Sheikh Hasina University, Netrokona

Course:

MATH-3105 (Multivariable Calculus & Geometry)

Textbook:

Calculus, Early Transcendentals

By Anton, Bivens, Davis (10th Edition)

Course Teacher:

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3rd Year 1st Semester

1

Vectors &

Geometry of Space

2

Chapter 11.2

Vectors

Vector

Definition

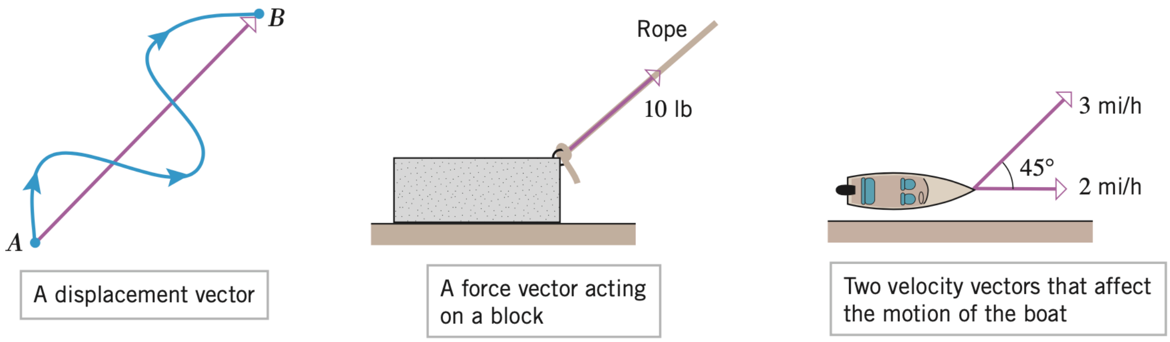
A vector is a quantity having both magnitude and direction.

Vectors can be represented geometrically by arrows in 2-space or 3-space; the direction of the arrow specifies the direction of the vector, and the length of the arrow describes its magnitude. The tail of the arrow is called the initial point of the vector, and the tip of the arrow the terminal point.

Example

Velocity, force, acceleration etc.

Vector



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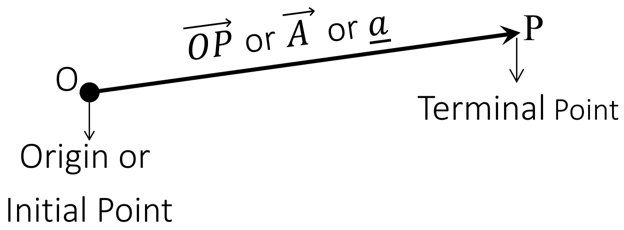
Vector Representation



- Graphically, a vector is represented by an arrow defining direction, the magnitude of the vector being indicated by the length of the arrow.
- Analytically, a vector is represented by a letter with an arrow over it as $\vec{A}$ and its magnitude is denoted by $|\vec{A}|$ or A.

6

Equal Vectors

Two vectors $\underline{a}$ and $\underline{b}$ are equal if they have the same magnitude and direction regardless of the position of their initial point.

Denoted by $\underline{a} = \underline{b}$.

Geometrically, two vectors are equal if they are translations of one another.

Null Vector and Proper Vector

Null or Zero Vector

If the initial and terminal points of a vector coincide, then the vector has length zero; we call this the zero vector and denote it by **0**. The zero vector does not have a specific direction, so we will agree that it can be assigned any convenient direction in a specific problem.

Proper Vector

A vector which is not null is a proper vector.

Vector Addition & Subtraction

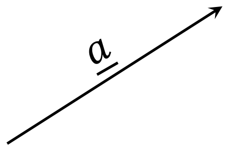


Fig. 1

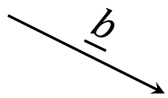


Fig. 2

- Two vector $\underline{a}$ and $\underline{b}$ are given (as shown in Figs. 1 & 2).
- We want to find $\underline{a} + \underline{b}$ and $\underline{a} - \underline{b}$.

Triangle Law for Vector Addition

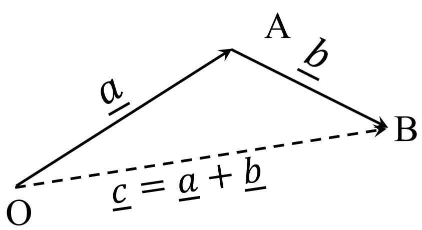


Fig. 3

i. Place the initial point of $\underline{b}$ on the terminal point on $\underline{a}$ (As shown in Fig. 3 $\overline{OA} = \underline{a}$ and $\overline{AB} = \underline{b}$).

ii. Join the initial point of $\underline{a}$ to the terminal point of $\underline{b}$ ($\overline{OB}$ in Fig. 3).

iii. As shown in Fig. 3 $\overline{OA} + \overline{AB} = \overline{OB}$ i.e. $\underline{a} + \underline{b} = \underline{c}$.

Parallelogram Law for Vector Addition

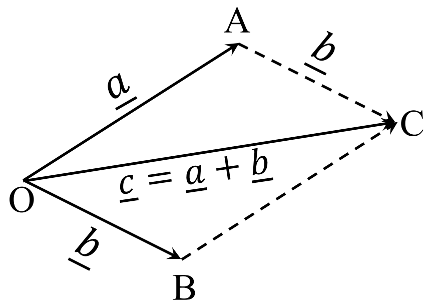


Fig.

If two vector quantities are represented by two adjacent sides or a parallelogram then the diagonal of parallelogram will be equal to the resultant of these two vectors.

Proof: Try yourself.

Vector Subtraction

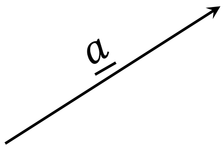


Fig. 1

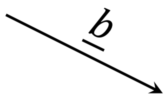


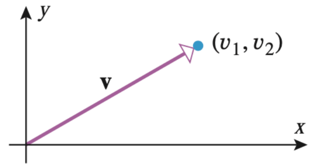
Fig. 2

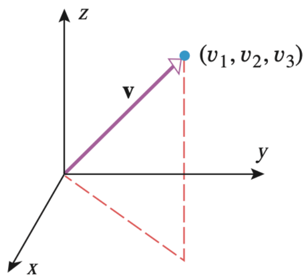
Try yourself

Two vector $\underline{a}$ and $\underline{b}$ are given (as shown in Figs. 1 & 2). Then Find $\underline{a} - \underline{b}$.

Vectors in Coordinate Systems

If a vector $\underline{v}$ is positioned with its initial point at the origin of a rectangular coordinate system, then its terminal point will have coordinates of the form (v_1, v_2) or (v_1, v_2, v_3), depending on whether the vector is in 2-space or 3-space. We call these coordinates the components of $\underline{v}$, and we write $\underline{v}$ in component form using the bracket notation





$$\underline{v} = (v_1, v_2) \quad \text{or} \quad \underline{v} = (v_1, v_2, v_3)$$

13

Vectors in Coordinate Systems

The zero vectors in 2-space and 3-space are

$$\underline{0} = (0,0) \quad \text{or} \quad \underline{0} = (0,0,0)$$

respectively.

14

Vectors in Coordinate Systems

Theorem 11.2.3

Two vectors are equivalent if and only if their corresponding components are equal.

Example

For example, consider the vectors $\underline{v} = (v_1, v_2)$ and $\underline{w} = (w_1, w_2)$ in 2-space. If $\underline{v} = \underline{w}$, then

$$v_1 = w_1 \text{ and } v_2 = w_2.$$

Arithmetic Operation on Vectors

Homework

11.2.4 Theorem (Textbook page 776)

Example 1 (Textbook page 776)

Vectors with Initial Point Not at the Origin

Theorem 11.2.5

If $\overrightarrow{P_1P_2}$ is a vector in 2-space with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$$

Similarly, if $\overrightarrow{P_1P_2}$ is a vector in 3-space with initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$, then

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

Rules of Vector Arithmetic

Homework

11.2.6 Theorem (Textbook page 777-778)

Example 2 (Textbook page 777)

Example 3 (Textbook page 778)

Norm of a Vector

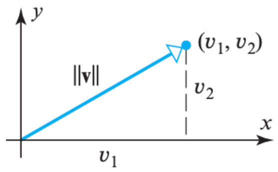
The distance between the initial and terminal points of a vector $\underline{\boldsymbol{v}}$ is called the length, the norm, or the magnitude of $\underline{\boldsymbol{v}}$ and is denoted by $\|\underline{\boldsymbol{v}}\|$.

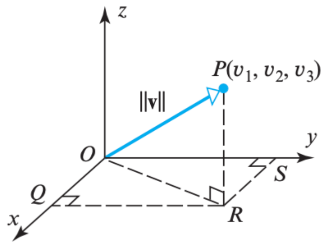
The norm of a vector $\underline{\boldsymbol{v}} = (v_1, v_2)$ in 2-space is

$$\|\underline{\boldsymbol{v}}\| = \sqrt{v_1^2 + v_2^2}$$

and norm of a vector $\underline{\boldsymbol{v}} = (v_1, v_2, v_3)$ in 3-space is

$$\|\underline{\boldsymbol{v}}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$





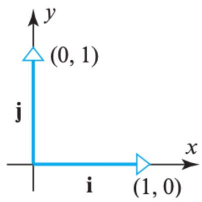
Unit Vectors

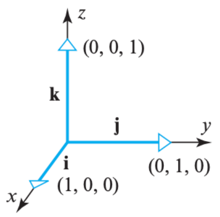
A vector of length 1 is called a unit vector.

The set of unit vectors are those having the directions of the positive $x-$, $y-$, and $z-$axes of three-dimensional rectangular coordinate system and are denoted respectively by $\hat{\imath}, \hat{\jmath}, \hat{k}$.

In 2-space: $\hat{\imath} = (1,0),\ \hat{\jmath} = (0,1)$

In 3-space: $\hat{\imath} = (1,0,0),\ \hat{\jmath} = (0,1,0),\ \hat{k} = (0,0,1)$





Unit Vectors

Every vector in 2-space is expressible uniquely in terms of $\hat{\imath}$ and $\hat{\jmath}$, and every vector in 3-space is expressible uniquely in terms of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ as follows:

$$\underline{\boldsymbol{v}} = (v_1, v_2) = (v_1, 0) + (0, v_2) = v_1(1,0) + v_2(0,1) = v_1\hat{\imath} + v_2\hat{\jmath}$$

$$\underline{\boldsymbol{v}} = (v_1, v_2, v_3) = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$$

Unit Vectors

Homework

Example 4 (Textbook page 779)

Normalizing a Vectors

If $\underline{\boldsymbol{v}}$ is a nonzero vector, then unit vector $\hat{\boldsymbol{u}}$ in the same direction as can $\underline{\boldsymbol{v}}$ be found by multiplying $\underline{\boldsymbol{v}}$ by the reciprocal of its length as

$$\hat{\boldsymbol{u}} = \frac{1}{\|\underline{\boldsymbol{v}}\|}\underline{\boldsymbol{v}}$$

The process of multiplying a vector $\underline{\boldsymbol{v}}$ by the reciprocal of its length to obtain a unit vector with the same direction is called normalizing $\underline{\boldsymbol{v}}$.

Normalizing a Vectors

Example 5 (Textbook p779)

Find the unit vector that has the same direction as $\underline{\boldsymbol{v}} = 2\hat{\imath} + 2\hat{\jmath} - \hat{k}$.

Solution

The length or norm of the vector $\underline{\boldsymbol{v}}$ is

$$\|v\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

the unit vector that has the same direction as $\underline{\boldsymbol{v}}$ is

$$\hat{\boldsymbol{u}} = \frac{1}{3}\underline{\boldsymbol{v}} = \frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} - \frac{1}{3}\hat{k}$$

Vectors Determined by Length and Angle

Homework

Example 6-8 (Textbook page 780-781)

Chapter 11.2

Homework

Exercise Set 11.2 (p782 –785)

Chapter 11.3

Dot Products; Projections

The Dot Product

Definition

If $\underline{\boldsymbol{u}} = (u_1, u_2, u_3)$ and $\underline{\boldsymbol{v}} = (v_1, v_2, v_3)$ are vectors in 3-space and θ is the angle between them, then the dot product of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ is written as $\underline{\boldsymbol{u}} \,.\, \underline{\boldsymbol{v}}$ and is define as

$$\underline{\boldsymbol{u}} \,.\, \underline{\boldsymbol{v}} = \|\underline{\boldsymbol{u}}\| \, \|\underline{\boldsymbol{v}}\| \cos\theta\,, \qquad 0 \le \theta \le \pi$$

$$= u_1v_1 + u_2v_2 + u_3v_3$$

The Dot Product

Example 1

$$(3,5).(-1,2) = 3(-1) + 5(2) = 7$$

$$(2,3).(-3,2) = 2(-3) + 3(2) = 0$$

$$(1,-3,4).(1,5,2) = 1(1) + (-3)(5) + 4(2) = -6$$

Another expression:

$$(3\hat{\imath} + 5\hat{\jmath}).(-\hat{\imath} + 2\hat{\jmath}) = 3(-1) + 5(2) = 7$$

$$(2\hat{\imath} + 3\hat{\jmath}).(-3\hat{\imath} + 2\hat{\jmath}) = 2(-3) + 3(2) = 0$$

$$(\hat{\imath} - 3\hat{\jmath} + 4\hat{k}).(\hat{\imath} + 5\hat{\jmath} + 2\hat{k}) = 1(1) + (-3)(5) + 4(2) = -6$$

The Dot Product

Algebraic Properties of the Dot Product

If $\underline{u}$, $\underline{v}$, and $\underline{w}$ are vectors in 2- or 3- space and k is a scalar, then:

(a) $\underline{u}\ .\ \underline{v} = \underline{v}\ .\ \underline{u}$; Comutative Law

(b) $\underline{u}\,.\left(\underline{v} + \underline{w}\right) = \underline{u}\,.\,\underline{v} + \underline{u}\,.\,\underline{w}$; Distributive Law

(c) $k\left(\underline{u}\ .\ \underline{v}\right) = \left(k\underline{u}\right).\ \underline{v} = \underline{u}\,.\left(k\underline{v}\right)$

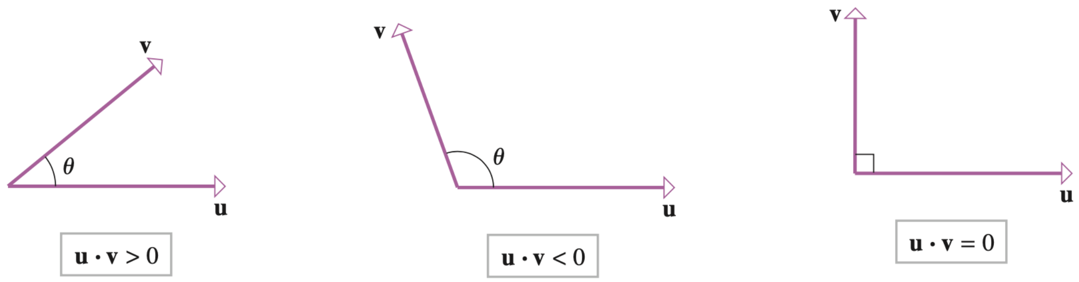
(d) $\underline{v}\,.\,\underline{v} = \left\|\underline{v}\right\|^2$

(e) $\underline{0}\,.\,\underline{v} = 0$

Proof: Homework p785-786

Interpreting the Sign of the Dot Product

$$\underline{\boldsymbol{u}} \,.\, \underline{\boldsymbol{v}} = \|\underline{\boldsymbol{u}}\| \, \|\underline{\boldsymbol{v}}\| \cos\theta \,, \qquad 0 \le \theta \le \pi$$



Angle between Vectors

Theorem

If $\underline{u}$ and $\underline{v}$ are nonzero vectors in 2-space or 3-space, and if $\boldsymbol{\theta}$ is the angle between them, then

$$\cos\theta = \frac{\underline{u} \,.\, \underline{v}}{\|\underline{\boldsymbol{u}}\| \, \|\underline{\boldsymbol{v}}\|}$$

Angle between Vectors

Example 2 (p786)

Find the angle between the vectors $\underline{u} = \hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ and

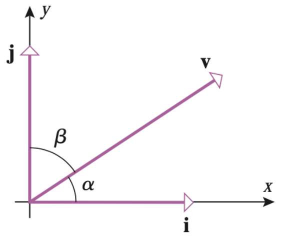
(a) $\underline{v} = -3\hat{\imath} + 6\hat{\jmath} + 2\hat{k}$ (b) $\underline{w} = 2\hat{\imath} + 7\hat{\jmath} + 6\hat{k}$ (c) $\underline{z} = -3\hat{\imath} + 6\hat{\jmath} - 6\hat{k}$

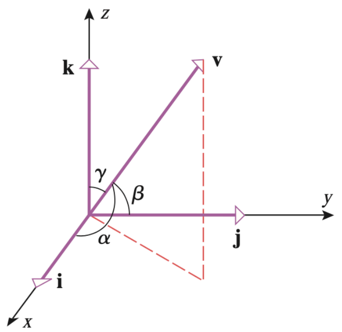
Solution 2(a)

$$\cos\theta = \frac{\underline{u}\,.\,\underline{v}}{\|\underline{\boldsymbol{u}}\|\,\|\underline{\boldsymbol{v}}\|} = \frac{1(-3) + (-2)(6) + 2(2)}{\sqrt{1^2 + (-2)^2 + 2^2}\sqrt{(-3)^2 + 6^2 + 2^2}} = -\frac{11}{21}$$

$$\therefore\ \theta = \cos^{-1}\left(-\frac{11}{21}\right) = 121.6^\circ$$

Direction Angles





In an xy −coordinate system, the direction of a nonzero vector $\underline{v}$ is completely determined by the angles α and β between $\underline{v}$ and the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

In an xyz −coordinate system the direction of a nonzero vector $\underline{v}$ is completely determined by the angles α, β, and γ between $\underline{v}$ and the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

Direction Angles

In both 2- and 3- spaces the angles between a nonzero vector $\underline{v}$ and the vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are called the direction angles of $\underline{v}$, and the cosines of those angles are called the direction cosines of $\underline{v}$. Then the direction cosines of a nonzero vector $\underline{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ can be obtained by

$$\cos\alpha = \frac{\underline{v}\,.\,\hat{\imath}}{\|\underline{v}\|\;\|\hat{\imath}\|} = \frac{v_1}{\|\underline{v}\|}$$

$$\cos\beta = \frac{\underline{v}\,.\,\hat{\jmath}}{\|\underline{v}\|\;\|\hat{\jmath}\|} = \frac{v_2}{\|\underline{v}\|}$$

$$\cos\gamma = \frac{\underline{v}\,.\,\hat{k}}{\|\underline{v}\|\;\|\hat{k}\|} = \frac{v_3}{\|\underline{v}\|}$$

Direction Cosine

Theorem

The direction cosine of a nonzero vector $\underline{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ are

$$\cos\alpha = \frac{v_1}{\|\underline{v}\|}, \qquad \cos\beta = \frac{v_2}{\|\underline{v}\|}, \qquad \cos\gamma = \frac{v_3}{\|\underline{v}\|}$$

Direction Cosine

Note 01

The direction cosine of a vector $\underline{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ can be computed by normalizing $\underline{v}$ and reading of the components of $\underline{v}/\|\underline{v}\|$, since

$$\frac{\underline{v}}{\|\underline{v}\|} = \frac{v_1}{\|\underline{v}\|}\hat{\imath} + \frac{v_2}{\|\underline{v}\|}\hat{\jmath} + \frac{v_3}{\|\underline{v}\|}\hat{k} = (\cos\alpha)\hat{\imath} + (\cos\beta)\hat{\jmath} + (\cos\gamma)\hat{k}$$

Note 02

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Angle between Vectors

Example 3 (p788)

Find the direction cosines of the vector $\underline{v} = 2\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$, and approximate the direction angles to the nearest degree.

Solution

The length of $\underline{v}$, $\|\underline{\boldsymbol{v}}\| = \sqrt{2^2 + (-4)^2 + 4^2} = 6$

Normalizing the vector $\underline{v}$ by its length

$$\frac{\underline{v}}{\|\underline{\boldsymbol{v}}\|} = \frac{1}{3}\hat{\imath} - \frac{2}{3}\hat{\jmath} + \frac{2}{3}\hat{k} = (\cos\alpha)\hat{\imath} + (\cos\beta)\hat{\jmath} + (\cos\gamma)\hat{k}$$

Angle between Vectors

Solution

$$\therefore \cos\alpha = \frac{1}{3}, \qquad \cos\beta = -\frac{2}{3}, \qquad \cos\gamma = \frac{2}{3}$$

That is

$$\alpha = \cos^{-1}\left(\frac{1}{3}\right) = 71°$$

$$\beta = \cos^{-1}\left(-\frac{2}{3}\right) = 132°$$

$$\gamma = \cos^{-1}\left(\frac{2}{3}\right) = 48°$$

Angle between Vectors

Example 4 (p788)

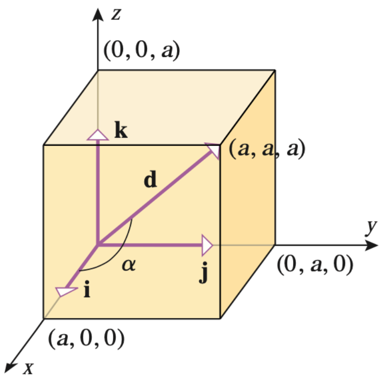
Find the angle between a diagonal of a cube and one of its edges.

Solution

Assume that the cube has side a, and introduce a coordinate system as shown in Figure. In this coordinate system the vector

$$\underline{d} = a\hat{\imath} + a\hat{\jmath} + a\hat{k}$$

is a diagonal of the cube and the unit vector $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ run along the edges.



Angle between Vectors

Solution

By symmetry the diagonal makes the same angle with each edge, so it is sufficient to find the angle between $\underline{d}$ and $\hat{\imath}$ (the direction angle α).

Thus

$$\cos\alpha = \frac{\underline{d}\,.\,\hat{\imath}}{\|\underline{d}\|\;\|\hat{\imath}\|} = \frac{a}{\|\underline{d}\|} = \frac{a}{\sqrt{3a^2}} = \frac{1}{\sqrt{3}}$$

and hence

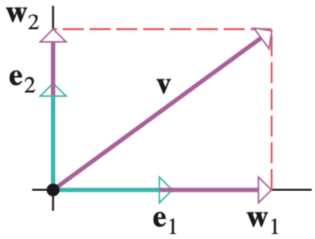
$$\therefore\ \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.955 \text{ radian} \approx 54.7°$$

Decomposing Vectors into Orthogonal Components

Suppose that $\hat{e}_1$ and $\hat{e}_2$ are two orthogonal unit vectors in 2-space. Then we can decompose the vector $\underline{v}$ into the sum of two orthogonal vector as

$$\underline{v} = (\underline{v}\ .\ \hat{e}_1)\hat{e}_1 + (\underline{v}\ .\ \hat{e}_2)\hat{e}_2$$

In this formula we call $(\underline{v}\ .\ \hat{e}_1)\hat{e}_1$ and $(\underline{v}\ .\ \hat{e}_2)\hat{e}_2$ the vector components of $\underline{v}$ along $\hat{e}_1$ and $\hat{e}_2$, respectively; and we call $(\underline{v}\ .\ \hat{e}_1)$ and $(\underline{v}\ .\ \hat{e}_2)$ the scalar components of $\underline{v}$ along $\hat{e}_1$ and $\hat{e}_2$, respectively.

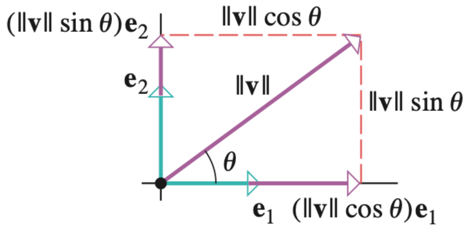


Decomposing Vectors into Orthogonal Components

If θ denotes the angle between $\underline{v}$ and $\hat{e}_1$ then $\underline{v}$ can be decomposed as

$$\underline{v} = (\|\underline{v}\| \cos\theta)\hat{e}_1 + (\|\underline{v}\| \sin\theta)\hat{e}_2$$

provided the angle between $\underline{v}$ and $\hat{e}_2$ is at most $\pi/2$.



Decomposing Vectors into Orthogonal Components

Homework

Example 5 – 6 (Textbook page 789-790)

Orthogonal Projections

The vector components of $\underline{v}$ along the orthogonal vectors $\hat{e}_1$ and $\hat{e}_2$ are called the orthogonal projections of $\underline{v}$ on $\hat{e}_1$ and $\hat{e}_2$ and are commonly denoted by

$$\text{proj}_{\hat{e}_1}\, \underline{v} = (\underline{v}\ .\ \hat{e}_1)\hat{e}_1 \quad \text{and} \quad \text{proj}_{\hat{e}_2}\, \underline{v} = (\underline{v}\ .\ \hat{e}_2)\hat{e}_2$$

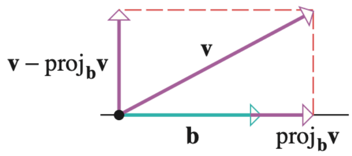
In general, if $\hat{e}$ is a unit vector, then the orthogonal projection of $\underline{v}$ on $\hat{e}$ to be

$$\text{proj}_{\hat{e}}\, \underline{v} = (\underline{v}\ .\ \hat{e})\hat{e}$$

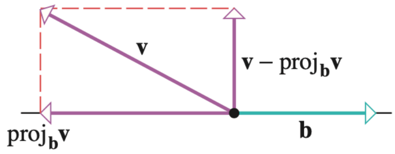
Orthogonal Projections

The orthogonal projection of $\underline{v}$ on an arbitrary nonzero vector $\underline{b}$ can be obtained by normalizing $\underline{b}$ as

$$\text{proj}_{\underline{b}}\, \underline{v} = \left(\underline{v}\ .\frac{\underline{b}}{\|\underline{b}\|}\right)\frac{\underline{b}}{\|\underline{b}\|} = \frac{\underline{v}\ .\ \underline{b}}{\|\underline{b}\|^2}\,\underline{b}$$

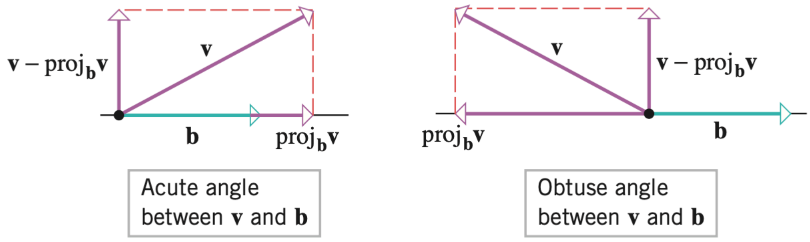


Acute angle between **v** and **b**



Obtuse angle between **v** and **b**

Orthogonal Projections



If we subtract $\text{proj}_{\underline{b}}\ \underline{v}$ from $\underline{v}$, then the resulting vector

$$\underline{v} - \text{proj}_{\underline{b}}\ \underline{v}$$

with be orthogonal to $\underline{b}$; we call this the vector component of $\underline{v}$ orthogonal to $\underline{b}$.

Orthogonal Projections

Example 7 (p788)

Find the orthogonal projection of $\underline{v} = \hat{\imath} + \hat{\jmath} + \hat{k}$ on $\underline{b} = 2\hat{\imath} + 2\hat{\jmath}$, and then find the vector component of $\underline{v}$ orthogonal to $\underline{b}$.

Solution

We have,

$$\underline{v}\ .\ \underline{b} = \left(\hat{\imath} + \hat{\jmath} + \hat{k}\right).\left(2\hat{\imath} + 2\hat{\jmath}\right) = 2 + 2 + 0 = 4$$

$$\left\|\underline{b}\right\|^2 = 2^2 + 2^2 = 8$$

Orthogonal Projections

Solution

Thus the orthogonal projection of $\underline{v}$ on $\underline{b}$ *is*

$$\text{proj}_{\underline{b}}\, \underline{v} = \frac{\underline{v}\ .\ \underline{b}}{\left\|\underline{b}\right\|^2}\,\underline{b} = \frac{4}{8}(2\hat{\imath} + 2\hat{\jmath}) = \hat{\imath} + \hat{\jmath}$$

and the vector components of $\underline{v}$ **orthogonal to** $\underline{b}$ is

$$\underline{v} - \text{proj}_{\underline{b}}\, \underline{v} = \left(\hat{\imath} + \hat{\jmath} + \hat{k}\right) - (\hat{\imath} + \hat{\jmath}) = \hat{k}$$

Work

Homework

Example 8 (Textbook page 792)

Chapter 11.3

Homework

Exercise Set 11.3 (p792 –794)

Chapter 11.4

Cross Products

Cross Product

Definition

If $\underline{\boldsymbol{u}} = (u_1, u_2, u_3)$ and $\underline{\boldsymbol{v}} = (v_1, v_2, v_3)$ are vectors in 3-space and θ is the angle between them, then the cross product $\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}$ is the vector defined by

$$\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}} = \|\underline{\boldsymbol{u}}\| \, \|\underline{\boldsymbol{v}}\| \sin\theta \, \hat{\boldsymbol{\eta}} \qquad ; 0 \leq \theta \leq \pi$$

;$\hat{\boldsymbol{\eta}}$ is a unit vector indicating the direction of $\underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}}$.

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}$$

$$= (u_2 v_3 - u_3 v_2)\hat{\imath} - (u_1 v_2 - u_3 v_1)\hat{\jmath} + (u_1 v_2 - u_2 v_1)\hat{k}$$

Cross Product

Example 1 (p796)

Let, $\underline{\boldsymbol{u}} = (1, 2, -2)$ and $\underline{\boldsymbol{v}} = (3, 0, 1)$. Find

(a) $\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}$ (b) $\underline{\boldsymbol{v}} \times \underline{\boldsymbol{u}}$

Solution (a)

$$\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

Cross Product

Solution (a)

$$\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \hat{k}$$

$$= 2\hat{\imath} - 7\hat{\jmath} - 6\hat{k}$$

Solution (b)

$$\underline{\boldsymbol{v}} \times \underline{\boldsymbol{u}} = -(\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}) = -2\hat{\imath} + 7\hat{\jmath} + 6\hat{k}$$

Cross Product

Example 2 (p797)

Show that $\underline{\boldsymbol{u}} \times \underline{\boldsymbol{u}} = \underline{\boldsymbol{0}}$ for any vector $\underline{\boldsymbol{u}}$ in 3-space.

Solution

Let, $\underline{\boldsymbol{u}} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$, then

$$\underline{\boldsymbol{u}} \times \underline{\boldsymbol{u}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$= 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k}$$

$$= \underline{\boldsymbol{0}}$$

Algebraic Properties of the Cross Product

11.4.3 THEOREM *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in 3-space and* k *is any scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

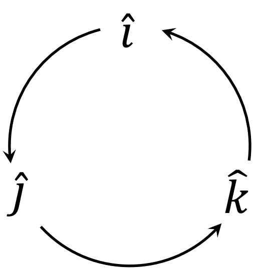
(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

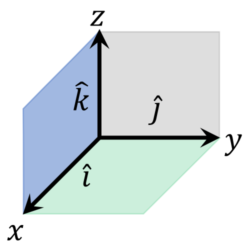
(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Algebraic Properties of the Cross Product

$$\hat{\imath} \times \hat{\jmath} = \hat{k}, \qquad \hat{\jmath} \times \hat{k} = \hat{\imath}, \qquad \hat{k} \times \hat{\imath} = \hat{\jmath}$$

$$\hat{\jmath} \times \hat{\imath} = -\hat{k}, \qquad \hat{k} \times \hat{\jmath} = -\hat{\imath}, \qquad \hat{\imath} \times \hat{k} = -\hat{\jmath}$$





Geometric Properties of the Cross Product

Theorem 11.4.4

If $\underline{u}$ and $\underline{v}$ are vectors in 3-space, then

(a) $\underline{\boldsymbol{u}} \,.\, (\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}) = 0$ ($\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}$ is orthogonal to $\underline{\boldsymbol{u}}$)

(b) $\underline{\boldsymbol{v}} \,.\, (\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}) = 0$ ($\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}$ is orthogonal to $\underline{\boldsymbol{v}}$)

Proof

Homework (Textbook p 798)

Geometric Properties of the Cross Product

Example 3 (p798)

Find a vector that is orthogonal to both of the vector $\underline{\boldsymbol{u}} = (2, -1, 3)$ and $\underline{\boldsymbol{v}} = (-7, 2, -1)$.

Solution

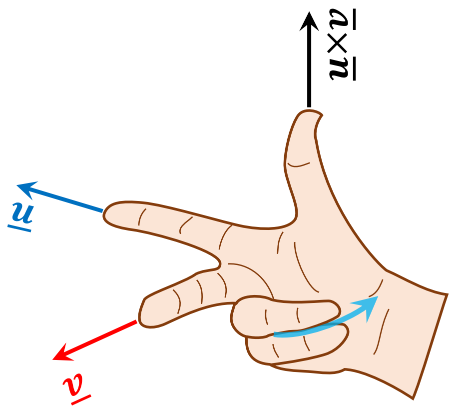
We know that the vector $\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}$ is orthogonal to both $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$. Therefore, a vector that is orthogonal to both $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ is

$$\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 3 \\ -7 & 2 & -1 \end{vmatrix} = -5\hat{\imath} - 19\hat{\jmath} - 3\hat{k}.$$

Geometric Properties of the Cross Product

Right-Handed Rule for the Direction of Cross Product

If $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ are nonzero and nonparallel vectors, then the direction of $\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}$ relative to $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ is determined by a right-hand rule; that is, if the fingers of the right hand are cupped so they curl from $\underline{\boldsymbol{u}}$ toward $\underline{\boldsymbol{v}}$ in the direction of rotation that takes $\underline{\boldsymbol{u}}$ into $\underline{\boldsymbol{v}}$ in less than $180°$, then the thumb will point (roughly) in the direction of $\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}$.



Geometric Properties of the Cross Product

Theorem 11.4.5

Let $\underline{u}$ and $\underline{v}$ be nonzero vectors in 3-space, and let θ be the angle between these vectors when they are positioned sol their initial points coincide.

(a) $\|\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}\| = \|\underline{\boldsymbol{u}}\| \, \|\underline{\boldsymbol{v}}\| \sin\theta$

(b) The area A of the parallelogram that has $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ as adjacent side is

$$A = \|\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}\|$$

(c) $\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$ if and only if $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ are parallel vectors, that is, if and only if they are scalar multiples of one another.

Proof Homework (Textbook p 799)

Geometric Properties of the Cross Product

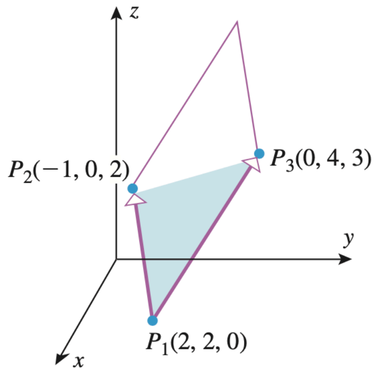
Example 4 (p799)

Find the area of the triangle that is determined by the points $P_1(2, 2, 0)$, $P_2(-1, 0, 2)$, and $P_3(0, 4, 3)$.

Solution

$$\overrightarrow{P_1P_2} = (-1-2, 0-2, 2-0) = (-3, -2, 2)$$

$$\overrightarrow{P_1P_3} = (0-2, 4-2, 3-0) = (-2, 2, 3)$$



$$\therefore \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3 & -2 & 2 \\ -2 & 2 & 3 \end{vmatrix}$$

$$= -10\hat{\imath} + 5\hat{\jmath} - 10\hat{k}$$

Geometric Properties of the Cross Product

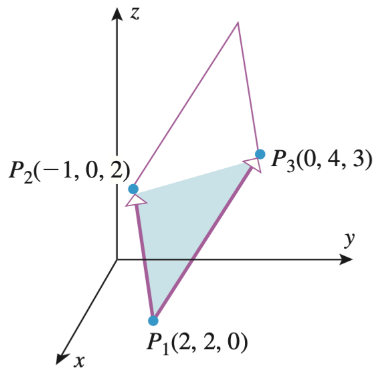
Solution

The area A of the triangle is equal to half of the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. Therefore

$$A = \frac{1}{2}\left\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right\|$$

$$= \frac{1}{2}\sqrt{(-10)^2 + 5^2 + (-10)^2}$$

$$= \frac{15}{2}$$



Scalar Triple Product

Definition

If $\underline{\boldsymbol{u}} = (u_1, u_2, u_3)$, $\underline{\boldsymbol{v}} = (v_1, v_2, v_3)$, and $\underline{\boldsymbol{w}} = (w_1, w_2, w_3)$ are vectors in 3-space then the number

$$\underline{\boldsymbol{u}} \,.\, (\underline{\boldsymbol{v}} \times \underline{\boldsymbol{w}})$$

is called the scaler triple product of $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$, and $\underline{\boldsymbol{w}}$. The formula to evaluate scalar triple product is given by

$$\underline{\boldsymbol{u}} \,.\, (\underline{\boldsymbol{v}} \times \underline{\boldsymbol{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Scalar Triple Product

Example 5 (p800)

Calculate the scalar triple product $\underline{\boldsymbol{u}} \,.\, (\underline{\boldsymbol{v}} \times \underline{\boldsymbol{w}})$ of the vectors

$$\underline{\boldsymbol{u}} = 3\hat{\imath} - 2\hat{\jmath} - 5\hat{k}, \qquad \underline{\boldsymbol{v}} = \hat{\imath} + 4\hat{\jmath} - 4\hat{k}, \qquad \underline{\boldsymbol{w}} = 3\hat{\jmath} + 2\hat{k}$$

Solution

$$\underline{\boldsymbol{u}} \,.\, (\underline{\boldsymbol{v}} \times \underline{\boldsymbol{w}}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 49$$

Geometric Properties of the S. T. P.

Theorem 11.4.6

Let $\underline{u}$, $\underline{v}$ and $\underline{w}$ be nonzero vectors in 3-space.

(a) The volume V of the parallelepiped that has $\underline{u}$, $\underline{v}$ and $\underline{w}$ as adjacent edges is

$$V = |\underline{\boldsymbol{u}} \, . \, (\underline{\boldsymbol{v}} \times \underline{\boldsymbol{w}})|$$

(b) $\underline{\boldsymbol{u}} \, . \, (\underline{\boldsymbol{v}} \times \underline{\boldsymbol{w}}) = \mathbf{0}$ if and only if $\underline{u}$, $\underline{v}$ and $\underline{w}$ lie in the same plane.

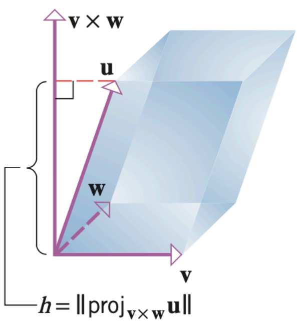
Proof

Homework (Textbook p 801)

Geometric Properties of the S. T. P.

Proof (Theorem 11.4.6)

Homework (Textbook p 801)



Algebraic Properties of the S. T. P.

$$\underline{\boldsymbol{u}}\,.\,(\underline{\boldsymbol{v}}\times\underline{\boldsymbol{w}}) = \underline{\boldsymbol{v}}\,.\,(\underline{\boldsymbol{w}}\times\underline{\boldsymbol{u}}) = \underline{\boldsymbol{w}}\,.\,(\underline{\boldsymbol{u}}\times\underline{\boldsymbol{v}})$$

$$\underline{\boldsymbol{u}}\,.\,(\underline{\boldsymbol{v}}\times\underline{\boldsymbol{w}}) = (\underline{\boldsymbol{u}}\times\underline{\boldsymbol{v}})\,.\,\underline{\boldsymbol{w}}$$

Chapter 11.4

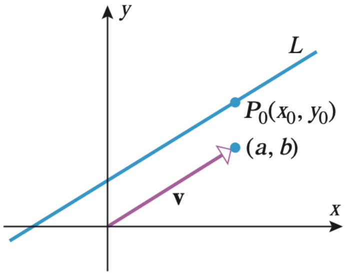
Homework

Exercise Set 11.4 (p803 –805)

Chapter 11.5

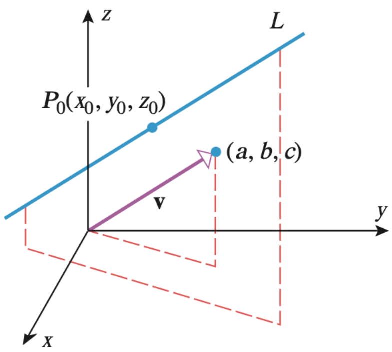
Parametric Equation of Lines

Lines Determined by a Point & a Vector

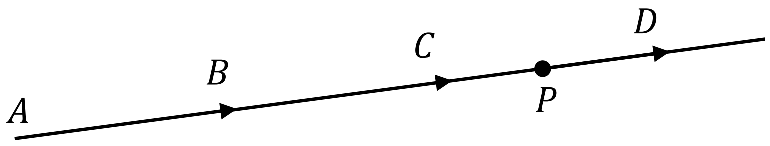


A line in 2-space or 3-space can be determined uniquely by specifying a point on the line and a nonzero vector parallel to the line.

For example, consider a line L in 3-space that passes through the point $P_0(x_0,\ y_0,\ z_0)$ and is parallel to the nonzero vector $\underline{\boldsymbol{v}} = (a, b, c)$.



Lines Determined by a Point & a Vector



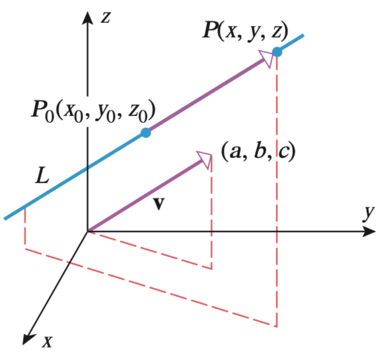
$$\overline{AB} = \underline{\boldsymbol{v}};\ \overline{AC} = 2\underline{\boldsymbol{v}};\ \overline{AD} = 3\underline{\boldsymbol{v}}$$

$$\overline{AP} = t\underline{\boldsymbol{v}};\ t \text{ is a real number}$$

Lines Determined by a Point & a Vector

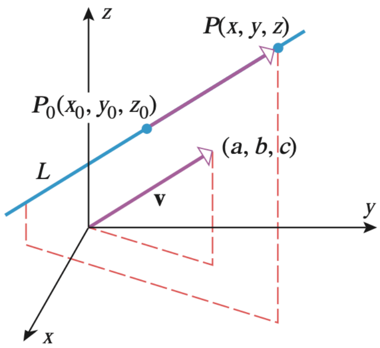
Consider a line L in 3-space that passes through the point $P_0(x_0,\ y_0,\ z_0)$ and is parallel to the nonzero vector $\underline{\boldsymbol{v}} = (a, b, c)$ .Then L consists precisely of those points $P(x, y, z)$ for which the vector $\overrightarrow{P_0P}$ is parallel to $\underline{\boldsymbol{v}}$ (see Figure). In other words, the point $P(x, y, z)$ is on L if and only $\overrightarrow{P_0P}$ is a scalar multiple of $\underline{\boldsymbol{v}}$, say

$$\overrightarrow{P_0P} = t\underline{\boldsymbol{v}}$$



Lines Determined by a Point & a Vector

$$\overrightarrow{P_0P} = t\underline{\boldsymbol{v}}$$



This equation can be written as

$$(x - x_0, y - y_0, z - z_0) = (ta, tb, tc)$$

which implies that

$$x - x_0 = ta, \qquad y - y_0 = tb, \qquad z - z_0 = tc$$

Thus, L can be describe by the parametric equations

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct$$

Lines Determined by a Point & a Vector

Theorem 11.5.1

(a) The line in 2-space that passes through the point $P_0(x_0,\ y_0)$ and is parallel to the nonzero vector $\underline{\boldsymbol{v}} = (a, b) = a\hat{\imath} + b\hat{\jmath}$ has parametric equations

$$x = x_0 + at, \qquad y = y_0 + bt$$

(b) The line in 3-space that passes through the point $P_0(x_0,\ y_0,\ z_0\ )$ and is parallel to the nonzero vector $\underline{\boldsymbol{v}} = (a, b, c) = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ has parametric equations

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct$$

Lines Determined by a Point & a Vector

Example 1 (p806)

Find parametric equation of the line

(a) passing through $(4, 2)$ and parallel to $\underline{\boldsymbol{v}} = (-1, 5)$;

(b) passing through $(1, 2, -3)$ and parallel to $\underline{\boldsymbol{v}} = 4\hat{\imath} + 5\hat{\jmath} - 7\hat{k}$;

(c) passing through the origin in 3-space and parallel to $\underline{\boldsymbol{v}} = (1, 1, 1)$.

Lines Determined by a Point & a Vector

Solution (a)

Given $(x_0, y_0) = (4, 2)$ and $\underline{\boldsymbol{v}} = (a, b) = (-1, 5)$, and therefore the parametric equation line $x = x_0 + at,\ y = y_0 + bt$ becomes

$$x = 4 - t, \qquad y = 2 + 5t$$

Solution (b)

Given $(x_0, y_0, z_0) = (1, 2, -3)$ and $\underline{\boldsymbol{v}} = (a, b, c) = (4, 5, -7)$, so the parametric equation line $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$ yields

$$x = 1 + 4t, \qquad y = 2 + 5t, \qquad z = -3 - 7t$$

Lines Determined by a Point & a Vector

Solution (c)

Given $(x_0, y_0, z_0) = (0, 0, 0)$ and $\underline{\boldsymbol{v}} = (a, b, c) = (1, 1, 1)$, so the parametric equation line $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$ yields

$$x = t, \qquad y = t, \qquad z = t$$

Lines Determined by a Point & a Vector

Example 2 (p807)

(a) Find parametric equation of the line L passing through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

(b) Where does the line intersect the $xy -$ plane?

Lines Determined by a Point & a Vector

Solution (a)

The vector $\overrightarrow{P_1P_2} = (3, -4, 8)$ is parallel to L and the point $P_1(2, 4, -1)$ lies on L, so the parametric equation of L is

$$x = 2 + 3t, \qquad y = 4 - 4t, \qquad z = -1 + 8t.$$

Also, if we use $P_2(5, 0, 7)$ as the point on L rather than $P_1(2, 4, -1)$, the parametric equation of L is

$$x = 5 + 3t, \qquad y = -4t, \qquad z = 7 + 8t.$$

Both equation are equivalent if we replace t by $t - 1$ in 2nd equation and $t + 1$ in first equation.

Lines Determined by a Point & a Vector

Solution (b)

The line intersects the $xy -$ plane at the point where $z = 0$, that is

$$-1 + 8t = 0 \Longrightarrow t = \frac{1}{8}$$

Substituting the value of $t = \frac{1}{8}$

$$(x, y, z) = \left(\frac{19}{8}, \frac{7}{2}, 0\right).$$

Lines Determined by a Point & a Vector

Example 3 (p807)

Let L_1 and L_2 be the lines

$$L_1: \quad x = 1 + 4t, \qquad y = 5 - 4t, \qquad z = -1 + 5t$$

$$L_2: \quad x = 2 + 8t, \qquad y = 4 - 3t, \qquad z = 5 + t$$

(a) Are the lines parallel?

(b) Do the lines intersects?

Lines Determined by a Point & a Vector

Solution (a)

The line L_1 is parallel to the vector $4\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$, and the line L_2 is parallel to the vector $8\hat{\imath} - 3\hat{\jmath} + \hat{k}$. These vectors are not parallel since neither is a scalar multiple of the other. Thus, the line are not parallel.

Lines Determined by a Point & a Vector

Solution (b)

The line L_1 and L_2 to intersect at some point (x_0, y_0, z_0) these coordinate would have to satisfy the equations of both lines. In other words, there would have to exist values t_1 and t_2 for the parameters such that

$$x_0 = 1 + 4t_1, \qquad y_0 = 5 - 4t_1, \qquad z_0 = -1 + 5t_1$$

and

$$x_0 = 2 + 8t_2, \qquad y_0 = 4 - 3t_2, \qquad z_0 = 5 + t_2$$

Lines Determined by a Point & a Vector

Solution (b)

This leads to three condition on t_1 and t_2

$$1 + 4t_1 = 2 + 8t_2$$
$$5 - 4t_1 = 4 - 3t_2$$
$$-1 + 5t_1 = 5 + t_2$$

Thus, the lines intersect if there are values of t_1 and t_2 that satisfy all three equations, and the lines do not intersect if there are no such values. Solving first two equation we get

$$t_1 = \frac{1}{4} \quad \text{and} \quad t_2 = 0$$

Lines Determined by a Point & a Vector

Solution (b)

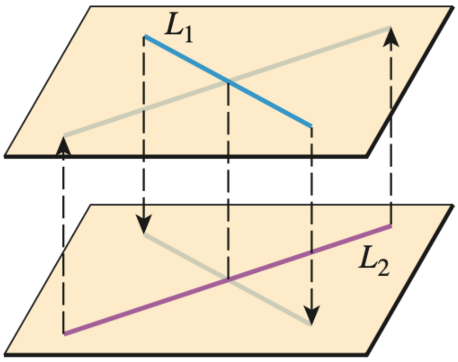
Solving first two equation we get

$$t_1 = \frac{1}{4} \quad \text{and} \quad t_2 = 0$$

However, these values do not satisfy the third equation.

Therefore, the lines do not intersect.

Lines Determined by a Point & a Vector



Parallel planes containing skew lines L_1 and L_2 can be determined by translating each line until it intersects the other.

Lines Segments

Example 4 (p808)

Find parametric equations describing the line segment joining the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

Solution

The vector $\overrightarrow{P_1P_2} = (3, -4, 8)$ is parallel to L and the point $P_1(2, 4, -1)$ lies on L, so the parametric equation of L is

$$x = 2 + 3t, \qquad y = 4 - 4t, \qquad z = -1 + 8t.$$

With these equation P_1 corresponds to $t = 0$ and P_2 to $t = 1$. Thus the line segment that joins P_1 and P_2 is given by

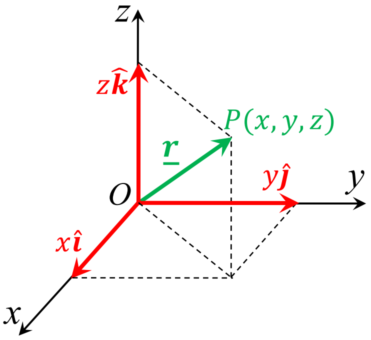
$$x = 2 + 3t, \qquad y = 4 - 4t, \qquad z = -1 + 8 \qquad (0 \leq t \leq 1)$$

Vector Equations of Lines

Position Vector of a Point

- Also known as location vector or radius vector.
- Represents the position of a point P in space in relation to an arbitrary reference origin O.
- corresponds to the straight-line segment from O to P.

i.e. $\underline{\boldsymbol{r}} = \overline{OP} = x\,\underline{\boldsymbol{i}} + y\,\underline{\boldsymbol{j}} + z\,\underline{\boldsymbol{k}} = (x, y, z)$



Vector Equations of Lines

The equation of the line that passes through the point $P_0(x_0, y_0, z_0\ )$ and is parallel to the nonzero vector $\underline{\boldsymbol{v}} = (a, b, c) = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ can be written in vector form as

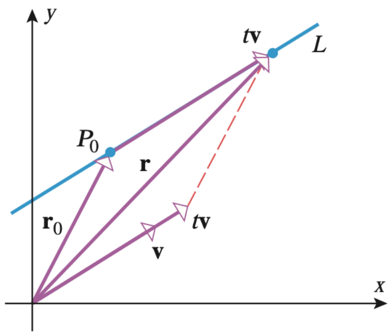
$$(x, y, z) = (x_0 + at,\ \ y_0 + bt,\ \ z_0 + ct)$$

or equivalently

$$(x, y, z) = (x_0, y_0, z_0\ ) + t(a, b, c)$$

Vector Equations of Lines

Let us define the vector,



$$\underline{\boldsymbol{r}} = x\ \underline{\boldsymbol{i}} + y\ \underline{\boldsymbol{j}} + z\ \underline{\boldsymbol{k}} = (x, y, z),$$

$$\underline{\boldsymbol{r_0}} = x_0\ \underline{\boldsymbol{i}} + y_0\ \underline{\boldsymbol{j}} + z_0\ \underline{\boldsymbol{k}} = (x_0, y_0, z_0).$$

$$\underline{\boldsymbol{v}} = a\ \underline{i} + b\ \underline{j} + c\ \underline{k} = (a, b, c).$$

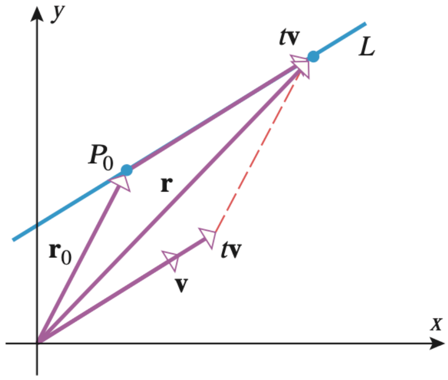
Then the equation $(x, y, z) = (x_0, y_0, z_0\ ) + t(a, b, c)$ becomes

$$\underline{\boldsymbol{r}} = \underline{\boldsymbol{r_0}} + t\ \underline{\boldsymbol{v}}$$

Vector Equations of Lines

The vector equation of straight line passing through the point $\underline{\boldsymbol{r}_0}$ and parallel to the nonzero vector $\underline{\boldsymbol{v}}$ is

$$\underline{\boldsymbol{r}} = \underline{\boldsymbol{r}_0} + t\,\underline{\boldsymbol{v}}$$



Vector Equations of Lines

Example 5

The equation

$$(x, y, z) = (-1, 0, 2\,) + t(1, 5, -4)$$

can be rewrite as

$$\underline{\boldsymbol{r}} = \underline{\boldsymbol{r}_0} + t\underline{\boldsymbol{v}}$$

with $\underline{\boldsymbol{r}_0} = (-1, 0, 2\,)$ and $\underline{\boldsymbol{v}} = (1, 5, -4)$.

Thus, the equation represents the line in 3-space that passes through the point $\underline{\boldsymbol{r}_0} = (-1, 0, 2\,)$ and is parallel to the vector $\underline{\boldsymbol{v}} = (1, 5, -4)$.

Vector Equations of Lines

Example 6

Find an equation of the line in 3-space that passes through the points $P_1(2,4,-1)$ and $P_2(5,0,7)$.

Solution

The vector $\underline{\boldsymbol{v}} = \overrightarrow{P_1P_2} = (3,-4,8)$ is parallel to the line and the line passes through the point $r_0 = (2,4,-1)$, therefore the vector equation of the line, $\underline{\boldsymbol{r}} = \underline{\boldsymbol{r_0}} + t\underline{\boldsymbol{v}}$ yields

$$(x,y,z) = (2,4,-1) + t(3,-4,8).$$

Chapter 11.5

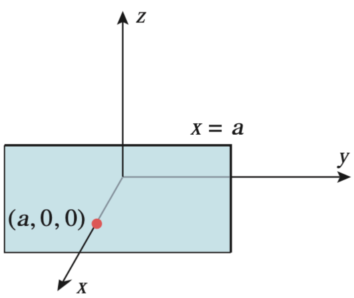
Homework

Exercise Set 11.5 (p810 – 812)

Chapter 11.6

Planes in 3-Space

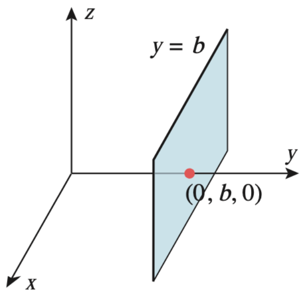
Planes Parallel to the Coordinate Planes

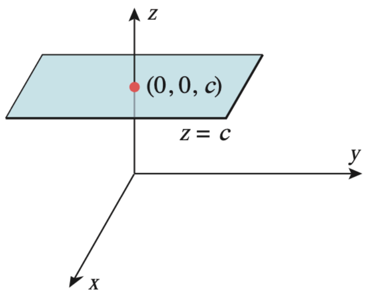


The graph of the equation $x = a$ in an xyz-coordinate system consists of all points of the form (a, y, z), where y and z are arbitrary. One such point is $(a, 0, 0)$, and all others are in the plane that passes through this point and is parallel to the yz −plane.

Planes Parallel to the Coordinate Planes

Similarly, the graph of $y = b$ is the plane through $(0, b, 0)$ that is parallel to the xz −plane, and the graph of $z = c$ is the plane through $(0, 0, c)$ that is parallel to the xy −plane.





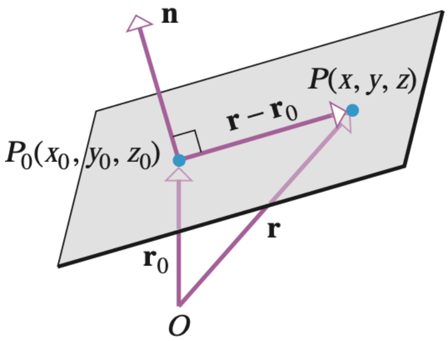
Planes Determined by a Point & a Normal Vector

Suppose that we want to find an equation of the plane passing through $P_0(x_0, y_0, z_0)$ and perpendicular to the vector $\underline{n} = (a, b, c)$. Define the vectors $\underline{r_0}$ and $\underline{r}$ as

$$\underline{r_0} = (x_0, y_0, z_0) \text{ and } \underline{r} = (x, y, z)$$

As shown in fig.

$$\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = \underline{r} - \underline{r_0}$$



Planes Determined by a Point & a Normal Vector

It should be evident from Figure that the plane consists precisely of those points $P(x, y, z)$ for which the vector $\underline{r} - \underline{r_0}$ is orthogonal to $\underline{n}$; or, expressed as an equation,

$$\underline{n} \,.\, (\underline{r} - \underline{r}_0) = 0$$

We can express this vector equation in terms of components as

$$(a, b, c) \,.\, (x - x_0, y - y_0, z - z_0) = 0$$

which yields

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is called the point-normal form of the equation of a plane.

Planes Determined by a Point & a Normal Vector

Example 1 (p814)

Find an equation of the plane passing through the points $(3, -1, 7)$ and perpendicular to the vector $\underline{n} = (4, 2, -5)$.

Solution

A point-normal form of the equation is

$$\underline{n} \,.\, (\underline{r} - \underline{r}_0) = 0$$

$$\Longrightarrow (4, 2, -5) \,.\, (x - 3, y + 1, z - 7) = 0$$

$$\Longrightarrow 4(x - 3) + 2(y + 1) - 5(z - 7) = 0$$

$$\Longrightarrow 4x + 2y - 5z + 25 = 0$$

Planes Determined by a Point & a Normal Vector

Theorem 11.6.1

If a, b, c, and d are constants, and a, b, and c are not all zero, then the graph of the equation

$$ax + by + cz + d = 0$$

is a plane that has the vector $\underline{n} = (a, b, c)$ as a normal.

Proof

Homework (Textbook p814)

Planes Determined by a Point & a Normal Vector

Example 2 (p814)

Determine whether the planes

$$3x - 4y + 5z = 0 \text{ and } -6x + 8y - 10z - 4 = 0$$

are parallel.

Planes Determined by a Point & a Normal Vector

Solution

It is clear geometrically that two planes are parallel if and only if their normals are parallel vectors. A normal to the first plane is

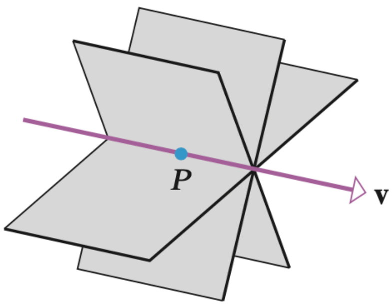
$$\underline{n}_1 = (3, -4, 5)$$

and a normal to the second plane is

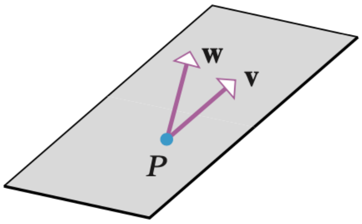
$$\underline{n}_2 = (-6, 8, -1)$$

Since $\underline{n}_2$ is a scalar multiple of $\underline{n}_1$, the normals are parallel, and hence so are the planes.

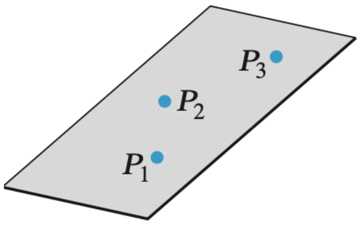
Planes Determined by a Point & a Normal Vector



There are infinitely many planes containing P and parallel to $\mathbf{v}$.



There is a unique plane through P that is parallel to both $\mathbf{v}$ and $\mathbf{w}$.



There is a unique plane through three noncollinear points.

Planes Determined by a Point & a Normal Vector

Example 3 (p815)

Find an equation of the plane through the points $P_1(1, 2, -1)$, $P_2(2, 3, 1)$, and $P_3(3, -1, 2)$.

Solution

Since the points P_1 , P_2 , and P_3 lie in the plane, the vectors

$$\overrightarrow{P_1P_2} = (1, 1, 2) \text{ and } \overrightarrow{P_1P_3} = (2, -3, 3)$$

are parallel to the plane.

Planes Determined by a Point & a Normal Vector

Solution

$$\therefore \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 2 \\ 2 & -3 & 3 \end{vmatrix} = 9\hat{\imath} + \hat{\jmath} - 5\hat{k}$$

is normal to the plane, since it is orthogonal to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$.

By using this normal and the point $P_1(1, 2, -1)$ in the plane, we obtain the point-normal form

$$9(x-1) + (y-2) - 5(z+1) = 0$$

$$\Longrightarrow 9x + y - 5z - 16 = 0$$

Planes Determined by a Point & a Normal Vector

Example 4 (p815)

Determine whether the line

$$x = 3 + 8t, \qquad y = 4 + 5t, \qquad z = -3 - t$$

is parallel to the plane $x - 3y + 5z = 12$.

Planes Determined by a Point & a Normal Vector

Solution

A parallel vector to the given line is $\underline{\boldsymbol{v}} = (8, 5, -1)$.

And a normal vector to the given plane is $\underline{\boldsymbol{n}} = (1, -3, 5)$.

For the line and plane to be parallel, the vectors $\underline{\boldsymbol{v}}$ and $\underline{\boldsymbol{n}}$ must be orthogonal. But

$$\underline{v} \,.\, \underline{n} = (8)(1) + (5)(-3) + (-1)(5) = -12 \neq 0$$

Thus, the line and plane are not parallel.

Planes Determined by a Point & a Normal Vector

Example 5 (p815)

Find the intersection of the line

$$x = 3 + 8t, \qquad y = 4 + 5t, \qquad z = -3 - t$$

and plane $x - 3y + 5z = 12$.

Solution

If we let (x_0, y_0, z_0) be the point of intersection, then the coordinates of this point satisfy both the equation of the plane and the parametric equations of the line.

Planes Determined by a Point & a Normal Vector

Solution

Thus,

$$x_0 - 3y_0 + 5z_0 = 12 \qquad (1)$$

and for some values of t, say $t = t_0$,

$$x_0 = 3 + 8t_0, \qquad y_0 = 4 + 5t_0, \qquad z_0 = -3 - t_0 \qquad (2)$$

Substituting (2) in (1)

$$(3 + 8t_0) - 3(4 + 5t_0) + 5(-3 - t_0) = 12$$

$$\Longrightarrow t_0 = -3$$

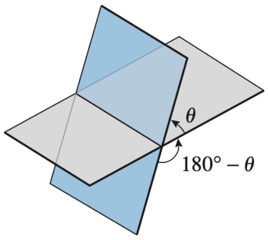
Using the value of t_0 in (2) we get

$$(x_0, y_0, z_0) = (-21, -11, 0)$$

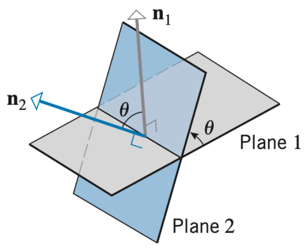
Intersecting Planes

Two distinct intersecting planes determine two positive angles of intersection—an (acute) angle θ that satisfies the condition $0 \leq \theta \leq \pi/2$ and the supplement of that angle (Fig. a).

If $\underline{n_1}$ and $\underline{n_2}$ are normals to the planes, then depending on the directions of $\underline{n_1}$ and $\underline{n_2}$, the angle θ is either the angle between $\underline{n_1}$ and $\underline{n_2}$ or the angle between $\underline{n_1}$ and $-\underline{n_2}$ (Fig. b).



(a)



(b)

Intersecting Planes

In both cases, the formula for the acute angle θ between the planes:

$$\cos\theta = \frac{|\underline{n_1} \cdot \underline{n_2}|}{\|\underline{n_1}\| \, \|\underline{n_2}\|}$$

Intersecting Planes

Example 6 (p816)

Find the acute angle of intersection between the two planes

$$2x - 4y + 4z = 6 \text{ and } 6x + 2y - 3z = 4.$$

Intersecting Planes

Solution

The normals of the given eq^{n}s are $\underline{n_1} = (2, -4, 4)$ and $\underline{n_2} = (6, 2, -3)$.

Thus, the angle between the planes

$$\cos\theta = \frac{|\underline{n_1} \cdot \underline{n_2}|}{\|\underline{n_1}\| \, \|\underline{n_2}\|} = \frac{|-8|}{\sqrt{36}\sqrt{49}} = \frac{4}{21}$$

$$\Longrightarrow \theta = \cos^{-1}\left(\frac{4}{21}\right) \approx 79°$$

Intersecting Planes

Example 7 (p816)

Find an equation for the line L of intersection of the planes

$$2x - 4y + 4z = 6 \text{ and } 6x + 2y - 3z = 4.$$

Solution

First compute $\underline{v} = \underline{n_1} \times \underline{n_2} = (2, -4, 4) \times (6, 2, -3) = (4, 30, 28)$.

Since $\underline{v}$ is orthogonal to $\underline{n_1}$, it is parallel to the first plane, and since $\underline{v}$ is orthogonal to $\underline{n_2}$, it is parallel to the second plane.

That is, $\underline{v}$ is parallel to L, the intersection of the two planes.

Intersecting Planes

Solution

To find a point on L we observe that L must intersect the xy −plane, $z = 0$, since $\underline{v} \,.\, (0, 0, 1) = 28 \neq 0$. Substituting $z = 0$ in the equations of both planes yields

$$2x - 4y = 6$$
$$6x + 2y = 4$$

with solution $x = 1, y = -1$. Thus, $P(1, -1, 0)$ is a point on L. A vector equation for L is

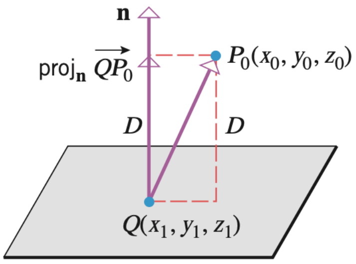
$$(x, y, z) = (1, -1, 0) + t(4, 30, 28)$$

Distance Between a Point & a Plane

Theorem 11.6.2

The distance D between a point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$



Proof

Homework (Textbook p817)

Distance Between a Point & a Plane

Example 8 (p816)

Find the distance D between the point $(1, -4, -3)$ and the plane

$$2x - 3y + 6z = -1.$$

Solution

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|(2)(1) + (-3)(-4) + 6(-3) + 1\,|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{3}{7}$$

Distance Between a Point & a Plane

Example 8 (p816)

Find the distance D between the point $(1, -4, -3)$ and the plane

$$2x - 3y + 6z = -1.$$

Solution

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|(2)(1) + (-3)(-4) + 6(-3) + 1\,|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{3}{7}$$

Distance Between a Point & a Plane

Example 9 (p816)

The planes

$$x + 2y - 2z = 3 \text{ and } 2x + 4y - 4z = 7$$

are parallel since their normals, $(1, 2, -2)$ and $(2, 4, -4)$, are parallel vectors. Find the distance between these planes.

Distance Between a Point & a Plane

Solution

To find the distance D between the planes, we can select an arbitrary point in one of the planes and compute its distance to the other plane.

By setting $y = z = 0$ in the equation $x + 2y - 2z = 3$, we obtain the point $P_0(3, 0, 0)$ in this plane.

Then the distance from $P_0(3, 0, 0)$ to the plane $2x + 4y - 4z = 7$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|(2)(3) + (4)(0) + (-4)(0) + (-7)\,|}{\sqrt{2^2 + 4^2 + (-4)^2}} = \frac{1}{6}$$

Distance Between a Point & a Plane

Example 10 (p816)

Find the distance between the lines

$$L_1:\quad x = 1 + 4t, \qquad y = 5 - 4t, \qquad z = -1 + 5t$$
$$L_2:\quad x = 2 + 8t, \qquad y = 4 - 3t, \qquad z = 5 + t$$

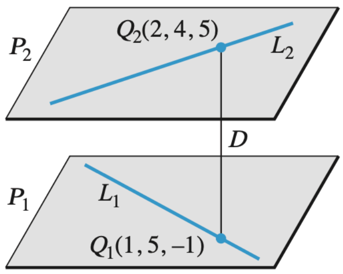
Distance Between a Point & a Plane

Solution

The line L_1 is parallel to the vector $4\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$, and the line L_2 is parallel to the vector $8\hat{\imath} - 3\hat{\jmath} + \hat{k}$. These vectors are not parallel since neither is a scalar multiple of the other. Thus, the line are not parallel.

Also, the lines do not intersect.

Therefore, the lines are skew.

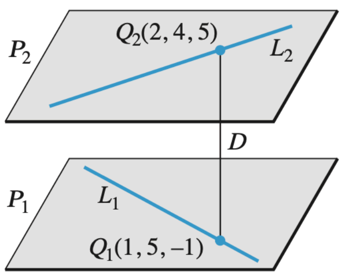


Distance Between a Point & a Plane

Solution

Let P_1 and P_2 denote parallel planes containing L_1 and L_2, respectively (as shown in Figure).

To find the distance D between L_1 and L_2 , we will calculate the distance from a point in P_1 to the plane P_2 . Since L_1 lies in plane P_1 , we can find a point in P_1 by finding a point on the line L_1 ; we can do this by substituting any convenient value of t in the parametric equations of L_1. The simplest choice is $t = 0$, which yields the point $Q_1(1, 5, -1)$.



Distance Between a Point & a Plane

Solution

The next step is to find an equation for the plane P_2. For this purpose, observe that the vector $\underline{u}_1 = (4, -4, 5)$ is parallel to line L_1, and therefore also parallel to planes P_1 and P_2. Similarly, $\underline{u}_2 = (8, -3, 1)$ is parallel to L_2 and hence parallel to P_1 and P_2. Therefore, the cross product

$$\underline{n} = \underline{u}_1 \times \underline{u}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} = 11\hat{\imath} + 36\hat{\jmath} + 20\hat{k}$$

is normal to both P_1 and P_2.

Distance Between a Point & a Plane

Solution

Using this normal and the point $Q_2(2, 4, 5)$ found by setting $t = 0$ in the equations of L_2, we obtain an equation for P_2:

$$11(x-2) + 36(y-4) + 20(z-5) = 0$$

$$\Longrightarrow 11x + 36y + 20z - 266 = 0$$

The distance between $Q_1(1, 5, -1)$ and this plane is

$$D = \frac{|(11)(1) + (36)(5) + (20)(-1) + (-266)\,|}{\sqrt{11^2 + 36^2 + 20^2}} = \frac{95}{\sqrt{1817}}$$

which is also the distance between L_1 and L_2.

Chapter 11.6

Homework

Exercise Set 11.6 (p818 – 821)