2nd List of examples:

1. $\displaystyle\int_1^3 \frac{dx}{(x^2+16)^{7/2}}$

2. $\displaystyle\int_0^3 \frac{dx}{(x+2)\sqrt{x+1}}$

3. $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sin x+\cos x}$

4. $\displaystyle\int_0^{\pi/2} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)}$

5. $\displaystyle\int_0^{\pi/2} e^x(\sin x+\cos x)\, dx$

6. $\displaystyle\int_a^b \frac{dx}{x\sqrt{(x-a)(b-x)}}$

Prove that $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Proof: $\displaystyle R.H.S. = \int_0^a f(a-x)\,dx$

let $a-x=z$

$\therefore -dx = dz$

$\Rightarrow dx = -dz.$

$\displaystyle\therefore R.H.S = \int_a^0 f(z)(-dz)$

When $x=a$ then $z=0$

" $x=0$ " $z=a$

$\displaystyle = -\int_a^0 f(z)\,dz = \int_0^a f(z)\,dz = \int_0^a f(x)\,dx$

$= L.H.S.$ (proved)

Examples:

1. $\int_0^{\pi/2} \frac{dx}{1+\cot x}$

2. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

3. $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}\,dx$

4. $\int_0^{\pi} \frac{x\,dx}{1+\sin x}$, 5. $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

6. $\int_0^{\pi} x\sin^2 x\,dx$ 7. $\int_0^{\pi/2} \ln(\tan x)\,dx$

8. $\int_0^{1} \frac{\ln(1+x)}{1+x^2}\,dx$ 9. $\int_0^{\pi/2} \ln \sin x\,dx$

prove that $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ if $f(2a-x) = f(x)$

proof: $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx \quad \cdots (1)$

let $x = 2a - z \quad \therefore dx = -dz$

when $x = a$ then $z = a$

" $x = 2a$ " $z = 0$

$\therefore (1) \Rightarrow \int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{0} f(2a-z)(-dz)$

$\Rightarrow \int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-z)\,dz = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx$

$= \int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx$

$= 2\int_0^{a} f(x)\,dx$

Solution:

i) $\int_0^3 \frac{dx}{(x+2)\sqrt{x+1}}$

putting $x+1 = z^2$

$\therefore dx = 2z\,dz$.

when $x=3$ then $z=2$

" $x=0$ " $z=1$

$\therefore I = \int_1^2 \frac{2z\,dz}{(z^2-1+2)z}$

$= 2\int_1^2 \frac{dz}{z^2+1}$

$= 2\left[\tan^{-1}z\right]_1^2$

$= 2\left(\tan^{-1}2 - \tan^{-1}1\right)$

$= 2\tan^{-1}2 - 2\cdot\frac{\pi}{4}$

$= 2\tan^{-1}2 - \frac{\pi}{2}$ Ans.

(ii) $\int_0^a \sqrt{\frac{a+x}{a-x}}\,dx$

putting $x = a\cos\theta$

$\therefore dx = -a\sin\theta\,d\theta$.

when $x=0$ then $a\cos\theta=0 \Rightarrow \cos\theta = 0 = \cos\pi/2$ $\therefore \theta = \pi/2$

" $x=a$ " $a\cos\theta = a \Rightarrow \cos\theta = 1 = \cos 0$ $\therefore \theta = 0$

$\therefore I = \int_{\pi/2}^{0} \sqrt{\frac{a+a\cos\theta}{a-a\cos\theta}}\,(-a\sin\theta\,d\theta)$

$= \int_0^{\pi/2} \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\; a\sin\theta\,d\theta$

$= \int_0^{\pi/2} \frac{\cos\theta/2}{\cancel{\sin\theta/2}}\; a\,2\cancel{\sin\theta/2}\cos\theta/2\,d\theta$.

$= a\int_0^{\pi/2} (1+\cos\theta)\,d\theta = a\left[\theta + \sin\theta\right]_0^{\pi/2} = a\left(\frac{\pi}{2}+1-0-0\right)$

$= a\left(\frac{\pi}{2}+1\right)$ Ans.

(iii) $$\int_{1/2}^{1} \frac{dx}{x\sqrt{1-x^2}}$$

putting $x = \sin\theta$

$\therefore dx = \cos\theta \, d\theta$

when $x = \frac{1}{2}$ then $\sin\theta = \frac{1}{2} = \sin\pi/6 \quad \therefore \theta = \pi/6$

" $x = 1$ " $\sin\theta = 1 = \sin\pi/2 \quad \therefore \theta = \pi/2$

$$\therefore I = \int_{\pi/6}^{\pi/2} \frac{\cancel{\cos\theta}\, d\theta}{\sin\theta \cancel{\cos\theta}} = \int_{\pi/6}^{\pi/2} \operatorname{cosec}\theta \, d\theta$$

$$= \Big[\ln(\operatorname{cosec} x - \cot x)\Big]_{\pi/6}^{\pi/2}$$

$$= \ln(\operatorname{cosec}\pi/2 - \cot\pi/2) - \ln(\operatorname{cosec}\pi/6 - \cot\pi/6)$$

$$= \ln(1-0) - \ln(2-\sqrt{3})$$

$$= -\ln(2-\sqrt{3}) \quad \text{Ans}$$

(iv) $$\int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}$$

putting $x = \sin\theta \quad \therefore dx = \cos\theta\, d\theta$.

when $x = 0$ then $\theta = 0$

" $x = 1$ " $\theta = \pi/2$

$$\therefore I = \int_0^{\pi/2} \frac{\cancel{\cos\theta}\, d\theta}{(1+\sin^2\theta)\cancel{\cos\theta}} = \int_0^{\pi/2} \frac{d\theta}{1+\sin^2\theta}$$

$$= \int_0^{\pi/2} \frac{\frac{1}{\cos^2\theta}\, d\theta}{\frac{1}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}}$$

$$= \int_0^{\pi/2} \frac{\sec^2\theta \, d\theta}{\sec^2\theta + \tan^2\theta}$$

$$= \int_0^{\pi/2} \frac{\sec^2\theta}{1+2\tan^2\theta} d\theta.$$

let $\tan\theta = z$

$\therefore \sec^2\theta \, d\theta = dz$

when $\theta = 0$ then $z = 0$

" $\theta = \pi/2$ " $z = \infty$

$$\therefore I = \int_0^{\infty} \frac{dz}{1+2z^2}$$

$$= \frac{1}{2}\int_0^{\infty} \frac{dz}{\frac{1}{2}+z^2}$$

$$= \frac{1}{2}\int_0^{\infty} \frac{dz}{\left(\frac{1}{\sqrt{2}}\right)^2+z^2}$$

$$= \frac{1}{2}\cdot\frac{1}{\frac{1}{\sqrt{2}}}\left[\tan^{-1}\left(\frac{z}{1/\sqrt{2}}\right)\right]_0^{\infty}$$

$$= \frac{\sqrt{2}}{2}\left(\tan^{-1}\sqrt{2}z\right)_0^{\infty}$$

$$= \frac{1}{\sqrt{2}}\left(\tan^{-1}\infty - \tan^{-1}0\right)$$

$$= \frac{1}{\sqrt{2}}\left(\tan^{-1}\tan\frac{\pi}{2} - \tan^{-1}\tan 0\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2\sqrt{2}} \quad \text{Ans}$$

2nd list:

1. $$\int_0^3 \frac{dx}{(x^2+16)^{3/2}}$$

putting $x = 4\tan\theta$

$\therefore dx = 4\sec^2\theta \, d\theta.$

when $x = 0$ then $\theta = 0$

" $x = 3$ " $\theta = \tan^{-1}\frac{3}{4}$

$$\therefore I = \int_0^{\tan^{-1}\frac{3}{4}} \frac{4\sec^2\theta \, d\theta}{16^{3/2}\cdot\sec^3\theta}$$

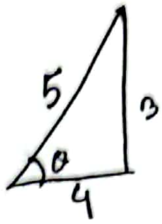
$$= \frac{1}{16}\int_0^{\tan^{-1}\frac{3}{4}} \frac{1}{\sec\theta} d\theta$$

$$= \frac{1}{16}\int_0^{\tan^{-1}\frac{3}{4}} \cos\theta \, d\theta.$$

$$= \frac{1}{16}\left[\sin\theta\right]_0^{\tan^{-1}\frac{3}{4}}$$

$$= \frac{1}{16}\left[\sin\left(\tan^{-1}\frac{3}{4}\right) - \sin 0\right]$$

$$= \frac{1}{16}\sin\left(\tan^{-1}\frac{3}{4}\right)$$



$$= \frac{1}{16}\sin\sin^{-1}\frac{3}{5}$$

$$\tan^{-1}\frac{3}{4} = \sin^{-1}\frac{3}{5}$$

$$= \frac{1}{16}\cdot\frac{3}{5} = \frac{3}{80}$$ Ans.

2. putting $x+1 = z^2$

3. $$\int_0^{\pi/2} \frac{dx}{1+\sin x+\cos x}$$

$$= \int_0^{\pi/2} \frac{dx}{1+\frac{2\tan x/2}{1+\tan^2 x/2}+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$$

$$= \int_0^{\pi/2} \frac{1+\tan^2 x/2 \; dx}{1+\cancel{\tan^2 x/2}+2\tan x/2+1-\cancel{\tan^2 x/2}}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x/2 \; dx}{2+2\tan x/2}$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sec^2 x/2}{1+\tan x/2}\,dx$$

putting $\tan x/2 = z$ $\therefore \sec^2 x/2 \, dx = 2\,dz$

when $x=0$ then $z=0$

" $x=\pi/2$ " $z=1$

$$\therefore\ I = \frac{1}{2}\int_0^1 \frac{2\,dz}{1+z}$$

$$= \ln(1+z)\Big]_0^1 = \ln(2) - \ln 1 = \ln 2$$

4. $\displaystyle\int_0^{\pi/2} \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$

putting $1+\sin x = z$

$\therefore \cos x\,dx = dz$

when $x = 0$ then $z = 1$

" $x = \pi/2$ " $z = 2$

$$I = \int_1^2 \frac{dz}{z(1+z)}$$

$$= \int_1^2 \left(\frac{1}{z} - \frac{1}{1+z}\right) dz$$

$$= \Big[\ln z - \ln(1+z)\Big]_1^2$$

$$= \ln \frac{z}{1+z}\Big]_1^2$$

$$= \ln\left(\frac{2}{3}\right) - \ln\frac{1}{2}$$

$$= \ln\left(\frac{2}{3}\times 2\right) = \ln\frac{4}{3}\quad \text{Ans.}$$

5. $\displaystyle\int_0^{\pi/2} e^x(\sin x + \cos x)\,dx$

$$= \Big[e^x \sin x\Big]_0^{\pi/2} = e^{\pi/2}\sin \pi/2 - e^0 \sin 0$$

$$= e^{\pi/2}\cdot 1 - 0 = e^{\pi/2}\quad \text{Ans}$$

Examples:

1. $$I = \int_0^{\pi/2} \frac{dx}{1+\cot x} = \int_0^{\pi/2} \frac{dx}{1+\frac{\cos x}{\sin x}} = \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx$$

$$\therefore I = \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x)+\cos(\pi/2 - x)}\,dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\,dx$$

$$\therefore I + I = \int_0^{\pi/2} \left(\frac{\sin x}{\sin x+\cos x} + \frac{\cos x}{\cos x+\sin x}\right) dx$$

$$= \int_0^{\pi/2} \frac{\sin x+\cos x}{\sin x+\cos x}\,dx$$

$$= \int_0^{\pi/2} 1\,dx = x\Big]_0^{\pi/2} = \pi/2$$

$$\Rightarrow 2I = \pi/2 \quad \therefore I = \frac{\pi}{4} \quad \text{Ans.}$$

4. $$I = \int_0^{\pi} \frac{x\,dx}{1+\sin x} = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\,dx$$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin x}\,dx$$

$$\therefore I + I = \int_0^{\pi} \left(\frac{x}{1+\sin x} + \frac{\pi - x}{1+\sin x} \right) dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1}{1+\sin x} dx$$

$$\Rightarrow I = \frac{\pi}{2} \int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx$$

$$= \frac{\pi}{2} \int_0^{\pi} \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos x} \frac{1}{\cos x} \right) dx$$

$$= \frac{\pi}{2} \int_0^{\pi} (\sec^2 x - \sec x \tan x) \, dx$$

$$= \frac{\pi}{2} \Big[\tan x - \sec x \Big]_0^{\pi}$$

$$= \frac{\pi}{2} \left(\tan \pi - \sec \pi - \tan 0 + \sec 0 \right)$$

$$= \frac{\pi}{2} (+1+1) = \pi \quad \underline{\underline{\text{Ans}}}$$

$\underline{\underline{5:}}$

$$I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x)}{1+\cos^2 (\pi - x)} dx$$

$$= \int_0^{\pi} \frac{(\pi - x) \sin x}{1+\cos^2 x} dx$$

$\therefore I + I = \int_0^{\pi} \left\{ \frac{x \sin x}{1+\cos^2 x} + \frac{(\pi - x)\sin x}{1+\cos^2 x} \right\} dx$

$\Rightarrow 2I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x} dx$

$= \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

putting $\cos x = z$

$\Rightarrow \sin x \, dx = -dz$

when $x=0$ then $z=1$

" $x=\pi$ " $z=-1$

$\therefore 2I = \pi \int_1^{-1} \frac{-dz}{1+z^2}$

$= \pi \int_{-1}^{1} \frac{dz}{1+z^2} = \pi \left[\tan^{-1} z\right]_{-1}^{1}$

$= \pi \left[\tan^{-1} 1 - \tan^{-1}(-1)\right]$

$= \pi \left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi^2}{2}$

$\therefore I = \frac{\pi^2}{4}$ Ans.

7. $I = \int_0^{\pi/2} \ln(\tan x) \, dx$

$\Rightarrow I = \int_0^{\pi/2} \ln \tan(\pi/2 - x) \, dx$

$= \int_0^{\pi/2} \ln \cot x \, dx$

$\therefore I + I = \int_0^{\pi/2} (\ln \tan x + \ln \cot x) \, dx$

$\Rightarrow 2I = \int_0^{\pi/2} \ln(\tan x \cot x) \, dx = \int_0^{\pi/2} \ln 1 \, dx = 0$

$\therefore I = 0$ Ans

8. $I = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx$

putting $x = \tan\theta$ $\therefore dx = \sec^2\theta \, d\theta$.

when $x=0$ $\tan\theta = 0$

" $x=1$ " $\theta = \pi/4$

$\therefore I = \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta} \cdot \sec^2\theta \, d\theta$.

$I = \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta$.

$= \int_0^{\pi/4} \ln\{1+\tan(\frac{\pi}{4}-\theta)\}\, d\theta$.

$= \int_0^{\pi/4} \ln\left(1+\frac{\tan\pi/4 - \tan\theta}{1+\tan\pi/4 \tan\theta}\right) d\theta$.

$= \int_0^{\pi/4} \ln\left(\frac{1+\cancel{\tan\theta}+1-\cancel{\tan\theta}}{1+\tan\theta}\right) d\theta$.

$= \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$

$= \int_0^{\pi/4} \{\ln 2 - \ln(1+\tan\theta)\}\, d\theta$.

$\therefore I+I = \int_0^{\pi/4} \{\cancel{\ln(1+\tan\theta)} + \ln 2 - \cancel{\ln(1+\tan\theta)}\}\, d\theta$.

$\Rightarrow 2I = \ln 2 \int_0^{\pi/4} d\theta = \ln 2\, [\theta]_0^{\pi/4} = \frac{\pi}{4}\ln 2$

$\therefore I = \frac{\pi}{8}\ln 2$ Ans

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9.

$$I = \int_0^{\pi/2} \ln \sin x \, dx \quad \cdots \text{(i)}$$

$$= \int_0^{\pi/2} \ln \sin\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \ln \cos x \, dx \quad \text{—— (ii)}$$

$$\text{(i)} + \text{(ii)} \Rightarrow \quad 2I = \int_0^{\pi/2} (\ln \sin x + \ln \cos x) \, dx$$

$$= \int_0^{\pi/2} \ln \sin x \cos x \, dx$$

$$= \int_0^{\pi/2} \ln \frac{\sin 2x}{2} \, dx$$

$$= \int_0^{\pi/2} \ln \sin 2x \, dx - \int_0^{\pi/2} \ln 2 \, dx$$

$$= \int_0^{\pi/2} \ln \sin 2x \, dx - \ln 2 \left[x\right]_0^{\pi/2}$$

let $2x = z$

$\therefore dx = dz/2$

when $x = 0$ then $z = 0$

" $x = \pi/2$ " $z = \pi$

$$\therefore 2I = \int_0^{\pi} \ln \sin z \, \frac{dz}{2} - \ln 2 \left(\frac{\pi}{2} - 0\right)$$

$$= \frac{1}{2} \int_0^{\pi} \ln \sin z \, dz - \frac{\pi}{2} \ln 2$$

$$\Rightarrow 2I = \int_0^{\pi/2} \ln \sin x \, dx - \frac{\pi}{2} \ln 2$$

$$\Rightarrow 2I = I - \frac{\pi}{2} \ln 2$$

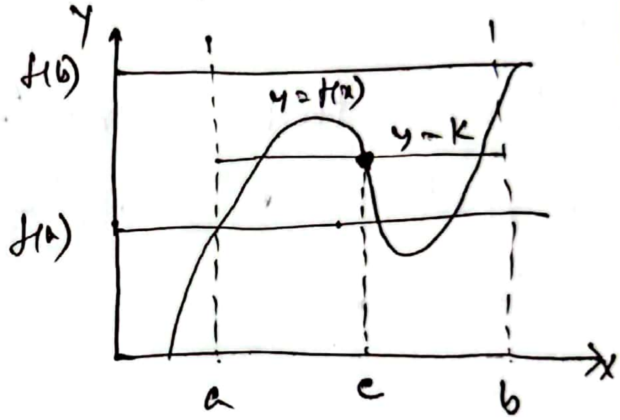
$$\Rightarrow I = -\frac{\pi}{2} \ln 2$$

$$\Rightarrow I = \frac{\pi}{2} \ln 2^{-1} = \frac{\pi}{2} \ln \frac{1}{2} \quad \underline{\text{Ans}}$$

State the intermediate value theorem.

Statement:

If k is a value between $f(a)$ and $f(b)$, i.e either $f(a) < k < f(b)$ or $f(a) > k > f(b)$ then there exists at least a number c within a to b i.e $c \in (a,b)$ in such a way that $f(c) = k$.



Question: check whether there is a solution to the ~~equation~~ ~~solution~~ equation $x^5 - 2x^3 - 2 = 0$ between the interval $[0, 2]$.

Sol^n let us find the values of the given function at the $x = 0$ and $x = 2$.

$$f(x) = x^5 - 2x^3 - 2 = 0$$

putting $x = 0$, $f(0) = -2$

" $x = 2$, $f(2) = 14$

Therefore, we conclude that $x=0$, the curve is below zero; while at $x=2$, it is above zero.

Since the given equation is a polynomial, its graph will be continuous.

Thus, applying the intermediate value Th^m, we can say that the graph must cross at some point between $(0,2)$.

Hence there exists a solution to the equation $x^5-2x^3-2=0$ between the interval $[0,2]$.

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