Sheikh Hasina University, Netrokona

Course:

MATH-3105 (Multivariable Calculus & Geometry)

Textbook:

Calculus, Early Transcendentals

By Anton, Bivens, Davis (10^{th} Edition)

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Vectors–Valued Functions

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Chapter 12.1

Introduction to Vector-Valued Functions

Parametric Curves in 3-Space

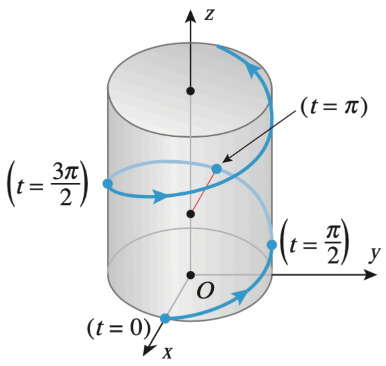
If f, g, and h are well-behaved functions, then the parametric equations

$$x = f(t), y = g(t), z = h(t)$$

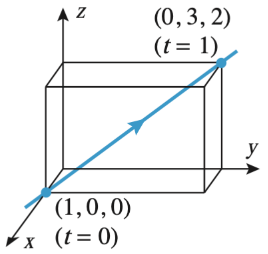
generate a curve in 3-space that is traced in a specific direction as t increases. This direction is called the orientation or direction of increasing parameter, and the curve together with its orientation is called the graph of the parametric equations or the parametric curve represented by the equations. If no restrictions are stated explicitly or are implied by the equations, then it will be understood that t varies over the interval $(-\infty, \infty)$.

Parametric Curves in 3-Space

Example (p842-843)



$x = a\cos t,\ \ y = a\sin t,\ \ z = ct$



$x = 1 - t,\ \ y = 3t,\ \ z = 2t$

The parametric equation

$$x = 1 - t, y = 3t, z = 3t$$

represents a line in 3-space.

The parametric equation

$$x = a\cos t\,, y = a\sin t\,, z = ct$$

represent a curve called a circular helix.

Vector-Valued Functions

Definition

The parametric equation $x = t, y = t^2, z = t^3$ is called twisted cubic is the set of points of the form (t, t^2, t^3) for real values of t. If we view each of these points as a terminal point for a vector $\underline{\boldsymbol{r}}$ whose initial point is at the origin,

$$\underline{\boldsymbol{r}} = (x, y, z) = (t, t^2, t^3) = t\,\hat{\boldsymbol{\imath}} + t^2\,\hat{\boldsymbol{\jmath}} + t^3\,\widehat{\boldsymbol{k}}$$

then we obtain $\underline{\boldsymbol{r}}$ as a function of the parameter t, that is, $\underline{r} = \underline{r}(t)$. Since this function produces a vector, we say that $\underline{\boldsymbol{r}} = \underline{\boldsymbol{r}}(t)$ defines $\underline{\boldsymbol{r}}$ as a vector-valued function of a real variable, or more simply, a vector-valued function.

Vector-Valued Functions

Definition

If $\underline{\boldsymbol{r}}(t)$ is a vector-valued function in 3-space, then for each allowable value of t the vector $\underline{\boldsymbol{r}} = \underline{\boldsymbol{r}}(t)$ can be represented in terms of components as

$$\underline{\boldsymbol{r}} = \underline{\boldsymbol{r}}(t) = \big(x(t), y(t), z(t)\big) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}} + z(t)\,\widehat{\boldsymbol{k}}$$

The functions $x(t)$, $y(t)$, and $z(t)$ are called the component functions or the components of $\underline{\boldsymbol{r}}(t)$.

Vector-Valued Functions

Domain

The domain of a vector-valued function $\underline{\boldsymbol{r}}(t)$ is the set of allowable values for t.

If $\underline{\boldsymbol{r}}(t)$ is defined in terms of component functions and the domain is not specified explicitly, then it will be understood that the domain is the intersection of the natural domains of the component functions; this is called the natural domain of $\underline{\boldsymbol{r}}(t)$.

Vector-Valued Functions

Homework

Example 1 – 6 (p842 –845)

Chapter 12.1

Homework

Exercise Set 12.1 (p845 –847)

Chapter 12.2

Calculus of Vector-Valued Functions

Limits and Continuity

Definition 12.2.1

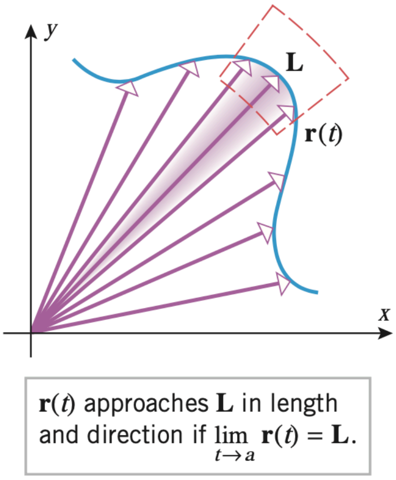
Let $\underline{\boldsymbol{r}}(t)$ be a vector-valued function that is defined for all t in some open interval containing the number a, except that $\underline{\boldsymbol{r}}(t)$ need not be defined at a. We will write

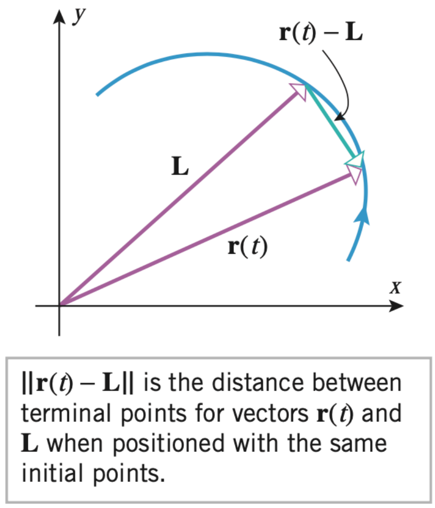
$$\lim_{t \to a} \underline{\boldsymbol{r}}(t) = \underline{\boldsymbol{L}}$$

if an only if

$$\lim_{t \to a} \left\| \underline{\boldsymbol{r}}(t) - \underline{\boldsymbol{L}} \right\| = 0$$

Limits and Continuity





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Limits and Continuity

Theorem 12.2.2

(a) If $\underline{\boldsymbol{r}} = \underline{\boldsymbol{r}}(t) = \big(x(t), y(t)\big) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}}$, then

$$\lim_{t\to a} \underline{\boldsymbol{r}}(t) = \left(\lim_{t\to a} x(t)\,, \lim_{t\to a} y(t)\right) = \lim_{t\to a} x(t)\,\hat{\boldsymbol{\imath}} + \lim_{t\to a} y(t)\,\hat{\boldsymbol{\jmath}}$$

provided the limits of the component functions exist. Conversely, the limits of the component functions exist provided $\underline{\boldsymbol{r}}(t)$ approaches a limiting vector as t approaches a.

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Limits and Continuity

Theorem 12.2.2

(b) If $\underline{\boldsymbol{r}} = \underline{\boldsymbol{r}}(t) = \big(x(t), y(t), z(t)\big) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}} + z(t)\,\hat{\boldsymbol{k}}$ then

$$\lim_{t\to a} \underline{\boldsymbol{r}}(t) = \left(\lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t)\right)$$

$$= \lim_{t\to a} x(t)\,\hat{\boldsymbol{\imath}} + \lim_{t\to a} y(t)\,\hat{\boldsymbol{\jmath}} + \lim_{t\to a} z(t)\,\hat{\boldsymbol{k}}$$

provided the limits of the component functions exist. Conversely, the limits of the component functions exist provided $\underline{\boldsymbol{r}}(t)$ approaches a limiting vector as t approaches a.

Limits and Continuity

Example 1 (Textbook p849)

Let $\underline{\boldsymbol{r}}(t) = t^2\,\hat{\boldsymbol{\imath}} + e^t\,\hat{\boldsymbol{\jmath}} - (2\cos \pi t)\,\hat{\boldsymbol{k}}$. Then

$$\lim_{t\to 0} \underline{\boldsymbol{r}}(t) = \left(\lim_{t\to 0} t^2\right)\hat{\boldsymbol{\imath}} + \left(\lim_{t\to 0} e^t\right)\hat{\boldsymbol{\jmath}} - \left(\lim_{t\to 0} 2\cos \pi t\right)\hat{\boldsymbol{k}}$$

$$= (0)\hat{\boldsymbol{\imath}} + (1)\hat{\boldsymbol{\jmath}} - (2)\,\hat{\boldsymbol{k}}$$

$$= \hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}$$

Continuity

Definition

We define a vector-valued function $\underline{\boldsymbol{r}}(t)$ to be continuous at $t = a$ if

$$\lim_{t \to a} \underline{\boldsymbol{r}}(t) = \underline{\boldsymbol{r}}(a)$$

That is, $\underline{\boldsymbol{r}}(a)$ is defined, the limit of $\underline{\boldsymbol{r}}(t)$ as $t \to a$ exists, and the two are equal.

Continuity

Definition

We say that $\underline{\boldsymbol{r}}(t)$ is continuous on an interval I if it is continuous at each point of I [with the understanding that at an endpoint in I the two-sided limit is replaced by the appropriate one-sided limit].

It follows that a vector-valued function is continuous at $t = a$ if and only if its component functions are continuous at $t = a$.

Derivatives

Definition

If $\underline{\boldsymbol{r}}(t)$ is a vector-valued function, we define the derivative of $\underline{\boldsymbol{r}}$ with respect to t to be the vector-valued function $\underline{\boldsymbol{r}}'$ given by

$$\underline{\boldsymbol{r}}'(t) = \lim_{h \to 0} \frac{\underline{\boldsymbol{r}}(t+h) - \underline{\boldsymbol{r}}(t)}{h}$$

The domain of $\underline{\boldsymbol{r}}'$ consists of all values of t in the domain of $\underline{\boldsymbol{r}}(t)$ for which the limit exists.

The function $\underline{\boldsymbol{r}}(t)$ is differentiable at t if the limit exists.

Derivatives

Notes:

- The derivative of $\underline{\boldsymbol{r}}(t)$ can be expressed as

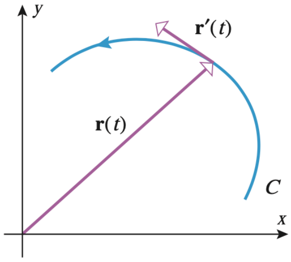
$$\frac{d}{dt}[\underline{\boldsymbol{r}}(t)], \qquad \frac{d\underline{\boldsymbol{r}}}{dt}, \qquad \underline{\boldsymbol{r}}'(t), \qquad \underline{\boldsymbol{r}}'$$

- It is important to keep in mind that $\underline{\boldsymbol{r}}'(t)$ is a vector, not a number, and hence has a magnitude and a direction for each value of t [except if $\underline{\boldsymbol{r}}'(t) = 0$, in which case $\underline{\boldsymbol{r}}'(t)$ has magnitude zero but no specific direction].

Derivatives

Geometric Interpretation of the Derivative

Suppose that C is the graph of a vector-valued function $\underline{\boldsymbol{r}}(t)$ in 2-space or 3-space and that $\underline{\boldsymbol{r}}'(t)$ exists and is nonzero for a given value of t. If the vector $\underline{\boldsymbol{r}}'(t)$ is positioned with its initial point at the terminal point of the radius vector $\underline{\boldsymbol{r}}(t)$, then $\underline{\boldsymbol{r}}'(t)$ is tangent to C and points in the direction of increasing parameter.



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Derivatives

Theorem 12.2.5

If $\underline{\boldsymbol{r}}(t)$ is a vector-valued function, then $\underline{\boldsymbol{r}}$ is differentiable at t if and only if each of its component functions is differentiable at t , in which case the component functions of $\underline{\boldsymbol{r}}'(t)$ are the derivatives of the corresponding component functions of $\underline{\boldsymbol{r}}(t)$. If $\underline{\boldsymbol{r}}(t) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}} + z(t)\,\hat{\boldsymbol{k}}$ then

$$\underline{\boldsymbol{r}}'(t) = \lim_{h\to 0}\frac{\underline{\boldsymbol{r}}(t+h) - \underline{\boldsymbol{r}}(t)}{h} = x'(t)\,\hat{\boldsymbol{\imath}} + y'(t)\,\hat{\boldsymbol{\jmath}} + z'(t)\,\hat{\boldsymbol{k}}$$

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Derivatives

Example 2 (Textbook p850)

Let $\underline{\boldsymbol{r}}(t) = t^2\,\hat{\boldsymbol{\imath}} + e^t\,\hat{\boldsymbol{\jmath}} - (2\cos\pi t)\,\widehat{\boldsymbol{k}}$. Then

$$\underline{\boldsymbol{r}}'(t) = \frac{d}{dt}(t^2)\,\hat{\boldsymbol{\imath}} + \frac{d}{dt}(e^t)\,\hat{\boldsymbol{\jmath}} - \frac{d}{dt}(2\cos\pi t)\widehat{\boldsymbol{k}}$$

$$= 2t\,\hat{\boldsymbol{\imath}} + e^t\,\hat{\boldsymbol{\jmath}} + (2\pi\sin\pi t)\,\widehat{\boldsymbol{k}}$$

Derivatives

12.2.6 THEOREM ***(Rules of Differentiation)*** *Let $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, and $\mathbf{r}_2(t)$ be differentiable vector-valued functions that are all in 2-space or all in 3-space, and let $f(t)$ be a differentiable real-valued function, k a scalar, and $\mathbf{c}$ a constant vector (that is, a vector whose value does not depend on t). Then the following rules of differentiation hold:*

(a) $\dfrac{d}{dt}[\mathbf{c}] = \mathbf{0}$

(b) $\dfrac{d}{dt}[k\mathbf{r}(t)] = k\dfrac{d}{dt}[\mathbf{r}(t)]$

(c) $\dfrac{d}{dt}[\mathbf{r}_1(t) + \mathbf{r}_2(t)] = \dfrac{d}{dt}[\mathbf{r}_1(t)] + \dfrac{d}{dt}[\mathbf{r}_2(t)]$

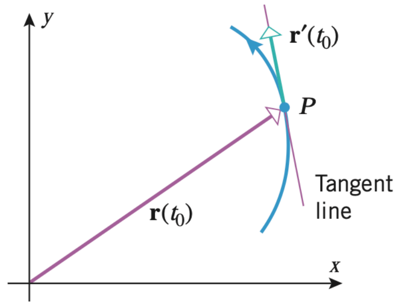
(d) $\dfrac{d}{dt}[\mathbf{r}_1(t) - \mathbf{r}_2(t)] = \dfrac{d}{dt}[\mathbf{r}_1(t)] - \dfrac{d}{dt}[\mathbf{r}_2(t)]$

(e) $\dfrac{d}{dt}[f(t)\mathbf{r}(t)] = f(t)\dfrac{d}{dt}[\mathbf{r}(t)] + \dfrac{d}{dt}[f(t)]\mathbf{r}(t)$

Tangent Lines

Definition

Let P be a point on the graph of a vector-valued function $\underline{\boldsymbol{r}}(t)$, and let $\underline{\boldsymbol{r}}(t_0)$ be the radius vector from the origin to P (Figure). If $\underline{\boldsymbol{r}}'(t_0)$ exists and $\underline{\boldsymbol{r}}'(t_0) \neq 0$, then we call $\underline{\boldsymbol{r}}'(t_0)$ a tangent vector to the graph of $\underline{\boldsymbol{r}}(t)$ at $\underline{\boldsymbol{r}}(t_0)$, and we call the line through P that is parallel to the tangent vector the tangent line to the graph of $\underline{\boldsymbol{r}}(t)$ at $\underline{\boldsymbol{r}}(t_0)$.



Tangent Lines

Definition

Let $\underline{\boldsymbol{r}}_0 = \underline{\boldsymbol{r}}(t_0)$ and $\underline{\boldsymbol{v}}_0 = \underline{\boldsymbol{r}}'(t_0)$. Then the tangent line to the graph of $\underline{\boldsymbol{r}}(t)$ at $\underline{\boldsymbol{r}}_0$ is given by the vector equation

$$\underline{\boldsymbol{r}} = \underline{\boldsymbol{r}}_0 + t\,\underline{\boldsymbol{v}}_0$$

Tangent Lines

Example 3 (p851)

Find parametric equations of the tangent line to the circular helix

$$x = \cos t\,, \qquad y = \sin t\,, \qquad z = t$$

where $t = t_0$, and use that result to find parametric equations for the tangent line at the point where $t = \pi$.

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Tangent Lines

Solution

The vector equation of the helix is

$$\underline{\boldsymbol{r}}(t) = \cos t\ \hat{\boldsymbol{\imath}} + \sin t\ \hat{\boldsymbol{\jmath}} + t\,\widehat{\boldsymbol{k}}$$

so we have

$$\underline{\boldsymbol{r}}_0 = \underline{\boldsymbol{r}}(t_0) = \cos t_0\ \hat{\boldsymbol{\imath}} + \sin t_0\ \hat{\boldsymbol{\jmath}} + t_0\,\widehat{\boldsymbol{k}}$$

$$\underline{\boldsymbol{v}}_0 = \underline{\boldsymbol{r}}'(t_0) = (-\sin t_0)\hat{\boldsymbol{\imath}} + \cos t_0\ \hat{\boldsymbol{\jmath}} + \widehat{\boldsymbol{k}}$$

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Tangent Lines

Solution

Therefore, the vector equation of the tangent line at $t = t_0$ is

$$\underline{\boldsymbol{r}}(t) = \underline{\boldsymbol{r}}_0 + t\,\underline{\boldsymbol{v}}_0$$

$$= \cos t_0\ \hat{\boldsymbol{\imath}} + \sin t_0\ \hat{\boldsymbol{\jmath}} + t_0\,\widehat{\boldsymbol{k}} + t\big[(-\sin t_0)\hat{\boldsymbol{\imath}} + \cos t_0\ \hat{\boldsymbol{\jmath}} + \widehat{\boldsymbol{k}}\big]$$

$$= (\cos t_0 - t\sin t_0)\hat{\boldsymbol{\imath}} + (\sin t_0 + t\cos t_0)\ \hat{\boldsymbol{\jmath}} + (t_0 + t)\widehat{\boldsymbol{k}}$$

Tangent Lines

Solution

Thus the parametric equations of the tangent line at $t = t_0$ are

$$x = \cos t_0 - t\sin t_0\,, \qquad y = \sin t_0 + t\cos t_0\,, \qquad z = t_0 + t$$

In particular, the tangent line at $t = \pi$ has parametric equations

$$x = -1, \qquad y = -t, \qquad z = \pi + t$$

Tangent Lines

Example 4 (p852)

Let,

$$\underline{\boldsymbol{r}}_1(t) = (\tan^{-1} t)\,\hat{\boldsymbol{\imath}} + (\sin t)\,\hat{\boldsymbol{\jmath}} + t^2\,\widehat{\boldsymbol{k}}$$

and

$$\underline{\boldsymbol{r}}_2(t) = (t^2 - t\,t)\,\hat{\boldsymbol{\imath}} + (2t - 2)\,\hat{\boldsymbol{\jmath}} + (\ln t)\,\widehat{\boldsymbol{k}}$$

The graphs of $\underline{\boldsymbol{r}}_1(t)$ and $\underline{\boldsymbol{r}}_2(t)$ intersect at the origin. Find the degree measure of the acute angle between the tangent lines to the graphs of $\underline{\boldsymbol{r}}_1(t)$ and $\underline{\boldsymbol{r}}_2(t)$ at the origin.

Tangent Lines

Solution

At the origin,

$$\tan^{-1} t = 0,\ \sin t = 0,\ t^2 = 0 \quad \Longrightarrow t = 0 \text{ for } \underline{\boldsymbol{r}}_1(t).$$

The graphs of $\underline{\boldsymbol{r}}_1(t)$ passes through the origin at $t = 0$, where its tangent vector is

$$\underline{\boldsymbol{r}}_1'(0) = \left(\frac{1}{1+t^2}, \cos t\,, 2t\right)\Bigg|_{t=0} = (1, 1, 0)$$

Tangent Lines

Solution

At the origin,

$$t^2 - t = 0,\ 2t - t = 0,\ \ln t = 0 \implies t = 1 \text{ for } \underline{\boldsymbol{r}}_2(t).$$

The graphs of $\underline{\boldsymbol{r}}_2(t)$ passes through the origin at $t = 1$, where its tangent vector is

$$\underline{\boldsymbol{r}}_2'(1) = \left(2t - 1, 2, \frac{1}{t}\right)\bigg|_{t=1} = (1, 2, 1)$$

Tangent Lines

Solution

The angle between two tangent vectors satisfies

$$\cos\theta = \frac{(1,1,0)\ .\ (1,2,1)}{\|(1,1,0)\|\ \ \|(1,2,1)\|} = \frac{1+2+0}{\sqrt{2}\,\sqrt{6}} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$$

$$\implies \theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{6}\text{radian}$$

$$= 30°$$

Derivatives of Dot & Cross Products

$$\frac{d}{dt}\left[\underline{\boldsymbol{r}}_1(t) \cdot \underline{\boldsymbol{r}}_2(t)\right] = \underline{\boldsymbol{r}}_1(t) \cdot \frac{d\underline{\boldsymbol{r}}_2}{dt} + \frac{d\underline{\boldsymbol{r}}_1}{dt} \cdot \underline{\boldsymbol{r}}_2(t)$$

$$\frac{d}{dt}\left[\underline{\boldsymbol{r}}_1(t) \times \underline{\boldsymbol{r}}_2(t)\right] = \underline{\boldsymbol{r}}_1(t) \times \frac{d\underline{\boldsymbol{r}}_2}{dt} + \frac{d\underline{\boldsymbol{r}}_1}{dt} \times \underline{\boldsymbol{r}}_2(t)$$

Derivatives of Dot & Cross Products

Theorem 12.2.8

If $\underline{\boldsymbol{r}}(t)$ is a differentiable vector-valued function in 2-space or 3-space and $\|\underline{\boldsymbol{r}}(t)\|$ is constant for all t, then

$$\underline{\boldsymbol{r}}(t) \cdot \underline{\boldsymbol{r}}'(t) = 0$$

that is, $\underline{\boldsymbol{r}}(t)$ and $\underline{\boldsymbol{r}}'(t)$ are orthogonal vectors for all t.

Proof

Homework (p853).

Definite Integrals of Vector-Valued F^{n}s

If $\underline{\boldsymbol{r}}(t)$ is a vector-valued function that is continuous on the interval $a \leq t \leq b$, then we define the definite integral of $\underline{\boldsymbol{r}}(t)$ over this interval as

$$\int_a^b \underline{\boldsymbol{r}}(t)\,dt = \left(\int_a^b x(t)\,dt\right)\hat{\boldsymbol{\imath}} + \left(\int_a^b y(t)\,dt\right)\hat{\boldsymbol{\jmath}} + \left(\int_a^b z(t)\,dt\right)\widehat{\boldsymbol{k}}$$

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Definite Integrals of Vector-Valued F^{n}s

Example 6 (p854)

Let $\underline{\boldsymbol{r}}(t) = t^{\mathbf{2}}\,\hat{\boldsymbol{\imath}} + e^t\hat{\boldsymbol{\jmath}} - (2\cos\pi t)\,\widehat{\boldsymbol{k}}$. Then

$$\int_0^1 \underline{\boldsymbol{r}}(t)\,dt = \left(\int_0^1 t^2\,dt\right)\hat{\boldsymbol{\imath}} + \left(\int_0^1 e^t\,dt\right)\hat{\boldsymbol{\jmath}} - \left(\int_a^b 2\cos\pi t\,dt\right)\widehat{\boldsymbol{k}}$$

$$= \frac{t^3}{3}\bigg|_0^1 \hat{\boldsymbol{\imath}} + e^t\Big|_0^1 \hat{\boldsymbol{\jmath}} - \frac{2}{\pi}\sin\pi t\bigg|_0^1 \widehat{\boldsymbol{k}}$$

$$= \frac{1}{3}\,\hat{\boldsymbol{\imath}} + (e-1)\,\hat{\boldsymbol{\jmath}}$$

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Rules of Integration

12.2.9 THEOREM (*Rules of Integration*) *Let* $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, *and* $\mathbf{r}_2(t)$ *be vector-valued functions in 2-space or 3-space that are continuous on the interval* $a \le t \le b$, *and let k be a scalar. Then the following rules of integration hold:*

(*a*) $\displaystyle\int_a^b k\mathbf{r}(t)\,dt = k\int_a^b \mathbf{r}(t)\,dt$

(*b*) $\displaystyle\int_a^b [\mathbf{r}_1(t) + \mathbf{r}_2(t)]\,dt = \int_a^b \mathbf{r}_1(t)\,dt + \int_a^b \mathbf{r}_2(t)\,dt$

(*c*) $\displaystyle\int_a^b [\mathbf{r}_1(t) - \mathbf{r}_2(t)]\,dt = \int_a^b \mathbf{r}_1(t)\,dt - \int_a^b \mathbf{r}_2(t)\,dt$

Antiderivatives of Vector-Valued Functions

An antiderivative for a vector-valued function $\underline{\boldsymbol{r}}(t)$ is a vector-valued function $\underline{\boldsymbol{R}}(t)$ such that

$$\underline{\boldsymbol{R}}'(t) = \underline{\boldsymbol{r}}(t)$$

We express this Equation using integral notation as

$$\int \underline{\boldsymbol{r}}(t)dt = \underline{\boldsymbol{R}}(t) + \underline{\boldsymbol{C}}$$

where $\underline{\boldsymbol{C}}$ represents an arbitrary constant vector.

Antiderivatives of Vector-Valued Functions

Example 7 (p855)

$$\int (2t\,\hat{\boldsymbol{\imath}} + 3t^2\,\hat{\boldsymbol{\jmath}})\,dt = \left(\int 2t\,dt\right)\hat{\boldsymbol{\imath}} + \left(\int 3t^2\,dt\right)\hat{\boldsymbol{\jmath}}$$

$$= (t^2 + C_1)\hat{\boldsymbol{\imath}} + (t^3 + C_2)\hat{\boldsymbol{\jmath}}$$

$$= t^2\hat{\boldsymbol{\imath}} + t^3\hat{\boldsymbol{\jmath}} + \underline{\boldsymbol{C}}$$

where $\underline{\boldsymbol{C}} = C_1\hat{\boldsymbol{\imath}} + C_2\hat{\boldsymbol{\jmath}}$ is an arbitrary vector constant of integration.

Antiderivatives of Vector-Valued Functions

$$\frac{d}{dt}\left[\int \underline{\boldsymbol{r}}(t)dt\right] = \underline{\boldsymbol{r}}(t)$$

$$\int \underline{\boldsymbol{r}}'(t)dt = \underline{\boldsymbol{r}}(t) + \underline{\boldsymbol{C}}$$

$$\int_a^b \underline{\boldsymbol{r}}(t)dt = \underline{\boldsymbol{R}}(t)\Big]_a^b = \underline{\boldsymbol{R}}(b) - \underline{\boldsymbol{R}}(a)$$

Antiderivatives of Vector-Valued Functions

Example 8 (p855)

Evaluate the definite integral

$$\int_0^2 (2t\,\hat{\boldsymbol{\imath}} + 3t^2\,\hat{\boldsymbol{\jmath}})\,dt.$$

Solution

$$\int_0^2 (2t\,\hat{\boldsymbol{\imath}} + 3t^2\,\hat{\boldsymbol{\jmath}})\,dt = t^2\,\hat{\boldsymbol{\imath}} + t^3\,\hat{\boldsymbol{\jmath}}\,\Big|_0^2$$
$$= 4\,\hat{\boldsymbol{\imath}} + 8\,\hat{\boldsymbol{\jmath}}$$

Antiderivatives of Vector-Valued Functions

Example 9 (p855)

Find $\underline{\boldsymbol{r}}(t)$ given that $\underline{\boldsymbol{r}}'(t) = (3, 2t)$ and $\underline{\boldsymbol{r}}(1) = (2,5)$.

Solution

We know

$$\underline{\boldsymbol{r}}(t) = \int \underline{\boldsymbol{r}}'(t)\,dt = \int (3, 2t)\,dt = (3t, t^2) + \underline{\boldsymbol{C}} \qquad \dots\dots\dots(1)$$

where $\underline{\boldsymbol{C}}$ is a vector constant of integration.

Antiderivatives of Vector-Valued Functions

Solution

Substituting $t = 1$ in equation (1) we get

$$\underline{\boldsymbol{r}}(1) = (3,1) + \underline{\boldsymbol{C}}$$

Given that $\underline{\boldsymbol{r}}(1) = (2,5)$, therefore comparing

$$(3,1) + \underline{\boldsymbol{C}} = (2,5)$$

$$\Longrightarrow \underline{\boldsymbol{C}} = (2,5) - (3,1) = (-1, 4)$$

Thus from equation (1) we get

$$\underline{\boldsymbol{r}}(t) = (3t, t^2) + (-1, 4) = (3t - 1, t^2 + 4)$$

Chapter 12.2

Homework

Exercise Set 12.2 (p856 –857)

Chapter 12.3

Change of Parameter; Arc Length

Smooth Parametrizations

Definition

We will say that a curve represented by $\underline{\boldsymbol{r}}(t)$ is smoothly parametrized by $\underline{\boldsymbol{r}}(t)$, or that $\underline{\boldsymbol{r}}(t)$ is a smooth function of t if $\underline{\boldsymbol{r}}'(t)$ is continuous and $\underline{\boldsymbol{r}}'(t) \neq 0$ for any allowable value of t.

Geometrically, this means that a smoothly parametrized curve can have no abrupt changes in direction as the parameter increases.

Smooth Parametrizations

Example 1 (p858)

Determine whether the following vector-valued functions are smooth.

(a) $\underline{\boldsymbol{r}}(t) = a\cos t\ \hat{\boldsymbol{\imath}} + a\sin t\ \hat{\boldsymbol{\jmath}} + ct\,\widehat{\boldsymbol{k}}\quad (a > 0, c > 0)$

(b) $\underline{\boldsymbol{r}}(t) = t^2\ \hat{\boldsymbol{\imath}} + t^3\ \hat{\boldsymbol{\jmath}}$

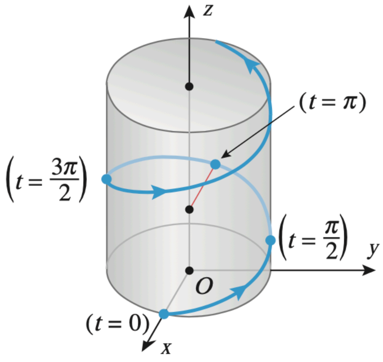
Smooth Parametrizations

Solution (a)

We have

$$\underline{\boldsymbol{r}}'(t) = -a\sin t\ \hat{\boldsymbol{\imath}} + a\cos t\ \hat{\boldsymbol{\jmath}} + c\,\widehat{\boldsymbol{k}}$$

The components are continuous functions of t, and there is no value of t for which all three of them are zero, so $\underline{\boldsymbol{r}}(t)$ is a smooth function.



$x = a\cos t,\ y = a\sin t,\ z = ct$

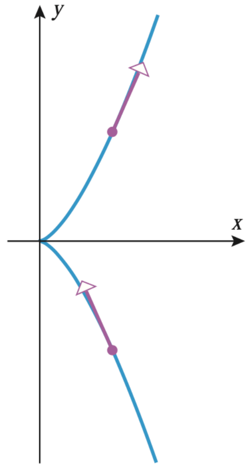
Smooth Parametrizations

Solution (b)

We have

$$\underline{\boldsymbol{r}}'(t) = 2t\,\hat{\boldsymbol{\imath}} + 3t^2\,\hat{\boldsymbol{\jmath}}$$

Although the components are continuous functions, they are both equal to zero if $t = 0$, so $\underline{\boldsymbol{r}}(t)$ is not a smooth function.



$\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

Arc Length from the Vector Viewpoint

Arc Length for Parametric Curve

The arc length L of a parametric curve

$$x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad (a \le t \le b)$$

in 3-space is given by the formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Arc Length from the Vector Viewpoint

Vector Form

Let us consider a vector-valued function

$$\underline{\boldsymbol{r}}(t) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}} + z(t)\,\hat{\boldsymbol{k}}$$

It follows that

$$\frac{d\underline{\boldsymbol{r}}}{dt} = \frac{dx}{dt}\,\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\,\hat{\boldsymbol{\jmath}} + \frac{dz}{dt}\,\hat{\boldsymbol{k}}$$

and hence

$$\left\|\frac{d\underline{\boldsymbol{r}}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Arc Length from the Vector Viewpoint

Theorem 12.3.1

If C is the graph of a smooth vector-valued function $\underline{\boldsymbol{r}}(t)$, then its arc length L from $t = a$ to $t = b$ is

$$L = \int_a^b \left\|\frac{d\underline{\boldsymbol{r}}}{dt}\right\| dt$$

Arc Length from the Vector Viewpoint

Example 2 (p859)

Find the arc length of that portion of the circular helix

$$x = \cos t\,, \qquad y = \sin t\,, \qquad z = t$$

from $t = 0$ to $t = \pi$.

Arc Length from the Vector Viewpoint

Solution

Let, $\underline{\boldsymbol{r}}(t) = \cos t\ \hat{\boldsymbol{\imath}} + \sin t\ \hat{\boldsymbol{\jmath}} + t\,\widehat{\boldsymbol{k}} = (\cos t\,, \sin t\,, t)$.

$$\therefore\ \underline{\boldsymbol{r}}'(t) = (-\sin t\,, \cos t\,, 1)$$

$$\Rightarrow \left\|\underline{\boldsymbol{r}}'(t)\right\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$$

Then the arc length of the helix is

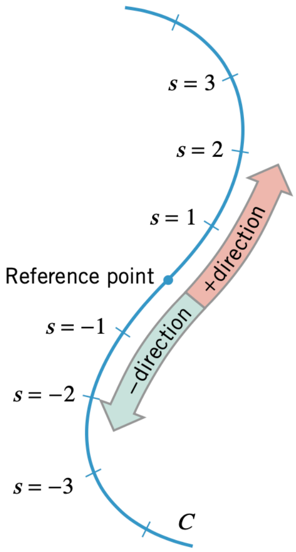
$$L = \int_0^{\pi} \left\|\frac{d\boldsymbol{r}}{dt}\right\| dt = \int_0^{\pi} \sqrt{2}\, dt = \sqrt{2}\pi$$

Arc Length As a Parameter

Step 1: Select an arbitrary point on the curve C to serve as a reference point.

Step 2: Starting from the reference point, choose one direction along the curve to be the positive direction and the other to be the negative direction.

Step 3: If P is a point on the curve, let s be the "signed" arc length along C from the reference point to P , where s is positive if P is in the positive direction from the reference point and s is negative if P is in the negative direction. Figure illustrates this idea.



Arc Length As a Parameter

By this procedure, a unique point P on the curve is determined when a value for s is given.

Let us now treat s as a variable. As the value of s changes, the corresponding point P moves along C and the coordinates of P become functions of s. Thus, the coordinates of P are $\big(x(s),\ y(s),\ z(s)\big)$. Therefore, the curve C is given by the parametric equations

$$x = x(s), \qquad y = y(s), \qquad z = z(s)$$

Arc Length As a Parameter

A parametric representation of a curve with arc length as the parameter is called an arc length parametrization of the curve.

Note that a given curve will generally have infinitely many different arc length parametrizations, since the reference point and orientation can be chosen arbitrarily.

Arc Length As a Parameter

Example 3 (p860)

Find the arc length parametrization of the circle $x^2 + y^2 = a^2$ with counterclockwise orientation and $(a, 0)$ as the reference point.

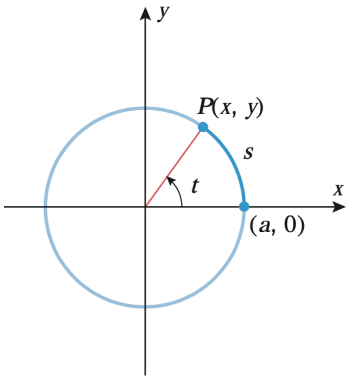
Arc Length As a Parameter

Solution

The circle with counterclockwise orientation can be represented by the parametric equations

$$x = a\cos t\,, y = a\sin t \quad (0 \le t \le 2\pi) \;\ldots\ldots\ldots (1)$$

in which t can be interpreted as the angle in radian measure from the positive x –axis to the radius from the origin to the point $P(x, y)$ (Figure).



Arc Length As a Parameter

Solution

If we take the positive direction for measuring the arc length to be counterclockwise, and we take $(a, 0)$ to be the reference point, then s and t are related by

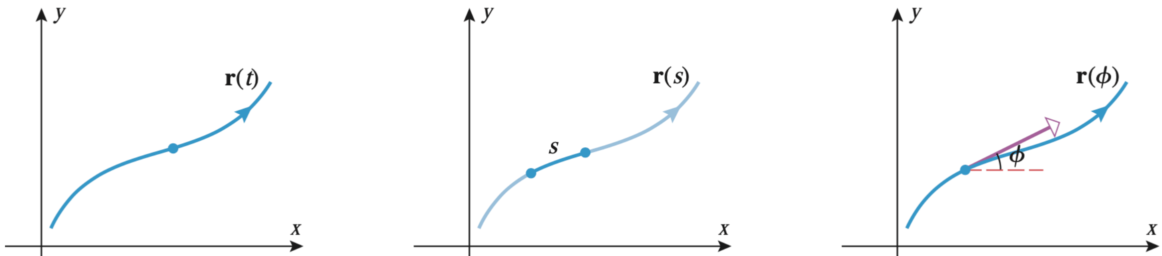
$$s = at \quad or \quad t = s/a\,.$$

Making this change of variable in (1) and noting that s increases from 0 to $2\pi a$ as t increases from 0 to 2π yields the following arc length parametrization of the circle:

$$x = a\cos(s/a)\,, \qquad y = a\sin(s/a) \qquad (0 \le s \le 2\pi a)$$

Change of Parameter

A change of parameter in a vector-valued function $\underline{\boldsymbol{r}}(t)$ is a substitution $t = g(\tau)$ that produces a new vector-valued function $\underline{\boldsymbol{r}}(g(\tau))$ having the same graph as $\underline{\boldsymbol{r}}(t)$, but possibly traced differently as the parameter τ increases.



Time as parameter | Arc length as parameter | ϕ as parameter

Change of Parameter

Example 4 (p861)

Find a change of parameter $t = g(\tau)$ for the circle

$$\underline{\boldsymbol{r}}(t) = \cos t\ \hat{\boldsymbol{\imath}} + \sin t\ \hat{\boldsymbol{\jmath}} \quad (0 \le t \le 2\pi)$$

such that

(a) the circle is traced counterclockwise as τ increases over the interval $[0, 1]$;

(b) the circle traced clockwise as τ increases over the interval $[0, 1]$.

Change of Parameter

Solution (a)

The given circle is traced counterclockwise as t increases.

Thus, if we choose g to be an increasing function, then it will follow from the relationship $t = g(\tau)$ that t increases when τ increases, thereby ensuring that the circle will be traced counterclockwise as τ increases.

We also want to choose g so that t increases from 0 to 2π as τ increases from 0 to 1.

A simple choice of g that satisfies all of the required criteria is the line as follows

$$t = g(\tau) = 2\pi\tau$$

Change of Parameter

Solution (a)

A simple choice of g that satisfies all of the required criteria is the line as follows

$$t = g(\tau) = 2\pi\tau$$

The resulting representation of the circle in terms of the parameter τ is

$$\underline{\boldsymbol{r}}\big(g(\tau)\big) = \cos(2\pi\tau)\ \hat{\boldsymbol{\imath}} + \sin(2\pi\tau)\ \hat{\boldsymbol{\jmath}} \quad (0 \le \tau \le 1).$$

Change of Parameter

Solution (b)

To ensure that the circle is traced clockwise, we will choose g to be a decreasing function such that t decreases from 2π to 0 as τ increases from 0 to 1. A simple choice of g that achieves this is the linear function

$$t = g(\tau) = 2\pi(1-\tau).$$

The resulting representation of the circle in terms of the parameter τ is

$$\underline{\boldsymbol{r}}\big(g(\tau)\big) = \cos\big(2\pi(1-\tau)\big)\ \hat{\boldsymbol{\imath}} + \sin\big(2\pi(1-\tau)\big)\ \hat{\boldsymbol{\jmath}} \quad (0 \le \tau \le 1).$$

which simplifies to

$$\underline{\boldsymbol{r}}\big(g(\tau)\big) = \cos(2\pi\tau)\ \hat{\boldsymbol{\imath}} - \sin(2\pi\tau)\ \hat{\boldsymbol{\jmath}} \quad (0 \le \tau \le 1).$$

Chain Rule

Theorem 12.3.2

Let $\underline{\boldsymbol{r}}(t)$ be a vector-valued function that is differentiable with respect to t. If $t = g(\tau)$ is a change of parameter in which g is differentiable with respect to τ, then $\underline{\boldsymbol{r}}\big(g(\tau)\big)$ is differentiable with respect to τ and

$$\frac{d\underline{\boldsymbol{r}}}{d\tau} = \frac{d\underline{\boldsymbol{r}}}{dt}\frac{dt}{d\tau}$$

Smooth Change of Parameter

A change of parameter $t = g(\tau)$ in which $\underline{\boldsymbol{r}}(g(\tau))$ is smooth if $\underline{\boldsymbol{r}}(t)$ is smooth is called a smooth change of parameter.

It follows from (9) that $t = g(\tau)$ will be a smooth change of parameter if $\frac{dt}{d\tau}$ is continuous and $\frac{dt}{d\tau} \neq 0$ for all values of τ, since these conditions imply that $\frac{d\underline{\boldsymbol{r}}}{d\tau}$ is continuous and nonzero if $\frac{d\underline{\boldsymbol{r}}}{dt}$ is continuous and nonzero.

Smooth Change of Parameter

Smooth changes of parameter fall into two categories—

those for which $dt/d\tau > 0$ for all τ (called positive changes of parameter) and

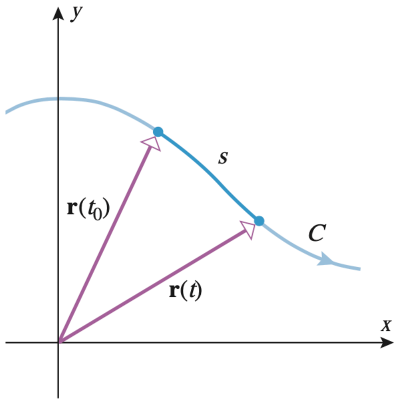
those for which $dt/d\tau < 0$ for all τ (called negative changes of parameter).

A positive change of parameter preserves the orientation of a parametric curve, and a negative change of parameter reverses it.

Finding Arc Length Parametrizations

Theorem 12.3.3

Let C be the graph of a smooth vector-valued function $\underline{\boldsymbol{r}}(t)$, and let $\underline{\boldsymbol{r}}(t_0)$ be any point on C. Then the following formula defines a positive change of parameter from t to s, where s is an arc length parameter having $\underline{\boldsymbol{r}}(t_0)$ as its reference point



$$s = \int_{t_0}^{t} \left\| \frac{d\underline{\boldsymbol{r}}}{du} \right\| du$$

Proof

Homework (p863).

Finding Arc Length Parametrizations

Example 6 (p863)

Find the arc length parametrization of the circular helix

$$\underline{\boldsymbol{r}}(t) = \cos t\ \hat{\boldsymbol{\imath}} + \sin t\ \hat{\boldsymbol{\jmath}} + t\,\hat{\boldsymbol{k}}$$

that has reference point $\underline{\boldsymbol{r}}(0) = (1, 0, 0)$ and the same orientation as the given helix.

Finding Arc Length Parametrizations

Solution (Example 6)

Replacing t by u in $\underline{\boldsymbol{r}}$ for integration purpose, we have

$$\underline{\boldsymbol{r}}(t) = \cos u\ \hat{\boldsymbol{\imath}} + \sin u\ \hat{\boldsymbol{\jmath}} + u\,\widehat{\boldsymbol{k}}$$

$$\therefore \left\|\frac{d\underline{\boldsymbol{r}}}{du}\right\| = \sqrt{(-\sin u)^2 + \cos^2 u + 1} = \sqrt{2}$$

Finding Arc Length Parametrizations

Solution (Example 6)

Now taking $t_0 = 0$

$$s = \int_0^t \left\|\frac{d\underline{\boldsymbol{r}}}{du}\right\| du = \int_0^t \sqrt{2}\, du = \sqrt{2}u\Big]_0^t = \sqrt{2}t$$

Thus, $t = s/\sqrt{2}$, so the given equation can be reparametrized in terms of s as

$$\underline{\boldsymbol{r}} = \cos\left(\frac{s}{\sqrt{2}}\right)\ \hat{\boldsymbol{\imath}} + \sin\left(\frac{s}{\sqrt{2}}\right)\ \hat{\boldsymbol{\jmath}} + \left(\frac{s}{\sqrt{2}}\right)\widehat{\boldsymbol{k}}$$

Finding Arc Length Parametrizations

Example 7 (p863)

A bug walks along the trunk of a tree following a path modeled by

$$\underline{\boldsymbol{r}}(t) = \cos t\ \hat{\boldsymbol{\imath}} + \sin t\ \hat{\boldsymbol{\jmath}} + t\,\widehat{\boldsymbol{k}}.$$

The bug starts at the reference point (1, 0, 0) and walks up the helix for a distance of 10 units. What are the bug's final coordinates?

Finding Arc Length Parametrizations

Solution (Example 7)

Replacing t by u in $\underline{\boldsymbol{r}}$ for integration purpose, we have

$$\underline{\boldsymbol{r}}(t) = \cos u\ \hat{\boldsymbol{\imath}} + \sin u\ \hat{\boldsymbol{\jmath}} + u\,\widehat{\boldsymbol{k}}$$

$$\therefore \left\|\frac{d\underline{\boldsymbol{r}}}{du}\right\| = \sqrt{(-\sin u)^2 + \cos^2 u + 1} = \sqrt{2}$$

Finding Arc Length Parametrizations

Solution (Example 6)

Now taking $t_0 = 0$

$$s = \int_0^t \left\|\frac{d\boldsymbol{r}}{du}\right\| du = \int_0^t \sqrt{2}\, du = \sqrt{2}u\Big]_0^t = \sqrt{2}t$$

Thus, $t = s/\sqrt{2}$, so the given equation can be reparametrized in terms of s as

$$\underline{\boldsymbol{r}} = \cos\left(\frac{s}{\sqrt{2}}\right)\hat{\boldsymbol{\imath}} + \sin\left(\frac{s}{\sqrt{2}}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{s}{\sqrt{2}}\right)\hat{\boldsymbol{k}}$$

Finding Arc Length Parametrizations

Solution (Example 6)

Parametrically,

$$x = \cos\left(\frac{s}{\sqrt{2}}\right), \qquad y = \sin\left(\frac{s}{\sqrt{2}}\right), \qquad z = \frac{s}{\sqrt{2}}$$

Thus, at $s = 10$ the coordinates are

$$(x, y, z) = \left(\cos\left(\frac{10}{\sqrt{2}}\right), \sin\left(\frac{10}{\sqrt{2}}\right), \frac{10}{\sqrt{2}}\right) \approx (0.705, 0.709, 7.07)$$

Finding Arc Length Parametrizations

Homework

Example 8 – 9 (p864)

Chapter 12.3

Homework

Exercise Set 12.3 (p866 –868)