Limit

Limit of a function:

A function $f(x)$ is said to have limit l as x approaches a, written $\lim_{x\to a} f(x) = l$, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x-a| < \delta \Rightarrow |f(x) - l| < \varepsilon$.

Right hand Limit (R.H.L) (one sided limits)

A function $f(x)$ is said to have right hand limit l_1 at $x = a$ written $\lim_{x\to a^+} f(x) = l_1$, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $a < x < a+\delta \Rightarrow |f(x) - l_1| < \varepsilon$

Left hand limit (L.H.L)

A function $f(x)$ is said to have left hand limit l_2 at $x = a$ written $\lim_{x\to a^-} f(x) = l_2$, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $a-\delta < x < a$ $\Rightarrow |f(x) - l_2| < \varepsilon$.

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If the limit of a function $f(x)$ i.e $\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exists and

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x).$$

or

$$L.H.L = R.H.L \Leftrightarrow \text{the limit of a function exists.}$$

Qus: What are the difference between $\lim_{x\to a} f(x)$ and $f(a)$:

Sol^n $f(a)$ means that the value of $f(x)$ when $x = a$. and $\lim_{x\to a} f(x)$ is a statement about the values of $f(x)$ when x assumes all values of x in the neighbourhood of a except $x = a$.

** when $\lim_{x\to a} f(x) = f(a)$, the function $f(x)$ is said to be continuous at $x = a$.

Def^n: Let $f(x)$ be defined for all x in some open interval containing the number a, with the possible exception that $f(x)$ need not be defined at a. We will write $\lim_{x\to a} f(x) = l$.

Summary on the fundamental Theorem on limits: (limit laws)

If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} \phi(x) = m$, where l and m are finite quantities then

i) $\lim_{x\to a} \{f(x) \pm \phi(x)\} = l \pm m$

ii) $\lim_{x\to a} \{f(x)\,\phi(x)\} = lm$

iii) $\lim_{x\to a} \left\{\frac{f(x)}{\phi(x)}\right\} = \frac{l}{m}$, if $\lim_{x\to a} \phi(x) \neq 0$

iv) $\lim_{x\to a} \phi\{f(x)\} = \phi\left(\lim_{x\to a} f(x)\right)$

v) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{l}$ provided $l > 0$ if n is even.

Meaning of $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} f(x) = -\infty$.

(Infinite limits)

A function $f(x)$ is said to tend to ∞ when x approaches a, if for any ~~ass~~ preassigned positive number N, however large, we can determine another positive number δ such that $f(x) > N$ for all values of x satisfying the inequality

$0 < |x-a| \leq \delta$ with $x > a$. or $a < x \leq a+\delta$.

A function $f(x)$ is said to tend to $-\infty$ when x approaches a, if for any given positive number N, however large, we can determine a positive number δ such that

$-f(x) \geq N$ or $f(x) \leq -N$ for all values of x satisfying the inequality $0 < x-a \leq \delta$.

Meaning of $\lim_{x\to\infty} f(x) = l$. (limits at infinity)

Let a be any positive numbers, that $f(x)$ is defined for all numbers $x \geq a$. We say that $f(x)$ approaches l as x tends to infinity, and we write $\lim_{x\to\infty} f(x) = l$.

If the following condition is satisfied, Given any $\varepsilon > 0$, there exists a positive number A, such that whenever $x > A$, we have

$|f(x) - l| < \varepsilon$.

Meaning of $\lim_{x\to-\infty} f(x) = l$. (limits at infinity)

For a given $\varepsilon > 0$, there exists a positive number A, such that whenever $x < -A$, we get $|f(x) - l| < \varepsilon$.

Find $\lim_{x\to 0} f(x)$ if $f(x) = \frac{|x|}{x}$

Solⁿ: we have $f(x) = \frac{|x|}{x}$

$$= \begin{cases} \frac{x}{x} & \text{if } x>0 \\ \frac{-x}{x} & \text{if } x<0 \end{cases}$$

$$= \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$

now $L.H.L = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-1) = -1$

$R.H.L = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1) = 1$.

Since ~~lim~~ $L.H.L \neq R.H.L$, therefore $\lim_{x\to 0} f(x)$ does not exist.

Ex: Find ~~limit~~ $\lim_{x\to a} f(x)$ if $f(x) = \frac{|x-a|}{x-a}$.

②

Solⁿ we have $f(x) = \frac{|x-a|}{x-a}$.

$$L.H.L = \lim_{x\to a^-} f(x) = \lim_{x\to a^-} \frac{|x-a|}{x-a}.$$

$$= \frac{-(x-a)}{x-a} = -1.$$

$$R.H.L = \lim_{x\to a^+} f(x) = \lim_{x\to a^+} \frac{x-a}{x-a} = 1.$$

Since $L.H.L \neq R.H.L$. therefore $\lim_{x\to a} f(x)$ does not exist.

Ex: If $f(x) = \frac{x-|x|}{x}$, then show that $\lim_{x\to 0} f(x)$ does not exist.

Solⁿ we have $f(x) = \frac{x-|x|}{x}$.

$$L.H.L = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x-|x|}{x} = \lim_{x\to 0^-} \frac{x-(-x)}{x}$$

$$= \lim_{x\to 0^-} \frac{2x}{x} = 2$$

$$R.H.L = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x-|x|}{x} = \lim_{x\to 0^+} \frac{x-x}{x} = 0,$$

Since L.H.L $\neq$ R.H.L.

So ~~the~~ $\lim_{x\to 0} f(x)$ does not exist.

Ex: $\lim_{x\to 0} \frac{x+|x|}{|x|}$ does not exist.

Ex: Show that $\lim_{x\to 0} \frac{3x}{|x|+x^2}$ does not exist.

Sol^n

We have $f(x) = \frac{3x}{|x|+x^2}$

L.H.L $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{3x}{|x|+x^2} = \lim_{x\to 0^-} \frac{3x}{-x+x^2} = \lim_{x\to 0^-} \frac{3}{-1+x}$

$= \frac{3}{-1} = -3$

R.H.L $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{3x}{|x|+x^2} = \lim_{x\to 0^+} \frac{3x}{x+x^2} = \lim_{x\to 0^+} \frac{3}{1+x} = 3$

Since L.H.L $\neq$ R.H.L, therefore $\lim_{x\to 0} \frac{3x}{|x|+x^2}$ does not exist.

Ex: Show that $\lim_{x\to 0} \frac{x}{|x|+x^2}$ does not exist.

Ex: If $f(x) = \begin{cases} \frac{|x-1|}{x-1} & \text{when } x \neq 1. \\ 1 & \text{"} \quad x = 1. \end{cases}$

Show that. $\lim_{x\to 1} f(x)$ does not exist but $\lim_{x\to 2} f(x)$ and $\lim_{x\to -3} f(x)$ exist.

Sol^n For part L.H.L $= \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{-(x-1)}{x-1} = -1.$

R.H.L $= \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x-1}{x-1} = 1.$

Since L.H.L $\neq$ R.H.L. So $\lim_{x\to 1} f(x)$. does not exist.

Again. L.H.L $= \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x-1}{x-1} = 1.$

R.H.L $= \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x-1}{x-1} = 1.$

$\therefore \lim_{x\to 2} f(x)$ exist.

$L.H.L = \lim_{x\to 5^-} f(x) = \lim_{x\to -5^-} \frac{|x-1|}{x-1} = \lim_{x\to 5^-} \frac{-(x-1)}{x-1} = -1$

$R.H.L = \lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} \frac{-(x-1)}{x-1} = -1.$

Since $L.H.L = R.H.L$. So $\lim_{x\to -5} f(x)$ exists.

Ex: If the function $f(x) = \begin{cases} 1+2x & ; -\frac{1}{2} \le x < 0 \\ 1-2x & : 0 \le x < \frac{1}{2} \\ -1+2x & : x > \frac{1}{2}. \end{cases}$

Find out $\lim_{x\to 0} f(x)$ & $\lim_{x\to \frac{1}{2}} f(x)$?

Sol^n

For $x = 0$.

$L.H.L. = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1+2x) = 1.$

$R.H.L = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1-2x) = 1.$

Since $L.H.L = R.H.L.$ therefore the $\lim_{x\to 0} f(x)$ exist

and $\lim_{x\to 0} f(x) = 1.$ (i)

For $x = \frac{1}{2}$

$L.H.L. = \lim_{x\to \frac{1}{2}^-} f(x) = \lim_{x\to \frac{1}{2}^-} (1-2x) = 1-2\cdot\frac{1}{2} = 0$

$R.H.L = \lim_{x\to \frac{1}{2}^+} f(x) = \lim_{x\to \frac{1}{2}^+} (-1+2x) = -1+2\cdot\frac{1}{2} = 0$

$\therefore \lim_{x\to \frac{1}{2}} f(x) = 0.$ Ans.

Ex: If $f(x) = \begin{cases} (1+2x)^{\frac{1}{x}} & \text{when } x \neq 0. \\ e^2 & \quad " \quad x = 0. \end{cases}$

Determine $\lim_{x\to 0} f(x)$.

Sol^n For $x = 0$.

$L.H.L. = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1+2x)^{1/x} = \lim_{x\to 0^-} \left\{(1+2x)^{\frac{1}{2x}}\right\}^2$

$= e^2.$

$$R.H.L = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1+2x)^{1/x} = \lim_{x\to 0^+}\left\{(1+2x)^{\frac{1}{2x}}\right\}^2$$
$$= e^2$$

Since $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

Therefore $\lim_{x\to 0} f(x) = e^2$. An.

Ex: If $f(x) = \dfrac{1}{5+e^{\frac{1}{x-2}}}$, then $\lim_{x\to 2} f(x) = ?$

Sol^n
$$L.H.L = \lim_{x\to 2^-} \frac{1}{5+e^{\frac{1}{x-2}}} = \lim \frac{1}{5+e^{\left[\frac{1}{2^- -2}\right]}}$$
$$= \frac{1}{5+e^{-\frac{1}{0}}}$$
$$= \frac{1}{5+e^{-\infty}} = \frac{1}{5+\frac{1}{e^{\infty}}} = \frac{1}{5+\frac{1}{\infty}}$$
$$= \frac{1}{5+0}$$
$$= \frac{1}{5}$$

$$R.H.L = \lim_{x\to 2^+} \frac{1}{5+e^{\frac{1}{x-2}}} = \cancel{\lim}$$
$$= \frac{1}{5+e^{\frac{1}{2^+-2}}} = \frac{1}{5+e^{\frac{1}{0}}} = \frac{1}{5+e^{\infty}} = \frac{1}{5+\infty}$$
$$= \frac{1}{\infty} = 0.$$

Since $L.H.L \neq R.H.L$.

Therefore $\lim_{x\to 2} f(x)$ does not exist. An.

Ex: Discuss about ~~$\lim_{x\to 0} f(x)$~~ the existness of $\lim_{x\to 0} f(x)$

where $f(x) = \begin{cases} e^{-\frac{|x|}{2}}, & -1<x<0 \\ x^2, & 0\le x<2. \end{cases}$

Sol^n
$$L.H.L = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{-\frac{|x|}{2}} = \lim_{x\to 0^-} e^{-\frac{-x}{2}} = \lim_{x\to 0^-} e^{\frac{x}{2}}$$
$$= e^0 = 1.$$

$$R.H.L = \cancel{\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{-\frac{|x|}{2}} = \lim_{x\to 0^+} e^{-\frac{x}{2}}} =$$
$$= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0.$$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

...mit does not exist.

Ex: If a function is defined $f(x) = \begin{cases} x+1 & , x<3 \\ 5 & , x=3 \\ 2(5-x) & , x>3 \end{cases}$

Find $\lim_{x\to 3} f(x)$.

Sol^n:

$$L.H.L = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x+1) = 3+1 = 4$$

$$R.H.L = \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \{2(5-x)\} = 2(5-3) = 2\cdot 2 = 4$$

Since $L.H.L = R.H.L$.

So $\lim_{x\to 3} f(x) = 4$. **Ans.**

Ex: If $f(x) = |x|$ or $f(x) = \begin{cases} x & , x>0 \\ 0 & , x=0 \\ -x & , x<0 \end{cases}$, Find $\lim_{x\to 0} f(x)$.

Sol^n:

$$L.H.L. = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x) = 0$$

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$$R.H.L = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x) = 0.$$

$\therefore$ $L.H.L = R.H.L$.

So the $\lim_{x\to 0} f(x) = 0$. **Ans.**

Ex. If $f(x) = \dfrac{1+2^{1/x}}{3+2^{1/x}}$. Does $\lim_{x\to 0} f(x)$ exist?

Sol^n:

$$L.H.L = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1+2^{\frac{1}{x}}}{3+2^{1/x}}$$

$$= \lim_{x\to 0^-} \frac{1+2^{\frac{1}{0^-}}}{3+2^{\frac{1}{0^-}}}$$

$$= \frac{1+2^{-\infty}}{3+2^{-\infty}} = \frac{1+\frac{1}{2^\infty}}{3+\frac{1}{2^\infty}}$$

$$= \frac{1+0}{3+0} = \frac{1}{3}.$$

$$R.H.L = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1+2^{1/x}}{3+2^{1/x}} = \lim_{x\to 0} \frac{1+2^{\infty}}{3+2^{\infty}} = \frac{1+\infty}{3+\infty}$$
$$= \frac{\infty}{\infty}.$$

Since $L.H.L \neq R.H.L.$

Therefore $\lim_{x\to 0^-} f(x)$ does not exist.

Ex: Show that $\lim_{x\to 2} f(x)$ does not exist. where $f(x) = \begin{cases} \frac{5}{7-x}, & x<2 \\ 7-x, & x>2 \end{cases}$

Sol^n

$$L.H.L = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{5}{7-x} = \frac{5}{7-2} = \frac{5}{5} = 1$$

$$R.H.L = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (7-x) = 7-2 = 5.$$

Since $L.H.L \neq R.H.L$

Therefore $\lim_{x\to 2} f(x)$ does not exist.

Ex: Show that $\lim_{x\to 3} f(x)$ exist, where $f(x) = \begin{cases} \frac{9}{6-x}, & x>3 \\ 6-x, & x<3 \end{cases}$

Sol^n

$$L.H.L = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (6-x) = 6-3 = 3$$

$$R.H.L = \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{9}{6-x} = \frac{9}{6-3} = \frac{9}{3} = 3$$

Since $L.H.L = R.H.L.$

Therefore $\lim_{x\to 3} f(x)$ exist.

Ex: If $f(x) = \begin{cases} \frac{|x-1|}{x-1}, & x \neq 1 \\ 1, & x=1 \end{cases}$ then show that $\lim_{x\to 1} f(x)$ does not exist but $\lim_{x\to 2} f(x)$ and $\lim_{x\to -}$

Show that $\lim_{x\to 0} x \sin\frac{1}{x} = 0$.

Sol^n

Now $\left|x\sin\frac{1}{x}\right| = |x|\left|\sin\frac{1}{x}\right| \le |x|\cdot 1$ $\quad \because \left|\sin\frac{1}{x}\right| \le 1$.

$\Rightarrow \left|x\sin\frac{1}{x}\right| \le |x|$

$\therefore \left|\lim_{x\to 0} x\sin\frac{1}{x}\right| \le \left|\lim_{x\to 0} x\right| = 0$.

$\Rightarrow \left|\lim_{x\to 0} x\sin\frac{1}{x}\right| \le 0$.

$\Rightarrow -0 \le \lim_{x\to 0} x\sin\frac{1}{x} \le 0$

$\Rightarrow \lim_{x\to 0} x\sin\frac{1}{x} = 0$. (proved).

Ex: Show that, if $f(x) = x^2 \sin\frac{1}{x}$ then $\lim_{x\to 0} f(x)$ exist. and the limit values is 0.

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Sol^n

Now, $f(x) = x^2\sin\frac{1}{x}$.

$$\Rightarrow |f(x)| = \left|x^2\sin\frac{1}{x}\right|$$
$$= |x^2|\left|\sin\frac{1}{x}\right|$$
$$\le |x^2|\cdot 1.$$

$\Rightarrow |f(x)| \le x^2$

$\therefore \left|\lim_{x\to 0} f(x)\right| \le \lim_{x\to 0} x^2 = 0$.

$\Rightarrow -0 \le \lim_{x\to 0} f(x) \le 0$.

$\therefore \lim_{x\to 0} f(x) = 0$. (proved)

Ex: If $f(x) = \begin{cases} x^2 \cos\frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{,, } x = 0. \end{cases}$ ~~then~~ ~~lim~~ Does $\lim_{x \to 0} f(x)$ exist?

Sol^n

we know $\left|x^2 \cos\frac{1}{x}\right| = |x^2| \left|\cos\frac{1}{x}\right| \leq |x^2| \cdot 1 = x^2$

$\therefore \left|\lim_{x \to 0} x^2 \cos\frac{1}{x}\right| \leq \lim_{x \to 0} x^2 = 0.$

$\Rightarrow \left|\lim_{x \to 0} x^2 \cos\frac{1}{x}\right| \leq 0.$

$\Rightarrow -0 \leq \lim_{x \to 0} x^2 \cos\frac{1}{x} \leq 0.$

$\therefore \lim_{x \to 0} x^2 \cos\frac{1}{x} = 0.$ ~~Ans~~.

Hence the limit exist and $\lim_{x \to 0} x^2 \cos\frac{1}{x} = 0.$ A

By using $(\delta\text{-}\varepsilon)$ def^n, prove that $\lim_{x\to 2}(3x+4)=10$.

Sol^n:

Here $f(x)=3x+4$, $a=2$, and $l=10$.

Now we shall show that for each $\varepsilon>0$ there exists a $\delta>0$ such that

$$0<|x-a|<\delta \Rightarrow |f(x)-l|<\varepsilon.$$

Now we have

$$|3x+4-10|<\varepsilon.$$
$$\Rightarrow |3x-6|<\varepsilon.$$
$$\Rightarrow 3|x-2|<\varepsilon.$$
$$\Rightarrow |x-2|<\varepsilon/3=\delta.$$
$$\Rightarrow |x-2|<\delta. \quad \text{where } \delta=\varepsilon/3$$

Thus $0<|x-a|<\delta \Rightarrow |f(x)-l|<\varepsilon$.

Hence $\lim_{x\to 2} f(3x+4)=10$. (proved)

Ex. By using $(\delta\text{-}\varepsilon)$ def^n, prove that $\lim_{x\to 3}\frac{x^2-9}{x-3}=6$.

Sol^n: Here $f(x)=\frac{x^2-9}{x-3}$, $a=3$, $l=6$.

Now we ~~should th~~ shall show that for each $\varepsilon>0$ there exists a $\delta>0$ such that

$$0<|x-a|<\delta \Rightarrow |f(x)-l|<\varepsilon.$$

Now we have,

$$|f(x)-l|<\varepsilon.$$
$$\Rightarrow \left|\frac{x^2-9}{x-3}-6\right|<\varepsilon.$$
$$\Rightarrow |x+3-6|<\varepsilon.$$
$$\Rightarrow |x-3|<\varepsilon=\delta.$$
$$\Rightarrow |x-3|<\delta.$$

Thus $0<|x-a|<\delta \Rightarrow |f(x)-l|<\varepsilon$. Hence $\lim_{x\to 3}\frac{x^2-9}{x-3}=6$. (proved)

Ex. By using $\delta-\varepsilon$ defⁿ, prove that $\lim_{x \to 1} \frac{x^3-1}{x-1} = 3$

Solⁿ: Here $f(x) = \frac{x^3-1}{x-1}$, $a=1$, $l=3$.

Now we shall show that, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $0 < |x-a| < \delta \Rightarrow |f(x)-l| < \varepsilon$.

Now we have, ~~$|f(x)-l| < \varepsilon$~~.

$$|f(x)-l| = \left|\frac{x^3-1}{x-1} - 3\right| = |x^2+x+1-3| = |x^2+x-2|$$

$$= |x^2-2x+1+3x-3|$$

$$= |(x-1)^2 + 3(x-1)| \leq |(x-1)^2| + |3(x-1)|$$

$$= |x-1|^2 + 3|x-1|$$

$\Rightarrow |f(x)-l| \leq |x-1|^2 + 3|x-1|$

$\Rightarrow |f(x)-l| < |x-1| + 3|x-1|$ $\quad [\because |x-1|^2 < |x-1|]$

$\Rightarrow |f(x)-l| < 4|x-1|$

$\Rightarrow |f(x)-l| < 4\delta$.

$\Rightarrow |f(x)-l| < \varepsilon$ where $4\delta = \varepsilon \Rightarrow \delta = \varepsilon/4$

Thus $0 < |x-a| < \delta \Rightarrow |f(x)-l| < \varepsilon$.

$\therefore \lim_{x \to 1} \frac{x^3-1}{x-1} = 3$. (proved)

By using $\delta-\varepsilon$ defn prove that $\lim_{x\to 2}(x^3-3x+7)=9$

Soln Here $f(x)=x^3-3x+7$, $a=2$, $l=9$.

Now we shall show that, for each $\varepsilon>0$, there there exists a δ s.t. $0<|x-a|<\delta \Rightarrow |f(x)-l|<\varepsilon$.

Now we have,

$$|f(x)-l| = |x^3-3x+7-9|$$

$$=|x^3-3x-2|$$

$$=|x^3-6x^2+12x-2^3+6x^2-15x+6|$$

$$=|(x-2)^3+6(x^2-4x+4)+9x-18|$$

$$=|(x-2)^3+6(x-2)^2+9(x-2)|$$

$$\leq |(x-2)^3|+|6(x-2)^2|+|9(x-2)|$$

$$=|x-2|^3+6|x-2|^2+9|x-2|$$

$$<|x-2|+6|x-2|+9|x-2|$$

$$=16|x-2|$$

$$\Rightarrow |f(x)-l|<16\delta.$$

$$\Rightarrow |f(x)-l|<\varepsilon \qquad \text{where } 16\delta=\varepsilon \Rightarrow \delta=\frac{\varepsilon}{16}.$$

Thus $0<|x-a|<\delta \Rightarrow |f(x)-l|<\varepsilon$

Hence $\lim_{x\to 2}(x^3-3x+7)=9$. (proved).

Ex: $\lim_{x\to 4}(2x-2)=6$.

Soln Here $f(x)=2x-2$, $a=4$, $l=6$.

Now we ~~show that~~ shall show that, for each $\varepsilon>0$ ~~given~~ there exists a δ s.t. $0<|x-a|<\delta \Rightarrow$ $|f(x)-l|<\varepsilon$.

Now we have

$$|f(x) - l| = |2x - 2 - 6| = |2x - 8| = 2|x - 4|$$

$$\Rightarrow |f(x) - l| < \varepsilon.$$

$$\Rightarrow \cancel{|f(x)} \; 2|x-4| < \varepsilon.$$

$$\Rightarrow |x-4| < \frac{\varepsilon}{2}$$

$$\Rightarrow |x-4| < \delta. \quad \text{where } \delta = \varepsilon/2$$

Thus $0 < |x-a| < \delta \Rightarrow |f(x) - l| < \varepsilon.$

Hence $\lim_{x \to 4} (2x - 2) = 6.$ (proved).

Ex. $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$

Soln Here $f(x) = \frac{x^2 - 1}{x - 1}$, $l = 2$, $a = 1.$

Now we shall show that, for each $\varepsilon > 0$ there exists a $\delta > 0$ s.t.

$$0 < |x - a| < \delta \Rightarrow |f(x) - l| < \varepsilon.$$

Now we have

$$|f(x) - l| < \varepsilon.$$

$$\Rightarrow \left| \frac{x^2 - 1}{x - 1} - 2 \right| < \varepsilon.$$

$$\Rightarrow |x + 1 - 2| < \varepsilon.$$

$$\Rightarrow |x - 1| < \varepsilon = \delta$$

$$\Rightarrow |x - 1| < \delta. \quad \text{where } \delta = \varepsilon.$$

Thus $0 < |x - a| < \delta \Rightarrow |f(x) - l| < \varepsilon.$

Hence $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$ (proved)

Ex: $\lim_{x \to a} \frac{x^2 - a^2}{x - a} = 2a.$

$$\lim_{x\to 2} \frac{2x^2-8}{x-2} = 8.$$

Sol^n Here $f(x) = \frac{2x^2-8}{x-2}$, $a = 2$, $l = 8$.

Now we shall show that, for each $\varepsilon > 0$ there exists a $\delta > 0$ s.t. $0 < |x-a| < \delta \Rightarrow |f(x)-l| < \varepsilon$.

Now we have,

$$|f(x)-l| = \left|\frac{2x^2-8}{x-2} - 8\right|$$

$$= |2(x+2) - 8|$$

$$= |2x-4|$$

$$= 2|x-2|$$

$\therefore$ $|f(x)-l| < \varepsilon$.

$\Rightarrow 2|x-2| < \varepsilon$.

$\Rightarrow |x-2| < \varepsilon/2 = \delta$.

$\therefore |x-2| < \delta$. where $\delta = \varepsilon/2$

Thus $0 < |x-a| < \delta \Rightarrow \lim_{x\to 2} |f(x)-l| < \varepsilon$.

Hence $\lim_{x\to 2} \frac{2x^2-8}{x-2} = 8$. (proved).

Ex: $\lim_{x\to 3} x^3 - 2x^2 + 1 = 10.$

Sol^n Here $f(x) = x^3 - 2x^2 + 1$, $a = 3$, $l = 10$.

Now we shall show that for each $\varepsilon > 0$, there exists a $\delta > 0$ s.t. $0 < |x-a| < \delta \Rightarrow |f(x)-l| < \varepsilon$.

Now we have $|f(x) - l| = |x^3 - 2x^2 + 1 - 10|$

$$= |x^3 - 2x^2 - 9|$$

$$= |x^3 - 3\cdot x^2\cdot 3 + 3\cdot x\cdot 3^2 - 3^3 + 7x^2 - 27x + 18|$$

$$= |(x-3)^3 + 7(x^2 - 6x + 3^2) + 15x - 45|$$

$$= |(x-3)^3 + 7(x-3)^2 + 15(x-3)|$$

$$\leq |x-3|^3 + 7|x-3|^2 + 15|x-3|$$

$$< |x-3| + 7|x-3| + 15|x-3|$$

$$= 23|x-3|$$

$$= 23\delta.$$

$$= \varepsilon.$$

$\therefore |f(x) - 1| < \varepsilon$ when $23\delta = \varepsilon$.

Thus $0 < |x-a| < \delta \Rightarrow |f(x) - 1| < \varepsilon$.

Hence $\lim_{x \to 3} x^3 - 2x^2 + 1 = 10.$ (proved).