Logarithmic Differentiation

If we have a function raised to a power which is also a function or if we have the product of a number of functions to differentiate such expressions it would be convenient first to take logarithm of the expression and then differentiate.

$y = \{f(x)\}^{g(x)} \quad \therefore lny = ln\{f(x)\}^{g(x)} = g(x)\ln\{f(x)\}$

$y = f_1(x)\, f_2(x)\, f_3(x) \quad \therefore lny = ln\,\{f_1(x)\, f_2(x)\, f_3(x)\} \;\Rightarrow lny = ln\, f_1(x) + ln\, f_2(x) + ln\, f_3(x)$

Example:

$i)\ x^{x^x} \quad ii)\ x^{x^{x^x}} \quad iii)\ x^{\cot^{-1}x} \quad iv)\ x^{\cos^{-1}x} \quad v)\ x^{2sinx} \quad vi)\ (sinx)^x \quad vii)\ \sqrt{x+1}^{\log(x+1)}$

$viii)\ (sinx)^{logx} \quad ix)\ (\sin^{-1}x)^{logx} \quad x)\ x^x + x^{\frac{1}{x}} \quad xi)\ \left(1+\frac{1}{x}\right)^x + x^{\frac{1}{x}} \quad xii)\ sinx^{logx}$

$+sin^2(\cos^{-1}x) \quad xiii)\ sinx^{cosx} + cosx^{sinx} \quad xiv)\ x^x + (sinx)^{logx}$

$xv)\ y = x^{x^{x^{\cdot^{\cdot^{\cdot^{\infty}}}}}} \quad xvi)\ y = \left(sin\sqrt{1+x^2}\right)^{lncosx} \quad xvii)\ (tanx)^{sinx} + (sinx)^{tanx}$

$xviii)\ x^{sinx} + (sinx)^x \quad xix)\ x^{tanx} + (sinx)^{cosx} \quad xx)\ x^{\cos^{-1}x} + (sinx)^{lnx}$
$xxi)\ x^{\sin^{-1}x} + (cosx)^{lnx} \quad xxii)\ y = e^{cosec^2\sqrt{x^2+3}}$

Sol^n (i) Let $y = x^{x^x}$

Taking ln on both sides

$\ln y = \ln x^{x^x}$

$\Rightarrow \ln y = x^x \ln x$

Again taking ln on both sides

$\ln(\ln y) = \ln(x^x \ln x)$

$\Rightarrow \ln(\ln y) = \ln x^x + \ln(\ln x)$

$\Rightarrow \ln(\ln y) = x\ln x + \ln(\ln x)$

Differentiate both sides w.r.to x we have

$\frac{d}{dx}[\ln(\ln y)] = \frac{d}{dx}(x\ln x) + \frac{d}{dx}(\ln(\ln x))$

$\Rightarrow \frac{1}{\ln y}\cdot\frac{d}{dx}(\ln y) = x\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}(x) + \frac{1}{\ln x}\cdot\frac{d}{dx}(\ln x)$

$\Rightarrow \frac{1}{\ln y}\cdot\frac{1}{y}\frac{dy}{dx} = x\cdot\frac{1}{x} + \ln x + \frac{1}{\ln x}\cdot\frac{1}{x}$

$\Rightarrow \frac{1}{y\ln y}\frac{dy}{dx} = 1 + \ln x + \frac{1}{x\ln x}$

$\Rightarrow \frac{dy}{dx} = y\ln y\left(1 + \ln x + \frac{1}{x\ln x}\right)$

$= x^{x^x}\cdot x^x\ln x\left(1 + \ln x + \frac{1}{x\ln x}\right)$

$\therefore \frac{dy}{dx} = x^{x^x}x^x\left\{\ln x + (\ln x)^2 + \frac{1}{x}\right\}$ Ans.

(iii) Let $y = x^{x^{x^x}}$

$\therefore \ln y = \ln x^{x^{x^x}}$

$\Rightarrow \ln y = x^{x^x}\ln x.$

$\therefore \frac{d}{dx}(\ln y) = \frac{d}{dx}\left(x^{x^x}\ln x\right)$

$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = x^{x^x}\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}\left(x^{x^x}\right)$

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$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x^{x^x}\frac{1}{x} + \ln x\left[x^{x^x}\cdot x^x\left\{\ln x + (\ln x)^2 + \frac{1}{x}\right\}\right]$ [From (i)]

$\therefore \frac{dy}{dx} = y\left[\frac{x^{x^x}}{x} + \ln x\left\{x^{x^x}x^x\left\{\ln x + (\ln x)^2 + \frac{1}{x}\right\}\right\}\right]$

$= x^{x^{x^x}}\left[\frac{x^{x^x}}{x} + \ln x\left\{x^{x^x}\cdot x^x\left\{\ln x + (\ln x)^2 + \frac{1}{x}\right\}\right\}\right]$ Ans

(iii) Let $y = x^{\cot^{-1}x}$

$\Rightarrow \ln y = \ln x^{\cot^{-1}x}$

$\Rightarrow \ln y = (\cot^{-1}x)(\ln x)$

$\therefore \frac{d}{dx}(\ln y) = \frac{d}{dx}\left[(\cot^{-1}x)\ln x\right]$

$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \cot^{-1}x \,\frac{d}{dx}(\ln x) + \ln x\,\frac{d}{dx}(\cot^{-1}x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (\cot^{-1}x)\,\frac{1}{x} + \ln x\cdot\frac{-1}{1+x^2}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\cot^{-1}x}{x} - \frac{\ln x}{1+x^2}$

$\Rightarrow \frac{dy}{dx} = y\left(\frac{\cot^{-1}x}{x} - \frac{\ln x}{1+x^2}\right)$

$\therefore \frac{dy}{dx} = x^{\cot^{-1}x}\left(\frac{\cot^{-1}x}{x} - \frac{\ln x}{1+x^2}\right)$ Ans. (2)

Same (iv), (v), (vi), (vii)

(viii) Let $y = \left(\sqrt{x+1}\right)^{\log(x+1)}$

$\therefore \ln y = \ln\left(\sqrt{x+1}\right)^{\log(x+1)} = \log(x+1)\,\ln\left(\sqrt{x+1}\right)$

$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\log(x+1)\cdot\ln(x+1)^{1/2}\right]$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(x+1)\,\frac{d}{dx}\left[\frac{1}{2}\ln(x+1)\right] + \frac{1}{2}\ln(x+1)\,\frac{d}{dx}\left[\log(x+1)\right]$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\log(x+1)\,\frac{1}{x+1}\cdot\frac{d}{dx}(x+1) + \frac{1}{2}\ln(x+1)\cdot\frac{1}{x+1}\,\frac{d}{dx}(x+1)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\log(x+1)}{2(x+1)}\cdot 1 + \frac{\ln(x+1)}{2(x+1)}\cdot 1$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{2\log(x+1)}{2(x+1)} \Rightarrow \frac{dy}{dx} = y\,\frac{\log(x+1)}{x+1}$

$= \left(\sqrt{x+1}\right)^{\log(x+1)}\cdot\frac{\log(x+1)}{(x+1)}$ Ans

xi) let $y = x^x + x^{\frac{1}{x}}$

$\Rightarrow y = u + v$ where $u = x^x$ and $v = x^{\frac{1}{x}}$

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Now $u = x^x$

$\therefore \ln u = \ln x^x$

$\Rightarrow \ln u = x \ln x$

$\therefore \frac{1}{y}\frac{du}{dx} = x \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(x)$

$\Rightarrow \frac{1}{y}\frac{du}{dx} = x \cdot \frac{1}{x} + \ln x \cdot 1$

$\Rightarrow \frac{1}{y}\frac{du}{dx} = 1 + \ln x$

$\Rightarrow \frac{du}{dx} = y(1+\ln x)$

$= x^x(1+\ln x)$

$v = x^{\frac{1}{x}}$

$\therefore \ln v = \ln x^{\frac{1}{x}}$

$\Rightarrow \ln v = \frac{1}{x} \ln x$

$\Rightarrow \ln v = \frac{\ln x}{x}$

$\therefore \frac{1}{v}\frac{dv}{dx} = \frac{x \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(x)}{x^2}$

$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{x \cdot \frac{1}{x} + \ln x}{x^2}$

$\Rightarrow \frac{dv}{dx} = v\left(\frac{1+\ln x}{x^2}\right)$

$= x^{\frac{1}{x}} \frac{(1+\ln x)}{x^2}$

$\therefore \frac{dy}{dx} = x^x(1+\ln x) + \frac{x^{\frac{1}{x}}(1+\ln x)}{x^2}$ Ans.

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xii Let $y = \left(1+\frac{1}{x}\right)^x + x^{\frac{1}{x}}$.

$\Rightarrow y = u + v$ where $u = \left(1+\frac{1}{x}\right)^x$ and $v = x^{\frac{1}{x}}$

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Now $u = \left(1+\frac{1}{x}\right)^x$

$\therefore \ln u = \ln\left(1+\frac{1}{x}\right)^x$

$\therefore \ln u = x \ln(1+\frac{1}{x})$

$\therefore \frac{1}{u}\frac{du}{dx} = x \frac{d}{dx}[\ln(1+\frac{1}{x})] + \ln(1+\frac{1}{x}) \frac{d}{dx}(x)$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{1+\frac{1}{x}} \frac{d}{dx}(1+\frac{1}{x}) + \ln(1+\frac{1}{x}) \cdot 1$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = x \cdot \frac{x}{1+x} \cdot \frac{-1}{x^2} + \ln(1+\frac{1}{x})$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = -\frac{1}{1+x} + \ln(1+\frac{1}{x})$

$\therefore \frac{du}{dx} = u\left[\ln(1+\frac{1}{x}) - \frac{1}{1+x}\right]$

$= (1+\frac{1}{x})^x \left\{\ln(1+\frac{1}{x}) - \frac{1}{1+x}\right]$

Again, $v = x^{\frac{1}{x}}$ (3)

$\therefore \ln v = \ln x^{\frac{1}{x}}$

$\Rightarrow \ln v = \frac{1}{x} \times \ln x$

$\Rightarrow \ln v = \frac{\ln x}{x}$

$\therefore \frac{1}{v}\frac{dv}{dx} = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2}$

$\Rightarrow \frac{dv}{dx} = \frac{v(1-\ln x)}{x^2}$

$= \frac{x^{\frac{1}{x}}(1-\ln x)}{x^2}$

$\therefore \frac{dy}{dx} = (1+\frac{1}{x})^x \left\{\ln(1+\frac{1}{x}) - \frac{1}{1+x}\right\} + \frac{x^{\frac{1}{x}}(1-\ln x)}{x^2}$ Ans.

xiii) Let $y = (\sin x)^{\log x} + \sin^2(\cos^{-1}x)$

$\Rightarrow y = u + v$ where $u = (\sin x)^{\log x}$, $v = \sin^2(\cos^{-1}x)$

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

xvii) $y = \left(\sin\sqrt{1+x^2}\right)^{\ln\cos x}$

$\therefore \ln y = \ln\left(\sin\sqrt{1+x^2}\right)^{\ln\cos x}$

$\Rightarrow \ln y = \ln\cos x \ \ln\left(\sin\sqrt{1+x^2}\right)$

$\therefore \frac{1}{y}\cdot\frac{dy}{dx} = \ln\cos x \ \frac{d}{dx}\left[\ln\left(\sin\sqrt{1+x^2}\right)\right] + \ln\left(\sin\sqrt{1+x^2}\right)\frac{d}{dx}(\ln\cos x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln\cos x \cdot \frac{1}{\sin\sqrt{1+x^2}} \frac{d}{dx}\left(\sin\sqrt{1+x^2}\right) + \ln\left(\sin\sqrt{1+x^2}\right)\cdot\frac{1}{\cos x}(-\sin x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\ln\cos x}{\sin\sqrt{1+x^2}} \cos\sqrt{1+x^2} \cdot \frac{1}{\cancel{2}\sqrt{1+x^2}}\cdot\cancel{2}x - \tan x \ln\left(\sin\sqrt{1+x^2}\right)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\cot\sqrt{1+x^2}\ \ln\cos x}{\sqrt{1+x^2}} - \tan x \ln\left(\sin\sqrt{1+x^2}\right)$

$\Rightarrow \frac{dy}{dx} = y\left\{\frac{\cot\sqrt{1+x^2}\ \ln\cos x}{\sqrt{1+x^2}} - \tan x \ln\left(\sin\sqrt{1+x^2}\right)\right\}$

$\therefore \frac{dy}{dx} = \left(\sin\sqrt{1+x^2}\right)^{\ln\cos x}\left\{\frac{\cot\sqrt{1+x^2}\ \ln\cos x}{\sqrt{1+x^2}} - \tan x \ln\left(\sin\sqrt{1+x^2}\right)\right\}$

Ans.

xxiii) $y = e^{\operatorname{cosec}^2\sqrt{x^2+3}}$

$\therefore \frac{dy}{dx} = e^{\operatorname{cosec}^2\sqrt{x^2+3}} \cdot \frac{d}{dx}\left[\left(\operatorname{cosec}\sqrt{x^2+3}\right)^2\right]$

$= e^{\operatorname{cosec}^2\sqrt{x^2+3}}\ 2\operatorname{cosec}\sqrt{x^2+3} \cdot \frac{d}{dx}\left(\operatorname{cosec}\sqrt{x^2+3}\right)$

$= -e^{\operatorname{cosec}^2\sqrt{x^2+3}} \cdot 2\operatorname{cosec}\sqrt{x^2+3}\ \operatorname{cosec}\sqrt{x^2+3}\ \cot\sqrt{x^2+3} \cdot \frac{1}{2\sqrt{x^2+3}}\cdot 2x$

$= -e^{\operatorname{cosec}^2\sqrt{x^2+3}} \cdot 2\operatorname{cosec}^2\sqrt{x^2+3}\ \cot\sqrt{x^2+3}\left(\frac{x}{\sqrt{x^2+3}}\right)$ Ans.