Lecture 7
Solving System of Non-Linear Equation using Newton-Raphson Method

DESIRED LEARNING OBJECTIVES : AFTER THIS LESSON STUDENTS WILL BE ABLE TO

- Solve system of non-linear equations using Newton Raphson method
- Establish an algorithm to implement Newton Raphson method

NEWTON'S METHOD (REVIEW)

Assumptions: $f(x)$, $f'(x)$, x_0 *are available,*

$$f'(x_0) \neq 0$$

Newton's Method new estimate:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Problem :

$f'(x_i)$ is not available,

or difficult to obtain analytically.

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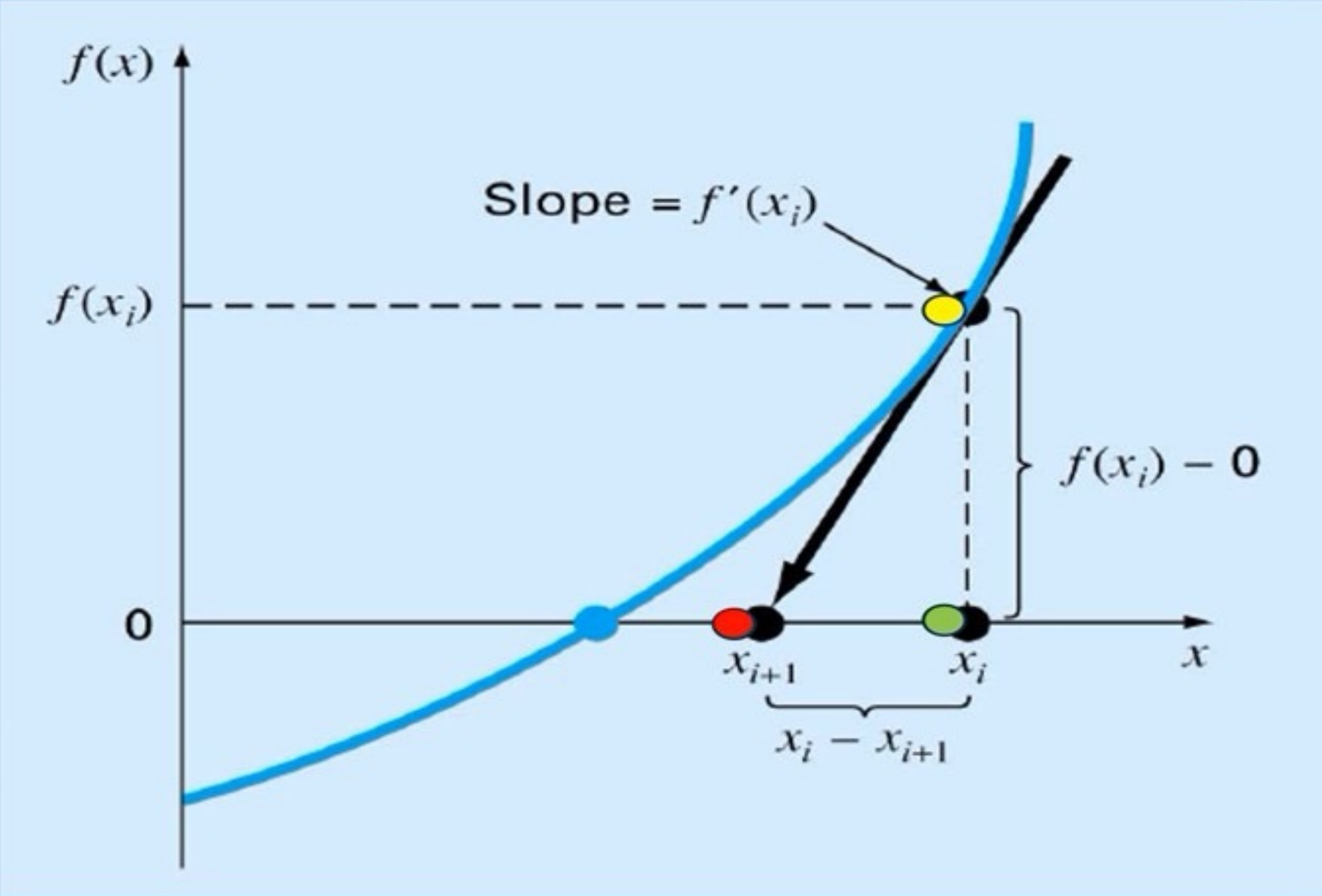
NEWTON RAPHSON METHOD - GRAPHICAL DEPICTION

Initial guess x_i, then the tangent $f'(x_i)$ is extended to find the estimate of the root at x_{i+1}



DERIVATION OF NEWTON'S METHOD

Given : x_i an initial guess of the root of $f(x) = 0$

Question : How do we obtain a better estimate x_{i+1}?

Taylor Therorem : $f(x+h) \approx f(x) + f'(x)h$

Find h such that $f(x+h) = 0$.

$$\Rightarrow h \approx -\frac{f(x)}{f'(x)}$$

Newton – Raphson Formula

A new guess of the root : $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

NEWTON'S METHOD

- Tangent line function

$$f(x) = f(x_0) + (x - x_0) f'(x_0)$$

- Iteration form

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- Tangent plane function

$$f(x, y) = f(x_0, y_0) + (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0)$$

$$g(x, y) = g(x_0, y_0) + (x - x_0) g_x(x_0, y_0) + (y - y_0) g_y(x_0, y_0)$$

- Iteration using r, s

$$x - x_0 = r$$

$$y - y_0 = s$$

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NEWTON'S METHOD (CONT')

$$f(x,y) = f(x_0,y_0) + (x - x_0) f_x(x_0,y_0) + (y - y_0) f_y(x_0,y_0)$$
$$g(x,y) = g(x_0,y_0) + (x - x_0) g_x(x_0,y_0) + (y - y_0) g_y(x_0,y_0)$$

$$x - x_0 = r$$
$$y - y_0 = s$$

- Step1

$$r\, f_x(x_0,y_0) + s\, f_y(x_0,y_0) = -f(x_0,y_0)$$
$$r\, g_x(x_0,y_0) + s\, g_y(x_0,y_0) = -g(x_0,y_0)$$

- Step2

$$x_1 = x_0 + r$$
$$y_1 = y_0 + s$$

MATRIX-VECTOR NOTATION

- Step1

$$\begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} -f(x_0, y_0) \\ -g(x_0, y_0) \end{bmatrix}$$

$$F'(x_{old})y = -F(x_{old}), \ (where \ \ y = x_{new} - x_{old})$$

- Step2

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} r \\ s \end{bmatrix}$$

$$x_{new} = x_{old} + y$$

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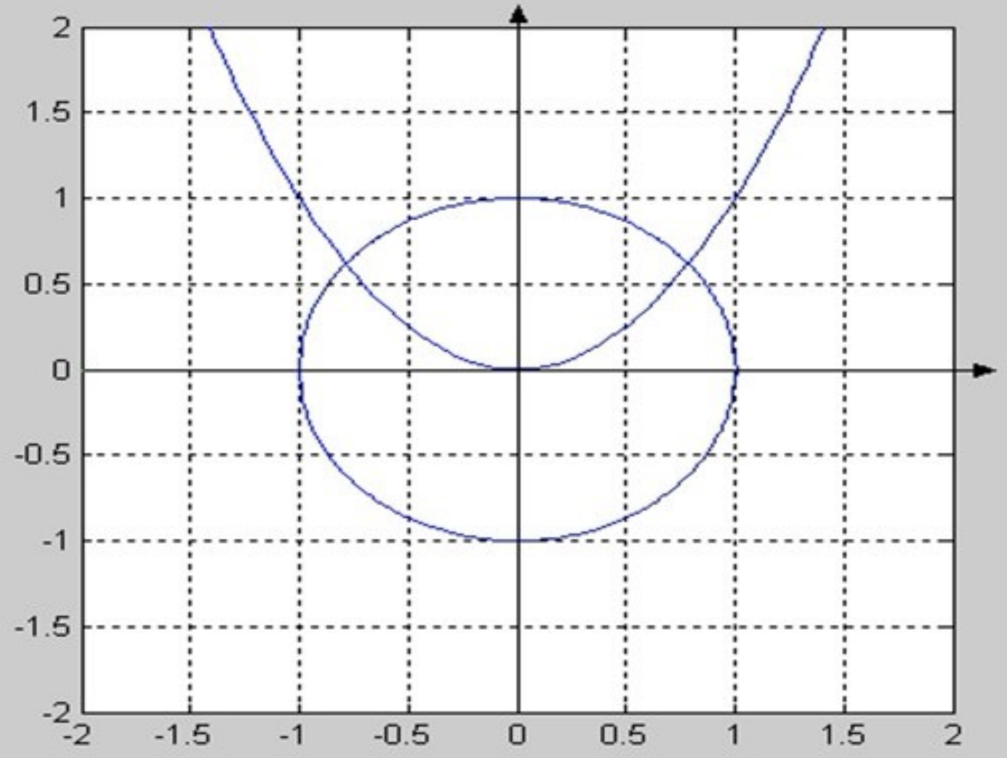
EXAMPLE

INTERSECTION OF A CIRCLE AND A PARABOLA



$$f(x, y) = \quad x^2 + y^2 - 1$$

$$g(x, y) = \quad x^2 - y$$

(Circle)

(Parabola)

EXAMPLE:
INTERSECTION OF A CIRCLE AND A PARABOLA

$$\begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} -f(x_0, y_0) \\ -g(x_0, y_0) \end{bmatrix}$$

- Step1 $$\begin{bmatrix} 2x & 2y \\ 2x & -1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} -f(x_0, y_0) \\ -g(x_0, y_0) \end{bmatrix}$$

- Initial estimate $(x_0, y_0) = (1/2, 1/2)$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix}$$

$$r = 3/8, \; s = 1/8$$

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EXAMPLE:
INTERSECTION OF A CIRCLE AND A PARABOLA

- Step2

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} r \\ s \end{bmatrix}$$

- The second approximate solution (x_1, y_1)

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 3/8 \\ 1/8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 7/8 \\ 5/8 \end{bmatrix}$$

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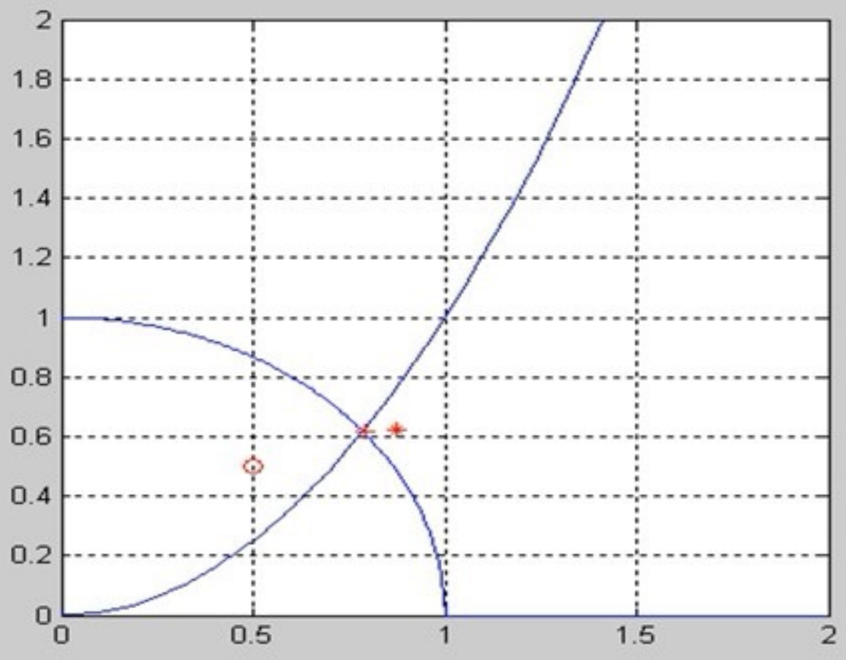
INITIAL ESTIMATE AND 2 ITERATIONS OF NEWTON'S METHOD



itr	x	y	|Δ|
0	0.5	0.5	
1	0.875	0.625	0.39528
2	0.79067	0.61806	0.084611

The true solution is
(x, y) = (0.78615, 0.61803)

NEWTON'S METHOD FOR SYSTEM OF NON-LINEAR EQUATIONS

$$Given : X_0 \text{ an initial guess of the root of } F(x) = 0$$

$$Newton\text{'s } Iteration$$

$$X_{k+1} = X_k - \left[F'(X_k)\right]^{-1} F(X_k)$$

$$F(X) = \begin{bmatrix} f_1(x_1, x_2, \ldots) \\ f_2(x_1, x_2, \ldots) \\ \vdots \end{bmatrix}, \quad F'(X) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \vdots \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \\ \vdots & & \end{bmatrix}$$

EXAMPLE

SOLVE THE FOLLOWING SYSTEM OF EQUATIONS

$$y + x^2 - x - 0.5 = 0$$

$$x^2 - 5xy - y = 0$$

Initial guess $x = 1, \quad y = 0$

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EXAMPLE

SOLVE THE FOLLOWING SYSTEM OF EQUATIONS

$$y + x^2 - x - 0.5 = 0$$

$$x^2 - 5xy - y = 0$$

$$\text{Initial guess } x = 1, \quad y = 0$$

$$F = \begin{bmatrix} y + x^2 - 0.5 - x \\ x^2 - 5xy - y \end{bmatrix}, \quad F' = \begin{bmatrix} 2x - 1 & 1 \\ 2x - 5y & -5x - 1 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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SOLUTION USING NEWTON'S METHOD

Iteration1

$$F = \begin{bmatrix} y + x^2 - 0.5 - x \\ x^2 - 5xy - y \end{bmatrix} = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$$

$$F' = \begin{bmatrix} 2x - 1 & 1 \\ 2x - 5y & -5x - 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix}^{-1} \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1.25 \\ 0.25 \end{bmatrix}$$

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SOLUTION USING NEWTON'S METHOD

Iteration 2:

$$F = \begin{bmatrix} 0.0625 \\ -0.25 \end{bmatrix}, \; F' = \begin{bmatrix} 1.5 & 1 \\ 1.25 & -7.25 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1.25 \\ 0.25 \end{bmatrix} - \begin{bmatrix} 1.5 & 1 \\ 1.25 & -7.25 \end{bmatrix}^{-1} \begin{bmatrix} 0.0625 \\ -0.25 \end{bmatrix} = \begin{bmatrix} 1.2332 \\ 0.2126 \end{bmatrix}$$

EXAMPLE

SOLVE THE FOLLOWING SYSTEM OF EQUATIONS

$$y + x^2 - 1 - x = 0$$

$$x^2 - 2y^2 - y = 0$$

$$\text{Initial guess } \; x = 0, \;\; y = 0$$

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EXAMPLE

SOLVE THE FOLLOWING SYSTEM OF EQUATIONS

$$y + x^2 - 1 - x = 0$$

$$x^2 - 2y^2 - y = 0$$

Initial guess $x = 0, \quad y = 0$

$$F = \begin{bmatrix} y + x^2 - 1 - x \\ x^2 - 2y^2 - y \end{bmatrix}, \quad F = \begin{bmatrix} 2x - 1 & 1 \\ 2x & -4y - 1 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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EXAMPLE
SOLUTION

Iteration	0	1	2	3	4	5
X_k	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}-1\\0\end{bmatrix}$	$\begin{bmatrix}-0.6\\0.2\end{bmatrix}$	$\begin{bmatrix}-0.5287\\0.1969\end{bmatrix}$	$\begin{bmatrix}-0.5257\\0.1980\end{bmatrix}$	$\begin{bmatrix}-0.5257\\0.1980\end{bmatrix}$

EXAMPLE

Estimates of the root of: $x-\cos(x)=0.$

0.60000000000000		Initial guess
0.744017319445 98	1	correct digit
0.73909047688624	4	correct digits
0.73908513322147	10	correct digits
0.73908513321516	14	correct digits

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EXAMPLE

In estimating the root of: ***x-cos(x)=0***, to get more than 13 correct digits:

- 4 iterations of Newton (x_0=0.8)
- 43 iterations of Bisection method
- (initial interval [0.6, 0.8])

Example:

Solve the system of non-linear equations

$$x_1^2 + x_2^2 + x_3^2 - 1 = 0$$

$$x_1^2 \qquad + x_3^2 = \frac{1}{4}$$

$$x_1^2 + x_2^2 - 4x_3 = 0$$

$$f_1(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 1$$

$$f_2(x_1, x_2, x_3) = x_1^2 \qquad + x_3^2 - \frac{1}{4}$$

$$f_3(x_1, x_2, x_3) = x_1^2 + x_2^2 - 4x_3$$

Example

$$F'(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \\ 2x_1 & 0 & 2x_3 \\ 2x_1 & 2x_2 & -4 \end{bmatrix}$$

Make an initial guess:

$$X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$F(X_0) = \begin{bmatrix} 2 \\ 1.75 \\ -2 \end{bmatrix}$$

$$F'(X_0) = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & -4 \end{bmatrix}$$

$$X_1 = X_0 - \left[F'(X_0)\right]^{-1} F\left[X_0\right]$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1.75 \\ -2 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 0.79167 \\ 0.87500 \\ 0.33333 \end{bmatrix}$$

$$X_2 = X_1 - \left[F'(X_1)\right]^{-1} F\left[X_1\right]$$

$$F(X_1) = \begin{bmatrix} 0.50348 \\ 0.48785 \\ 0.05905 \end{bmatrix}$$

$$F'(X_1) = \begin{bmatrix} 1.58334 & 1.75000 & 0.66666 \\ 1.58334 & 0 & 0.66666 \\ 1.58334 & 1.75000 & -4 \end{bmatrix}$$

$$X_2 = X_1 - \left[F'(X_1)\right]^{-1} F\left[X_1\right]$$

$$X_2 = \begin{bmatrix} 0.79167 \\ 0.87500 \\ 0.33333 \end{bmatrix} - \begin{bmatrix} 1.58334 & 1.75000 & 0.66666 \\ 1.58334 & 0 & 0.66666 \\ 1.58334 & 1.75000 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 0.50348 \\ 0.48785 \\ 0.05905 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 0.52365 \\ 0.86607 \\ 0.23810 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 0.44733 \\ 0.86603 \\ 0.23607 \end{bmatrix}$$

$$X_4 = \begin{bmatrix} 0.44081 \\ 0.86603 \\ 0.23607 \end{bmatrix}$$

EXAMPLE

A SYSTEM OF THREE EQUATIONS

$$\begin{aligned} f_1(x_1,x_2,x_3) &= x_1^2 + x_2^2 + x_3^2 - 1 = 0, \\ f_2(x_1,x_2,x_3) &= x_1^2 + x_3^2 - 1/4 = 0, \\ f_3(x_1,x_2,x_3) &= x_1^2 + x_2^2 - 4x_3 = 0 \end{aligned}$$

$$F(x) = \begin{bmatrix} x_1^2 + x_2^2 + x_3^2 - 1 \\ x_1^2 + x_3^2 - 1/4 \\ x_1^2 + x_2^2 - 4x_3 \end{bmatrix}$$

$$F'(x) = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \\ 2x_1 & 0 & 2x_3 \\ 2x_1 & 2x_2 & 4 \end{bmatrix}$$

```
>> Newton_sys('three_equation','three_equation_j',[1 1 1],0.00001,10);
    0     1     1     1

    1.0000    0.7917    0.8750    0.3333    0.7096

    2.0000    0.5237    0.8661    0.2381    0.2846

    3.0000    0.4473    0.8660    0.2361    0.0764

    4.0000    0.4408    0.8660    0.2361    0.0065

    5.0000    0.4408    0.8660    0.2361    0.0000

    6.0000    0.4408    0.8660    0.2361    0.0000

Newton method has converged
```

Thank you