Math-3105

Multivariable Calculus & Geometry

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Ref: Calculus, Early Transcendentals
by Anton, Bivens, Davis

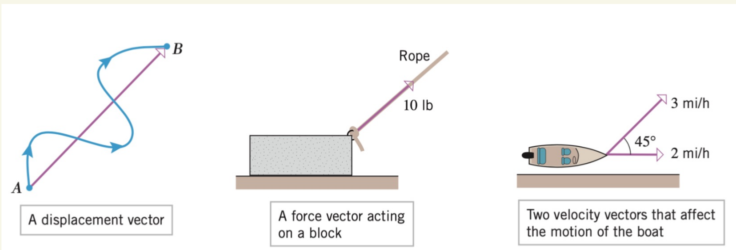
Chapter 11.2 - Vector

A vector is a quantity having both magnitude and direction.

vectors can be represented geometrically by arrow in 2-space or 3-space;
set of vector

the direction of the arrow specifies the direction of the vector, and the length of the arrow describes its magnitude. The tail of the arrow is called the initial point of the vector, and the tip of the arrow the terminal point.

Ex. velocity, force, acceleration etc.



A displacement vector | A force vector acting on a block | Two velocity vectors that affect the motion of the boat

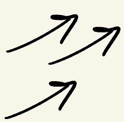
Graphically, a vector is represented by an arrow defining direction, the magnitude of the vector being indicated by the length of the arrow.

Analytically, a vector is represented by a letter with an arrow over it as $\vec{A}$ and its magnitude is denoted by $|\vec{A}|$ or A.

Equal vectors: two vectors $\underline{a}$ and $\underline{b}$ are equal if they have the same magnitude and direction regardless of the position of their initial point.

denoted by $\underline{a} = \underline{b}$

Geometrically, two vectors are equal if they are translations of one another.



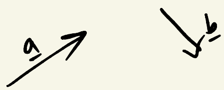
Null or Zero vector

If the initial and terminal points of a vector coincide, then the vector has length zero; we call this the zero vector and denote it by $\underline{0}$. The zero vector does not have a specific direction, so we will agree that it can be assigned any convenient direction in a specific problem.

Proper vector

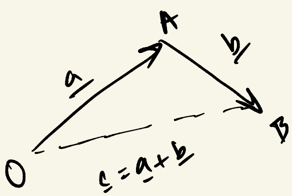
A vector which is not null is a proper vector.

Vector Addition & Subtraction



We want to find $\underline{a}+\underline{b}$ and $\underline{a}-\underline{b}$ $(\underline{a}+(-\underline{b}))$

Triangle law for vector addition

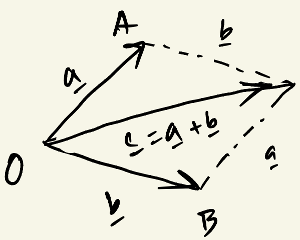


i. place the initial point of $\underline{b}$ on the terminal point on $\underline{a}$

ii. Join the initial point of $\underline{a}$ to the terminal point of $\underline{b}$

iii. As shown, $\vec{OA}+\vec{AB}=\vec{OB}$ i.e. $\underline{a}+\underline{b}=\underline{c}$

Parallelogram law for vector addition

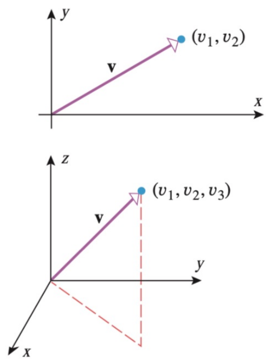


If two vector quantities are represented by two adjacent sides or a parallelogram then the diagonal of parallelogram will be equal to the resultant of these two vectors.

Vectors in Coordinate Systems

If a vector $\underline{v}$ is positioned with its initial point at the origin of a rectangular coordinate system, then its terminal point will have coordinates of the form (v_1, v_2) or (v_1, v_2, v_3), depending on whether the vector is in 2-space or 3-space. We call these coordinates the components of $\underline{v}$, and we write $\underline{v}$ in component form using the bracket notation

$$\underline{v} = (v_1, v_2) \quad \text{or} \quad \underline{v} = (v_1, v_2, v_3)$$



The zero vectors in 2-space and 3-space are,

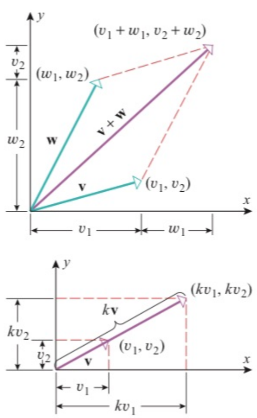
$\underline{0} = \langle 0, 0 \rangle$ or $\underline{0} = \langle 0,0,0 \rangle$ respectively

Theorem 11.2.3

Two vectors are equivalent if and only if their corresponding components are equal.

Ex. consider the vectors $\underline{v} = (v_1, v_2)$ and $\underline{w} = (w_1, w_2)$ in 2space. if $\underline{v} = \underline{w}$, then $v_1 = w_1$ and $v_2 = w_2$

Arithmetic Operations on vectors



▲ Figure 11.2.8

11.2.4 THEOREM *If* $\mathbf{v} = \langle v_1, v_2 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2 \rangle$ *are vectors in 2-space and k is any scalar, then*

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2 \rangle \tag{1}$$
$$\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2 \rangle \tag{2}$$
$$k\mathbf{v} = \langle kv_1, kv_2 \rangle \tag{3}$$

Similarly, if $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ *are vectors in 3-space and k is any scalar, then*

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \tag{4}$$
$$\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle \tag{5}$$
$$k\mathbf{v} = \langle kv_1, kv_2, kv_3 \rangle \tag{6}$$

We will not prove this theorem. However, results (1) and (3) should be evident from Figure 11.2.8. Similar figures in 3-space can be used to motivate (4) and (6). Formulas (2) and (5) can be obtained by writing $\mathbf{v} + \mathbf{w} = \mathbf{v} + (-1)\mathbf{w}$.

Ex. 1

if $v = \langle -2, 0, 1 \rangle$ and $w = \langle 3, 5, -4 \rangle$ then,

$v + w = \langle -2, 0, 1 \rangle + \langle 3, 5, -4 \rangle = \langle 1, 5, -3 \rangle$

$3v = \langle -6, 0, 3 \rangle$

$w - 2v = \langle 3, 5, -4 \rangle - \langle -4, 0, 2 \rangle = \langle 7, 5, -6 \rangle$

Vector with Initial point not at the Origin

11.2.5 THEOREM *If $\overrightarrow{P_1P_2}$ is a vector in 2-space with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then*

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1 \rangle \tag{7}$$

Similarly, if $\overrightarrow{P_1P_2}$ is a vector in 3-space with initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$, then

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \tag{8}$$

Ex. 2 In 2-space the vector from $P_1(1, 3)$ to $P_2(4, -2)$ is

$\overrightarrow{P_1P_2} = \langle 4 - 1, -2 - 3 \rangle = \langle 3, -5 \rangle$

and in 3-space the vector from $A(0, -2, 5)$ to $B(3, 4, -1)$ is

$\overrightarrow{AB} = \langle 3 - 0, 4 + 2, -1 - 5 \rangle = \langle 3, 6, -6 \rangle$

Rules of vector Arithmetic

11.2.6 THEOREM *For any vectors* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *and any scalars k and l, the following relationships hold:*

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(e) $k(l\mathbf{u}) = (kl)\mathbf{u}$

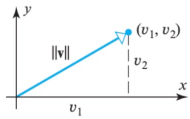
(f) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(g) $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$

(h) $1\mathbf{u} = \mathbf{u}$

Norm of a vector

The distance between the initial and terminal points of a vector $\underline{\boldsymbol{v}}$ is called the length, the norm, or the magnitude of $\underline{\boldsymbol{v}}$ and is denoted by $\|\underline{\boldsymbol{v}}\|$.

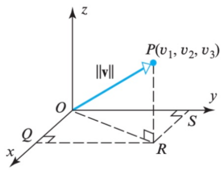


The norm of a vector $\underline{\boldsymbol{v}} = (v_1, v_2)$ in 2-space is

$$\|\underline{\boldsymbol{v}}\| = \sqrt{v_1^2 + v_2^2}$$

and norm of a vector $\underline{\boldsymbol{v}} = (v_1, v_2, v_3)$ in 3-space is

$$\|\underline{\boldsymbol{v}}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$



Ex-3
Find the norm of $v = \langle -2, 3\rangle$, $10v = \langle -20, 30\rangle$ and $w = \langle 2, 3, 6\rangle$

$$\|v\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$$

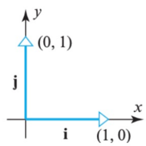
$$\|10v\| = \sqrt{(-20)^2 + 30^2} = \sqrt{1300} = 10\sqrt{13}$$

$$\|w\| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$$

Unit vectors

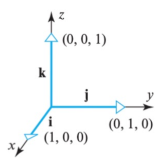
A vector of length 1 is called a unit vector.

The set of unit vectors are those having the directions of the positive $x-$, $y-$, and $z-$axes of three-dimensional rectangular coordinate system and are denoted respectively by $\hat{\imath}, \hat{\jmath}, \hat{k}$.



In 2-space: $\hat{\imath} = (1,0)$, $\hat{\jmath} = (0,1)$

In 3-space: $\hat{\imath} = (1,0,0)$, $\hat{\jmath} = (0,1,0)$, $\hat{k} = (0,0,1)$



Every vector in 2-space is expressible uniquely in terms of $\hat{\imath}$, and $\hat{\jmath}$ and every vector in 3-space is expressible uniquely in terms of $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ as follows,

$$\underline{v} = (v_1, v_2) = (v_1, 0) + (0, v_2) = v_1(1,0) + v_2(0,1) = v_1\hat{\imath} + v_2\hat{\jmath}$$

$$\underline{v} = (v_1, v_2, v_3) = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$$

▶ **Example 4** The following table provides some examples of vector notation in 2-space and 3-space.

2-SPACE	3-SPACE
$\langle 2, 3\rangle = 2\mathbf{i} + 3\mathbf{j}$	$\langle 2, -3, 4\rangle = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$
$\langle -4, 0\rangle = -4\mathbf{i} + 0\mathbf{j} = -4\mathbf{i}$	$\langle 0, 3, 0\rangle = 3\mathbf{j}$
$\langle 0, 0\rangle = 0\mathbf{i} + 0\mathbf{j} = \mathbf{0}$	$\langle 0, 0, 0\rangle = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$
$(3\mathbf{i} + 2\mathbf{j}) + (4\mathbf{i} + \mathbf{j}) = 7\mathbf{i} + 3\mathbf{j}$	$(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$
$5(6\mathbf{i} - 2\mathbf{j}) = 30\mathbf{i} - 10\mathbf{j}$	$2(\mathbf{i} + \mathbf{j} - \mathbf{k}) + 4(\mathbf{i} - \mathbf{j}) = 6\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$
$\lVert 2\mathbf{i} - 3\mathbf{j}\rVert = \sqrt{2^2 + (-3)^2} = \sqrt{13}$	$\lVert \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\rVert = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$
$\lVert v_1\mathbf{i} + v_2\mathbf{j}\rVert = \sqrt{v_1^2 + v_2^2}$	$\lVert\langle v_1, v_2, v_3\rangle\rVert = \sqrt{v_1^2 + v_2^2 + v_3^2}$

◀

Normalizing a vectors

unitless

no unit

$\frac{F_1}{F_2} = \dots$

If $\boldsymbol{v}$ is a nonzero vector, then unit vector $\hat{\boldsymbol{u}}$ in the same direction as can $\boldsymbol{v}$ be found by multiplying $\boldsymbol{v}$ by the reciprocal of its length as

$$\hat{\boldsymbol{u}} = \frac{1}{\lVert\boldsymbol{v}\rVert}\boldsymbol{v}$$

The process of multiplying a vector $\boldsymbol{v}$ by the reciprocal of its length to obtain a unit vector with the same direction is called normalizing $\boldsymbol{v}$.

Ex. 5 Find the unit vector that has the same direction as $\underline{v} = 2\hat{i} + 2\hat{j} - \hat{k}$

Sol^n

The length or norm of the vector $\underline{v}$ is,

$$\lVert v\rVert = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

the unit vector that has the same direction as $\underline{v}$ is

$$\hat{u} = \frac{1}{3}\underline{v} = \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$$

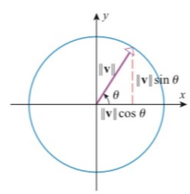
vector determined by length and angle

If $\mathbf{v}$ is a nonzero vector with its initial point at the origin of an xy-coordinate system, and if θ is the angle from the positive x-axis to the radial line through $\mathbf{v}$, then the x-component of $\mathbf{v}$ can be written as $\lVert\mathbf{v}\rVert\cos\theta$ and the y-component as $\lVert\mathbf{v}\rVert\sin\theta$ (Figure 11.2.14); and hence $\mathbf{v}$ can be expressed in trigonometric form as

$$\mathbf{v} = \lVert\mathbf{v}\rVert\langle\cos\theta, \sin\theta\rangle \quad \text{or} \quad \mathbf{v} = \lVert\mathbf{v}\rVert\cos\theta\,\mathbf{i} + \lVert\mathbf{v}\rVert\sin\theta\,\mathbf{j} \tag{12}$$

In the special case of a unit vector $\mathbf{u}$ this simplifies to

$$\mathbf{u} = \langle\cos\theta, \sin\theta\rangle \quad \text{or} \quad \mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \tag{13}$$



▲ Figure 11.2.14

Ex. 6

a. Find the vector of length 2 that makes an angle of $\frac{\pi}{4}$ with the positive x-axis.

$\Rightarrow$ $v = 2\cos\frac{\pi}{4}\, i + 2\sin\frac{\pi}{4}\, j = \sqrt{2}\, i + \sqrt{2}\, j$

b. Find the angle that the vector $v = -\sqrt{3}\, i + j$ makes with the positive x-axis.

$\Rightarrow$ We will normalize v, then use to find $\sin\theta$ and $\cos\theta$, and then use these values to find θ. Normalizing v yields,

$$\frac{v}{\|v\|} = \frac{-\sqrt{3}\, i + j}{\sqrt{(\sqrt{3})^2 + 1^2}} = -\frac{\sqrt{3}}{2} i + \frac{1}{2} j$$

Thus, $\cos\theta = -\frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$, from which we conclude that $\theta = \frac{5\pi}{6}$

vectors determined by length and a vector in the same direction.

It is a common problem in many applications that a direction in 2-space or 3-space is determined by some known unit vector **u**, and it is of interest to find the components of a vector **v** that has the same direction as **u** and some specified length $\|\mathbf{v}\|$. This can be done by expressing **v** as

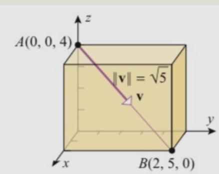
$$\mathbf{v} = \|\mathbf{v}\|\mathbf{u}$$

v is equal to its length times a unit vector in the same direction.

and then reading off the components of $\|\mathbf{v}\|\mathbf{u}$.

► **Example 7** Figure 11.2.15 shows a vector **v** of length $\sqrt{5}$ that extends along the line through A and B. Find the components of **v**.

Solution. First we will find the components of the vector $\overrightarrow{AB}$, then we will normalize this vector to obtain a unit vector in the direction of **v**, and then we will multiply this unit vector by $\|\mathbf{v}\|$ to obtain the vector **v**. The computations are as follows:



▲ **Figure 11.2.15**

$$\overrightarrow{AB} \overset{B-A}{=} \langle 2,5,0\rangle - \langle 0,0,4\rangle = \langle 2,5,-4\rangle$$

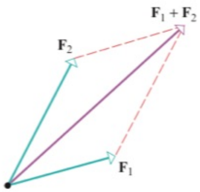
$$\|\overrightarrow{AB}\| = \sqrt{2^2+5^2+(-4)^2} = \sqrt{45} = 3\sqrt{5}$$

$$\frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \left\langle \frac{2}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}, \frac{-4}{3\sqrt{5}} \right\rangle$$

$$v = \|v\| \underbrace{\left(\frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|}\right)}_{u} = \sqrt{5}\left\langle \frac{2}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}, \frac{-4}{3\sqrt{5}} \right\rangle = \left\langle \frac{2}{3}, \frac{5}{3}, \frac{-4}{3} \right\rangle$$

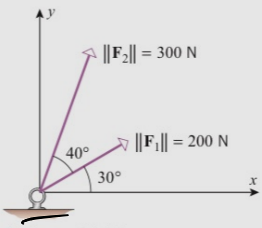
Resultant of two Concurrent Forces

The effect that a force has on an object depends on the magnitude and direction of the force and the point at which it is applied. Thus, forces are regarded to be vector quantities and, indeed, the algebraic operations on vectors that we have defined in this section have their origin in the study of forces. For example, it is a fact of physics that if two forces $\mathbf{F}_1$ and

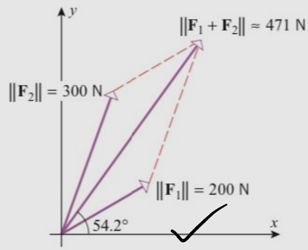


$\mathbf{F}_2$ are applied at the same point on an object, then the two forces have the same effect on the object as the single force $\mathbf{F}_1 + \mathbf{F}_2$ applied at the point (Figure 11.2.16). Physicists and engineers call $\mathbf{F}_1 + \mathbf{F}_2$ the ***resultant*** of $\mathbf{F}_1$ and $\mathbf{F}_2$, and they say that the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are ***concurrent*** to indicate that they are applied at the same point.

In many applications, the magnitudes of two concurrent forces and the angle between them are known, and the problem is to find the magnitude and direction of the resultant. One approach to solving this problem is to use (12) to find the components of the concurrent forces, and then use (1) to find the components of the resultant. The next example illustrates this method.



▲ **Figure 11.2.17**



▲ **Figure 11.2.18**

Ex. 6

Suppose that two forces are applied to an eye bracket, as shown in Fig. 11.2.17. Find the magnitude of the resultant and the angle θ that it makes with the positive x-axis.

$\Rightarrow$ Note that, F_1 makes an angle $30°$, F_2 makes an angle $30°+40° = 70°$ with the positive x-axis. Since we are given that, $\|F_1\| = 200$N, $\|F_2\| = 300$N, yields

$$F_1 = 200\langle \cos 30°, \sin 30°\rangle = \langle 100\sqrt{3}, 100\rangle$$

$$F_2 = 300\langle \cos 70°, \sin 70°\rangle = \langle 300\cos 70°, 300\sin 70°\rangle$$

therefor, the resultant,

$$F = F_1 + F_2 = \langle 100\sqrt{3}, 100\rangle + \langle 300\cos 70°, 300\sin 70°\rangle$$

$$= \langle 100\sqrt{3} + 300\cos 70°,\ 100 + 300\sin 70°\rangle$$

$$F = \|F\|\langle \cos\theta, \sin\theta\rangle \qquad = 100\langle \sqrt{3} + 3\cos 70°,\ 1 + 3\sin 70°\rangle$$

The mangnitude of the resultant is then,

$$\|F\| = 100\sqrt{(\sqrt{3}+3\cos 70^\circ)^2 + (1+3\sin 70^\circ)^2} \approx 471 \text{ N}$$

Let θ denote the angle F makes with the positive x-axis when the initial point of F is at the origin.

$$\|F\| \cos\theta = 100\,(\sqrt{3}+3\cos 70^\circ)$$

$$= 100\sqrt{3} + 300\cos 70^\circ$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{100\sqrt{3}+300\cos 70^\circ}{\|F\|}\right) \approx 54.2^\circ$$

Chapter 11.3 – Dot Product, Projection

If $\underline{\boldsymbol{u}} = (u_1, u_2, u_3)$ and $\underline{\boldsymbol{v}} = (v_1, v_2, v_3)$ are vectors in 3-space and θ is the angle between them, then the dot product of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ is written as $\underline{\boldsymbol{u}} . \underline{\boldsymbol{v}}$ and is define as

$$\underline{\boldsymbol{u}} . \underline{\boldsymbol{v}} = \|\underline{\boldsymbol{u}}\| \, \|\underline{\boldsymbol{v}}\| \cos\theta, \qquad 0 \le \theta \le \pi$$

$$= u_1v_1 + u_2v_2 + u_3v_3$$

http://www.malinc.se/math/trigonometry/unitcircleen.php

Example 1

$$(3,5).(-1,2) = 3(-1) + 5(2) = 7$$
$$(2,3).(-3,2) = 2(-3) + 3(2) = 0$$
$$(1,-3,4).(1,5,2) = 1(1) + (-3)(5) + 4(2) = -6$$

Another expression:

$$(3\hat{\imath} + 5\hat{\jmath}).(-\hat{\imath} + 2\hat{\jmath}) = 3(-1) + 5(2) = 7$$
$$(2\hat{\imath} + 3\hat{\jmath}).(-3\hat{\imath} + 2\hat{\jmath}) = 2(-3) + 3(2) = 0$$
$$(\hat{\imath} - 3\hat{\jmath} + 4\hat{k}).(\hat{\imath} + 5\hat{\jmath} + 2\hat{k}) = 1(1) + (-3)(5) + 4(2) = -6$$

Algebraic Properties of the Dot Product

If $\underline{u}$, $\underline{v}$, and $\underline{w}$ are vectors in 2- or 3- space and k is a scalar, then:

(a) $\underline{u} . \underline{v} = \underline{v} . \underline{u}$; Comutative Law

(b) $\underline{u} . (\underline{v} + \underline{w}) = \underline{u} . \underline{v} + \underline{u} . \underline{w}$; Distributive Law

(c) $k(\underline{u} . \underline{v}) = (k\underline{u}) . \underline{v} = \underline{u} . (k\underline{v})$

(d) $\underline{v} . \underline{v} = \|\underline{v}\|^2$

(e) $\underline{0} . \underline{v} = 0$

PROOF (*c*) Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$k(\mathbf{u} \cdot \mathbf{v}) = k(u_1v_1 + u_2v_2 + u_3v_3) = (ku_1)v_1 + (ku_2)v_2 + (ku_3)v_3 = (k\mathbf{u}) \cdot \mathbf{v}$$

Similarly, $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$.

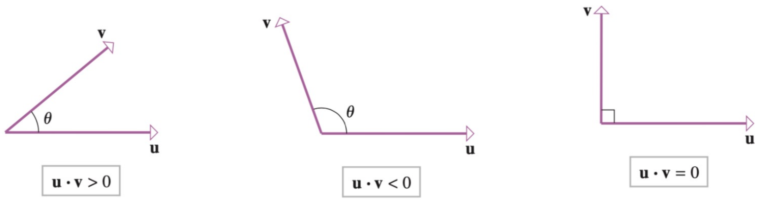
PROOF (*d*) $\mathbf{v} \cdot \mathbf{v} = v_1v_1 + v_2v_2 + v_3v_3 = v_1^2 + v_2^2 + v_3^2 = \|\mathbf{v}\|^2$. ■

The following alternative form of the formula in part (*d*) of Theorem 11.3.2 provides a useful way of expressing the norm of a vector in terms of a dot product:

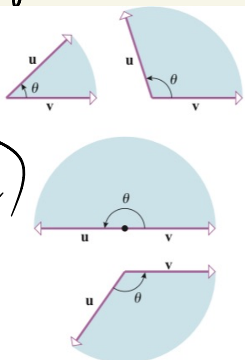
$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \tag{1}$$

Interpreting the sign of the dot product

$$\underline{u} \cdot \underline{v} = \|\underline{u}\| \, \|\underline{v}\| \cos\theta \, , \qquad 0 \leq \theta \leq \pi$$



Angles between vectors



smaller angle

Suppose that **u** and **v** are nonzero vectors in 2-space or 3-space that are positioned so their initial points coincide. We define the ***angle between* u *and* v** to be the angle θ determined by the vectors that satisfies the condition $0 \leq \theta \leq \pi$ (Figure 11.3.1). In 2-space, θ is the smallest counterclockwise angle through which one of the vectors can be rotated until it aligns with the other.

The next theorem provides a way of calculating the angle between two vectors from their components.

11.3.3 THEOREM *If* **u** *and* **v** *are nonzero vectors in 2-space or 3-space, and if* θ *is the angle between them, then*

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} \tag{2}$$

Ex. 2

Find the angle between the vector $u = i - 2j + 2k$ and

(a) $v = -3i + 6j + 2k$ (b) $w = 2i + 7j + 6k$ (c) $z = -3i + 6j - 6k = -3u$

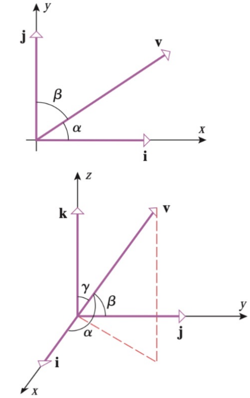
(a) $\Rightarrow \cos\theta = \dfrac{u \cdot v}{\|u\| \, \|v\|} = \dfrac{-11}{3 \times 7} = -\dfrac{11}{21}$

$\theta = \cos^{-1}\left(-11/21\right) \approx 121.6^\circ$

(b) $\Rightarrow \cos\theta = \dfrac{u \cdot w}{\|u\| \cdot \|w\|} = \dfrac{0}{\|u\| \, \|w\|} = 0$, thus $\theta = \pi/2$, which means that the vectors are perpendicular

(c) $\Rightarrow \cos\theta = \dfrac{u \cdot z}{\|u\| \, \|z\|} = \dfrac{-27}{3 \times 9} = -1$, thus $\theta = \pi$, which means that the vectors are oppositely directed.

* Direction Angles:



In an xy –coordinate system, the direction of a nonzero vector $\underline{v}$ is completely determined by the angles α and β between $\underline{v}$ and the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

In an xyz –coordinate system the direction of a nonzero vector $\underline{v}$ is completely determined by the angles α, β, and γ between $\underline{v}$ and the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

α, β, γ

In both 2- and 3- spaces the angles between a nonzero vector $\underline{v}$ and the vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are called the direction angles of $\underline{v}$, and the cosines of those angles are called the direction cosines of $\underline{v}$. Then the direction cosines of a nonzero vector $\underline{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ can be obtained by

$\cos\alpha, \cos\beta, \cos\gamma$

$$\cos\alpha = \frac{\underline{v} \cdot \hat{\imath}}{\|\underline{v}\| \, \|\hat{\imath}\|} = \frac{v_1}{\|\underline{v}\|}$$

$$\cos\beta = \frac{\underline{v} \cdot \hat{\jmath}}{\|\underline{v}\| \, \|\hat{\jmath}\|} = \frac{v_2}{\|\underline{v}\|}$$

$$\cos\gamma = \frac{\underline{v} \cdot \hat{k}}{\|\underline{v}\| \, \|\hat{k}\|} = \frac{v_3}{\|\underline{v}\|}$$

11.3.4 THEOREM *The direction cosines of a nonzero vector* $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ *are*

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$

The direction cosines of a vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ can be computed by normalizing $\mathbf{v}$ and reading off the components of $\mathbf{v}/\|\mathbf{v}\|$, since

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{v_1}{\|\mathbf{v}\|}\mathbf{i} + \frac{v_2}{\|\mathbf{v}\|}\mathbf{j} + \frac{v_3}{\|\mathbf{v}\|}\mathbf{k} = (\cos\alpha)\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k}$$

We leave it as an exercise for you to show that the direction cosines of a vector satisfy the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{5}$$

Ex-3 find the direction cosines of the vector $v = 2i - 4j + 4k$, and approximate the direction angles to the nearest degree.

Soln:

First we normalize the vector v and then readoff the components.

we have, $v = \sqrt{2^2 + (-4)^2 + 4^2} = 6$

so that, $\frac{v}{\|v\|} = \frac{2}{6}i - \frac{-4}{6}j + \frac{4}{6}k = \frac{1}{3}i + \frac{2}{3}j + \frac{2}{3}k$

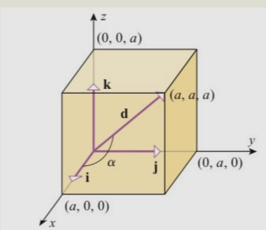
thus, $\cos\alpha = \frac{1}{3}$, $\cos\beta = -\frac{2}{3}$, $\cos\gamma = \frac{2}{3}$

With the help of calculating utility, we obtain,

$$\left.\begin{aligned} \alpha &= \cos^{-1}\left(\tfrac{1}{3}\right) \approx 71^\circ \\ \beta &= \cos^{-1}\left(\tfrac{-2}{3}\right) \approx 132^\circ \\ \gamma &= \cos^{-1}\left(\tfrac{2}{3}\right) \approx 48^\circ \end{aligned}\right\} \text{Ans}$$

Ex 4

Find the angle between a diamond of a cube and one of its degree.



Soln

Assume that, cube has a side a, and introduce a coordinate system as shown. In this coordinate system the vector,

$$d = ai + aj + ak$$

In a diagonal of the cube and the unit vectors i, j and k run along the edges. By symmetry, the diagonal makes the same angle with each edge, so it is sufficient to find the angle between d and i (the direction angle α), Thus,

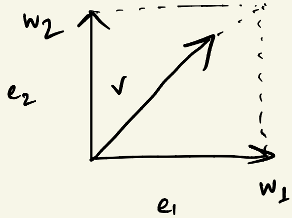
$$\cos\alpha = \frac{d \cdot i}{\|d\| \, \|i\|} = \frac{a}{d} = \frac{a}{\sqrt{3a^2}} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$$

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.955 \text{ radian} \approx 54.7^\circ$$

Decomposing vectors into Orthogonal Components

we have to decomposing a vector into sum of orthogonal vectors. For this purpose, suppose that e_1 and e_2 are two orthogonal unit vectors in 2-space, and suppose that we want to express a given vector v as a sum of,

$$v = w_1 + w_2$$



so that, w_1 is a scalar multiple e_1 and w_2 is a scalar multiple of e_2. that is we want to find scalars k_1 and k_2 such that,

$$v = k_1 e_1 + k_2 e_2$$

we can find k_1 by taking the dot product of v with e_1. This yields

$$v \cdot e_1 = (k_1 e_1 + k_2 e_2) \cdot e_1 = k_1 (e_1 \cdot e_2) + k_2 (e_2 \cdot e_1) = k_1 \|e_1\|^2 + 0 = k_1$$

Similarly,

$$v \cdot e_2 = k_2$$

then we get,

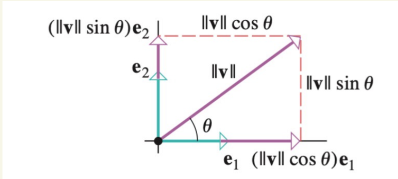
$$\boxed{v = (v \cdot e_1) e_1 + (v \cdot e_2) e_2}$$

* In this formula we call $(\underline{v} \cdot \hat{e}_1)\hat{e}_1$ and $(\underline{v} \cdot \hat{e}_2)\hat{e}_2$ the <u>vector components</u> of $\underline{v}$ along $\hat{e}_1$ and $\hat{e}_2$, respectively; and we call $\underline{v} \cdot \hat{e}_1$ and $\underline{v} \cdot \hat{e}_2$ the <u>scalar components</u> of $\underline{v}$ along $\hat{e}_1$ and $\hat{e}_2$ respectively.

If θ denotes the angle between $\underline{v}$ and $\hat{e}_1$ then $\underline{v}$ can be decomposed as.

$$\underline{v} = (\|\underline{v}\| \cos\theta)\,\hat{e}_1 + (\|\underline{v}\| \sin\theta)\,\hat{e}_2$$

Provided the angle between $\underline{v}$ and $\hat{e}_2$ is at most $\pi/2$.



Ex. 5

Let $v = \langle 2,3 \rangle$, $e_1 = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, and $e_2 = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, find the scalar components of v along e_1 and e_2 and the vector components of v along e_1 and e_2.

Soln

The scalar components of v along e_1 and e_2 are,

$$v \cdot e_1 = 2\left(\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{5}{\sqrt{2}}$$

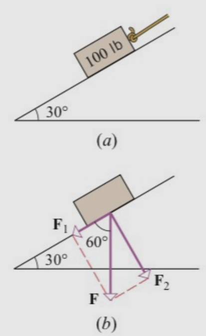
$$v \cdot e_2 = 2\left(-\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

So the vector components are,

$$(v \cdot e_1)e_1 = \frac{5}{\sqrt{2}} \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle \frac{5}{2}, \frac{5}{2} \right\rangle$$

$$(v \cdot e_2)e_2 = \frac{1}{\sqrt{2}} \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle -\frac{1}{2}, \frac{1}{2} \right\rangle$$

Ex 6



▲ Figure 11.3.9

▶ Example 6 A rope is attached to a 100 lb block on a ramp that is inclined at an angle of 30° with the ground (Figure 11.3.9*a*). How much force does the block exert against the ramp, and how much force must be applied to the rope in a direction parallel to the ramp to prevent the block from sliding down the ramp? (Assume that the ramp is smooth, that is, exerts no frictional forces.)

Soln Let F denote the downward force of gravity on the block (so $\|F\| = 100$ lb), and let, F_1 and F_2 be the vector components of F parallel and perpendicular to the ramp. The lengths of F_1 and F_2 are

$$\|F_1\| = \|F\| \cos 60° = 100 \times \frac{1}{2} = 50 \text{ lb}$$

$$\|F_2\| = \|F\| \sin 60° = 100 \times \frac{\sqrt{3}}{2} \approx 86.6 \text{ lb}$$

Thus, the block exerts a force of approximately 86.6 lb against the ramp, and it requires a force of 50 lb to prevent the block from sliding down the ramp.

Orthogonal Projections

The vector components of v along $\hat{e}_1$ and $\hat{e}_2$ are called the orthogonal projections of v on $\hat{e}_1$ and $\hat{e}_2$, are commonly denoted by,

$$\text{proj}_{\hat{e}_1} \underline{v} = (\underline{v} \cdot \hat{e}_1)\hat{e}_1 \quad \text{and} \quad \text{proj}_{\hat{e}_2} \underline{v} = (\underline{v} \cdot \hat{e}_2)\hat{e}_2$$

In general, if $\hat{e}$ is a unit vector, then the orthogonal projection of $\underline{v}$ on $\hat{e}$ to be,

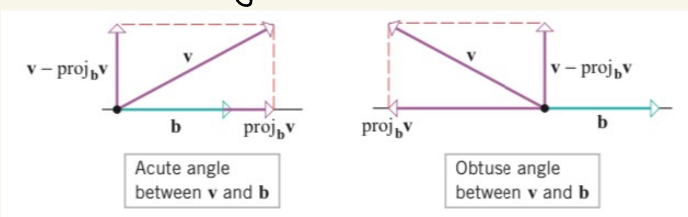
$$\boxed{\text{proj}_{\hat{e}} \underline{v} = (\underline{v} \cdot \hat{e})\hat{e}}$$

the orthogonal projection of $\underline{v}$ on an arbitrary nonzero vector $\underline{b}$ can be obtained by normalizing $\underline{b}$, that is,

$$\boxed{\text{proj}_{\underline{b}} \underline{v} = \left(\underline{v} \cdot \frac{\underline{b}}{\|\underline{b}\|}\right) \frac{\underline{b}}{\|\underline{b}\|} = \frac{\underline{v} \cdot \underline{b}}{\|\underline{b}\|^2} \underline{b}}$$

dot product using smallest angle

Geometrically, if $\underline{b}$ and $\underline{v}$ have a common initial point, then $\text{proj}_{\underline{b}} \underline{v}$ is the vector that is determined when a perpendicular is dropped from the terminal point of $\underline{v}$ to the line through b.



Moreover, if we subtract $\text{proj}_{\underline{b}} \bar{v}$ from v, then the resulting vector,

$$\underline{v} - \text{proj}_{\underline{b}} \underline{v}$$

will be orthogonal to b; we call this <u>the vector component of v orthogonal to b</u>

Ex. 7

Find the orthogonal projection of $\underline{v} = \hat{i} + \hat{j} + \hat{k}$ on $\underline{b} = 2\hat{i} + 2\hat{j}$, and then find the vector component of v orthogonal to b.

Sol^n We have,

$$\underline{v} \cdot \underline{b} = (\hat{i} + \hat{j} + \hat{k}) \cdot (2\hat{i} + 2\hat{j}) = 2 + 2 + 0 = 4$$

$$\|\underline{b}\|^2 = 2^2 + 2^2 = 8$$

Thus, the orthogonal projection of $\underline{v}$ on $\underline{b}$ is,

$$\text{proj}_{\underline{b}} \underline{v} = \frac{\underline{v} \cdot \underline{b}}{\|\underline{b}\|^2} \underline{b} = \frac{4}{8} (2\hat{i} + 2\hat{j}) = \hat{i} + \hat{j}$$

and the vector component of v, orthogonal to $\underline{b}$ is,

$$\underline{v} - \text{proj}_{\underline{b}} \underline{v} = (\hat{i} + \hat{j} + \hat{k}) - (\hat{i} + \hat{j}) = k$$

Work

$$W = \underline{F} \cdot \underline{d} = Fd\cos\theta = \|\vec{F}\| \cos\theta \, \|\vec{PQ}\|$$

Ex. 8

(a) A wagon is pulled horizontally by exerting a constant force of 10 lb on the handle at an angle of 60° with the horizontal. How much work is done in moving the wagon 50 ft?

(b) A force of $\mathbf{F} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ lb is applied to a point that moves on a line from $P(-1, 1, 2)$ to $Q(3, 0, -2)$. If distance is measured in feet, how much work is done?

Sol^n

a

With $\|F\| = 10$, $\theta = 60°$, $\|\vec{PQ}\| = 50$, it follows that the work done is

$$W = \|\vec{F}\| \cos\theta \, \|\vec{PQ}\| = 10 \times \frac{1}{2} \times 50 = 250 \text{ ft.lb}$$

b

$$\vec{PQ} = (3 - (-1))\hat{i} + (0 - 1)\hat{j} + (-2 - 2)\hat{k} = 4\hat{i} - \hat{j} - 4\hat{k}$$, the work done is

$$W = \vec{F} \cdot \vec{PQ} = (3\hat{i} - \hat{j} + 2\hat{k}) \cdot (4\hat{i} - \hat{j} - 4\hat{k}) = 12 + 1 - 8 = 5 \text{ ft.lb}$$

Cross Product

If $\underline{\boldsymbol{u}} = (u_1, u_2, u_3)$ and $\underline{\boldsymbol{v}} = (v_1, v_2, v_3)$ are vectors in 3-space and $\boldsymbol{\theta}$ is the angle between them, then the cross product $\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}$ is the vector defined by

$$\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}} = \|\underline{\boldsymbol{u}}\| \, \|\underline{\boldsymbol{v}}\| \sin\theta \, \hat{\boldsymbol{\eta}} \qquad ; 0 \le \theta \le \pi$$

;$\hat{\boldsymbol{\eta}}$ is a unit vector indicating the direction of $\underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}}$.

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}$$

$$= (u_2 v_3 - u_3 v_2)\hat{\imath} - (u_1 v_2 - u_3 v_1)\hat{\jmath} + (u_1 v_2 - u_2 v_1)\hat{k}$$

Ex. 1

Let $u = \langle 1, 2, -2 \rangle$ and $v = \langle 3, 0, 1 \rangle$ find (a) $u \times v$ (b) $v \times u$

(a) Sol^n

$$\underline{u} \times \underline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \hat{i}(2) - \hat{j}(1+6) + \hat{k}(-6) = 2\hat{i} - 7\hat{j} - 6\hat{k}$$

(b) Sol^n

$$\underline{v} \times \underline{u} = -(u \times v) = -(2\hat{i} - 7\hat{j} - 6\hat{k}) = -2\hat{i} + 7\hat{j} + 6\hat{k}$$

Ex. 2

Show that $\underline{u} \times \underline{u} = 0$ for any vector u in 3 space.

Sol^n Let, $\underline{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$

$$\underline{u} \times \underline{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = 0\hat{i} - 0\hat{j} + 0\hat{k} = \underline{0}$$

Algebraic Properties of the Cross Product

11.4.3 THEOREM *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in 3-space and* k *is any scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

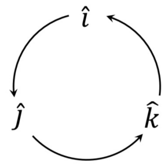
(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

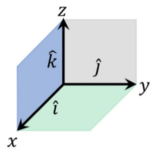
(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

$\hat{\imath} \times \hat{\jmath} = \hat{k}, \qquad \hat{\jmath} \times \hat{k} = \hat{\imath}, \qquad \hat{k} \times \hat{\imath} = \hat{\jmath}$

$\hat{\jmath} \times \hat{\imath} = -\hat{k}, \qquad \hat{k} \times \hat{\jmath} = -\hat{\imath}, \qquad \hat{\imath} \times \hat{k} = -\hat{\jmath}$





Geometric Properties of the Cross Product

If $\underline{u}$ and $\underline{v}$ are vectors in 3-space, then

(a) $\underline{u} \cdot (\underline{u} \times \underline{v}) = 0$ ($\underline{u} \times \underline{v}$ is orthogonal to $\underline{u}$)

(b) $\underline{v} \cdot (\underline{u} \times \underline{v}) = 0$ ($\underline{u} \times \underline{v}$ is orthogonal to $\underline{v}$)

Proof (a)

Let $\underline{u} = \langle u_1, u_2, u_3 \rangle$ and $\underline{v} = \langle v_1, v_2, v_3 \rangle$ then,

$$\underline{u} \times \underline{v} = \langle u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1 \rangle$$

so that,

$$\underline{u} \cdot (\underline{u} \times \underline{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0$$

Ex3 Find a vector that is orthogonal to both of the vector $\underline{u} = \langle 2, -1, 3 \rangle$ and $\underline{v} = \langle -7, 2, -1 \rangle$

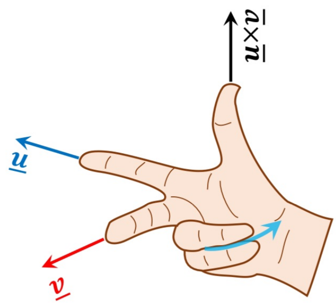
Sol^n

We know that the vector $\underline{u} \times \underline{v}$ is orthogonal to both $\underline{u}$ and $\underline{v}$. Therefore, a vector that is orthogonal to both $\underline{u}$ and $\underline{v}$ is,

$$\underline{u} \times \underline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ -7 & 2 & -1 \end{vmatrix} = -5\hat{i} - 19\hat{j} - 3\hat{k}$$

Right-Handed Rule for the Direction of Cross Product

If $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ are nonzero and nonparallel vectors, then the direction of $\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}$ relative to $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ is determined by a right-hand rule; that is, if the fingers of the right hand are cupped so they curl from $\underline{\boldsymbol{u}}$ toward $\underline{\boldsymbol{v}}$ in the direction of rotation that takes $\underline{\boldsymbol{u}}$ into $\underline{\boldsymbol{v}}$ in less than $180°$, then the thumb will point (roughly) in the direction of $\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}$.



Theorem 11.4.5

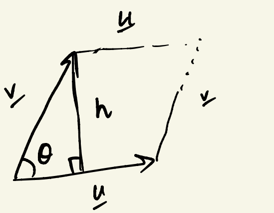
Let $\underline{u}$ and $\underline{v}$ be nonzero vectors in 3-space, and let θ be the angle between these vectors when they are positioned sol their initial points coincide.

(a) $\|\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}\| = \|\underline{\boldsymbol{u}}\|\,\|\underline{\boldsymbol{v}}\| \sin\theta$

(b) The area A of the parallelogram that has $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ as adjacent side is

$$A = \|\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}}\|$$

(c) $\underline{\boldsymbol{u}} \times \underline{\boldsymbol{v}} = \mathbf{0}$ if and only if $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ are parallel vectors, that is, if and only if they are scalar multiples of one another.



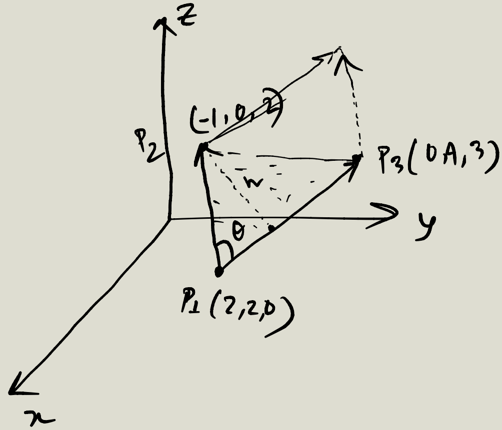
Area = base × height
$= u \times v \sin\theta$
$= \|\underline{u} \times \underline{v}\|$

Proof (a)

$$\|\underline{u}\|\,\|\underline{v}\| \sin\theta = \|u\|\,\|v\| \sqrt{1-\cos^2\theta}$$
$$= \|u\|\,\|v\| \sqrt{1 - \frac{(\underline{u}\cdot\underline{v})^2}{\|u\|^2\,\|v\|^2}}$$
$$= \sqrt{\|u\|^2\,\|v\|^2 - (\underline{u}\cdot\underline{v})^2}$$
$$= \sqrt{(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2}$$
$$= \sqrt{(u_2v_3-u_3v_2)^2 + (u_1v_3-u_3v_1)^2 + (u_1v_2-u_2v_1)^2}$$
$$= \|u \times v\|$$

Proof (c) Since $\underline{u}$ and $\underline{v}$ are assumed to be nonzero vectors, it follows from that, $\underline{u} \times \underline{v} = \underline{0}$ if and only if $\sin\theta = 0$; this is true if and only if $\theta = 0$ or $\theta = \pi$ since $0 \le \theta \le \pi$. Geometrically this means that, $\underline{u} \times \underline{v} = \underline{0}$ if $\underline{u}$ and $\underline{v}$ are parallel vectors.

Ex.4 Find the area of the triangle that is determined by the points $P_1(2,2,0)$, $P_2(-1,0,2)$ and $P_3(0,4,3)$



Soln

$$\overrightarrow{P_1P_2} = \langle -1-2 \,, 0-2, 2-0 \rangle = \langle -3,-2,2 \rangle$$

$$\overrightarrow{P_1P_3} = \langle 0-2, 4-2, 3-0 \rangle = \langle -2,2,3 \rangle$$

$$\therefore \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -2 & 2 \\ -2 & 2 & 3 \end{vmatrix}$$

$$= -10\hat{i} + 5\hat{j} - 10\hat{k}$$

$$\therefore A = \frac{1}{2} \| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \| = 15/2 \text{ (Ans)}$$

\# The area A of the triangle is equal to half of the area of the parallelogram.

Scalar Triple Product

If $\underline{\boldsymbol{u}} = (u_1, u_2, u_3)$, $\underline{\boldsymbol{v}} = (v_1, v_2, v_3)$, and $\underline{\boldsymbol{w}} = (w_1, w_2, w_3)$ are vectors in 3-space then the number

$$\underline{\boldsymbol{u}} \,.\, (\underline{\boldsymbol{v}} \times \underline{\boldsymbol{w}})$$

is called the scaler triple product of $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$, and $\underline{\boldsymbol{w}}$. The formula to evaluate scalar triple product is given by

$$\underline{\boldsymbol{u}} \,.\, (\underline{\boldsymbol{v}} \times \underline{\boldsymbol{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Ex.5 Calculate the scalar triple product $\underline{u} \cdot (\underline{v} \times \underline{w})$ of the vectors
$\underline{u} = 3\hat{i} - 2\hat{j} - 5\hat{k}$, $\underline{v} = \hat{i} + 4\hat{j} - 4\hat{k}$, $\underline{w} = 3\hat{j} + 2\hat{k}$

Soln

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 49$$

Geometric properties of the scalar triple product

Theorem 11.4.6

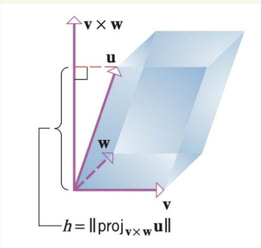
Let $\underline{u}$, $\underline{v}$ and $\underline{w}$ be nonzero vectors in 3-space.

ঘনবস্তু

(a) The volume V of the parallelepiped that has $\underline{u}$, $\underline{v}$ and $\underline{w}$ as adjacent edges is

$$V = |\underline{u} \,.\, (\underline{v} \times \underline{w})|$$

(b) $\underline{u} \,.\, (\underline{v} \times \underline{w}) = \mathbf{0}$ if and only if $\underline{u}$, $\underline{v}$ and $\underline{w}$ lie in the same plane.



Proof (a)

Let us regard the base of the parallelpiped with u, v and w as adjacent sides to be the parallelogram determined by v and w. Thus the area of the base is $\|\underline{v} \times \underline{w}\|$ and the altitude h of the parallelpiped is the length of the orthogonal projection of $\underline{u}$ on the vector $\underline{v} \times \underline{w}$. Thus from formula, we have,

$$h = \|\text{proj}_{\underline{v}\times\underline{w}}\, \underline{u}\| = \frac{|\underline{u} \cdot \underline{v}\times\underline{w}|}{\|\underline{v}\times\underline{w}\|^2}\|\underline{v}\times\underline{w}\| = \frac{|\underline{u}\cdot(\underline{v}\times\underline{w})|}{\|\underline{v}\times\underline{w}\|}$$

It now follows that the volume of the parallelpiped is,

$$V = (\text{area of base})(\text{height}) = \|\underline{v}\times\underline{w}\|\, h = |\underline{u}\cdot(\underline{v}\times\underline{w})|$$

Algebraic properties of the scalar triple product

$$\underline{u} \,.\, (\underline{v} \times \underline{w}) = \underline{v} \,.\, (\underline{w} \times \underline{u}) = \underline{w} \,.\, (\underline{u} \times \underline{v})$$

$$\underline{u} \,.\, (\underline{v} \times \underline{w}) = (\underline{u} \times \underline{v}) \,.\, \underline{w}$$

Parametric Equation of Lines

variable ⇒ it represent a model state, and may change during simulation.

parameter ⇒ a constant in a single simulation and is changed only when you need to adjust your model behavior.

3D case

$x=a \rightarrow$ plane

$x=a, y=b, z=c \rightarrow$ point

$x=a, y=b$ if intersect → line

$\left.\begin{array}{l} ax+by+cz+d=0 \rightarrow \text{plane} \\ - - - - \\ - - - - \end{array}\right\}$ combinely

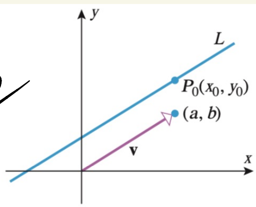
plane
2 → line
3 → point

2D case

$\left.\begin{array}{l} x+y=2 \rightarrow \text{line eq.} \\ x-y=0 \rightarrow \text{line eq.} \end{array}\right\}$ combinely → intersect point

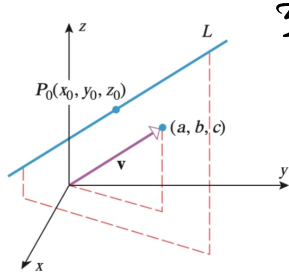
Line determined by a point & a vector

2D

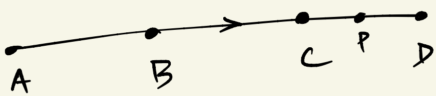


A line in 2-space or 3-space can be determined uniquely by specifying a point on the line and a nonzero vector parallel to the line.

3D



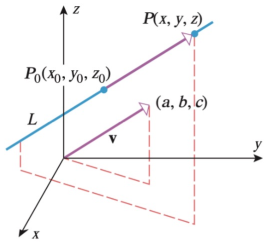
For example, consider a line L in 3-space that passes through the point $P_0(x_0,\ y_0,\ z_0)$ and is parallel to the nonzero vector $\underline{v} = (a, b, c)$.



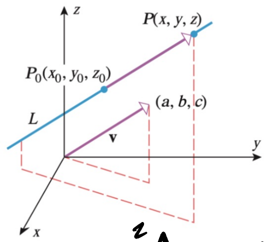
$\vec{AB} = \vec{BC} = \vec{CD} = \underline{v}$, $\vec{AC} = 2\underline{v}$, $\vec{AD} = 3\underline{v}$

$\vec{AP} = t\underline{v}$; t is a real number

Consider a line L in 3-space that passes through the point $P_0(x_0,\ y_0,\ z_0\ )$ and is parallel to the nonzero vector $\underline{\boldsymbol{v}} = (a, b, c)$.Then L consists precisely of those points $P(x, y, z)$ for which the vector $\overrightarrow{P_0P}$ is parallel to $\underline{\boldsymbol{v}}$ (see Figure). In other words, the point $P(x, y, z)$ is on L if and only $\overrightarrow{P_0P}$ is a scalar multiple of $\underline{\boldsymbol{v}}$, say



$$\overrightarrow{P_0P} = t\underline{\boldsymbol{v}}$$



This equation can be written as

$$(x - x_0, y - y_0, z - z_0) = (ta, tb, tc)$$

which implies that

$$x - x_0 = ta, \qquad y - y_0 = tb, \qquad z - z_0 = tc$$

Thus, L can be describe by the parametric equations

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct$$

z y that way x in that direction

Proof this theorem *

Theorem 11.5.1

(a) The line in 2-space that passes through the point $P_0(x_0,\ y_0)$ and is parallel to the nonzero vector $\underline{\boldsymbol{v}} = (a, b) = a\hat{\imath} + b\hat{\jmath}$ has parametric equations

$$x = x_0 + at, \qquad y = y_0 + bt$$

(b) The line in 3-space that passes through the point $P_0(x_0,\ y_0,\ z_0\ )$ and is parallel to the nonzero vector $\underline{\boldsymbol{v}} = (a, b, c) = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ has parametric equations

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct$$

Example 1 (p806)

Find parametric equation of the line

(a) passing through $(4, 2)$ and parallel to $\underline{\boldsymbol{v}} = (-1, 5)$;

(b) passing through $(1, 2, -3)$ and parallel to $\underline{\boldsymbol{v}} = 4\hat{\imath} + 5\hat{\jmath} - 7\hat{k}$;

(c) passing through the origin in 3-space and parallel to $\underline{\boldsymbol{v}} = (1, 1, 1)$.

Solution (a).

Given $(x_0, y_0) = (4, 2)$ and $\underline{v} = (a, b) = (-1, 5)$ and therefore the parametric equation line $x = x_0 + at$, $y = y_0 + bt$ becomes

$x = 4 - t$, $y = 2 + 5t$

Solution(b)

Given $(x_0, y_0, z_0) = (1, 2, 3)$ and $\underline{v} = (a, b, c) = (4, 5, -7)$, so the parametric equation line $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$ yields

$$x = 1 + 4t, \quad y = 2 + 5t, \quad z = -3 - 7t$$

Solution (c)

Given $(x_0, y_0, z_0) = (0, 0, 0)$ and $\underline{v} = (a, b, c) = (1, 1, 1)$, so the parametric equation line $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$ yields

$$x = t, \quad y = t, \quad z = t$$

Example 2 (p807)

(a) Find parametric equation of the line L passing through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

(b) Where does the line intersect the $xy-$plane?

Solution (a):

The vector $\overrightarrow{P_1P_2} = (3, -4, 8)$ is parallel to L and the point $P_1(2, 4, -1)$ lies on L,

So, Parametric equation of L is,

$$x = 2 + 3t, \quad y = 4 - 4t, \quad z = -1 + 8t \longrightarrow (i)$$

Had we used P_2 as a point on L rather than P_1, we would have obtained the equation,

$$x = 5 + 3t, \quad y = -4t, \quad z = 7 + 8t \longrightarrow (ii)$$

Both equation are equivalent if we replace $t = t - 1$ in 2nd equation and $t = t + 1$ in 1st equation. ✓

Although these equations look different from those obtained using P_1, the two sets of equations are actually equivalent in that both generate L as t varies from $-\infty$ to $+\infty$. To see this, note that if t_1 gives a point

$$t_1 (x_1, y_1, z_1) = (2 + 3t_1, 4 - 4t_1, -1 + 8t_1)$$

On L using the first set of equations then $t_2 = t_1 - 1$ gives the same point

$$t_2 (x_2, y_2, z_2) = (5 + 3t_2, -4t_2, 7 + 8t_2)$$

$$= (5 + 3(t_1 - 1), -4(t_1 - 1), 7 + 8(t_1 - 1))$$

$$= (2 + 3t_1, 4 - 4t_1, -1 + 8t_1)$$

on L using the second set of equation. Conversely, if t_2 gives a point on L using the second set of equations, then $t_1 = t_2 + 1$ gives the same point using the first set.

Solution(b):

The line intersects the xy-plance at the point where $z = 0$, that is,

$$-1 + 8t = 0$$

$$\Rightarrow t = \frac{1}{8}$$

substituting the value of t,

$$(x, y, z) = \left(\frac{19}{8}, \frac{7}{2}, 0\right)$$

Example 3 (p807) *

Let L_1 and L_2 be the lines

L_1: $x = 1 + 4t$, $y = 5 - 4t$, $z = -1 + 5t$

L_2: $x = 2 + 8t$, $y = 4 - 3t$, $z = 5 + t$

(a) Are the lines parallel?

*(b) Do the lines intersects?

In 3D

যদি vector parallel না হয়েও
Intersect নাও করতে পারে

Solution(a): The line L_1 is parallel to the vector $4\hat{i} - 4\hat{j} + 5\hat{k}$, and the line L_2 is parallel to the vector $8\hat{i} - 3\hat{j} + \hat{k}$. These vectors are not parallel since neither is a scalar multiple of the other. Thus the line are not parallel.

Solution(b):

The line l_1 and l_2 to intersect at some point (x_0, y_0, z_0) these coordinate would have to satisfy the equation of both lines. In other words, there would have to exist values t_1 and t_2 for the parameters such that,

$$x_0 = 1+4t_1, \quad y_0 = 5-4t_1, \quad z_0 = -1+5t_1$$

and

$$x_0 = 2+8t_2, \quad y_0 = 4-3t_2, \quad z_0 = 5+t_2$$

this leads to three condition on t_1 and t_2

$$1+4t_1 = 2+8t_2 \quad \text{—— (i)}$$

$$5-4t_1 = 4-3t_2 \quad \text{—— (ii)}$$

$$-1+5t_1 = 5+t_2 \quad \text{—— (iii)}$$

Thus, the lines intersects if there are values of t_1 and t_2 that satisfy all three equations, and the line do not intersect if there are no such values. Solving first two equation we get,

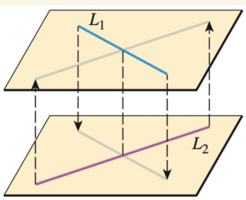
$$t_1 = \frac{1}{4} \text{ and } t_2 = 0$$

If we replace t_1, t_2 in (iii) equation, $-1 + \frac{5}{4} = 5$

$$\Rightarrow \frac{-4+5}{4} = 5 \Rightarrow 1 \neq 20$$

However thes values do not satisfy the third equation.

therefore, the lines do not intersect.



Parallel planes containing skew lines L_1 and L_2 can be determined by translating each line until it intersects the other.

In 2D → if two lines are not parallel then they are intersect in some point.

But in 3D → it can intersect or not

Line Segments

Parametric equations of a line segment can be obtained by finding parametric equations for the entire line, and then restricting the parameter appropriately so that, only the desired segment is generated.

Ex 4 Find parametric equations describing the line segment joining the points $P_1(2,4,-1)$ and $P_2(5,0,7)$

Solution:

The vector $\overrightarrow{P_1P_2} = (3,-4,8)$ is parallel to L and the point $P_1(2,4,-1)$ lies on L, so the parametric equation of L is,

$$x = 2+3t, \quad y = 4-4t, \quad z = -1+8t$$

P_1: $2 = 2+3t \Rightarrow t = 0$, $\quad P_2$: $2+3t = 5$

$$\Rightarrow 3t = 5-2 = 3$$

$$\Rightarrow t = 1$$

With these equation P_1 corresponds to $t=0$ and P_2 to $t=1$. Thus the line segment that joins P_1 and P_2 is given by,

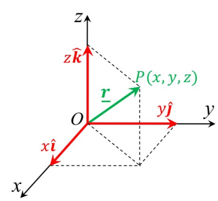
$$x = 2+3t, \quad y = 4-4t, \quad z = -1+8t \quad (0 \le t \le 1)$$

Vector Equations of lines

Position Vector of a Point

- Also known as location vector or radius vector.
- Represents the position of a point P in space in relation to an arbitrary reference origin O.
- corresponds to the straight-line segment from O to P.

i.e. $\underline{r} = \overline{OP} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k} = (x,y,z)$



The equation of the line that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\underline{v} = (a, b, c) = a\hat{i} + b\hat{j} + c\hat{k}$ can be written in vector form as,

$$(x, y, z) = (x_0 + at,\ y_0 + bt,\ z_0 + ct)$$

or equivalently,

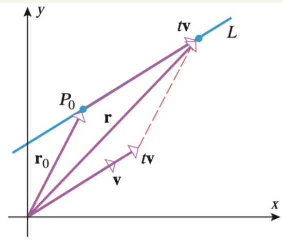
$$\underbrace{(x, y, z)}_{\overrightarrow{OP}} = \underbrace{(x_0, y_0, z_0)}_{\overrightarrow{OP_0}} + \underbrace{t(a, b, c)}_{t\underline{v}}$$

Let us define the vector,

$$\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k} = (x, y, z),$$

$$\underline{r_0} = x_0\,\underline{i} + y_0\,\underline{j} + z_0\,\underline{k} = (x_0, y_0, z_0).$$

$$\underline{v} = a\,\underline{i} + b\,\underline{j} + c\,\underline{k} = (a, b, c).$$



Then the equation $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$ becomes

$$\underline{r} = \underline{r_0} + t\,\underline{v}$$

The vector equation of straight line passing through the point $\underline{r_0}$ and parallel to the nonzero vector $\underline{v}$ is

$$\underline{r} = \underline{r_0} + t\underline{v}$$

Ex-5 The equation $(x, y, z) = (-1, 0, 2) + t(1, 5, -4)$ can be written as $\underline{r} = \underline{r_0} + t\underline{v}$ with, $\underline{r_0} = (-1, 0, 2)$ and $\underline{v} = (1, 5, -4)$.

Thus, the equation represents the line in 3-space that passes through the point $\underline{r_0} = (-1, 0, 2)$ and is parallel to the vector $\underline{v} = (1, 5, -4)$

Ex 6 find an equation of the line in 3 space that passes through the points $P_1(2,4,-1)$ and $P_2(5,0,7)$

Solution:
the vector $\overrightarrow{P_1P_2} = (3,-4,8)$ is parallel to the line, so it can be used as v.
For r_0 we can use either the vector from the origin P_1 or from the origin to P_2. So,

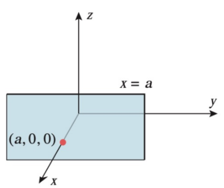
$$r_0 = (2,4,-1) \quad \text{(if we use } P_1 \text{ as } r_0\text{)}$$

Thus, a vector equation of the line through P_1 and P_2 is

$$\underline{r} = \underline{r_0} + t\underline{v}$$
$$(x,y,z) = (2,4,-1) + t(3,-4,8).$$

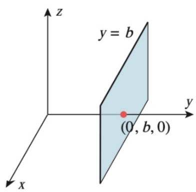
Planes in 3-Space

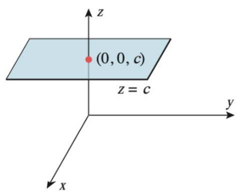
Planes parallel to the Coordinate planes



The graph of the equation $x = a$ in an xyz-coordinate system consists of all points of the form (a, y, z), where y and z are arbitrary. One such point is $(a, 0, 0)$, and all others are in the plane that passes through this point and is parallel to the yz –plane.

Similarly, the graph of $y = b$ is the plane through $(0, b, 0)$ that is parallel to the xz –plane, and the graph of $z = c$ is the plane through $(0, 0, c)$ that is parallel to the xy –plane.





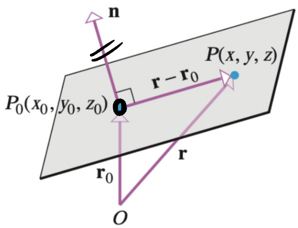
Planes Determined by a point & a normal vector

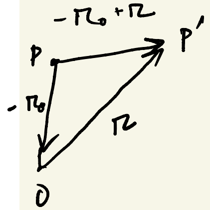
Suppose that we want to find an $\boxed{\text{equation of the plane}}$ passing through $P_0(x_0, y_0, z_0)$ and perpendicular to the vector $\underline{n} = (a, b, c)$. Define the vectors $\underline{r_0}$ and $\underline{r}$ as

$$\underline{r_0} = (x_0, y_0, z_0) \text{ and } \underline{r} = (x, y, z)$$

As shown in fig.

$$\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = \underline{r} - \underline{r_0}$$





It should be evident from Figure that the plane consists precisely of those points $P(x, y, z)$ for which the vector $\underline{r} - \underline{r_0}$ is orthogonal to $\underline{n}$; or, expressed as an equation,

$$\underline{n} \,.\, (\underline{r} - \underline{r_0}) = 0$$ → vector equation

We can express this vector equation in terms of components as

$$(a, b, c) \,.\, (x - x_0, y - y_0, z - z_0) = 0$$

which yields

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$ → point-normal form

This is called the * point-normal form of the equation of a plane.

Example 1 (p814)

Find an equation of the plane passing through the points $(3, -1, 7)$ and perpendicular to the vector $\underline{n} = (4, 2, -5)$.

Solution:

A point normal form of the equation is,

$$\underline{n} \,.\, (\underline{r} - \underline{r_0}) = 0$$

$$\Rightarrow (4, 2, -5) \,.\, ((x, y, z) - (3, -1, 7)) = 0$$

$$\Rightarrow (4, 2, -5) \,.\, (x-3, y+1, z-7) = 0$$

$$\Rightarrow 4(x-3) + 2(y+1) - 5(z-7) = 0$$

$$\Rightarrow 4x + 2y - 5z + 25 = 0$$

Theorem 11.6.1

If a, b, c, and d are constants, and a, b, and c are not all zero, then the graph of the equation

$$ax + by + cz + d = 0 \quad \text{— \#}$$

is a plane that has the vector $\underline{n} = (a, b, c)$ as a normal.

Proof: Since a, b and c are not all zero, there is at least one point (x_0, y_0, z_0) whose coordinates satisfy that equation. For example, if $a \neq 0$, then such a point is $(-d/a, 0, 0)$, and similarly if $b \neq 0$ or $c \neq 0$ (verify). Thus let (x_0, y_0, z_0) be any point whose coordinates satisfy that equation. That is,

$$ax_0 + by_0 + cz_0 + d = 0$$

Subtracting this equation by that equation,

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

which is point-normal-form of a plane with normal $\underline{n} = (a, b, c)$

$\boxed{ax + by + cz + d = 0 \text{ is called the general form of the equation of a plane}}$

Example 2 (p814)

Determine whether the planes

$$3x - 4y + 5z = 0 \text{ and } -6x + 8y - 10z - 4 = 0$$

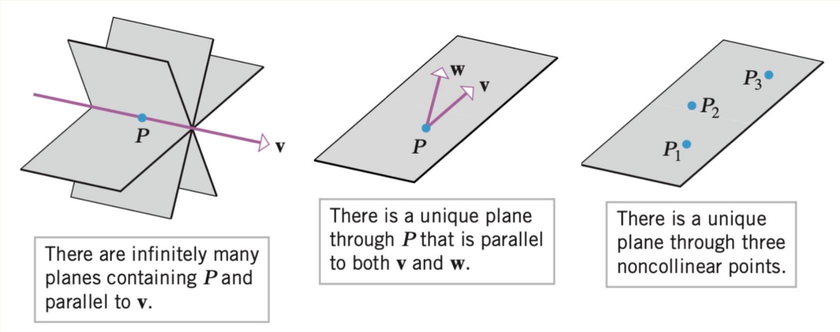
are parallel.

Solution: It is clear geometrically that, two planes are parallel if and only if their normals are parallel vectors. A normal to the first plane is,

$$\underline{n_1} = (3, -4, 5)$$

and the normal to the second plane is, $\underline{n_2} = (-6, 8, -1)$

Since $\underline{n_2}$ is a scalar multiple of $\underline{n_1}$, the normals are parallel, and hence so are the planes.



Example 3 (p815)

Find an equation of the plane through the points $P_1(1,2,-1)$, $P_2(2,3,1)$, and $P_3(3,-1,2)$.

Solution:

Since the points P_1, P_2 and P_3 lie in the plane, the vectors

$\overrightarrow{P_1P_2} = (1,1,2)$ and $\overrightarrow{P_1P_3} = (2,-3,3)$ are parallel to the plane.

$$\therefore \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 2 & -3 & 3 \end{vmatrix} = 9\hat{i} + \hat{j} - 5\hat{k}$$

is normal to the plane, since it is orthogonal to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$

By using this normal and the point $P_1(1,2,-1)$ in the plane, we obtain the point-normal form,

$$9(x-1) + (y-2) - 5(z+1) = 0$$

$$\Rightarrow 9x + y - 5z - 16 = 0$$

Example 4 (p815)

Determine whether the line

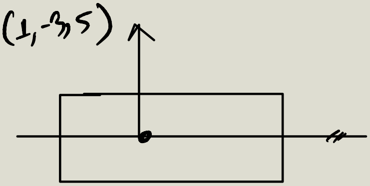
$$x = 3 + 8t, \qquad y = 4 + 5t, \qquad z = -3 - t$$

is parallel to the plane $x - 3y + 5z = 12$.

Solution:

A parallel vector to the given line is, $\underline{v} = (8, 5, -1)$

and the normal vector to the given plane is, $\underline{n} = (1, -3, 5)$



For the line and plane to be parallel,

The vectors $\underline{v}$ and $\underline{n}$ must be orthogonal. But

$$\underline{v} \cdot \underline{n} = 8 \times 1 + 5 \times -3 + -1 \times 5 = -12 \neq 0$$

Thus, the line and plane are not parallel.

Example 5 (p815)

Find the intersection of the line

$$x = 3 + 8t, \qquad y = 4 + 5t, \qquad z = -3 - t$$

and plane $x - 3y + 5z = 12$.

Solution:

If we let (x_0, y_0, z_0) be the point of intersection, then the coordinates of this point satisfy both the equation of the plane and the parametric equations of the line.

Thus,

$$x_0 - 3y_0 + 5z_0 = 12 \quad \text{— (i)}$$

and for some values of t, say $t = t_0$,

$$x_0 = 3 + 8t_0, \quad y_0 = 4 + 5t_0, \quad z = -3 - t_0 \quad \text{— (ii)}$$

Substituting (ii) in (i),

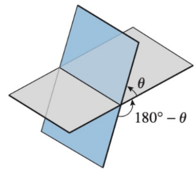
$$(3 + 8t_0) - 3(4 + 5t_0) + 5(-3 - t_0) = 12$$

$$\Rightarrow t_0 = -3$$

Using the value of t_0 in (ii) we get, $(x_0, y_0, z_0) = (-21, -11, 0)$

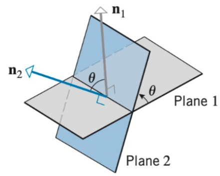
Intersecting Planes

Two distinct intersecting planes determine two positive angles of intersection—an (acute) angle θ that satisfies the condition $0 \le \theta \le \pi/2$ and the supplement of that angle (Fig. a).



(*a*)

If $\underline{n_1}$ and $\underline{n_2}$ are normals to the planes, then depending on the directions of $\underline{n_1}$ and $\underline{n_2}$, the angle θ is either the angle between $\underline{n_1}$ and $\underline{n_2}$ or the angle between $\underline{n_1}$ and $-\underline{n_2}$ (Fig. b).



(*b*)

In both cases, the formula for the acute angle θ between the planes,

$$\cos\theta = \frac{|\underline{n_1} \cdot \underline{n_2}|}{\|\underline{n_1}\| \, \|\underline{n_2}\|}$$