obtained using the approach described for Thévenin ' s theorem will determine the proper value of RN.

4. IN:

Calculate IN by first returning all sources to their original position and then finding the short-circuit current between the marked terminals.

It is the same current that would be measured by an ammeter placed between the marked terminals.

Norton Theorem

9.4 NORTON'S THEOREM
Norton's Theorem Procedure

5. Conclusion:

Draw the Norton equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the equivalent circuit.

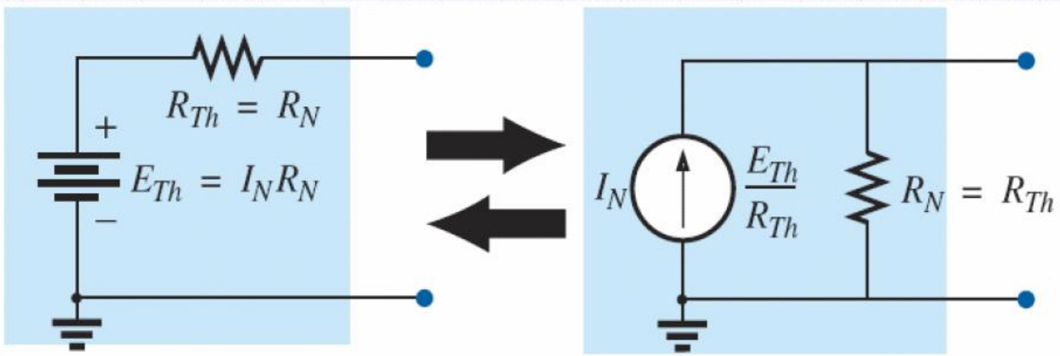


FIG. 9.66 *Converting between Thévenin and Norton equivalent circuits.*

Norton Theorem

9.4 NORTON'S THEOREM

EXAMPLE 9.12 Find the Norton equivalent circuit for the network in the shaded area in Fig. 9.67.

Solution:

Steps 1 and 2: See Fig. 9.68.

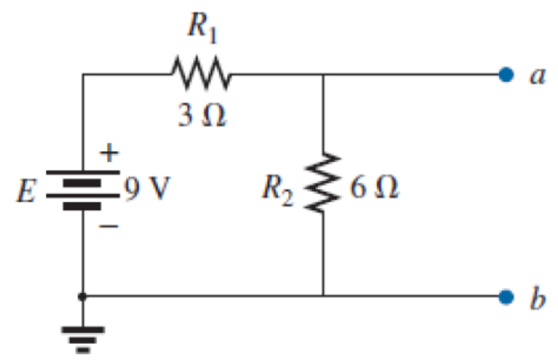


FIG. 9.68

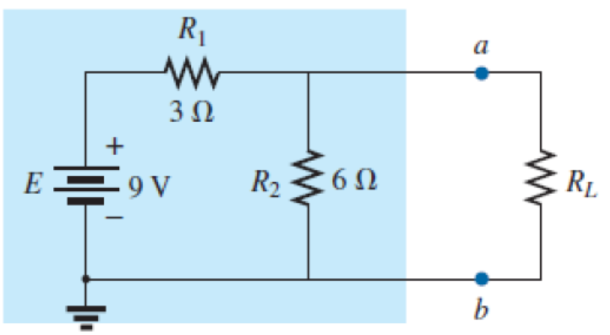


FIG. 9.67

Norton Theorem

9.4 NORTON'S THEOREM

Step 3: See Fig. 9.69, and

$$R_N = R_1 \| R_2 = 3\ \Omega \| 6\ \Omega = \frac{(3\ \Omega)(6\ \Omega)}{3\ \Omega + 6\ \Omega} = \frac{18\ \Omega}{9} = \mathbf{2\ \Omega}$$

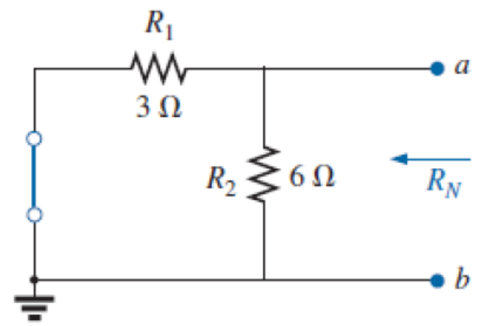


FIG. 9.69

Step 4: See Fig. 9.70, which clearly indicates that the short-circuit connection between terminals a and b is in parallel with R2 and eliminates its effect. IN is therefore the same as through R1, and the full battery voltage appears across R1 since

$$V_2 = I_2R_2 = (0)6\ \Omega = 0\ \text{V}$$

Therefore,

$$I_N = \frac{E}{R_1} = \frac{9\ \text{V}}{3\ \Omega} = \mathbf{3\ A}$$

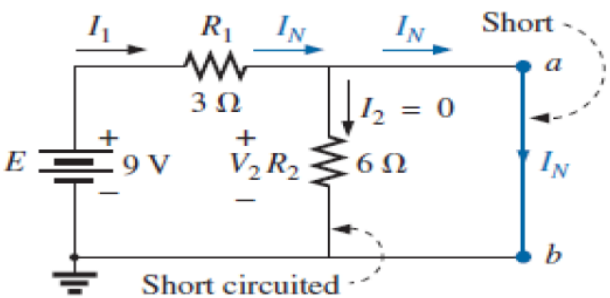


FIG. 9.70

Norton Theorem

Step 5: See Fig. 9.71. This circuit is the same as the first one considered in the development of Thévenin's theorem. A simple conversion indicates that the Thévenin circuits are, in fact, the same (Fig. 9.72).

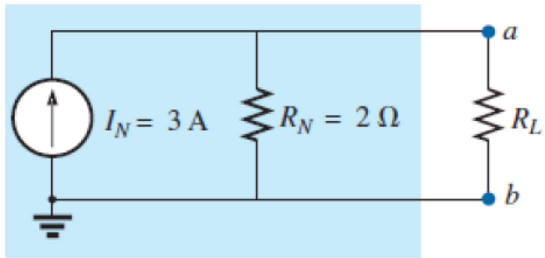


FIG. 9.71

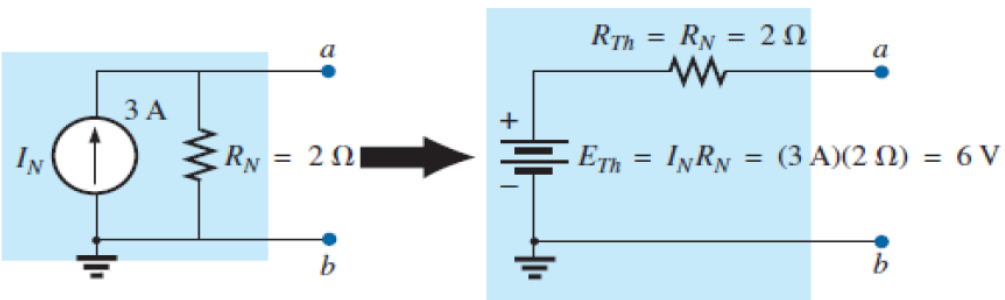


FIG. 9.72

Norton Theorem

9.4 NORTON'S THEOREM

EXAMPLE 9.13 Find the Norton equivalent circuit for the network external to the 9 Ω resistor in Fig. 9.73

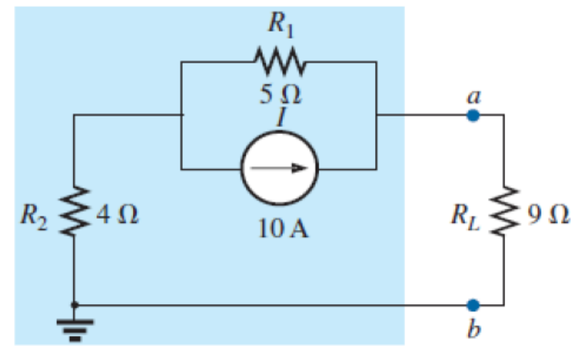
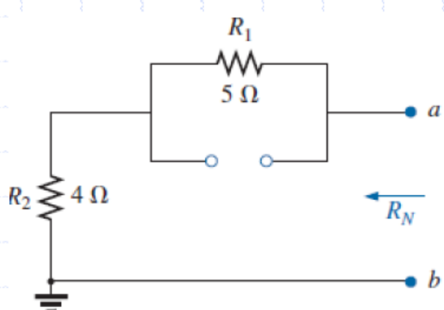


FIG. 9.73

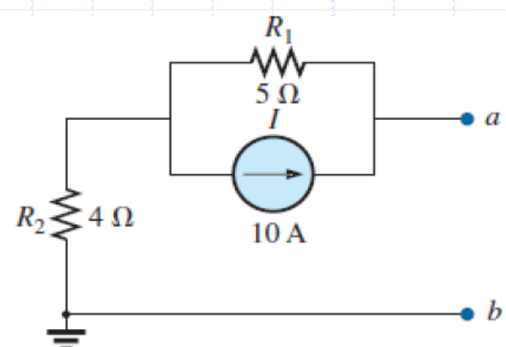
Solution:

Steps 1 and 2: See Fig. 9.74.

Step 3: See Fig. 9.75, and



$$R_N = R_1 + R_2 = 5\ \Omega + 4\ \Omega = 9\ \Omega$$



Norton Theorem

Step 4: As shown in Fig. 9.76, the Norton current is the same as the current through the 4Ω resistor. Applying the current divider rule gives

$$I_N = \frac{R_1 I}{R_1 + R_2} = \frac{(5\ \Omega)(10\ \text{A})}{5\ \Omega + 4\ \Omega} = \frac{50\ \text{A}}{9} = \mathbf{5.56\ A}$$

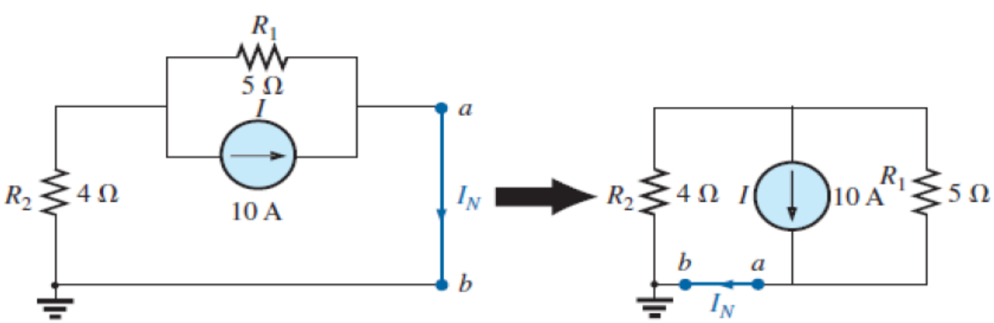


FIG. 9.76

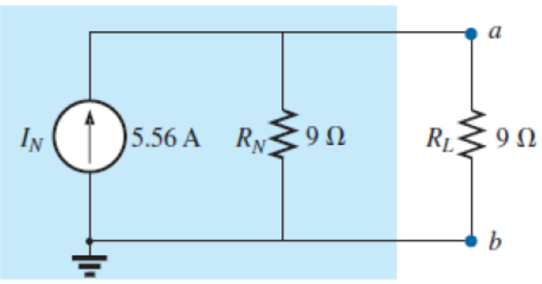


FIG. 9.77

Norton Theorem

9.4 NORTON'S THEOREM

EXAMPLE 9.14 (Two sources) Find the Norton equivalent circuit for the portion of the network to the left of a-b in Fig. 9.78.

Solution:

Steps 1 and 2: See Fig. 9.79

Step 3: See Fig. 9.80, and

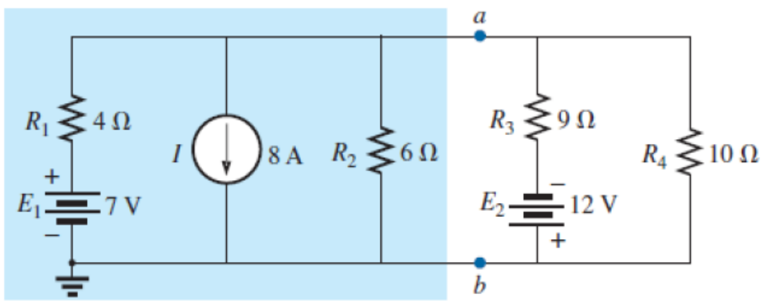


FIG. 9.78

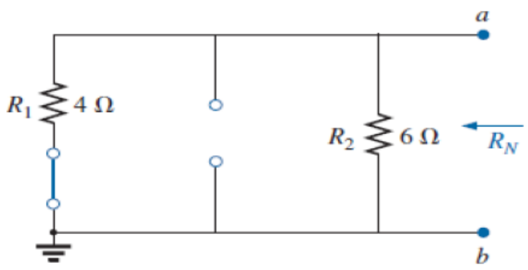


FIG. 9.80

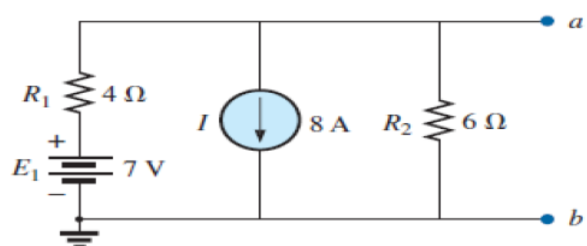


FIG. 9.79

$$R_N = R_1 \| R_2 = 4\ \Omega \| 6\ \Omega = \frac{(4\ \Omega)(6\ \Omega)}{4\ \Omega + 6\ \Omega} = \frac{24\ \Omega}{10} = 2.4\ \Omega$$

Norton Theorem

Step 4: (Using superposition) For the 7 V battery (Fig. 9.81),

$$I'_N = \frac{E_1}{R_1} = \frac{7\text{ V}}{4\ \Omega} = 1.75\text{ A}$$

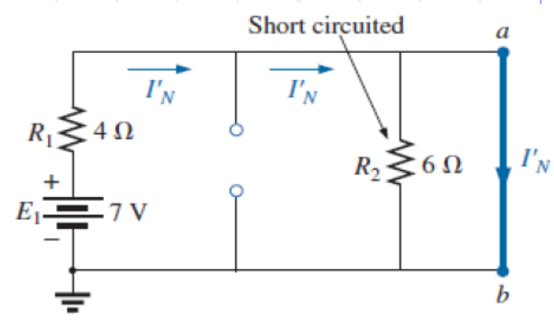


FIG. 9.81

For the 8 A source (Fig. 9.82), we find that both R1 and R2 have been "short circuited" by the direct connection between a and b, and

$$I''_N = I = 8\text{ A}$$

The result is

$$I_N = I''_N - I'_N = 8\text{ A} - 1.75\text{ A} = \mathbf{6.25\text{ A}}$$

Norton Theorem

Step 5: See Fig. 9.83

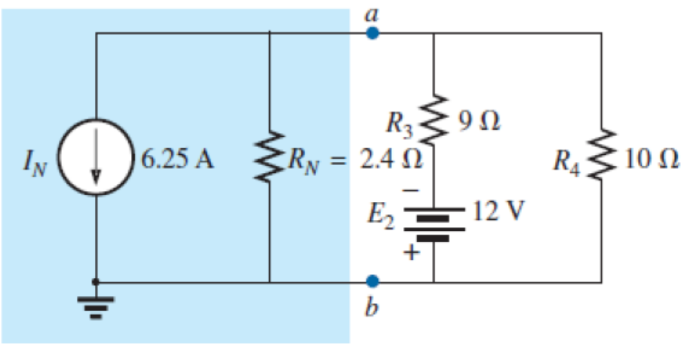
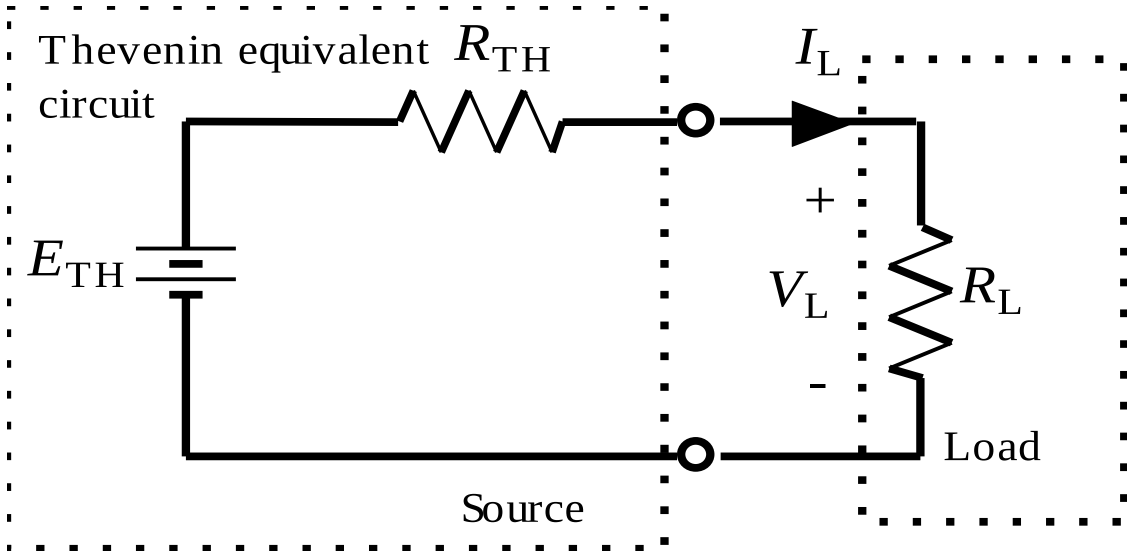


FIG. 9.83

POWER DELIVERED TO LOAD



$$P_L = V_L I_L = I_L^2 R_L = V_L^2 / R_L$$

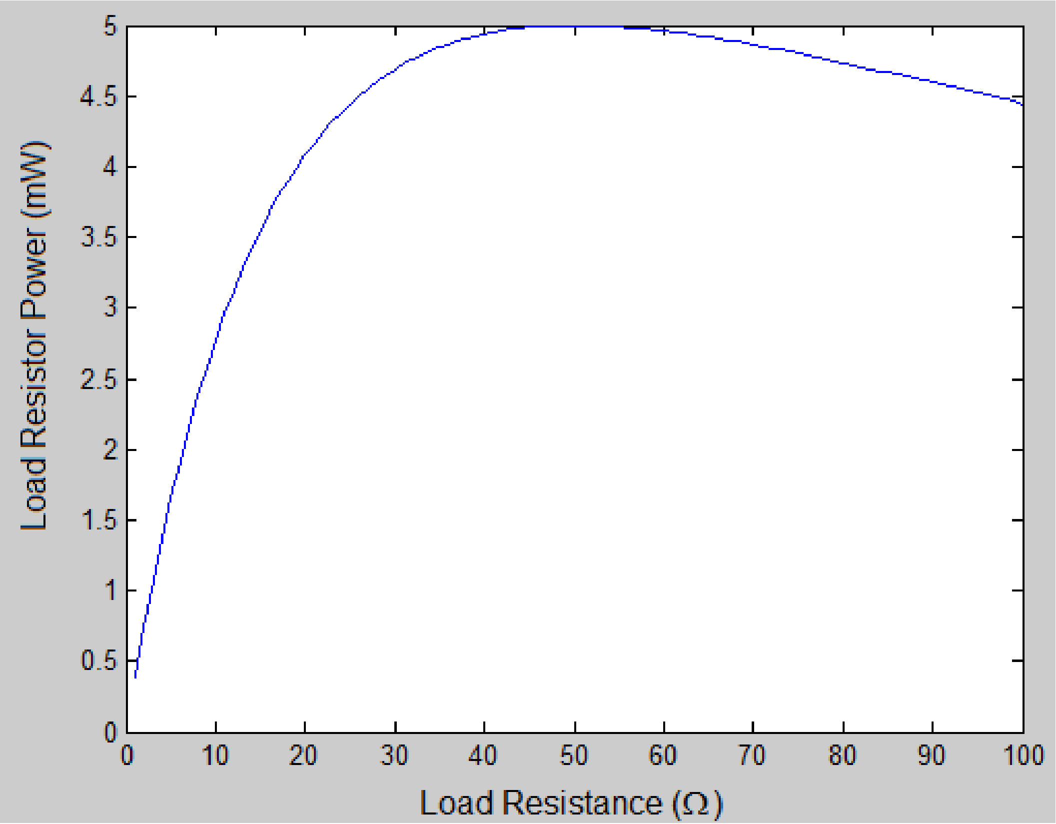
- The source develops a voltage V_L across the load and enables current I_L to flow into it
- The power delivered to the load resistance (R_L) depends on the value of R_L

MAXIMUM POWER, CURRENT AND VOLTAGE CONDITIONS

- Maximum current I_L occurs when $R_L = 0$ (shorted terminals)
- The maximum voltage V_L occurs when $R_L = \infty$ (open circuited terminals)
- Yet load power $P_L = 0$ for both cases
- P_L is maximum when R_L equals the Thevenin equivalent resistance of the source, I.e. when $R_L = R_{TH}$

- The maximum power transfer theorem is thus:
  - **Maximum power is developed in a load when the load resistance equals the Thevenin resistance of the source to which it is connected**

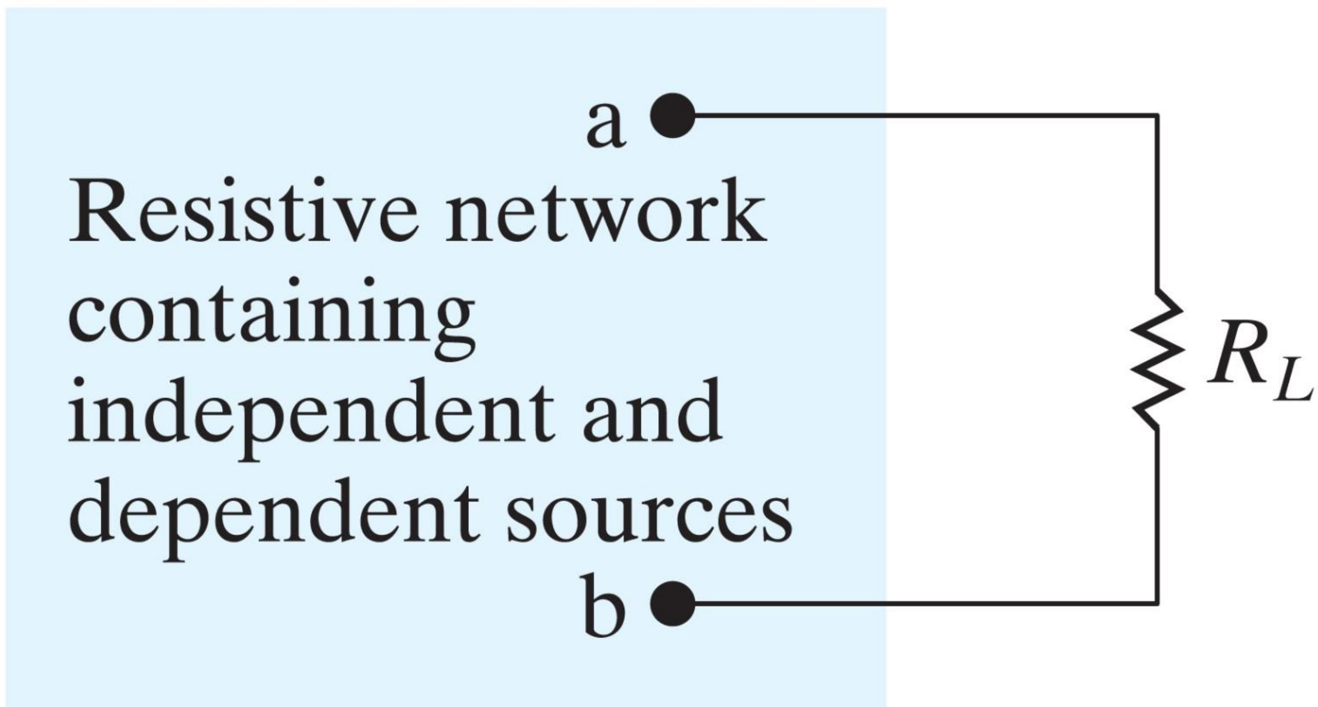
MAXIMUM POWER, CURRENT AND VOLTAGE CONDITIONS



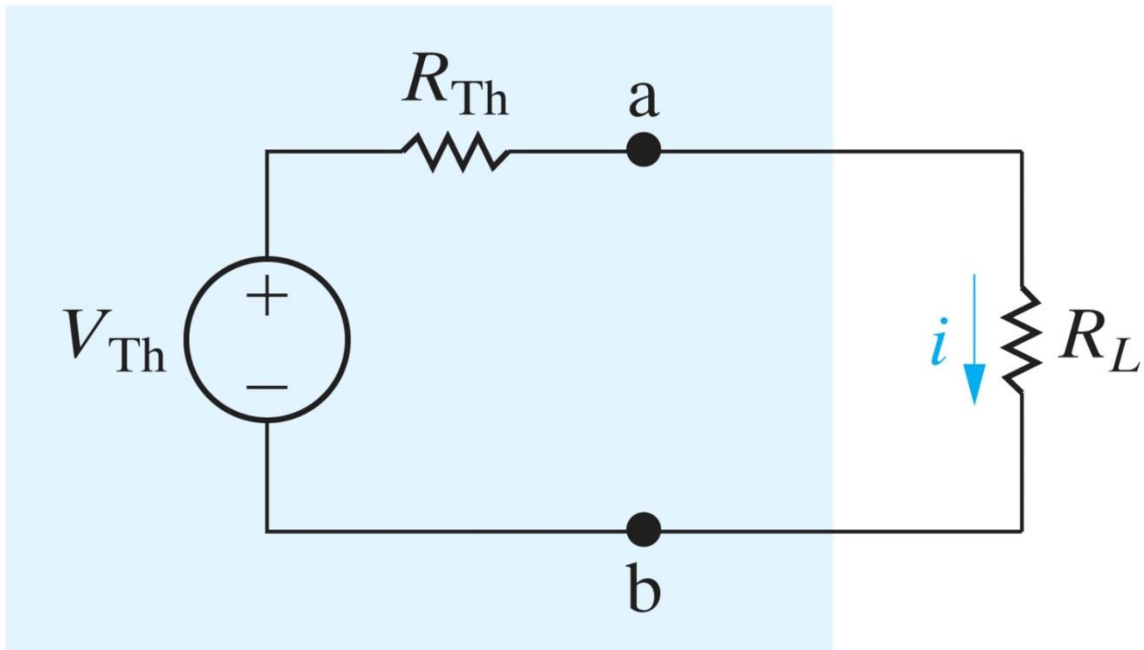
Consider the General Case

- A resistive network contains independent and dependent sources.
- A load is connected to a pair of terminals labeled a – b.
- What value of load resistance permits maximum power delivery to the load?

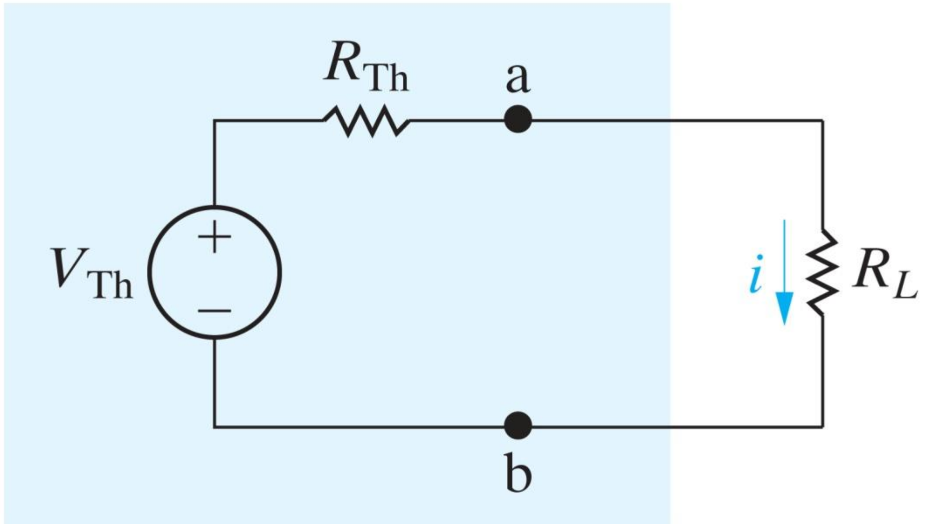
General Case (continued)



Take the Thevenin equivalent of the circuit



Look at the power developed in the load



$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L$$

Find the value of R_{load} that maximizes power

$$\frac{dp}{dR_L} = V_{Th}^2 \left(\frac{(R_{Th} + R_L)^2 - 2R_L(R_{Th} + R_L)}{(R_{Th} + R_L)^4} \right) = 0$$

$$(R_{Th} + R_L)^2 = 2R_{load}(R_{Th} + R_L)$$

$$\boxed{R_L = R_{Th}}$$

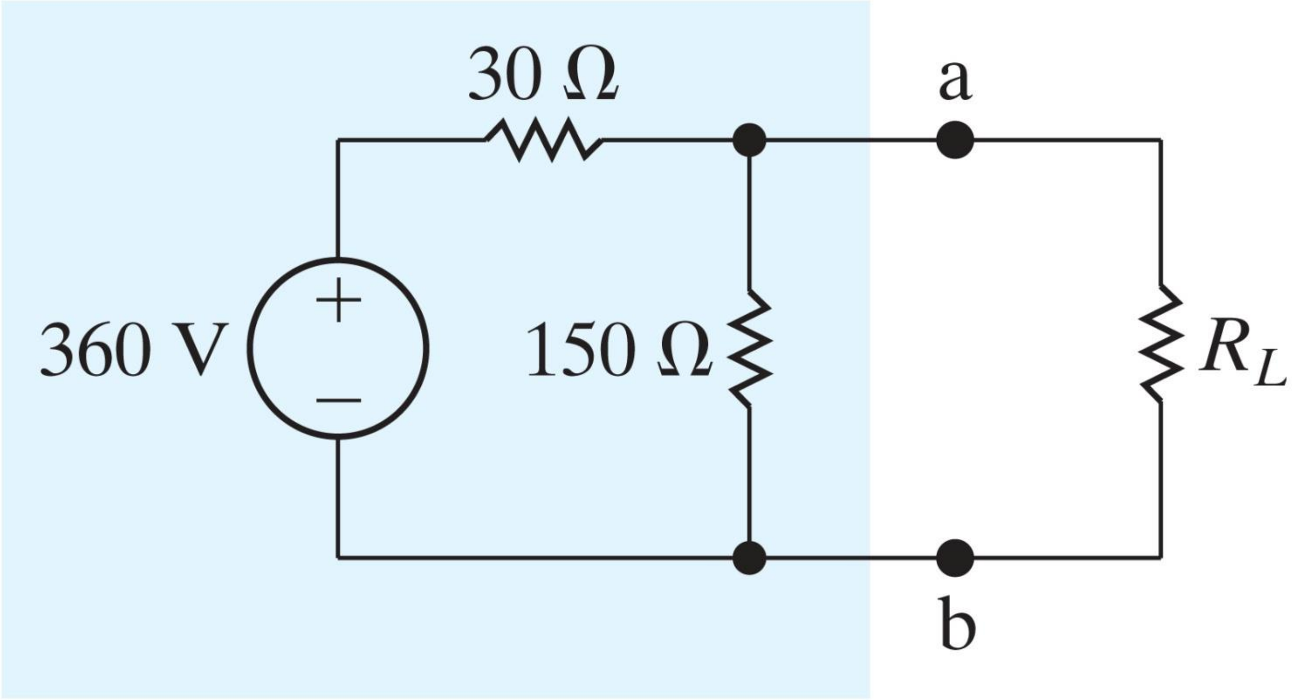
The maximum power delivered to the load

$$p_{max} = I^2 R_L = \frac{V_{Th}^2}{(2R_L)^2} R_L$$

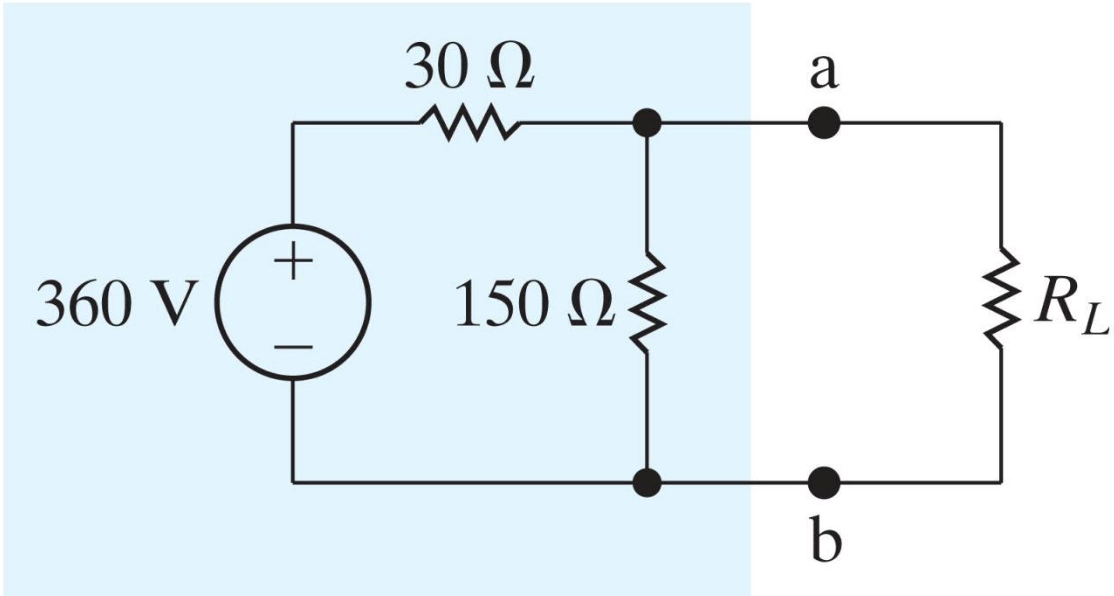
$$p_{max} = \frac{V_{Th}^2}{4R_L}$$

Example 4.12

- a) Find the value of R_L for maximum power transfer to R_L.

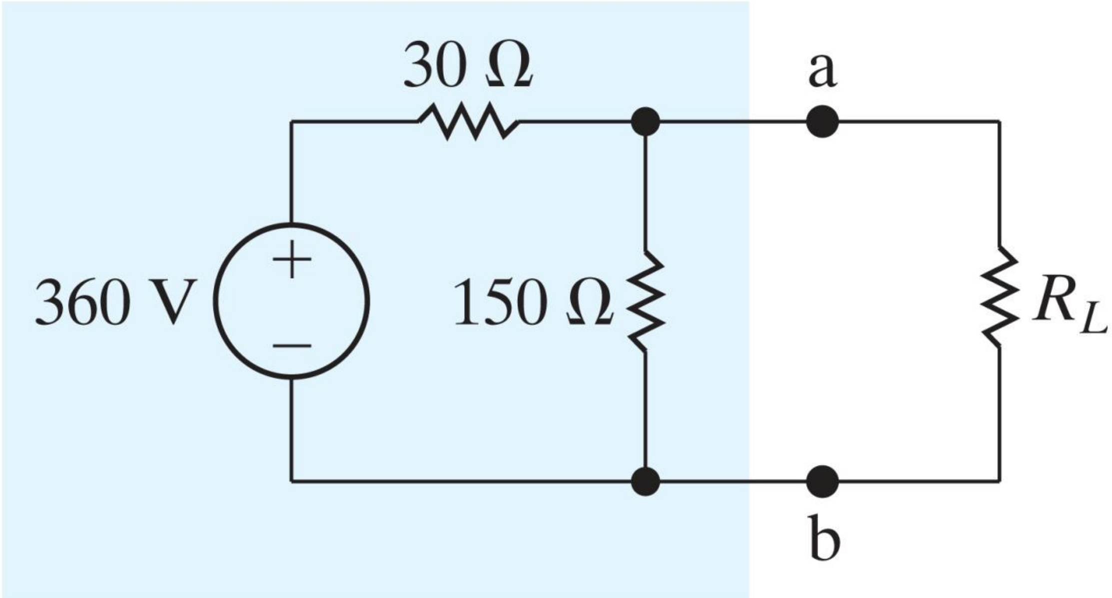


Determine the Thevenin Equivalent

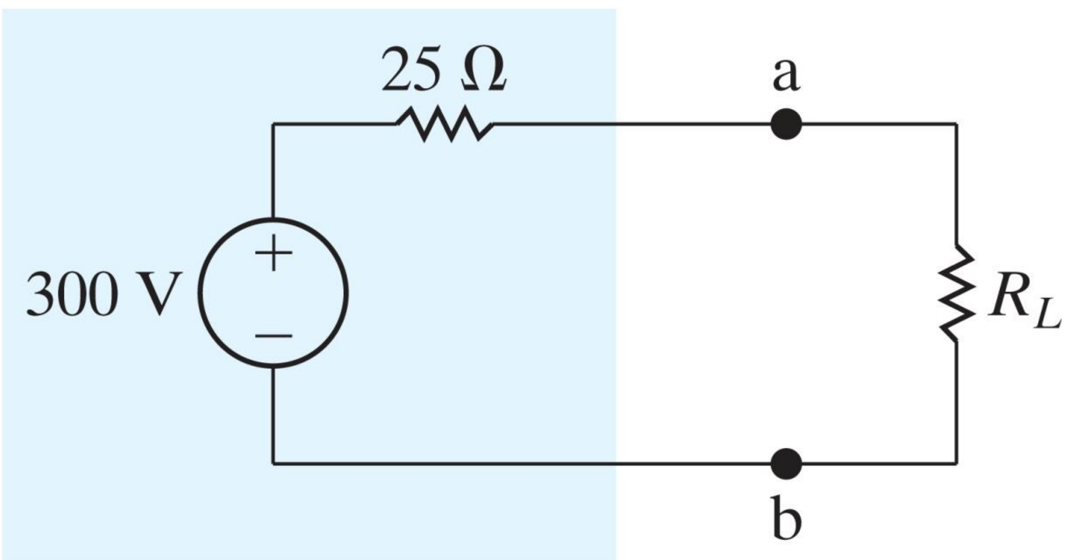


$$V_{Th} = \frac{150}{180}(360) = 300\text{V}$$

$$R_{Th} = \frac{(150)(30)}{150 + 30} = 25\Omega$$

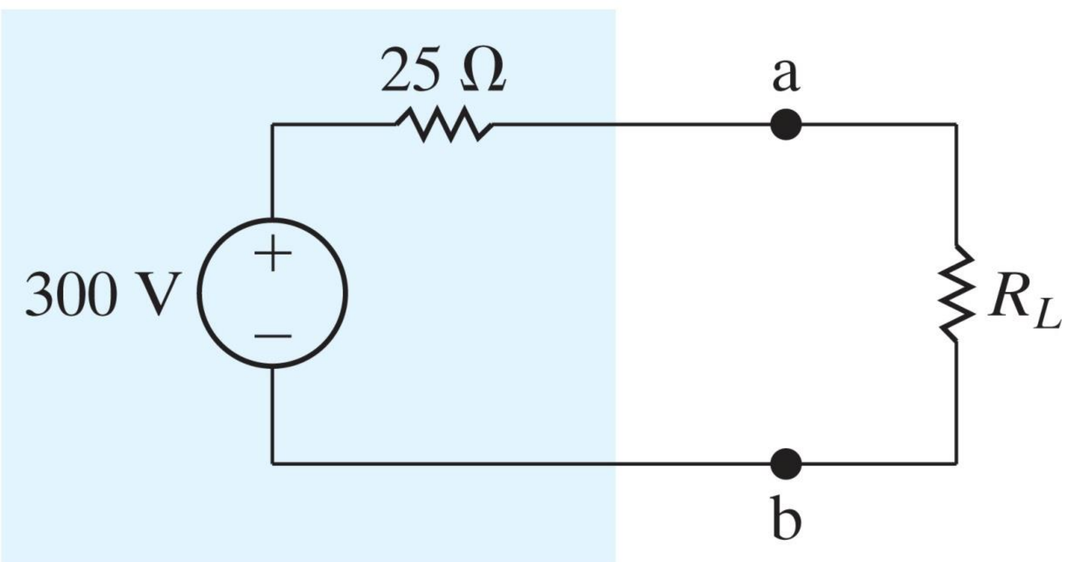


$R_L = 25\Omega$



Example 4.12 continued

- b) Calculate the maximum power that can be delivered to R_L.



$$p = i^2 R_L = \left(\frac{300}{50}\right)^2 (25)$$

$$p = 900W$$

Millman's Theorem

- Through the application of Millman's theorem, **any number of parallel voltage sources can be reduced to one.**

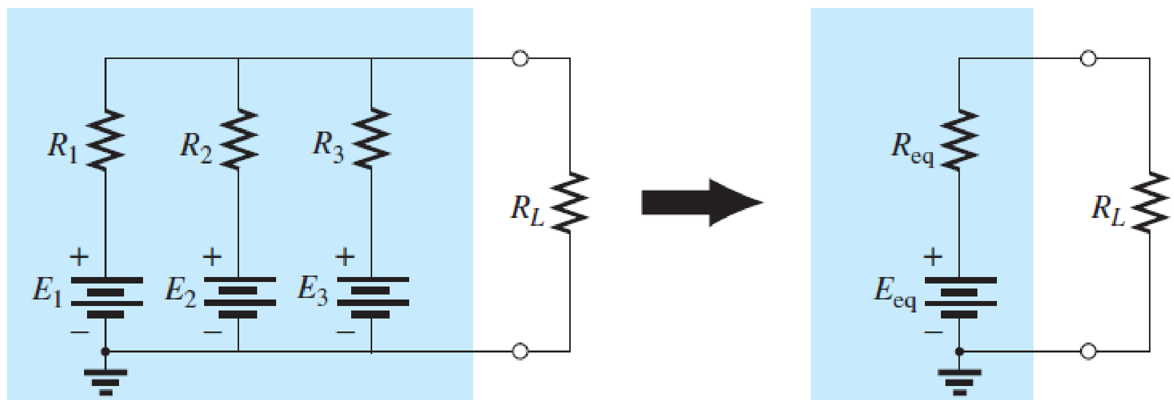


FIG. 91
Demonstrating the effect of applying Millman's theorem.

Millman's Theorem

- Through the application of Millman's theorem, any number of parallel voltage sources can be reduced to one.
- *Step 1:* Convert all voltage sources to current sources.

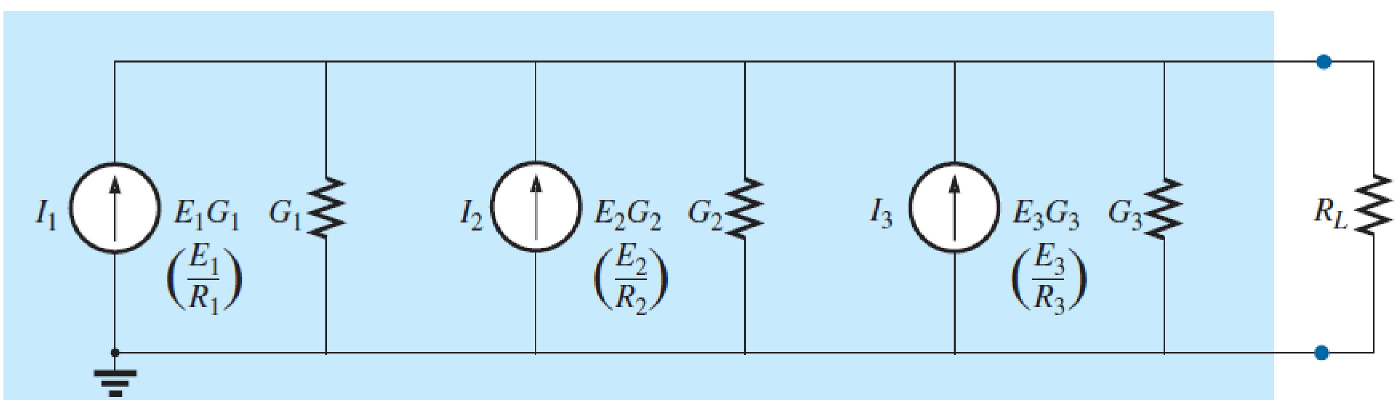


FIG. 92
Converting all the sources in Fig. 91 to current sources.

Millman’s Theorem

- *Step 2:* Combine parallel current sources.

$$I_T = I_1 + I_2 + I_3 \qquad \text{and} \qquad G_T = G_1 + G_2 + G_3$$

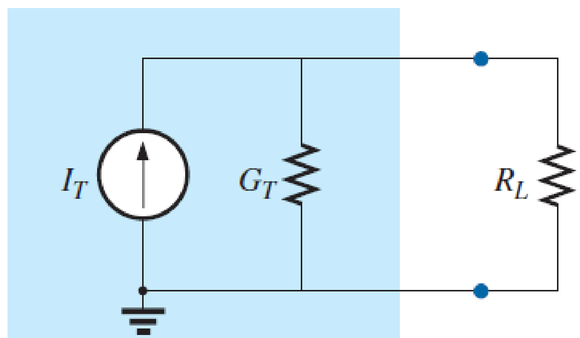


FIG. 93
Reducing all the current sources in Fig. 92 to a single current source.

Millman's Theorem

- *Step 3:* Convert the resulting current source to a voltage source, and the desired single-source network is obtained, as shown in Fig. 94.

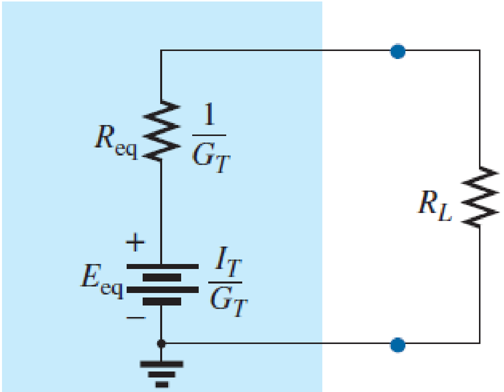


FIG. 94

Converting the current source in Fig. 93 to a voltage source.

Millman's Theorem

- In general, Millman's theorem states that for any number of parallel voltage sources

$$E_{eq} = \frac{I_T}{G_T} = \frac{\pm I_1 \pm I_2 \pm I_3 \pm \cdots \pm I_N}{G_1 + G_2 + G_3 + \cdots + G_N}$$

or

$$\boxed{E_{eq} = \frac{\pm E_1G_1 \pm E_2G_2 \pm E_3G_3 \pm \cdots \pm E_NG_N}{G_1 + G_2 + G_3 + \cdots + G_N}} \qquad \textbf{(8)}$$

- The plus-and-minus signs appear in Eq. (8) to include those cases where the sources may not be supplying energy in the same direction. (Note Example 18.)

Millman's Theorem

- The equivalent resistance is-

$$R_{eq} = \frac{1}{G_T} = \frac{1}{G_1 + G_2 + G_3 + \cdots + G_N} \tag{9}$$

- In terms of the resistance values,

$$E_{eq} = \frac{\pm \dfrac{E_1}{R_1} \pm \dfrac{E_2}{R_2} \pm \dfrac{E_3}{R_3} \pm \cdots \pm \dfrac{E_N}{R_N}}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \cdots + \dfrac{1}{R_N}} \tag{10}$$

and

$$R_{eq} = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \cdots + \dfrac{1}{R_N}} \tag{11}$$

Millman's Theorem

- In general, Millman's theorem states that for any number of parallel voltage sources

$$E_{eq} = \frac{I_T}{G_T} = \frac{\pm I_1 \pm I_2 \pm I_3 \pm \cdots \pm I_N}{G_1 + G_2 + G_3 + \cdots + G_N}$$

or

$$\boxed{E_{eq} = \frac{\pm E_1G_1 \pm E_2G_2 \pm E_3G_3 \pm \cdots \pm E_NG_N}{G_1 + G_2 + G_3 + \cdots + G_N}} \qquad \textbf{(8)}$$

- The plus-and-minus signs appear in Eq. (8) to include those cases where the sources may not be supplying energy in the same direction. (Note Example 18.)

Millman's Theorem

EXAMPLE 18 Using Millman's theorem, find the current through and voltage across the resistor R_L in Fig. 95.

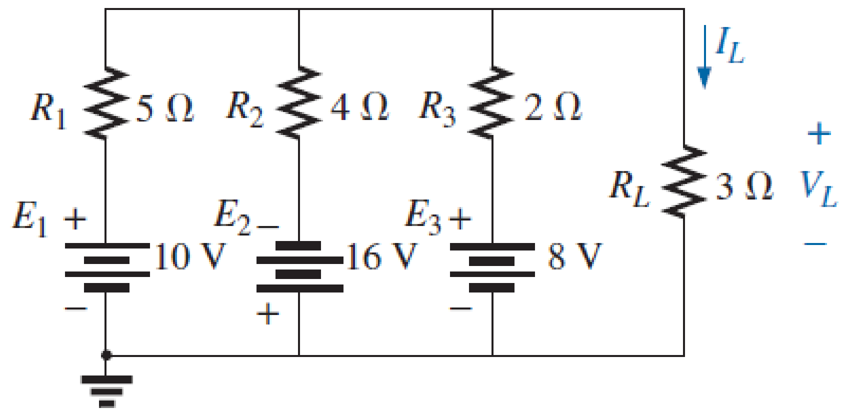


FIG. 95
Example 18.

Millman's Theorem

EXAMPLE 18 Using Millman's theorem, find the current through and voltage across the resistor R_L in Fig. 95.

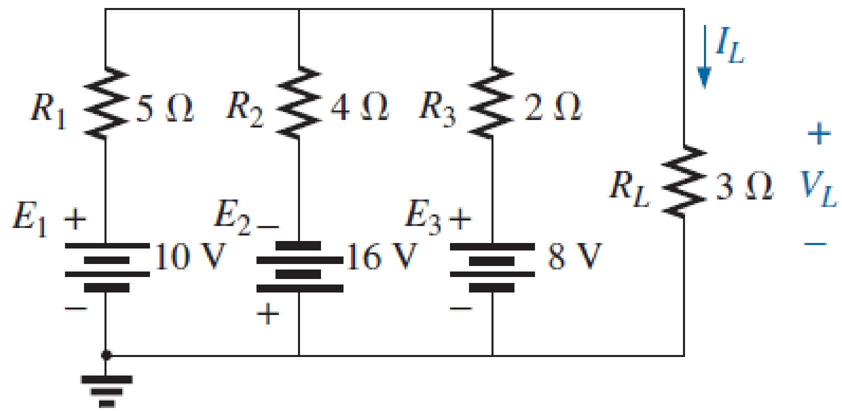


FIG. 95
Example 18.

Millman's Theorem

Solution: By Eq. (10),

$$E_{eq} = \frac{+\dfrac{E_1}{R_1} - \dfrac{E_2}{R_2} + \dfrac{E_3}{R_3}}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}}$$

The minus sign is used for E_2/R_2 because that supply has the opposite polarity of the other two. The chosen reference direction is therefore that of E_1 and E_3. The total conductance is unaffected by the direction, and

Millman's Theorem

$$E_{eq} = \frac{+\dfrac{10\text{ V}}{5\,\Omega} - \dfrac{16\text{ V}}{4\,\Omega} + \dfrac{8\text{ V}}{2\,\Omega}}{\dfrac{1}{5\,\Omega} + \dfrac{1}{4\,\Omega} + \dfrac{1}{2\,\Omega}} = \frac{2\text{ A} - 4\text{ A} + 4\text{ A}}{0.2\text{ S} + 0.25\text{ S} + 0.5\text{ S}}$$

$$= \frac{2\text{ A}}{0.95\text{ S}} = \mathbf{2.11\ V}$$

with $$R_{eq} = \frac{1}{\dfrac{1}{5\,\Omega} + \dfrac{1}{4\,\Omega} + \dfrac{1}{2\,\Omega}} = \frac{1}{0.95\text{ S}} = \mathbf{1.05\ \Omega}$$

The resultant source is shown in Fig. 96, and

$$I_L = \frac{2.11\text{ V}}{1.05\,\Omega + 3\,\Omega} = \frac{2.11\text{ V}}{4.05\,\Omega} = \mathbf{0.52\ A}$$

with $$V_L = I_L R_L = (0.52\text{ A})(3\,\Omega) = \mathbf{1.56\ V}$$

Substitution Theorem

- Statement:

If the voltage across and the current through any branch of a dc bilateral network are known, this branch can be replaced by any combination of elements that will maintain the same voltage across and current through the chosen branch.

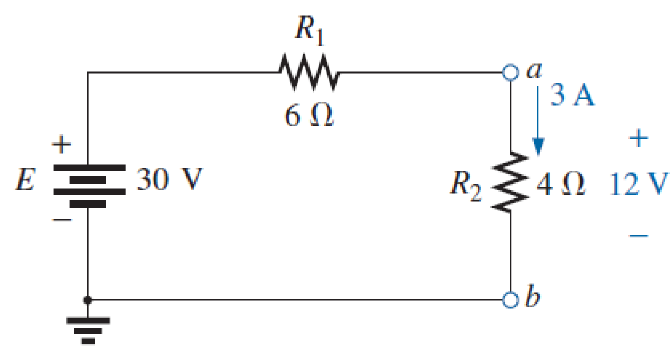


FIG. 102
Demonstrating the effect of the substitution theorem.

Substitution Theorem

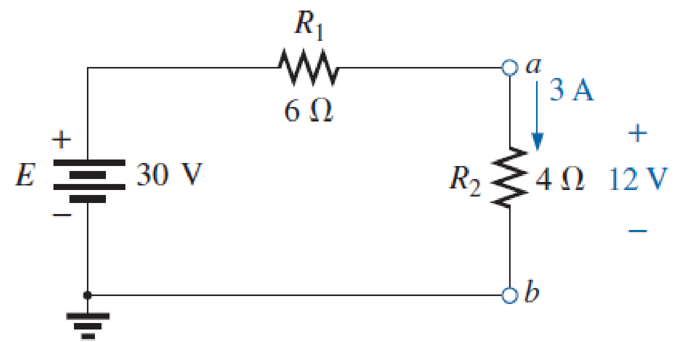


FIG. 102
Demonstrating the effect of the substitution theorem.

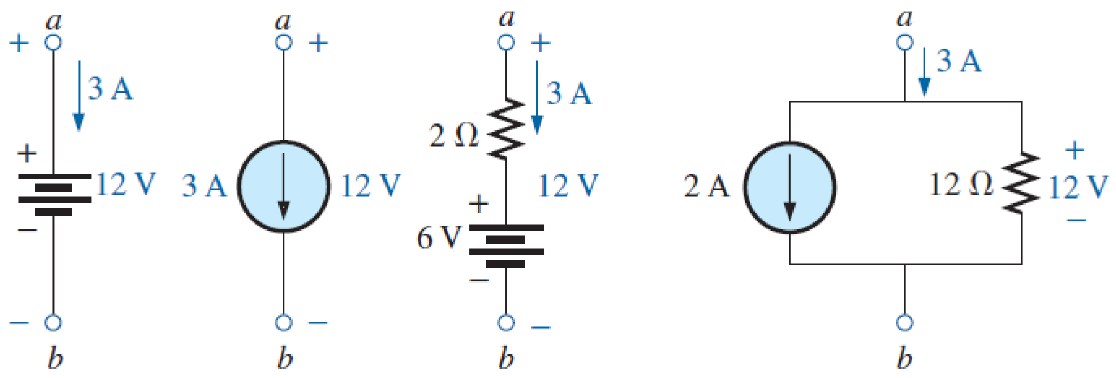


FIG. 103
Equivalent branches for the branch a-b in Fig. 102.

Substitution Theorem

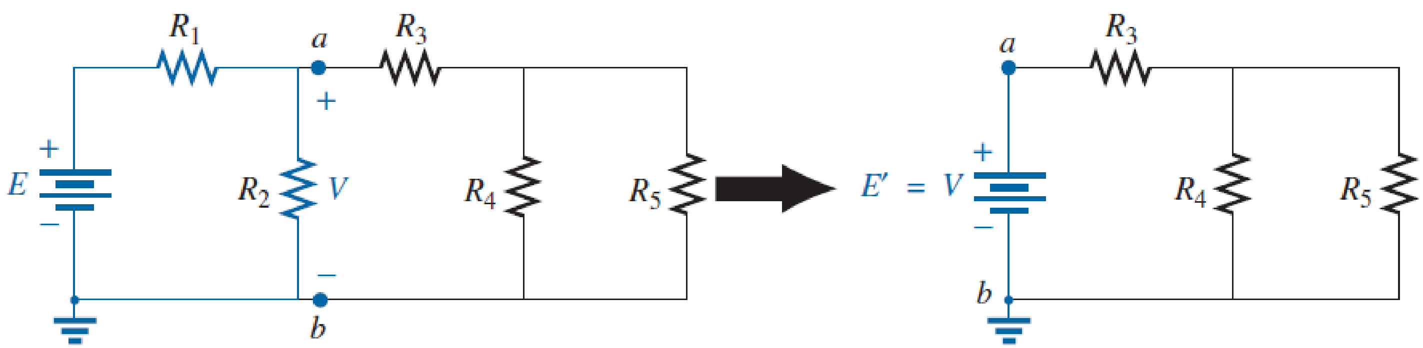


FIG. 104
Demonstrating the effect of knowing a voltage at some point in a complex network.

Substitution Theorem

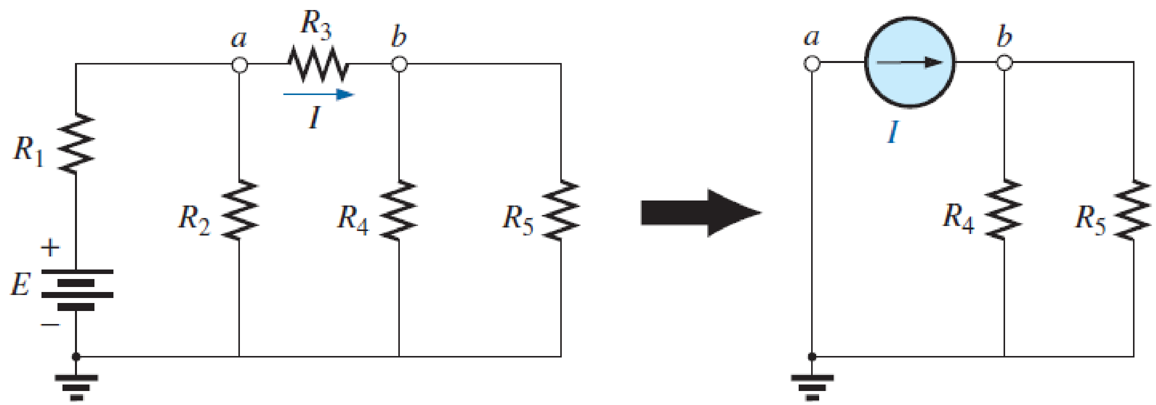


FIG. 105

Demonstrating the effect of knowing a current at some point in a complex network.

RECIPROCITY Theorem

- **Statement:**
  - The current I in any branch of a network due to a single voltage source E anywhere else in the network will equal the current through the branch in which the source was originally located if the source is placed in the branch in which the current I was originally measured.

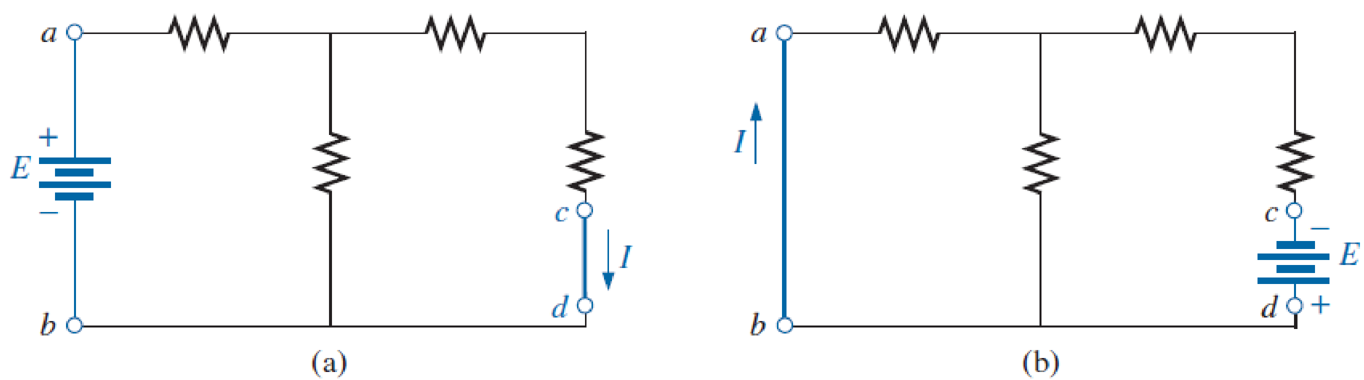


FIG. 106
Demonstrating the impact of the reciprocity theorem.

RECIPROCITY Theorem

- To demonstrate the validity of this statement and the theorem, consider the network in Fig. 107, in which values for the elements of Fig. 106(a) have been assigned.

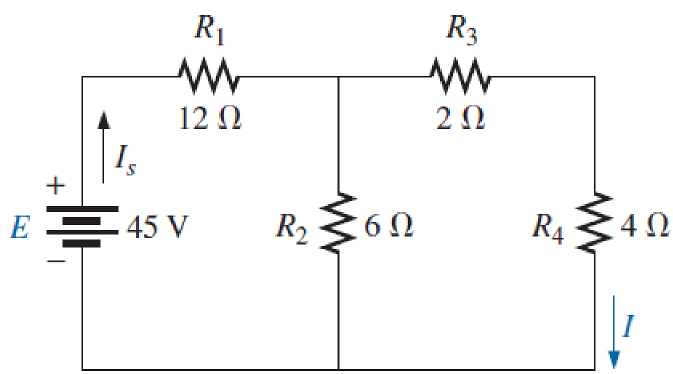


FIG. 107
Finding the current I due to a source E.

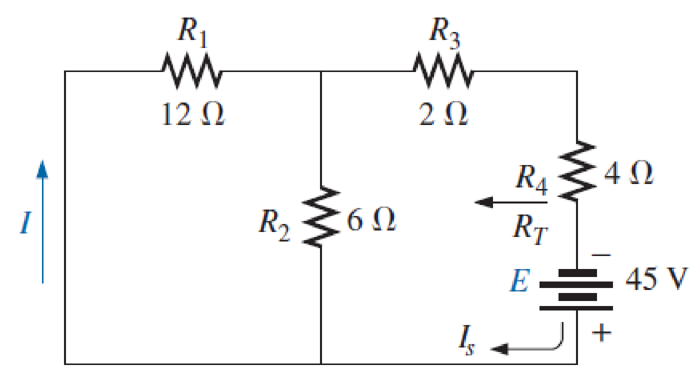


FIG. 108
Interchanging the location of E and I of Fig. 107 to demonstrate the validity of the reciprocity theorem.

RECIPROCITY Theorem

The total resistance is

$$R_T = R_1 + R_2 \| (R_3 + R_4) = 12\ \Omega + 6\ \Omega \| (2\ \Omega + 4\ \Omega)$$
$$= 12\ \Omega + 6\ \Omega \| 6\ \Omega = 12\ \Omega + 3\ \Omega = 15\ \Omega$$

and
$$I_s = \frac{E}{R_T} = \frac{45\ \text{V}}{15\ \Omega} = 3\ \text{A}$$

with
$$I = \frac{3\ \text{A}}{2} = \mathbf{1.5\ A}$$

For the network in Fig. 108, which corresponds to that in Fig. 106(b), we find

$$R_T = R_4 + R_3 + R_1 \| R_2$$
$$= 4\ \Omega + 2\ \Omega + 12\ \Omega \| 6\ \Omega = 10\ \Omega$$

and
$$I_s = \frac{E}{R_T} = \frac{45\ \text{V}}{10\ \Omega} = 4.5\ \text{A}$$

so that
$$I = \frac{(6\ \Omega)(4.5\ \text{A})}{12\ \Omega + 6\ \Omega} = \frac{4.5\ \text{A}}{3} = \mathbf{1.5\ A}$$

which agrees with the above.