Lecture 4
Roots of Non-Linear Equation Using Bisection Method

Objectives

- Solve an algebraic or transcendental equation using Bisection method

- Establish an algorithm to implement Bisection method

Roots of an Equation

- The **root of an equation** is a value of the variable that makes the equation **true**

- For the equation

$$x^2 - 4 = 0$$

- If we solve it

$$x^2 = 4 \;=> x = \pm 2$$

So, the roots are **x = 2** and **x = -2**

Types of Equation

- An equation $f(x) = 0$ belong to one of the following types
  - Algebraic equations
  - Polynomial equations
  - Transcendental equations

Types of Equation

- Algebraic equations
  - General syntax $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$
  - Examples:
    - Linear (n = 1):

$$a_1 x + a_0 = 0$$

    - Quadratic (n = 2):

$$a_2 x^2 + a_1 x + a_0 = 0$$

    - Cubic (n = 3):

$$a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0$$

    - Quartic (n = 4):

$$a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0$$

- Polynomial equations
  - General syntax

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0$$

  - Examples: $2x^3 - 3x^2 + 4x - 5 = 0$

Types of Equation

- Transcendental equations
  - A transcendental equation is **not algebraic**, meaning it **cannot** be written in the form of a polynomial equation
  - Examples:

$e^x = 5$ → Exponential equation

$\log(x) + x = 3$ → Logarithmic + linear

$\sin(x) = x - 1$ → Trigonometric + linear

$x^2 = \ln(x)$ → Polynomial = logarithmic

$\cos(x) = x$ → Trigonometric equation

$x^x = 10$ → Power with variable exponent

Methods of Solution

- There are number of ways to find the roots on non-linear equations
  - Direct analytical methods
  - Graphical methods
  - Trial and error methods
  - Iterative methods

Methods of Solution

- There are number of ways to find the roots on non-linear equations
  - Direct analytical methods
    - $f(x) = \quad ax^2 + bx + c = 0 \quad \text{where } a \neq 0$
    - Root: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
    - How about $2\sin(x) - x = 0$

Methods of Solution

- There are number of ways to find the roots on non-linear equations
    - Graphical methods

      The idea is to plot the **graph** of the function $f(x) = 0$ (or rearranged as $f(x) = g(x)$) and visually identify the **points of intersection** with the **x-axis**. These intersections represent the approximate **roots** of the equation.

Time-Consuming, Lack of precision, difficult for complex equations

Methods of Solution

- There are number of ways to find the roots on non-linear equations

  - Trial and error methods

    In this method, you **guess** values for x and **substitute** them into the equation to check if they satisfy the equation $f(x) = 0$.

    You start by choosing a range of values, test them, and **refine your guesses** based on the results.

Time-Consuming, Lack of precision, difficult for complex equations

Methods of Solution

- There are number of ways to find the roots on non-linear equations
    - Iterative methods
        - With the advent of computers, algorithmic approaches known as methods ***iterative*** have become popular
        - These methods start with an **initial guess** and iteratively refine the solution until it converges to a desired level of accuracy.

Iterative methods

- Based on the number of guesses they use, can he grouped into two categories:
  - Bracketing methods
  - Open end methods

Iterative methods

- Bracketing methods
  - Enclose the root within a specific interval $[a,b]$
  - Iteratively reduces the interval size until the root is found with sufficient accuracy
  - Examples:
    - Bisection Method
    - False Position

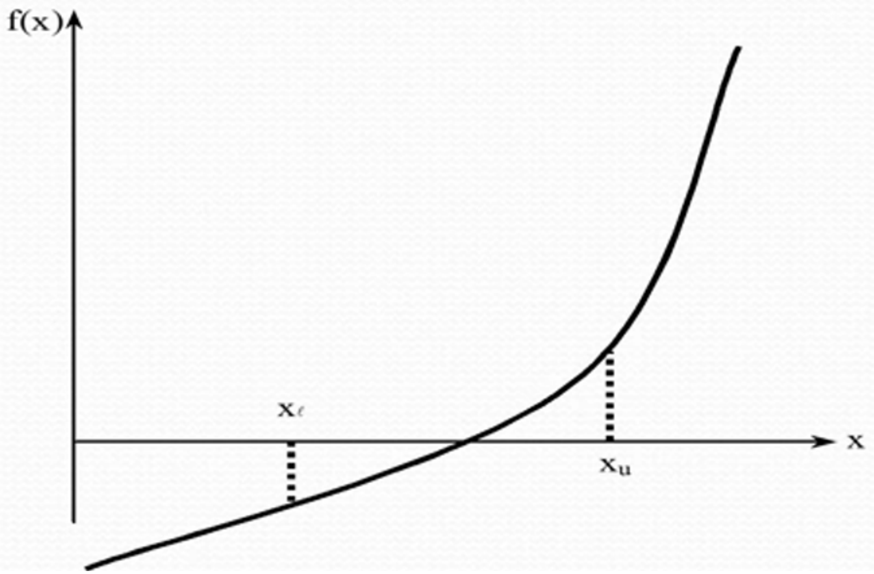
Iterative methods

- Open end methods
  - Used when we have an initial guess or approximation for the root but do not necessarily know the bounds
  - When the function may not have a sign change over an interval
  - Examples:
    - Newton-Raphson Method
    - Secant Method
    - Fixed-Point Iteration Method
    - Muller's Method

Bisection Method

Basis of Bisection Method

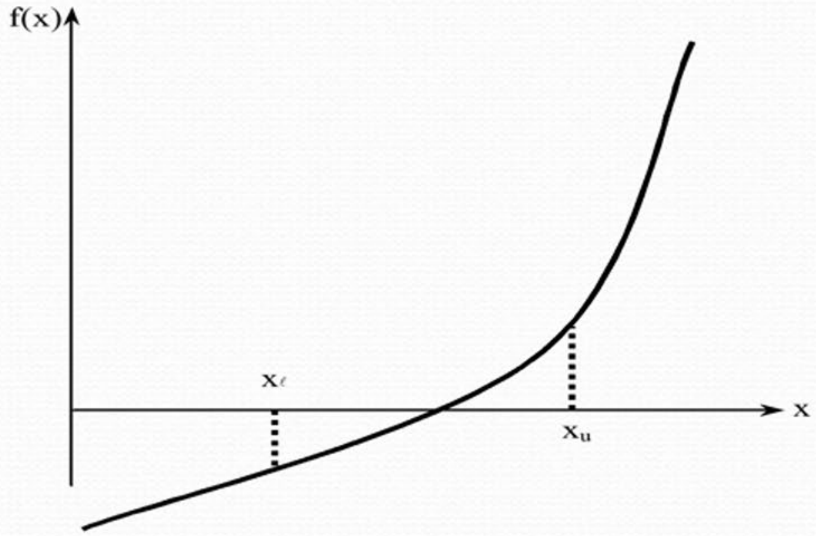
An equation $f(x)=0$, where $f(x)$ is a real continuous function, has at least one root between x_l and x_u if $f(x_l)\ f(x_u) < 0$.



Algorithm for Bisection Method

Step 1

- Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)\, f(x_u) < 0$, or in other words, $f(x)$ changes sign between x_ℓ and x_u.

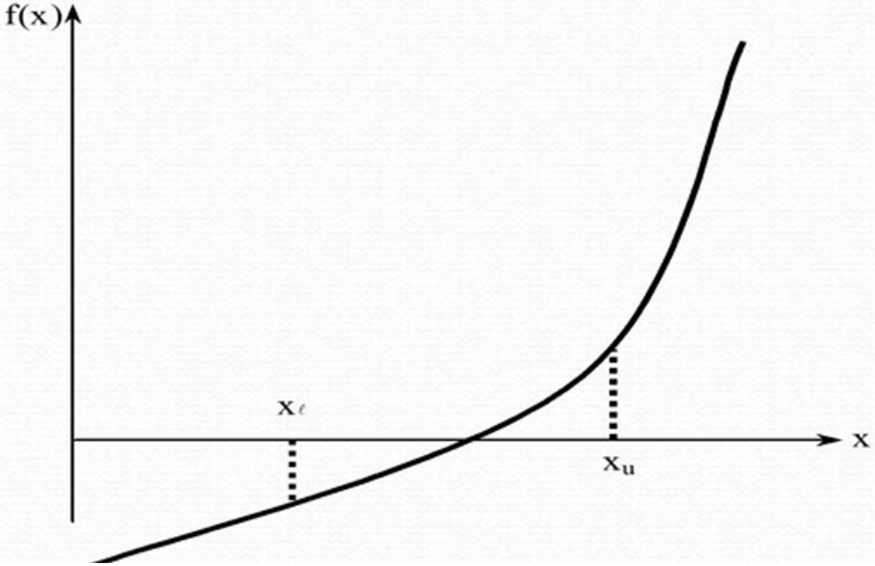


Algorithm for Bisection Method

Step 2

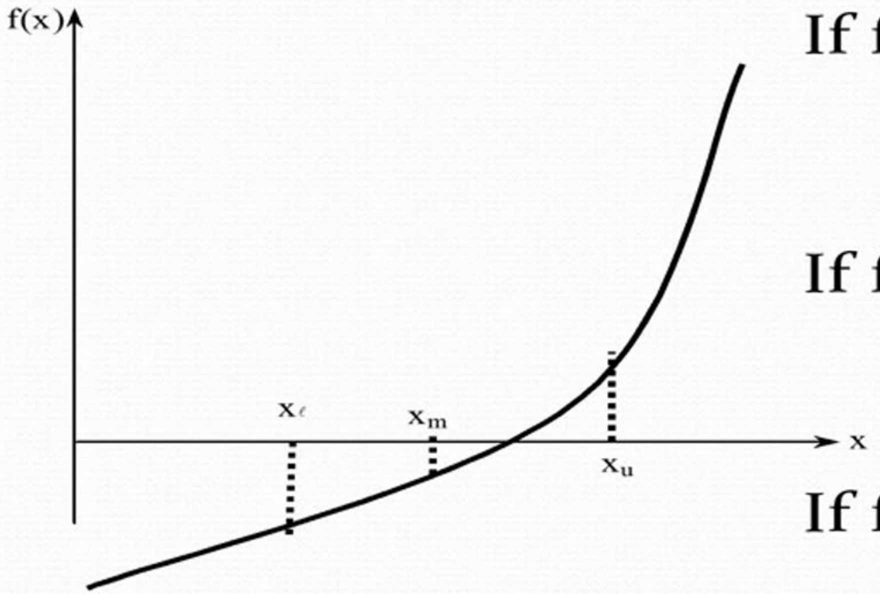
Estimate the root, x_m of the equation $f(x) = 0$ as the mid-point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$



Algorithm for Bisection Method

Step 3



Now check the following

If $f(x_\ell)\, f(x_m) < 0$, then the root lies between x_ℓ and x_m; then $x_\ell = x_\ell$; $x_u = x_m$.

If $f(x_\ell)\, f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\ell = x_m$; $x_u = x_u$.

If $f(x_\ell)\, f(x_m) = 0$; then the root is x_m. Stop the algorithm if this is true.

Algorithm for Bisection Method

Step 4

New estimate

$$x_m = \frac{x_\ell + x_u}{2}$$

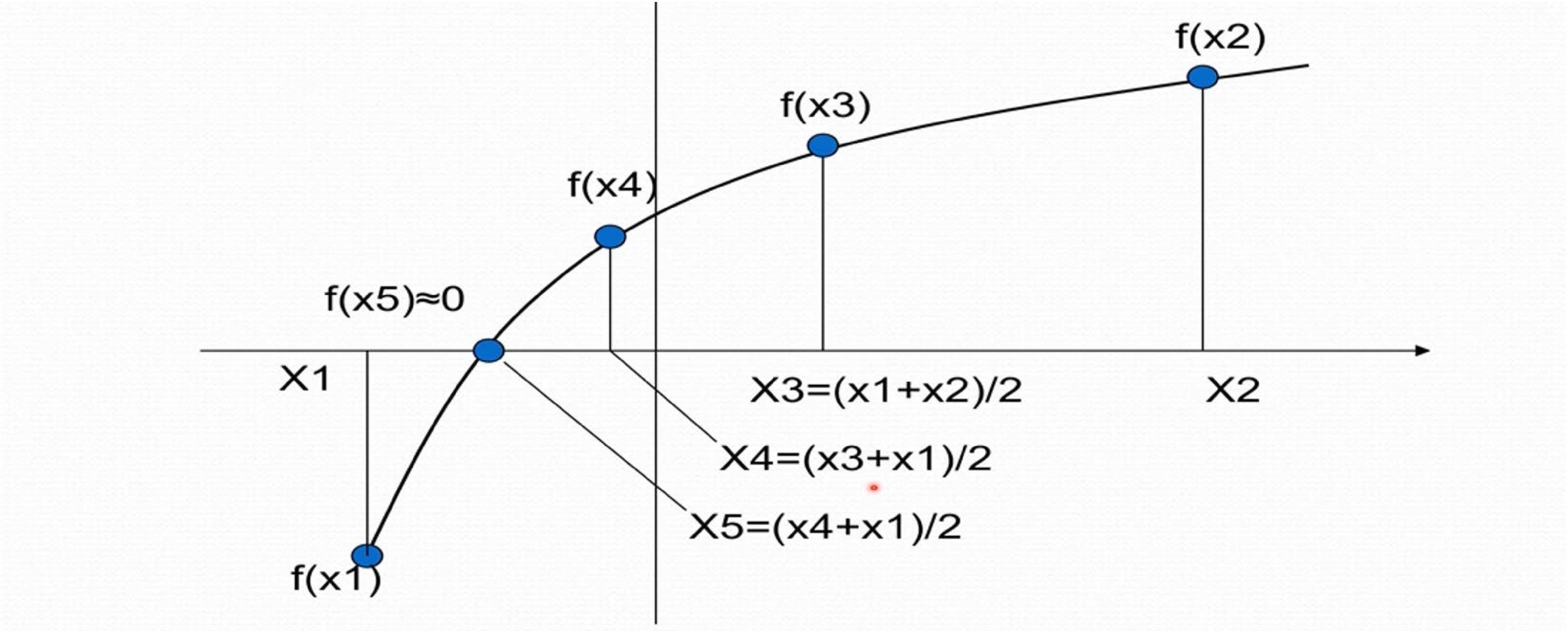
Absolute Relative Approximate Error

$$|\in_a| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100$$

$$x_m^{old} = \text{previous estimate of root}$$

$$x_m^{new} = \text{current estimate of root}$$

Example



- Cauchy's Bound

Let a_n be the leading coefficient (assumed ≠ 0), and all coefficients are real.

The absolute value of **all real roots** of $f(x) = 0$ is less than or equal to:

$$R = 1 + \frac{\max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}}{|a_n|}$$

This means all real roots lie in:

$$x \in [-R, R]$$

- Cauchy's Bound
  - Assume a function $f(x) = 2x^3 - 4x^2 + 5x - 10$
    - $a_n = 2$
    - $\max\{|a_0|, |a_1|, |a_2|\} = \max\{10, 5, 4\} = 10$

Then:

$$R = 1 + \frac{10}{2} = 6$$

Task

Find a root of $f(x) = x^3 - 3x + 1$,

In the interval [0,1]

f(x) is continuous

$f(0) = 1$, $f(1) = -1 \implies f(0)f(1) < 0$

We can use Bisection Method

Bisection Method – Example

- $f(x) = x^3 - 3x + 1$

- Search bracket
  - By applying Cauchy's boundary

$$\max\{1, 3, 0\} = 3$$

$$R = 1 + \frac{3}{1} = 4$$

  - Bracket = [-4,4]

Bisection Method – Example

- $f(x) = x^3 - 3x + 1$

- Check for Sign Change
  - By applying Cauchy's boundary

$$f(-4) = (-4)^3 - 3(-4) + 1 = -64 + 12 + 1 = -51$$

$$f(4) = (4)^3 - 3(4) + 1 = 64 - 12 + 1 = 53$$

  - Sign changes. So, at least one root exists in $[-4,4]$

Bisection Method – Example

- $f(x) = x^3 - 3x + 1$

Let the interval be:

$$a = -4, \quad b = 4$$

We compute midpoint:

$$c = \frac{a+b}{2}$$

And check sign of $f(c)$ to decide the new interval.

Bisection Method – $f(x) = x^3 - 3x + 1$

Iter	a	b	c (midpoint)	$f(c)$	Sign of $f(c)$	New Interval
1	-4	4	0.0	f(0)=1f(0) = 1f(0)=1	+	[−4, 0]
2	-4	0	-2.0	f(−2)=−1f(-2) = -1f(−2)=−1	−	[−2, 0]
3	-2	0	-1.0	f(−1)=3f(-1) = 3f(−1)=3	+	[−2, −1]
4	-2	-1	-1.5	f≈2.125f ≈ 2.125f≈2.125	+	[−2, −1.5]
5	-2	-1.5	-1.75	f≈0.89f ≈ 0.89f≈0.89	+	[−2, −1.75]
6	-2	-1.75	-1.875	f≈0.035f ≈ 0.035f≈0.035	+	[−2, −1.875]
7	-2	-1.875	-1.9375	f≈−0.457f ≈ -0.457f≈−0.457	−	[−1.9375, −1.875]
8	-1.9375	-1.875	-1.90625	f≈−0.21f ≈ -0.21f≈−0.21	−	[−1.90625, −1.875]
9	-1.90625	-1.875	-1.890625	f≈−0.088f ≈ -0.088f≈−0.088	−	[−1.890625, −1.875]
10	-1.890625	-1.875	-1.8828	f≈−0.026f ≈ -0.026f≈−0.026	−	[−1.8828, −1.875]
11	-1.8828	-1.875	-1.8789	f≈0.004f ≈ 0.004f≈0.004	+	[−1.8828, −1.8789]
12	-1.8828	-1.8789	-1.8809	f≈−0.011f ≈ -0.011f≈−0.011	−	[−1.8809, −1.8789]
13	-1.8809	-1.8789	-1.8799	f≈−0.0036f ≈ -0.0036f≈−0.0036	−	[−1.8799, −1.8789]
14	-1.8799	-1.8789	-1.8794	f≈0.0002f ≈ 0.0002f≈0.0002	+	[−1.8799, −1.8794]
15	-1.8799	-1.8794	-1.8796	f≈−0.0017f ≈ -0.0017f≈−0.0017	−	[−1.8796, −1.8794]
16	-1.8796	-1.8794	-1.8795	f≈−0.0007f ≈ -0.0007f≈−0.0007	−	[−1.8795, −1.8794]
17	-1.8795	-1.8794	-1.87945	f≈−0.0002f ≈ -0.0002f≈−0.0002	−	[−1.87945, −1.8794]
18	-1.87945	-1.8794	-1.87943	f≈−0.00001f ≈ -0.00001f≈−0.00001	-	Stop — desired precision

Thank you

Question and Suggestion