Sheikh Hasina University, Netrokona

Course:
MATH-3105 (Multivariable Calculus & Geometry)

Textbook:
Calculus, Early Transcendentals
By Anton, Bivens, Davis (10th Edition)

Course Teacher:
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3rd Year 1st Semester

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Partial Derivatives

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Chapter 13.1

Functions of Two or More Variables

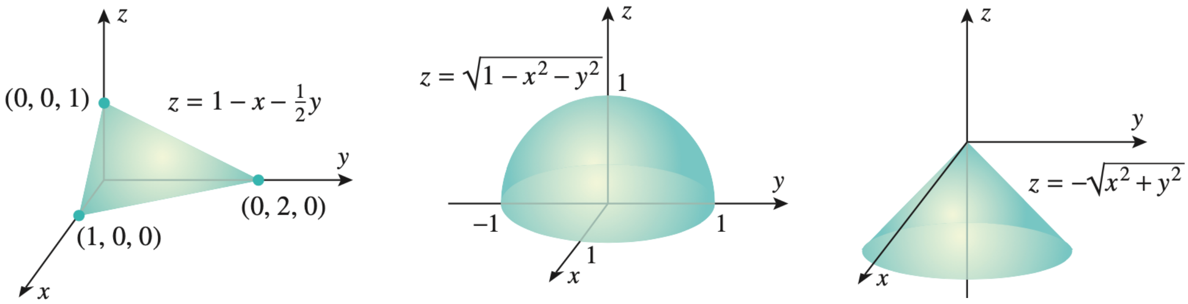
Function of Multiple Variables

A function f of two variables, x and y, is a rule that assigns a unique real number $f(x, y)$ to each point (x, y) in some set D in the xy −plane.

A function f of three variables, x, y and z, is a rule that assigns a unique real number $f(x, y, z)$ to each point (x, y, z) in some set D in three dimensional space.

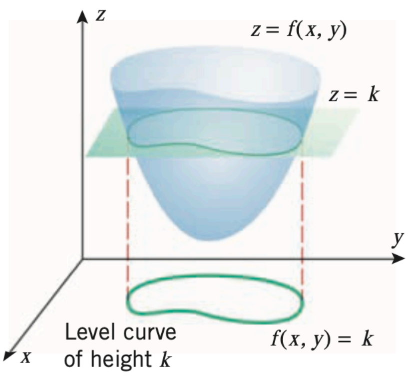
Graphs of Function of Two Variables

If f is a function of two variables, we define the graph of $f(x, y)$ in xyz −space to be the graph of the equation $z = f(x, y)$. In general, such a graph will be a surface in 3-space.

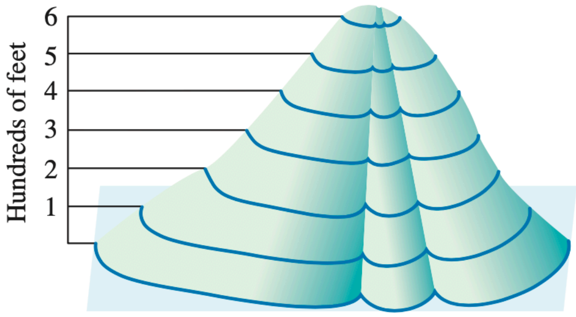


Level Curves & Contour Plot

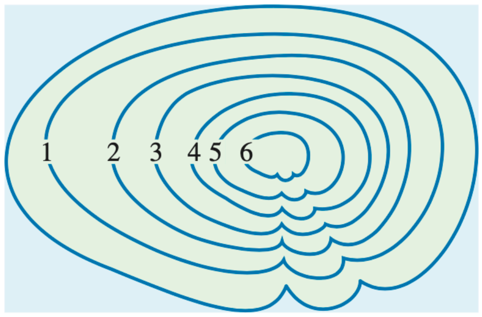
If the surface $z = f(x, y)$ is cut by the horizontal plane $z = k$, then at all points on the intersection we have $f(x, y) = k$. The projection of this intersection onto the xy −plane is called the level curve of height k or the level curve with constant k (Figure). A set of level curves for $z = f(x, y)$ is called a contour plot or contour map of f.



Level Curves & Contour Plot



A perspective view of a model hill with two gullies



A contour map of the model hill

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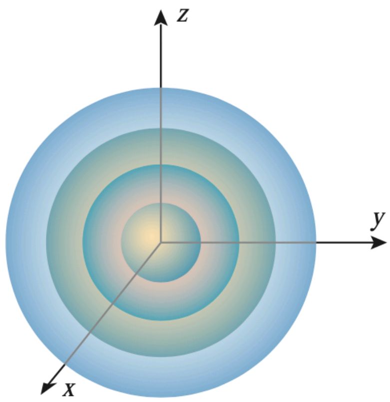
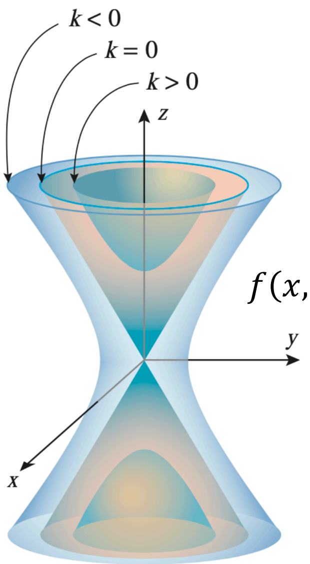
Level Curves & Contour Plot



Level surfaces of
$f(x, y, z) = x^2 + y^2 + z^2$



Level surfaces of
$f(x, y, z) = z^2 - x^2 - y^2$

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Level Curves & Contour Plot

Except in the simplest cases, contour plots can be difficult to produce without the help of a graphing utility.

Chapter 13.2

Limits and Continuity

Limits along Curves

For a function of one variable there are two one-sided limits at a point x_0, namely,

$$\lim_{x \to x_0^+} f(x) \quad \text{and} \quad \lim_{x \to x_0^-} f(x)$$

reflecting the fact that there are only two directions from which x can approach x_0, the right or the left.

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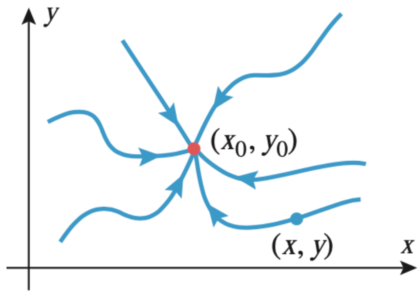
Limits along Curves

For functions of two or three variables the situation is more complicated because there are infinitely many different curves along which one point can approach another (Figure).

Our first objective in this section is to define the limit of $f(x, y)$ as (x, y) approaches a point (x_0, y_0) along a curve C (and similarly for functions of three variables).



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Limits along Curves

If C is a smooth parametric curve in 2-space that is represented by the equations

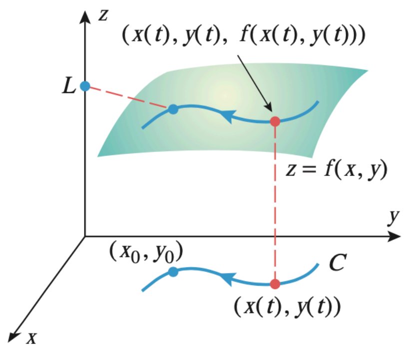
$$x = x(t), \qquad y = y(t)$$

and if $x_0 = x(t_0)$, $y_0 = y(t_0)$ then the limits

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ \text{(along C)}}} f(x,y)$$

are defined by

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ \text{(along C)}}} f(x,y) = \lim_{t\to t_0} f\big(x(t), y(t)\big)$$



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Limits along Curves

Example 1 (p918)

Let

$$f(x,y) = -\frac{xy}{x^2 + y^2}$$

Find the limit of $f(x,y)$ as (x,y) approacheds $(0,0)$ along

(a) the x −axis (b) the y −axis (c) the line $y = x$

(d) the line $y = -x$ (e) the parabola $y = x^2$

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Limits along Curves

Solution (a)

The parametric equations of x −axis is $x = t$, $y = 0$.

And $(x, y) = (0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0) \\ (\text{along } y=0)}} f(x,y) = \lim_{t\to 0} f(t,0) = \lim_{t\to 0}\left(-\frac{t\cdot 0}{t^2+0^2}\right) = \lim_{t\to 0} 0 = 0$$

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Limits along Curves

Solution (b)

The parametric equations of y −axis is $x = 0$, $y = t$.

And $(x, y) = (0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0) \\ (\text{along } x=0)}} f(x,y) = \lim_{t\to 0} f(0,t) = \lim_{t\to 0}\left(-\frac{0\cdot t}{0^2+t^2}\right) = \lim_{t\to 0} 0 = 0$$

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Limits along Curves

Solution (c)

The parametric equations of the line $y = x$ is $x = t, y = t$.

And $(x, y) = (0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x)}} f(x,y) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0}\left(-\frac{t\cdot t}{t^2+t^2}\right) = \lim_{t\to 0}\left(-\frac{t^2}{2t^2}\right)$$

$$= \lim_{t\to 0}\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

Limits along Curves

Solution (d)

The parametric equations of the line $y = -x$ is $x = t, y = -t$.

And $(x, y) = (0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=-x)}} f(x,y) = \lim_{t\to 0} f(t,-t) = \lim_{t\to 0}\left(-\frac{t\cdot(-t)}{t^2+(-t)^2}\right) = \lim_{t\to 0}\left(\frac{t^2}{2t^2}\right)$$

$$= \lim_{t\to 0}\left(\frac{1}{2}\right) = \frac{1}{2}$$

Limits along Curves

Solution (e)

The parametric equations of the parabola $y = x^2$ is $x = t, y = t^2$.

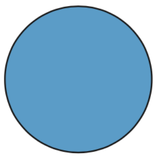
And $(x, y) = (0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x^2)}} f(x,y) = \lim_{t\to 0} f(t,t^2) = \lim_{t\to 0}\left(-\frac{t\cdot t^2}{t^2+(t^2)^2}\right) = \lim_{t\to 0}\left(\frac{t^3}{2t^2}\right)$$

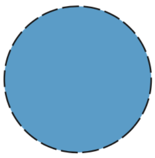
$$= \lim_{t\to 0}\left(\frac{1}{2}\right) = \frac{1}{2}$$

Open & Closed Disks

Let C be a circle in 2-space that is centered at (x_0, y_0) and has positive radius δ. The set of points that are enclosed by the circle, but do not lie on the circle, is called the open disk of radius δ centered at (x_0, y_0), and the set of points that lie on the circle together with those enclosed by the circle is called the closed disk of radius δ centered at (x_0, y_0).



A closed disk includes all of the points on its bounding circle.



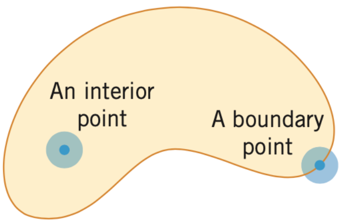
An open disk contains none of the points on its bounding circle.

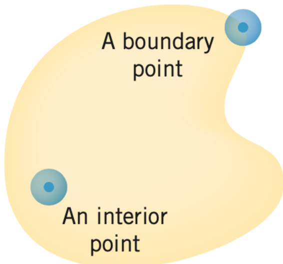
Open & Closed Balls

If S is a sphere in 3-space that is centered at(x_0, y_0, z_0) and has positive radius δ, then the set of points that are enclosed by the sphere, but do not lie on the sphere, is called the open ball of radius δ centered at (x_0, y_0, z_0), and the set of points that lie on the sphere together with those enclosed by the sphere is called the closed ball of radius δ centered at (x_0, y_0, z_0).

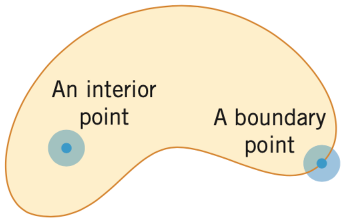
Open & Closed Sets

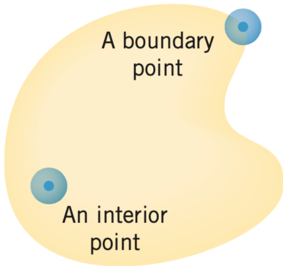
If D is a set of points in 2-space, then a point (x_0, y_0) is called an interior point of D if there is some open disk centered at (x_0, y_0) that contains only points of D, and (x_0, y_0) is called a boundary point of D if every open disk centered at (x_0, y_0) contains both points in D and points not in D.





Open & Closed Sets





For a set D in either 2-space or 3-space, the set of all interior points is called the interior of D and the set of all boundary points is called the boundary of D. Moreover, just as for disks, we say that D is closed if it contains all of its boundary points and open if it contains none of its boundary points.

General Limits of Functions of 2-Variables

Definition 13.2.1

Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (x_0, y_0), except possibly at (x_0, y_0). We will write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if given any number $\varepsilon > 0$, we can find a number $\delta > 0$ such that $f(x,y)$ satisfies

$$|f(x,y) - L| < \varepsilon$$

whenever the distance between (x,y) and (x_0,y_0) satisfies

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

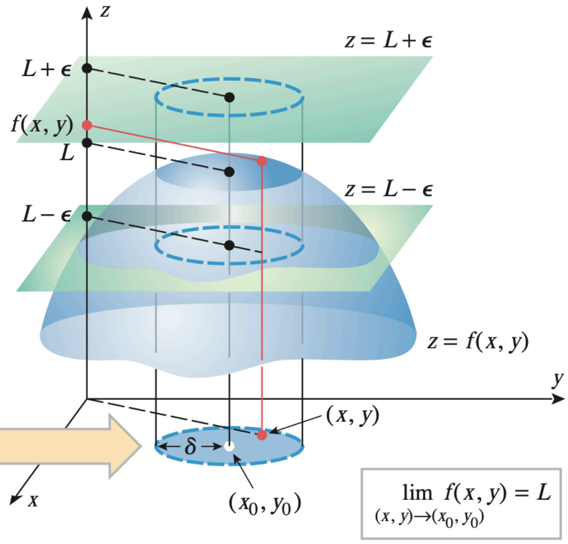
General Limits of Functions of 2-Variables

In Figure the condition

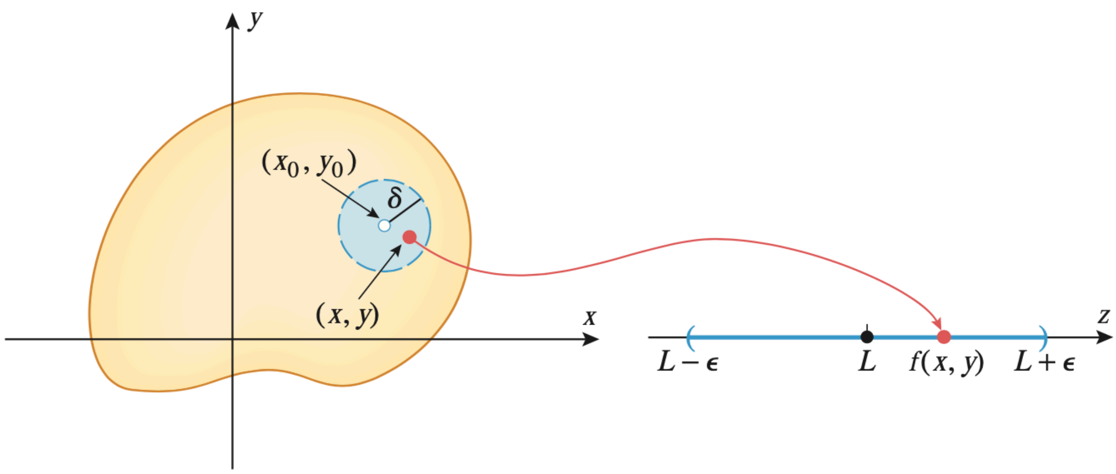
$$|f(x, y) - L| < \epsilon$$

is satisfied at each point (x, y) within the circular region. However, the fact that this condition is satisfied at the center of the circular region is not relevant to the limit.

This circular region with the center removed consists of all points (x, y) that satisfy $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.



General Limits of Functions of 2-Variables



General Limits of Functions of 2-Variables

Example 2 (p921)

$$\lim_{(x,y)\to(1,4)}[5x^3y^2 - 9] = \lim_{(x,y)\to(1,4)}[5x^3y^2] - \lim_{(x,y)\to(1,4)}[9]$$
$$= 5\cdot(1)^3\cdot(4)^2 - 9$$
$$= 71$$

Relation between General Limits & Limits along Smooth Curve

Theorem 13.2.2

(a) If $f(x,y)\to L$ as $(x,y)\to(x_0,y_0)$, then $f(x,y)\to L$ as $(x,y)\to(x_0,y_0)$ along any smooth curve.

(b) If the limit of $f(x,y)$ fails to exist as $(x,y)\to(x_0,y_0)$ along some smooth curve, or if $f(x,y)$ has different limits as $(x,y)\to(x_0,y_0)$ along two different smooth curves, then the limit of $f(x,y)$ does not exist as $(x,y)\to(x_0,y_0)$.

General Limits of Functions of 2-Variables

Example 3 (p922)

The limit

$$\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2+y^2}$$

does not exist. Because,

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } x=0)}} -\frac{xy}{x^2+y^2} = 0 \quad \text{and} \quad \lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x)}} -\frac{xy}{x^2+y^2} = -\frac{1}{2}$$

i.e., two different smooth curves along which this limit has different values.

Continuity

Definition 13.2.3

A function $f(x,y)$ is said to be continuous at (x_0, y_0) if $f(x_0, y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$$

In addition, if f is continuous at every point in an open set D, then we say that f is continuous on D, and if f is continuous at every point in the xy −plane, then we say that f is continuous everywhere.

Continuity

Theorem 13.2.4

(a) If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x,y) = g(x)h(y)$ is continuous at (x_0, y_0).

(b) If $h(x,y)$ is continuous at (x_0, y_0) and $g(u)$ is continuous at $u = h(x_0, y_0)$, then the composition $f(x,y) = g\big(h(x,y)\big)$ is continuous at (x_0, y_0).

(c) If $f(x,y)$ is continuous at (x_0, y_0), and if $x(t)$ and $y(t)$ are continuous at t_0 with $x(t_0) = x_0$ and $y(t_0) = y_0$, then the composition $f\big(x(t), y(t)\big)$ is continuous at t_0.

Continuity

Example 4 (p922)

Show that the functions $f(x,y) = 3x^2y^5$ and $f_1(x,y) = \sin(3x^2y^5)$ are continuous everywhere.

Solution

The polynomials $g(x) = 3x^2$ is continuous at every real number $x \in \mathbb{R}$ and $h(y) = y^5$ is continuous at every real number $y \in \mathbb{R}$.

Therefore, the function

$$f(x,y) = g(x)h(y) = 3x^2y^5$$

is continuous at every point $(x,y) \in \mathbb{R}^2$ in the xy −plane.

Continuity

Solution

Since, $f(x, y) = 3x^2y^5$ is continuous at every point in the xy −plane and $g(u) = \sin u$ is continuous at every real number $u \in \mathbb{R}$.

It follows that the composition

$$f_1(x, y) = g_1\big(f(x, y)\big) = g_1(3x^2y^5) = \sin(3x^2y^5)$$

is continuous everywhere.

Continuity

Recognizing Continuous Functions

- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

Continuity

Example 5 (p923)

Evaluate $\lim\limits_{(x,y)\to(-1,2)} \dfrac{xy}{x^2+y^2}$

Solution

Since $f(x,y) = xy/(x^2+y^2)$ is continuous at $(-1,2)$. It follows from the definition of continuity

$$\lim_{(x,y)\to(-1,2)} f(x,y) = f(-1,2) = \frac{(-1)(2)}{(-1)^2+(2)^2} = -\frac{2}{5}$$

Continuity

Example 6 (p923)

Determine the condition where the following function is continuous:

$$f(x,y) = \frac{x^3y^2}{1-xy}$$

Solution

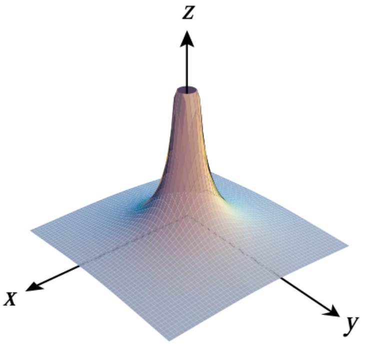
x^3y^2 and $1-xy$ are continuous at every point $(x,y) \in \mathbb{R}^2$ in the xy −plane.

Therefore, $f(x,y)$ is continuous except where $1-xy=0$. Thus, $f(x,y)$ is continuous everywhere except on the hyperbola $xy=1$.

Limits at Discontinuity

Example (p923)

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = +\infty$$



$$z = \frac{1}{x^2+y^2}$$

Continuity

Example 7 (p923)

Find $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

Solution

Let, (r,θ) be polar coordinates of the point (x,y) with $r \geq 0$. Then we have

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2+y^2$$

Moreover, Since $r \geq 0$ we have $r = \sqrt{x^2+y^2}$, so that $r \to 0^+$ if and only if $(x,y) \to (0,0)$.

Continuity

Solution

Thus, we can rewrite the given limit as

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim_{r\to 0^+} r^2 \ln r^2$$
$$= \lim_{r\to 0^+} \frac{2\ln r}{1/r^2}$$
$$= \lim_{r\to 0^+} \frac{2/r}{-2/r^3}$$
$$= \lim_{r\to 0^+} (-r^2)$$
$$= 0$$

Chapter 13.2

Homework

Exercise Set 13.2 (p925 –926)

Chapter 13.3

Partial Derivatives

Partial Derivatives of F^{n}s of Two Variables

Definition 13.3.1

If $z = f(x, y)$ and (x_0, y_0) is a point in the domain of f, then the partial derivative of f with respect to x at (x_0, y_0) [also called the partial derivative of z with respect to x at (x_0, y_0)] is the derivative at x_0 of the function that results when $y = y_0$ is held fixed and x is allowed to vary. This partial derivative is denoted by $f_x(x_0, y_0)$ and is given by

$$f_x(x_0, y_0) = \frac{d}{dx}[f(x, y_0]\bigg|_{x=x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

Partial Derivatives of F^{n}s of Two Variables

Definition 13.3.1

Similarly, the partial derivative of f with respect to y at (x_0, y_0) [also called the partial derivative of z with respect to y at (x_0, y_0)] is the derivative at y_0 of the function that results when $x = x_0$ is held fixed and y is allowed to vary. This partial derivative is denoted by $f_y(x_0, y_0)$ and is given by

$$f_y(x_0, y_0) = \frac{d}{dy}[f(x_0, y)]\bigg|_{y=y_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

Partial Derivatives of F^{n}s of Two Variables

Definition 13.3.1

Geometrically, $f_x(x_0, y_0)$ is the slope of the surface in the x −direction at (x_0, y_0) and $f_y(x_0, y_0)$ the slope of the surface in the y −direction at (x_0, y_0).

Partial Derivatives of F^{n}s of Two Variables

Example 1 (p928)

Find $f_x(1,3)$ and $f_y(1,3)$ for the function $f(x,y) = 2x^3y^2 + 2y + 4x$.

Partial Derivatives of F^{n}s of Two Variables

Solution

$$f_x(x,3) = \frac{d}{dx}[f(x,3)] = \frac{d}{dx}[18x^3 + 4x + 6] = 54x^2 + 4$$

$$\therefore f_x(1,3) = 54 \cdot (1)^2 + 4 = 58$$

And,

$$f_y(1,y) = \frac{d}{dy}[f(1,y)] = \frac{d}{dy}[2y^2 + 2y + 4] = 4y + 2$$

$$\therefore f_y(1,3) = 4 \cdot 3 + 2 = 14$$

Partial Derivative Functions

The partial derivatives as functions of the variables x and y are

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Partial Derivative Functions

Example 2 (p928)

Find $f_x(x,y)$ and $f_y(x,y)$ for the function $f(x,y) = 2x^3y^2 + 2y + 4x$ and use those partial derivatives to compute $f_x(1,3)$ and $f_y(1,3)$.

Partial Derivative Functions

Solution

Keeping y fixed and differentiating with respect to x yields

$$f_x(x, y) = \frac{\partial}{\partial x}[2x^3y^2 + 2y + 4x] = 6x^2y^2 + 4$$

Keeping x fixed and differentiating with respect to x yields

$$f_y(x, y) = \frac{\partial}{\partial y}[2x^3y^2 + 2y + 4x] = 4x^3y + 2$$

Thus,

$$f_x(1, 3) = 6(1^2)(3^2) + 4 = 58 \quad \text{and} \quad f_y(1, 3) = 4(1^3)((3) + 2 = 14$$

Partial Derivative Notation

If $z = f(x, y)$, then the partial derivatives f_x and f_y are also denoted by the symbols

$$\frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x}$$

Some typical notations for the partial derivatives of $z = f(x, y)$ at a point (x_0, y_0) are

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0, y=y_0}, \quad \left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)}, \quad \frac{\partial f}{\partial x}(x_0, y_0)$$

Partial Derivative

Example 3 (p929)

Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = x^4 \sin(xy^3)$.

Solution

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[x^4 \sin(xy^3)]$$

$$= x^4 \frac{\partial}{\partial x}[\sin(xy^3)] + \sin(xy^3) \frac{\partial}{\partial x}(x^4)$$

$$= x^4 \cos(xy^3) \,.\, y^3 + \sin(xy^3) \,.\, 4x^3$$

$$= x^4 y^3 \cos(xy^3) + 4x^3 \sin(xy^3)$$

Partial Derivative

Solution

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[x^4 \sin(xy^3)]$$

$$= x^4 \frac{\partial}{\partial y}[\sin(xy^3)]$$

$$= x^4 \cos(xy^3) \,.\, 3xy^2$$

$$= 3x^5 y^2 \cos(xy^3)$$

Partial Derivative

Example 4 (p929)

The wind chill temperature index is given by the formula

$$W = 35.74 + 0.6215T + (0.4275T - 35.75)v^{0.16}$$

Compute the partial derivative of W with respect to v at the point $(T, v) = (25, 10)$ and interpret this partial derivative as a rate of change.

Partial Derivative

Solution

Holding T fixed and differentiating with respect to v yields

$$\frac{\partial W}{\partial v}(T, v) = 0 + 0 + (0.4275T - 35.75)(0.16)v^{0.16-1}$$

$$= (0.4275T - 35.75)(0.16)v^{0.16-1}$$

Substituting $T = 25$ and $v = 10$ gives

$$\frac{\partial W}{\partial v}(T, v) = (0.4275 \times 25 - 35.75)(0.16)(10)^{0.16-1}$$

$$\approx -0.58 \frac{°\mathrm{F}}{\mathrm{mi/h}}$$

Partial Derivative

Solution

$$\frac{\partial W}{\partial v}(T, v) \approx -0.58 \frac{°\mathrm{F}}{\mathrm{mi/h}}$$

That is, the instantaneous rate of change of W with respect to v at $(T, v) = (25, 10)$ is about $-0.58\ °\mathrm{F}/(\mathrm{mi/h})$.

We conclude that if the air temperature is a constant $25\ °\mathrm{F}$ and the wind speed changes by a small amount from an initial speed of $10\ \mathrm{mi/h}$, then the ratio of the change in the wind chill index to the change in wind speed should be about $-0.58\ °\mathrm{F}/(\mathrm{mi/h})$.

Partial Derivative

Example 5 (p931)

Let $f(x, y) = x^2 y + 5y^3$.

(a) Find the slope of the surface $z = f(x, y)$ in the x −direction at the point $(1, -2)$.

(b) Find the slope of the surface $z = f(x, y)$ in the y −direction at the point $(1, -2)$.

Partial Derivative

Solution

(a) Differentiating f with respect to x with y held fixed yields

$$f_x(x, y) = 2xy$$

Thus, the slope in the x-direction is $f_x(1, -2) = -4$; that is, z is decreasing at the rate of 4 units per unit increase in x.

(b) Differentiating f with respect to y with x held fixed yields

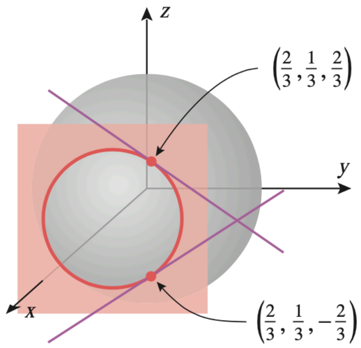
$$f_y(x, y) = x^2 + 15y^2$$

Thus, the slope in the x-direction is $f_y(1, -2) = 61$; that is, z is increasing at the rate of 61 units per unit increase in y.

Implicit Partial Differentiation

Example 7 (p931)

Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the y −direction at the points $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$.



Implicit Partial Differentiation

Solution

The point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ lies on the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$, and the point $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ lies on the lower hemisphere $z = -\sqrt{1 - x^2 - y^2}$. We could find the slopes by differentiating each expression for z separately with respect to y and then evaluating the derivatives at $x = \frac{2}{3}$ and $y = \frac{1}{3}$. However, it is more efficient to differentiate the

$$x^2 + y^2 + z^2 = 1$$

implicitly with respect to y, since this will give us both slopes with one differentiation.

Implicit Partial Differentiation

Solution

To perform the implicit differentiation, we view z as a function of x and y and differentiate both sides with respect to y, taking x to be fixed. This follows that

$$\frac{\partial}{\partial y}[x^2 + y^2 + z^2] = \frac{\partial}{\partial y}[1]$$

$$\Longrightarrow 0 + 2y + 2z\frac{\partial z}{\partial y} = 0$$

$$\Longrightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}$$

Implicit Partial Differentiation

Solution

The slope at the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

$$\left.\frac{\partial z}{\partial y}\right|_{(x,y,z)=\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)} = -\frac{1/3}{2/3} = -\frac{1}{2}$$

And the slope at the point $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$

$$\left.\frac{\partial z}{\partial y}\right|_{(x,y,z)=\left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right)} = -\frac{1/3}{-2/3} = \frac{1}{2}$$

Implicit Partial Differentiation

Example 8 (p931)

Suppose that $D = \sqrt{x^2 + y^2}$ is the length of the diagonal of a rectangle whose sides have lengths x and y that are allowed to vary. Find a formula for the rate of change of D with respect to x if x varies with y held constant, and use this formula to find the rate of change of D with respect to x at the point where $x = 3$ and $y = 4$.

Implicit Partial Differentiation

Solution

Differentiating both sides of the equation $D^2 = x^2 + y^2$ w. r. to x yields

$$2D\frac{\partial D}{\partial x} = 2x \Longrightarrow D\frac{\partial D}{\partial x} = x$$

At $x = 3$ and $y = 4$ we have $D = \sqrt{3^2 + 4^2} = 5$, it follows that

$$5\left.\frac{\partial D}{\partial x}\right|_{x=3, y=4} = 3 \Longrightarrow \left.\frac{\partial D}{\partial x}\right|_{x=3, y=4} = \frac{3}{5}$$

Thus, D is increasing at a rate of $\frac{3}{5}$ unit per unit increase in x at $(3, 4)$.

Partial Derivatives & Continuity

In contrast to the case of functions of a single variable, the existence of partial derivatives for a multivariable function does not guarantee the continuity of the function.

Functions with More that 2 Variables

For a function $f(x,y,z)$ of three variables, there are three partial derivatives:

$$f_x(x,y,z), \qquad f_y(x,y,z), \qquad f_z(x,y,z)$$

The partial derivative f_x is calculated by holding y and z constant and differentiating with respect to x. For f_y the variables x and z are held constant, and for f_z the variables x and y are held constant. If a dependent variable

$$w = f(x,y,z)$$

is used, then the three partial derivatives of f can be denoted by

$$\frac{\partial w}{\partial x}, \qquad \frac{\partial w}{\partial y}, \qquad \frac{\partial w}{\partial z}$$

Partial Differentiation

Example 10 (p933)

If $f(x,y,z) = x^3y^2z^4 + 2xy + z$, then

$$f_x(x,y,z) = ?$$

$$f_y(x,y,z) = ?$$

$$f_z(x,y,z) = ?$$

$$f_z(-1,1,2) = ?$$

Partial Differentiation

Example 11 (p933)

If $f(\rho, \theta, \phi) = \rho^2 \cos\phi \sin\theta$, then

$$f_\rho(\rho, \theta, \phi) = ?$$

$$f_\theta(\rho, \theta, \phi) = ?$$

$$f_\phi(\rho, \theta, \phi) = ?$$

Second Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

Differentiate twice with respect to x.

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Differentiate twice with respect to y.

Mixed Second Order Partial Derivatives

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

Differentiate first with respect to x and then with respect to y.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

Differentiate first with respect to y and then with respect to x.

This two cases are called the mixed second-order partial derivatives or the mixed second partials.

Mixed Second Order Partial Derivatives

Observe that the two notations for the mixed second partials have opposite conventions for the order of differentiation. In the "∂" notation the derivatives are taken right to left, and in the "subscript" notation they are taken left to right. The conventions are logical if you insert parentheses:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Right to left.

$$f_{xy} = (f_x)_y$$

Left to right.

Differentiate inside the parenthesis first.

Higher Order Partial Derivatives

Third-order, Fourth-order, Higher Order Partial Derivative

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = f_{xxx}, \qquad \frac{\partial^4 f}{\partial y^4} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y^3}\right) = f_{yyyy}$$

$$\frac{\partial^3 f}{\partial y^2\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right) = f_{xyy}, \qquad \frac{\partial^4 f}{\partial y^2\,\partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y\,\partial x^2}\right) = f_{xxyy}$$

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Higher Order Partial Derivatives

Example 12 (p934)

Find the second-order partial derivatives of $f(x, y) = x^2y^3 + x^4y$.

Solution

We have

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y$$

and $$\frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

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Higher Order Partial Derivatives

Solution

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy^3 + 4x^3y) = 2y^3 + 12x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2y^2 + x^4) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2 + x^4) = 6xy^2 + 4x^3$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy^3 + 4x^3y) = 6xy^2 + 4x^3$$

Higher Order Partial Derivatives

Example 13 (p934)

Let $f(x, y) = y^2e^x + y$. Find f_{xyy}.

Solution

$$f_{xyy} = \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y^2}(y^2e^x) = \frac{\partial}{\partial y}(2ye^x) = 2e^x$$

Equality of Mixed Partials

Theorem 13.3.2

Let f be a function of two variables. If f_{xy} and f_{yx} are continuous on some open disk, then $f_{xy} = f_{yx}$ on that disk.

Higher Order Partial Derivatives

Example 14 (p935)

Show that the function $u(x, t) = \sin(x - ct)$ is a solution of equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Solution

$$\frac{\partial u}{\partial x} = \cos(x - ct)$$

$$\frac{\partial u}{\partial t} = -c\cos(x - ct)$$

Higher Order Partial Derivatives

Solution

LHS:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}(-c\cos(x - ct)) = -c^2 \sin(x - ct)$$

RHS:

$$c^2 \frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = c^2 \frac{\partial}{\partial x}(\cos(x - ct)) = -c^2 \sin(x - ct)$$

Thus, $u(x, t)$ satisfied the given equation.

Chapter 13.3

Homework

Exercise Set 13.3 (p936 –940)

Chapter 13.4

Differentiability, Differentials, and Local Linearity

Differentiability

For a function $f(x, y)$, the symbol Δf, called the increment of f , denotes the change in the value of $f(x, y)$ that results when (x, y) varies from some initial position (x_0, y_0) to some new position $(x_0 + \Delta x, y_0 + \Delta y)$; thus

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

Let us assume that both $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and make the approximation

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

Differentiability

For Δx and Δy close to 0, we would like the error

$$\Delta f - f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

in this approximation to be much smaller than the distance $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ between (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$. We can guarantee this by requiring that

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

Differentiability

Definition 13.4.1

A function f of two variables is said to be differentiable at (x_0, y_0) provided $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist and

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

Differentiability

Example 1 (p942)

Prove that the function $f(x, y) = x^2 + y^2$ is differentiable at $(0, 0)$.

Solution

The increment is

$$\Delta f = f(0 + \Delta x, 0 + \Delta y) - f(0, 0) = (\Delta x)^2 + (\Delta y)^2$$

Since $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$,

we have $f_x(0,0) = 0$ and $f_y(0,0) = 0$.

Differentiability

Solution

and $$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{(\Delta x)^2 + (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= 0$$

Therefore, f is differentiable at $(0, 0)$.

Differentiability

Definition 13.4.2

A function f of three variables is said to be differentiable at (x_0, y_0, z_0) provided $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$ and $f_z(x_0, y_0, z_0)$ exist and

$$\lim_{(\Delta x, \Delta y, \Delta z)\to(0,0,0)} \frac{\Delta f - f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y - f_z(x_0, y_0, z_0)\Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}} = 0$$

Differentiability and Continuity

Theorem 13.4.3

If a function is differentiable at a point, then it is continuous at that point.

Proof

Homework

Differentiability and Continuity

Theorem 13.4.4

If all first-order partial derivatives of f exist and are continuous at a point, then f is differentiable at that point.

Differentials

If $z = f(x, y)$ is differentiable at a point (x_0, y_0), we let

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

denote a new function with dependent variable dz and independent variables dx and dy. We refer to this function (also denoted df) as the total differential of z at (x_0, y_0) or as the total differential of f at (x_0, y_0). Similarly, for a function $w = f(x, y, z)$ of three variables we have the total differential of w at (x_0, y_0, z_0),

$$dw = f_x(x_0, y_0, z_0)dx + f_y(x_0, y_0, z_0)dy + f_z(x_0, y_0, z_0)dz$$

which is also referred to as the total differential of f at (x_0, y_0, z_0).

Differentials

Commonly the **total differential** for a function of two variable

$$dz = f_x(x,y)dx + f_y(x,y)dy$$

and for three variables

$$dw = f_x(x,y,z)dx + f_y(x,y,z)dy + f_z\big((x,y,z)\big)dz.$$

Chapter 13.4

Homework

Exercise Set 13.4 (p947 –949)

Chapter 13.5

The Chain Rule

Chain Rules for Derivatives

Theorem 13.5.1

If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x, y) = \big(x(t),\ y(t)\big)$, then $z = f\big(x(t), y(t)\big)$ is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

Chain Rules for Derivatives

Theorem 13.5.1

If each of the functions $x = x(t)$, $y = y(t)$ and $z = z(t)$ is differentiable at t, and if $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(t), y(t), z(t))$, then the function $w = f(x(t), y(t), z(t))$ is differentiable at t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y, z).

Chain Rules for Derivatives

Example 1 (p951)

Suppose that

$$z = x^2 y, \qquad x = t^2, \qquad y = t^3$$

Use the chain rule to find dz/dt, and check the result by expressing z as a function of t and differentiating directly.

Chain Rules for Derivatives

Solution

Since $z = z(x, y)$ and $x = x(t), y = y(t)$ by the chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= (2xy)(2t) + (x^2)(3t^2)$$
$$= (2t^5)(2t) + (t^4)(3t^2)$$
$$= 7t^6$$

Chain Rules for Derivatives

Alternative Solution

Alternatively, we can express z directly as a function of t,

$$z = x^2y = (t^2)^2(t^3) = t^7$$

$$\therefore \frac{dz}{dt} = 7t^6$$

Chain Rules for Derivatives

Example 2 (p951)

Suppose that

$$w = \sqrt{x^2 + y^2 + z^2}, \qquad x = \cos\theta\,, \qquad y = \sin\theta\,, \qquad z = \tan\theta$$

Use the chain rule to find $dw/d\theta$ when $\theta = \pi/4$.

Chain Rules for Derivatives

Solution

Since $w = w(x, y, z)$ and x, y, z are function of θ by the chain rule

$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\frac{dz}{d\theta}$$

$$= \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x)(-\sin\theta) + \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2y)(\cos\theta)$$

$$+ \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2z)(\sec^2\theta)$$

$$= \frac{-x\sin\theta + y\cos\theta + z\sec^2\theta}{\sqrt{x^2 + y^2 + z^2}}$$

Chain Rules for Derivatives

Solution

When $\theta = \pi/4$, we have

$$x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \qquad y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \qquad z = \tan\frac{\pi}{4} = 1$$

Substituting these values in $\frac{dw}{d\theta}$ yield

$$\left.\frac{dw}{d\theta}\right|_{\theta=\pi/4} = \left[\frac{-x\sin\theta + y\cos\theta + z\sec^2\theta}{\sqrt{x^2+y^2+z^2}}\right]_{\theta=\pi/4}$$

Chain Rules for Derivatives

Solution

Substituting this value in $\frac{dw}{d\theta}$ yields

$$\left.\frac{dw}{d\theta}\right|_{\theta=\pi/4} = \left[\frac{-x\sin\theta + y\cos\theta + z\sec^2\theta}{\sqrt{x^2+y^2+z^2}}\right]_{\theta=\pi/4}$$

$$= \frac{-\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + 1\cdot 2}{\sqrt{2}}$$

$$= \sqrt{2}$$

Chain Rules for Partial Derivatives

Theorem 13.5.2

If $x = x(u, v)$ and $y = y(u, v)$ have first-order partial derivatives at the point (u, v), and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(u, v), y(u, v))$, then $z = f(x(u, v), y(u, v))$ has first-order partial derivatives at the point (u, v) given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \qquad \text{and} \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Chain Rules for Partial Derivatives

Theorem 13.5.2

If each function $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ has first-order partial deriva- tives at the point (u, v), and if the function $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(u, v), y(u, v), z(u, v))$, then $w = f(x(u, v), y(u, v), z(u, v))$ has first- order partial derivatives at the point (u, v) given by

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \quad \text{and} \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

Chain Rules for Partial Derivatives

Example 3 (p953)

Given that

$$z = e^{xy}, \qquad x = 2u + v, \qquad y = \frac{u}{v}$$

find $\partial z/\partial u$ and $\partial z/\partial v$ using chain rule.

Chain Rules for Partial Derivatives

Solution

Since $z = z(x, y)$ and x, y are function of u and v by the chain rule

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= (ye^{xy})(2) + (xe^{xy})\left(\frac{1}{v}\right)$$

$$= \left[2y + \frac{x}{v}\right]e^{xy}$$

$$= \left[\frac{2u}{v} + \frac{2u + v}{v}\right]e^{(2u+v)(u/v)}$$

$$= \left[\frac{4u}{v} + 1\right]e^{(2u+v)(u/v)}$$

Chain Rules for Partial Derivatives

Solution

Similarly,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= (ye^{xy})(1) + (xe^{xy})\left(-\frac{u}{v^2}\right)$$

$$= \left[y - x\left(\frac{u}{v^2}\right)\right]e^{xy}$$

$$= \left[\frac{u}{v} - (2u+v)\left(\frac{u}{v^2}\right)\right]e^{(2u+v)(u/v)}$$

$$= -\frac{2u^2}{v^2}e^{(2u+v)(u/v)}$$

Chain Rules for Partial Derivatives

Example 4 (p953)

Given that

$$w = e^{xyz}, \qquad x = 3u + v, \qquad y = 3u - v, \qquad z = u^2 v$$

Use appropriate forms of the chain rule to find $\partial w/\partial u$ and $\partial w/\partial v$.

Chain Rules for Partial Derivatives

Solution

Since $w = w(x, y, z)$ and x, y, z are function of u and v by the chain rule

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} = e^{xyz}(3yz + 3xz + 2xyuv)$$

and

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} = e^{xyz}(yz - xz + xyu^2)$$

Chain Rules for Partial Derivatives

Example 5 (p954)

Suppose that $w = x^2 + y^2 - z^2$ and

$$x = \rho \sin\phi \cos\theta\,, \qquad y = \rho \sin\phi \sin\theta\,, \qquad z = \rho\cos\phi$$

Use appropriate forms of the chain rule to find $\partial w/\partial\rho$ and $\partial w/\partial\theta$.

Chain Rules for Partial Derivatives

Solution

Since $w = w(x, y, z)$ and $x = x(\rho, \phi, \theta)$, $y = y(\rho, \phi, \theta)$, $z = z(\rho, \phi)$ by the chain rule

$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho} = ??$$

and

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} = ??$$

Other Version of Chain Rule

The chain rule extends to functions $w = f(v_1, v_2, \dots, v_n)$ of n variables. For example, if each v_i is a function of t , $i = 1, 2, \dots, n$, the relevant formula is

$$\frac{dw}{dt} = \frac{\partial w}{\partial v_1}\frac{dv_1}{dt} + \frac{\partial w}{\partial v_2}\frac{dv_2}{dt} + \cdots + \frac{\partial w}{\partial v_n}\frac{dv_n}{dt}$$

Chain Rules for Partial Derivatives

Example 6 (p954)

Suppose that

$$w = xy + yz, \qquad y = \sin x\,, \qquad z = e^x$$

Use appropriate forms of the chain rule to find dw/dx.

Chain Rules for Partial Derivatives

Solution

Since $w = w(x, y, z)$ and $y = y(x)$, $z = z(x)$ by the chain rule

$$\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx} + \frac{\partial w}{\partial z}\frac{dz}{dx}$$

$$= y + (x + z)\cos x + ye^x$$

$$= \sin x + (x + e^x)\cos x + e^x \sin x$$

Implicit Differentiation

Theorem 13.5.3

If the equation $f(x, y) = c$ defines y implicitly as a differentiable function of x, and if $\partial f/\partial x \neq 0$, then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

Implicit Differentiation

Example 7 (p955)

Given that

$$x^3 + y^2x - 3 = 0$$

Find dy/dx.

Solution

Let, $f(x, y) = x^3 + y^2x - 3$.

$$\therefore \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2 + y^2}{2yx}$$

Implicit Differentiation

Alternative Solution

Implicit differentiation of $x^3 + y^3x - 3 = 0$ with respect to x yields

$$3x^2 + y^2 + x\left(2y\frac{dy}{dx}\right) - 0 = 0$$

$$\therefore \frac{dy}{dx} = -\frac{3x^2 + y^2}{2yx}$$

Implicit Differentiation

Theorem 13.5.4

If the equation $f(x, y, z) = c$ defines z implicitly as a differentiable function of x and y, and if $\partial f/\partial z \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

Implicit Differentiation

Example 8 (p956)

Consider the sphere $x^2 + y^2 + z^2 = 1$. Find $\partial z/\partial x$ and $\partial z/\partial y$ at the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

Implicit Differentiation

Solution

Let, $f(x, y, z) = x^2 + y^2 + z^2$.

$$\therefore \frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{2x}{2z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{2y}{2z} = -\frac{y}{z}$$

At the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$,

$$\frac{\partial z}{\partial x} = -1 \quad and \quad \frac{\partial z}{\partial y} = -\frac{1}{2}$$

Chapter 13.5

Homework

Exercise Set 13.5 (p956 – 959)