Continuity

Def^n:

A function $f(x)$ is said to be continuous for $x = a$ provided $\lim_{x \to a} f(x)$ exists, is finite and is equal to $f(a)$.

OR

A function $f(x)$ is said to be continuous at $x = a$ if

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$$

OR

Briefly $f(a+0) = f(a-0) = f(a)$

OR

$$f(a+h) = f(a-h) = f(a) \text{ as } h \to 0.$$

Ex: $f(x) = |x|$ is cont^s at $x = 0$.

Continuity in an interval:

A function $f(x)$ is said to be continuous in an interval if it is cont^s at all points of the interval

In particular:

If $f(x)$ is defined in the closed interval $a \leq x \leq b$ then $f(x)$ is cont^s in the interval iff

$$\lim_{x \to x_0} f(x) = f(x_0) \text{ for } a < x_0 < b$$

Ex: $f(x) = \sqrt{9 - x^2}$ is cont^s in the closed interval $[-3, 3]$.

Discontinuous:

A function $f(x)$ is said to be discont^{s} for $x=a$ if $\lim_{x \to a} f(x) \neq f(a)$

Ex: $f(x) = \cos\frac{1}{x}$ is discont^{s} at $x=0$. since $\lim_{x \to 0} \cos\frac{1}{x}$ does not exist.

Ex: A function $f(x)$ is defined as follows

$$f(x) = x \sin\frac{1}{x} \quad \text{for } x \neq 0$$
$$= 0 \quad \text{for } x = 0$$

Show that $f(x)$ is cont^{s} at $x=0$

Sol^{n} Since $|\sin\frac{1}{x}| \leq 1$

now $|x \sin\frac{1}{x}| \leq |x| \cdot 1 = |x|$

$\Rightarrow |x \sin\frac{1}{x}| \leq |x|$

$\therefore \lim_{x \to 0} x \sin\frac{1}{x} = 0.$

Also $f(0) = 0$ as defined

Thus $\lim_{x \to 0} f(x) = f(0)$

$\therefore f(x)$ is cont^{s} at $x=0$.

Differentiation

Differentiability of a function:

A function $f(x)$ is said to be differentiable at $x=a$ if

i) Both $a+h$ and a belong to the domain of f as $h \to 0$

ii) $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists.

We write
$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$ provided the limit exists.

Now $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists mean

$$\lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0^-} \frac{f(a+h)-f(a)}{h}$$

\# $R \cdot H \cdot D = \lim\limits_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$

$L \cdot H \cdot D = \lim\limits_{h \to 0^-} \frac{f(a+h)-f(a)}{h}$ / $\lim\limits_{h \to 0^+} \frac{f(a-h)-f(a)}{-h}$.

Qus: prove that, Every finitely derivable function is continuous.

or

If $f'(a)$ is finite, $f(x)$ must be cont^s at $x=a$.

proof: Since the function $f(x)$ is differentiable at $x=a$

$$\therefore f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Now we can write

$$f(a+h)-f(a) = \frac{f(a+h)-f(a)}{h} \times h$$

$$\therefore \lim_{h\to 0}\{f(a+h)-f(a)\} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \times h$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \times \lim_{h\to 0} h$$

$$= f'(a) \times 0$$

$$= 0. \quad \text{Since } f'(a) \text{ is finite.}$$

$$\Rightarrow \lim_{h\to 0}\{f(a+h)-f(a)\} = 0$$

$$\therefore \lim_{h\to 0} f(a+h) = f(a).$$

Hence for the differentiable coefficient of $f(x)$ to exist finitely for any value of x, the function $f(x)$ must be continuous at the point.

The converse however is not always true. We shall explain by following examples.

Ex: A function $f(x)$ is defined in the following way $f(x) = |x|$ or $f(x) = \begin{cases} x & \text{when } x > 0 \\ 0 & \quad '' \quad x = 0 \\ -x & \quad '' \quad x < 0 \end{cases}$

then the function is cont^s at $x=0$ but not differentiable.

proof: $\lim_{h \to 0^+} f(0+h) = \lim_{h \to 0^+} (0+h) = 0$

$\lim_{h \to 0^-} f(0+h) = \lim_{h \to 0^-} -(0+h) = 0$

$f(0) = 0$

So $f(x)$ is continuous at $x=0$.

Again,

$$R.H.D = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{0+h-0}{h}$$

$$= \lim_{h \to 0^+} \frac{h}{h} = 1$$

$$L.H.D = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^-} \frac{-(0+h) - 0}{h} = -1$$

Since $L.H.D \neq R.H.D.$ at $x=0$

$\therefore f(x)$ is not differentiable

Ex: A function $f(x)$ is defined in the following way:

$$f(x) = \begin{cases} x \sin\frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Show that $f'(0)$ does not exist.

Soln:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(0+h)\sin\frac{1}{0+h} - 0}{h}$$

$$= \lim_{h \to 0} \sin\frac{1}{h}.$$ which does not exist.

$\therefore f'(0)$ does not exist.

Ex: A function $f(x)$ is defined in the following way:

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & " \quad x = 0 \end{cases}$$

Find $f'(0)$

Ex: $$f(x) = \begin{cases} x^2 \cos\frac{1}{x} & \text{when } x \neq 0 \\ 0 & " \text{ when } x = 0 \end{cases}$$

Find $f'(0)$.

Ex: A function $f(x)$ is defined as follows:

$$f(x) = \frac{1}{2} - x \quad \text{when } 0 < x < \frac{1}{2}$$
$$= \frac{1}{2} \quad \text{"} \quad x = \frac{1}{2}$$
$$= \frac{3}{2} - x \quad \text{"} \quad \frac{1}{2} < x < 1$$

Show that $f(x)$ is discont^{s} at $x = \frac{1}{2}$.

Sol^n

$$L.H.L = \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^-} \left(\frac{1}{2} - x\right) = \frac{1}{2} - \frac{1}{2} = 0$$

$$R.H.L = \lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^+} \left(\frac{3}{2} - x\right) = \frac{3}{2} - \frac{1}{2} = 1$$

Since L.H.L $\neq$ R.H.L

$\therefore \lim_{x \to \frac{1}{2}} f(x)$ does not exist.

Hence $f(x)$ is discont^{s} at $x = \frac{1}{2}$.

Ex:
$$f(x) = \begin{cases} x^2, & 0 < x < 1 \\ x, & 1 \leq x < 2 \\ \frac{1}{4}x^3, & 2 \leq x < 3 \end{cases}$$

Investigate continuity at $x = 1, x = 2$.

Ex: $f(x) = x - |x|$, $x = 0$ (Investigate)

Ex:
$$f(x) = \begin{cases} \sin x, & 0 < x < 1 \\ \ln x, & 1 \leq x < 2 \end{cases}$$
Investigate at $x = 1$.

Ex: Is the following function cont at the origin

$$f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Ex: $f(x) = \begin{cases} x \cos\frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{"} \ x = 0. \end{cases}$ | Investigate at $x = 0$

Ex: Is the following function cont^{s} at the origin

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Ex: $f(x) = \begin{cases} x \cos \frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{,, } x = 0. \end{cases}$ | Investigate at $x = 0$

Ex:

$$f(x) = \begin{cases} 1 & \text{for}, x < 0 \\ 1+\sin x, \text{ "}, & 0 \le x < \frac{\pi}{2} \\ 2+(x-\frac{\pi}{2})^2, & \frac{1}{2}\pi \le x \end{cases}$$

Show that $f'(x)$ exists at $x=\frac{\pi}{2}$ but does not exists at $x=0$.

For $x=\frac{\pi}{2}$

$$L.H.D = \lim_{h\to 0^-} \frac{f(\frac{\pi}{2}+h)-f(\frac{\pi}{2})}{h}$$

$$= \lim_{h\to 0^-} \frac{1+\sin(\frac{\pi}{2}+h)-2-(\frac{\pi}{2}-\frac{\pi}{2})^2}{h}$$

$$= \lim_{h\to 0^-} \frac{1+\cos h-2-0}{h}$$

$$= \lim_{h\to 0^-} \frac{-1+\cos h}{h}$$

$$= \lim_{h\to 0^-} \frac{-(1-\cos 2\cdot\frac{h}{2})}{h}$$

$$= \lim_{h\to 0^-} \frac{-2\sin^2 h/2}{h}$$

$$= -2\lim_{h\to 0^-} \left(\frac{\sin h/2}{h/2}\right)^2 \times \frac{h}{4}$$

$$= -2\cdot 1^2 \cdot \lim_{h\to 0^-} \frac{h}{4}$$

$$= -2\cdot 1\cdot 0$$

$$= 0$$

$$R.H.D = \lim_{h\to 0^+} \frac{f(\frac{\pi}{2}+h) - f(\frac{\pi}{2})}{h}$$

$$= \lim_{h\to 0^+} \frac{\cancel{2} + (\frac{\pi}{2}+h-\frac{\pi}{2})^2 - \cancel{2}}{h}$$

$$= \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h$$

$$= \frac{0}{2} = 0$$

Since $L.H.D = R.H.D$

Hence the function $f'(x)$ exists at $x = \pi/2$

For at $x = 0$

$$L.H.D = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{1-(1+\sin 0)}{h}$$

$$= \lim_{h\to 0^-} \frac{0}{h} = 0$$

$$R.H.D = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{1+\sin(0+h) - (1+\sin 0)}{h}$$

$$= \lim_{h\to 0^+} \frac{1+\sin h - 1}{h}$$

$$= \lim_{h\to 0^+} \frac{\sin h}{h} = 1.$$

Since $Rf'(0) \neq Lf'(0)$

Hence $f'(0)$ does not exist.

…t of example:

1. Discuss the continuity and differentiability of the function $f(x) = |x-1| + |x+3|$ at $x=-3$ and $x=1$.

2. Show that the function $f(x) = \begin{cases} x & \text{when } 0 \le x < \frac{1}{2} \\ 1-x & \text{"} \ \frac{1}{2} \le x < 1 \end{cases}$

is continuous at $x = \frac{1}{2}$ but not differentiable.

3. Examine the continuity of the following function at $x=0$ and $x=1$ where

$$f(x) = \begin{cases} x^2+1 & ; \text{ when } x < 0 \\ x & ; \quad \text{"} \quad 0 \le x \le 1 \\ \frac{1}{x} & ; \quad \text{"} \quad x > 1 \end{cases}$$

4. If $f(x) = \begin{cases} x^2, & \text{when } x \le 0 \\ x, & \text{"} \ 0 < x < 1 \\ \frac{1}{x}, & \text{"} \ x \ge 1 \end{cases}$ then discuss the continuity and differentiability of $f(x)$ at $x=0$ and $x=1$.

5. Find $f'(1)$, if it exists, where

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & \text{when } x \ne 1 \\ 2, & \text{"} \quad x = 1 \end{cases}$$

6. If $f(x) = \begin{cases} -x & \text{when } x \le 0 \\ x & \text{"} \quad 0 < x < 1 \\ 1-x & \text{"} \quad x \ge 1 \end{cases}$ then show that $f(x)$ is continuous at $x=0$ but discontinuous at $x=1$.

7. A function $f(x)$ is defined as follows

$$f(x) = \begin{cases} 3+2x & \text{when } -\frac{3}{2} \le x < 0 \\ 3-2x & \text{"} \quad 0 \le x \le \frac{3}{2} \\ -3-2x & \text{"} \quad x > \frac{3}{2} \end{cases}$$

prove that $f(x)$ is continuous at $x=0$ and discuss discontinuous at $x = \frac{3}{2}$.

8. If $f(x) = \begin{cases} \dfrac{x^3-8}{x-2} & \text{when } x \neq 2 \\ 10 & \text{"} \quad x = 2 \end{cases}$

prove that the function $f(x)$ is discontinuous at $x=2$. The definition of $f(x)$ for $x=2$ be the modified so as to make it continuous.

9. $f(x) = \begin{cases} \dfrac{x+|x|}{x} & \text{when } x \neq 0 \\ 2 & \text{"} \quad x = 0. \end{cases}$

Examine the continuity of the function at $x=0$.

10. $f(x) = \begin{cases} e^{-\frac{1}{(x-2)^2}} & \text{when } x \neq 2 \\ 0 & \text{"} \quad x = 2 \end{cases}$

Discuss the continuity of the function at $x=2$.

11. $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{when } x \neq 2 \\ 4 & \text{"} \quad x = 2 \end{cases}$

Discuss the continuity at $x=2$

12. $f(x) = \begin{cases} \dfrac{|x-3|}{x-3} & \text{when } x \neq 3 \\ 0 & \text{"} \quad x = 3 \end{cases}$

Discuss the continuity at $x=3$.

13. $f(x) = \begin{cases} e^x & \text{when } x > 0 \\ 2 & \text{"} \quad x = 0 \\ 1 & \text{"} \quad x < 0 \end{cases}$

Discuss the continuity at $x=0$.

14. $f(x) = \begin{cases} \dfrac{\sin(x-a)}{x-a} & \text{when } x \neq a \\ 0 & \text{"} \quad x = a. \end{cases}$

Discuss the continuity at $x=a$.

15. $f(x) = \begin{cases} (x-a)\sin\dfrac{1}{x-a} & \text{when } x \neq a \\ 0 & \text{"} \quad x = a \end{cases}$

Discuss the continuity at $x=a$.

16. Find constant a and b so that the given function will continuous for all x.

2 i) $f(x) = \begin{cases} ax + 3 & \text{when } x > 5 \\ 8 & \text{when } x = 5 \\ x^2 + bx + 1 & \text{when } x < 5. \end{cases}$

ii) $f(x) = \begin{cases} x^2 - 4x + b + 3 & \text{when } x < 1 \\ 3 & \text{when } x = 1 \\ ax + b & \text{when } x > 1 \end{cases}$

17. $f(x) = \begin{cases} \dfrac{\sin^2 ax}{x^2} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0. \end{cases}$

Prove that $f(x)$ function is discontinuous at $x = 0$ unless $a = \pm 1$

18. Test the continuity and differentiability of the function $f(x) = \begin{cases} 1 + x & \text{when } x < 0 \\ x & \text{when } 0 < x < 1 \\ 2 - x & \text{when } 1 \le x \le 2 \\ 2x - x^2 & \text{when } x > 2 \end{cases}$

at $x = 1, 2$.

19. Test the continuity of $f(x) = \begin{cases} 3 + 2x & \text{when } -\frac{3}{2} \le x < 0 \\ 3 - 2x & \text{when } 0 \le x < \frac{3}{2} \\ -3 - 2x & \text{when } x \geq \frac{3}{2}. \end{cases}$

at

19. Discuss the continuity and differentiability of $f(x) = |x-1| + 2|x-2|$ or $f(x) = \begin{cases} 3x - 5, & x > 2 \\ -x + 3, & 1 \le x \le 2 \\ -3x + 5, & x < 1 \end{cases}$

at $x = 1, 2$.

20. Discuss the continuity and differentiability at $x = 1, 2$ of

i) $f(x) = |x-1| + |x-2| = \begin{cases} 2x - 3, & x \geq 2 \\ 1, & 1 \le x < 2 \\ -2x + 3, & x < 1 \end{cases}$

ii) $f(x) = |x-2| + 2|x-3|$ or $f(x) = \begin{cases} 3x - 8, & x \geq 3 \\ -x + 4, & 2 \le x < 3 \\ -3x + 8, & x < 2 \end{cases}$

at $x = 2, 3$

iii) $f(x) = |x-2| + |x| + |x+2|$ or $f(x) = \begin{cases} 3x, & x \geq 2 \\ x + 4, & 0 \le x < 2 \\ -x + 4, & -2 \le x < 0 \\ -3x, & x < -2 \end{cases}$

at $x = -2, 0, 2$

iv) $f(x) = \begin{cases} (x-a)\sin\frac{1}{x-a}, & x \neq a \\ 0, & x = a. \end{cases}$ at $x = a$.

v) $f(x) = \begin{cases} \sqrt{|x|}, & \text{when } x \geq 0. \\ -\sqrt{|x|}, & \text{when } x < 0. \end{cases}$ at $x = 0$.

21. Function $f(x) = \begin{cases} x^2+3x+a, & \text{when } x \leq 1 \\ bx+2, & \text{when } x > 1. \end{cases}$ is derivable for every x. Find the values of a and b

22. $f(x) = \begin{cases} ax-b, & x > 1 \\ bx^2, & x \leq 1 \end{cases}$ is derivable for every values of x. Find a and b.

23. Is the function is continuous at $x=2$. If not, how may the function be defined to make it continuous at this point.

i) $f(x) = \dfrac{\sin(\pi x)}{x-2}$

(ii) $f(x) = \begin{cases} 2x+5, & x > 2 \\ 15-x^2, & x < 2 \end{cases}$

iii) $f(x) = \dfrac{x^2-x-2}{x-2}$

24. ~~that $f'(x)$~~. Show that $f(x) = |x| + |x-1|$ is cont^d but not differentiable at $x = 0, 1$

$f(x) = \begin{cases} 2x-1 & \text{for } x \geq 1 \\ 1 & \text{for } 0 \leq x < 1 \\ -2x+1 & \text{for } x < 0 \end{cases}$

9

1. Given that the function $f(x) = |x-1| + |x+3|$

when $x < -3$ then $f(x) = -(x-1) - (x+3)$ $-3 \quad 1$

$= -x+1-x-3 = -2x-2$

" $-3 \leq x \leq 1$ then $f(x) = -(x-1) + x+3 = -x+1+x+3 = 4$

" $x > 1$ " $f(x) = x-1+x+3 = 2x+2$

$$\therefore f(x) = \begin{cases} -2x-2 & \text{when } -\infty < x < -3 \\ 4 & \text{" } -3 \leq x \leq 1 \\ 2x+2 & \text{" } 1 < x < \infty. \end{cases}$$

For $x = -3$

L.H.L $= \lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (-2x-2) = -2(-3)-2 = 6-2 = 4$

R.H.L $= \lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} 4 = 4$

$f(-3) = 4$ (1)

Since L.H.L = R.H.L = $f(-3)$

$\therefore f(x)$ is cont^s at $x = -3$

For $x = 1$

L.H.L $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 4 = 4$

R.H.L $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x+2) = 2 \cdot 1 + 2 = 4$

$f(1) = 4$

Since L.H.L = R.H.L = $f(1)$

$\therefore f(x)$ is cont^s at $x = 1$.

Again for differentiability at $x = -3$

$$\text{L.H.D} = Lf'(-3) = \lim_{h \to 0^-} \frac{f(-3+h) - f(-3)}{h}$$

$$= \lim_{h \to 0^-} \frac{-2(-3+h)-2-4}{h}$$

$$= \lim_{h \to 0^-} \frac{6 - 2h - 2 - 4}{h}$$

$$= \lim_{h \to 0^-} \frac{-2h}{h} = \lim_{h \to 0^-} (-2) = -2$$

$$R.H.D = \lim_{h \to 0^+} \frac{f(-3+h) - f(-3)}{h}$$

$$= \lim_{h \to 0^+} \frac{4 - 4}{h} = 0.$$

Since $L.H.D \neq R.H.D.$

$\therefore f(x)$ is not differentiable at $x = -3$

For $x = 1$

$$L.H.D = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{4 - 4}{h} = 0.$$

$$R.H.D = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{2(1+h) + 2 - 4}{h}$$

$$= \lim_{h \to 0^+} \frac{2 + 2h + 2 - 4}{h}$$

$$= \lim_{h \to 0^+} \frac{4 + 2h - 4}{h}$$

$$= \lim_{h \to 0^+} \frac{2h}{h} = \lim_{h \to 0^+} 2 = 2$$

Since $L.H.D \neq R.H.D.$

$\therefore f(x)$ is not differential at $x = 1.$

2. Given that the function $f(x) = \begin{cases} x, & 0 \leq x < \frac{1}{2} \\ 1-x, & \frac{1}{2} \leq x < 1 \end{cases}$

For continuity at $x = \frac{1}{2}$.

$$L.H.L = \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^-} x = \frac{1}{2}$$

$$R.H.L = \lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^+} (1-x) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$f\left(\tfrac{1}{2}\right) = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

Since L.H.L = R.H.L = $f(\frac{1}{2})$

$\therefore$ f(x) is cont^{d} at $x = \frac{1}{2}$

For differentiability at $x = \frac{1}{2}$

$$L.H.D = \lim_{h\to 0^-} \frac{f(\frac{1}{2}+h) - f(\frac{1}{2})}{h}$$

$$= \lim_{h\to 0^-} \frac{\frac{1}{2}+h - (1-\frac{1}{2})}{h}$$

$$= \lim_{h\to 0^-} \frac{\cancel{\frac{1}{2}}+h - \cancel{\frac{1}{2}}}{h} = \lim_{h\to 0^-} \frac{h}{h} = 1$$

$$R.H.D = \lim_{h\to 0^+} \frac{f(\frac{1}{2}+h) - f(\frac{1}{2})}{h}$$

$$= \lim_{h\to 0^+} \frac{1-(\frac{1}{2}+h) - (1-\frac{1}{2})}{h}$$

$$= \lim_{h\to 0^+} \frac{1-\frac{1}{2}-h-\frac{1}{2}}{h} = \lim_{h\to 0^+} \frac{1-1-h}{h} = \lim_{h\to 0^+} \frac{-h}{h}$$

$$= -1.$$

Since L.H.D $\neq$ R.H.D.

$\therefore$ f(x) is not differentiable at $x = 1$

(proved)

3. Given that the function $f(x) = \begin{cases} x^2+1, & x < 0 \\ x, & 0 \le x \le 1 \\ \frac{1}{x}, & x > 1 \end{cases}$

For $x = 0$

$$L.H.L = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2+1) = 0+1 = 1$$

$$R.H.L = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x) = 0.$$

Since L.H.L $\neq$ R.H.L.

$\therefore$ f(x) is not continuous at $x = 0$

For $x = 1$

$$L.H.L = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x) = 1.$$

$$R.H.L = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x} = \frac{1}{1} = 1$$

$f(1) = 1$

Since $L.H.L = R.H.L = f(1)$. Ans.

$\therefore f(x)$ is continuous at $x = 1$.

4. Given that the function $f(x) = \begin{cases} x^2, & x \le 0 \\ x, & 0 < x < 1 \\ \frac{1}{x}, & x \geq 1 \end{cases}$

For continuity at $x = 0$

$$L.H.L = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0^2 = 0$$

$$R.H.L = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x) = 0 = 0$$

$$f(0) = 0^2 = 0$$

Since $L.H.L = R.H.L = f(0)$

$\therefore f(x)$ is cont^{s} at $x = 0$.

For differentiability at $x = 0$

$$L.H.D = \lim_{x \to 0^-} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{x \to 0^-} \frac{(0+h)^2 - 0^2}{h}$$

$$= \lim_{h \to 0^-} \frac{h^2}{h} = \lim_{h \to 0^-} h = 0.$$

$$R.H.D = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{0+h - 0}{h} = 1.$$

Since $L.H.D \neq R.H.D$, so $f(x)$ is not differentiable at $x = 0$.

For continuity at $x = 1$

$$L.H.L = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$$

$$R.H.L = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{1}{x} = \frac{1}{1} = 1$$

$$f(1) = \frac{1}{1} = 1$$

Since L.H.L = R.H.L = $f(1)$.

$\therefore f(x)$ is continuous at $x = 1$

For differentiability at $x=1$

$$L.H.D = \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0^-} \frac{1+h - \frac{1}{1}}{h}$$

$$= \lim_{h\to 0^-} \frac{1+h-1}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = 1$$

$$R.H.D = \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{\frac{1}{1+h} - \frac{1}{1}}{h}$$

$$= \lim_{h\to 0^-} \frac{\frac{\cancel{1}-\cancel{1}-h}{1+h}}{h}$$

$$= \lim_{h\to 0^-} \frac{-\cancel{h}}{(1+h)\cancel{h}} = \lim_{h\to 0^-} \frac{-1}{1+h} = \frac{-1}{1+0} = \frac{-1}{1}$$

$$= -1.$$

Since L.H.D $\neq$ R.H.D

$\therefore f(x)$ is not differentiable at $x = 1$. Ans.

5. Given that the function $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1. \\ 2, & x = 1. \end{cases}$

$$L.H.D = Lf'(1) = \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0^-} \frac{\frac{(1+h)^2-1}{\cancel{1}+h-\cancel{1}} - 2}{h}$$

$$= \lim_{h\to 0^-} \frac{\frac{\cancel{1}+2h+h^2-\cancel{1}}{h} - 2}{h}$$

$$= \lim_{h\to 0^-} \frac{\cancel{2}+h-\cancel{2}}{h} = \lim_{h\to 0^-} 1 = 1$$

$$R.H.D = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{\frac{(1+h)^2-1}{1+h-1} - 2}{h}$$

$$= 1.$$

Since L.H.D = R.H.D. So $f'(1)$ exists.

6. Given that the function $f(x) = \begin{cases} -x & , x \le 0 \\ x & , 0 < x < 1 \\ 1-x & , x \geq 1. \end{cases}$

For continuity at $x=0$:

$$L.H.L = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x) = 0$$

$$R.H.L = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$$

$$f(0) = -0 = 0.$$

Since L.H.L = R.H.L = $f(0)$

So $f(x)$ is cont^d at $x=0$.

For continuity at $x=1$

$$L.H.L = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1.$$

$$R.H.L = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1-x) = 1-1 = 0.$$

Since L.H.L $\neq$ R.H.L.

$\therefore f(x)$ is discontinuous at $x=1$. (proved)

7. For Continuity at $x=0$

$L.H.L = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (3+2x) = 3+2\cdot 0 = 3$

$R.H.L = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (3-2x) = 3-2\cdot 0 = 3$

$f(0) = 3-2\cdot 0 = 3$

Since $L.H.L = R.H.L = f(0)$.

$\therefore$ $f(x)$ is continuous at $x=0$.

(4)

For continuity at $x = \frac{3}{2}$

$L.H.L = \lim_{x\to \frac{3}{2}^-} f(x) = \lim_{x\to \frac{3}{2}^-} (3-2x) = 3-2\cdot\frac{3}{2} = 0$.

$R.H.L = \lim_{x\to \frac{3}{2}^+} f(x) = \lim_{x\to \frac{3}{2}^+} (-3-2x) = -3-2\cdot\frac{3}{2} = -6$

Since $L.H.L \neq R.H.L$.

$\therefore$ $f(x)$ is discontinuous at $x=\frac{3}{2}$. (proved)

8.

$\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)}$

$= \lim_{x\to 2} (x^2+2x+4) = 2^2+2\cdot 2+4 = 12$

Again, $f(2) = 10$.

Since $f(2) \neq \lim_{x\to 2} f(x)$

$\therefore$ $f(x)$ is discontinuous at $x=2$

2nd part:

If the function is redefined $f(x) = \begin{cases} \frac{x^3-8}{x-2}, & x\neq 2 \\ 12, & x=2 \end{cases}$

then $\lim_{x\to 2} f(x) = f(2)$

Then the function is continuous at $x=2$.

9.

$$L.H.L = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x+|x|}{x}$$

$$= \lim_{x\to 0^-} \frac{x-x}{x} = 0.$$

$$R.H.L = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x+|x|}{x}$$

$$= \lim_{x\to 0^+} \frac{x+x}{x}$$

$$= \lim_{x\to 0^+} \frac{2x}{x} = 2$$

Since $L.H.L \neq R.H.L.$ So $f(x)$ is discontinuous at $x=0$

10.

$$L.H.L = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} e^{-\frac{1}{(x-2)^2}} = e^{-\frac{1}{(2^- -2)^2}} = e^{-\frac{1}{0}} = e^{-\infty} = \frac{1}{e^{\infty}}$$

$$= \frac{1}{\infty} = 0$$

$$R.H.L = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} e^{-\frac{1}{(x-2)^2}} =$$

$$= e^{\frac{-1}{(2^+-2)^2}} = e^{\frac{-1}{0}} = e^{-\infty} = 0$$

$f(2) = 0.$

Since $L.H.L = R.H.L = f(2)$

So $f(x)$ is cont^d at $x=2$.

11. Given that the function $f(x) = \begin{cases} \frac{x^2-4}{x-2} & , x \neq 2 \\ 4 & , x = 2 \end{cases}$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} (x+2) = 2+2 = 4$$

$f(2) = 4$

Since $\lim_{x\to 2} f(x) = f(2)$

$\therefore$ $f(x)$ is cont^d at $x = 2$. Ans.

12 $L.H.L = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{|x-3|}{x-3} = \lim_{x\to 3^-} \frac{-(x-3)}{x-3} = -1$

$R.H.L = \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{|x-3|}{x-3} = \lim_{x\to 3^+} \frac{x-3}{x-3} = 1$

Since $L.H.L \neq R.H.L$

So $f(x)$ is not continuous at $x=3$.

13 $L.H.L = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1 = 1$

$R.H.L = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^x = e^0 = 1$

$f(0) = 2$

Since $L.H.L = R.H.L \neq f(0)$

So $f(x)$ is not cont^n at $x=0$.

(5)

14. ~~Same as~~

~~L.H.L~~ $\lim_{x\to a} f(x) = \lim_{(x-a)\to 0} \frac{\sin(x-a)}{x-a}$

$= 1.$

$f(a) = 0.$

Since $\lim_{x\to a} f(x) \neq f(a)$

$\therefore f(x)$ is ~~con~~ not cont^n at $x=a$.

15. $\left|(x-a)\sin\frac{1}{x-a}\right| = |x-a| \left|\sin\frac{1}{x-a}\right| \le |x-a|$

$\therefore \lim_{x\to a} \left|(x-a)\sin\frac{1}{x-a}\right| \le \lim_{x\to a} |x-a| = 0$

$\Rightarrow \lim_{x\to a} (x-a)\sin\frac{1}{x-a} = 0.$

Again, $f(a) = 0.$

Since ~~$\lim f(x)$~~ $\lim_{x\to a} f(x) = f(a)$ $\therefore$ the function is continuous. proved Ans

16. (i) Given that the function $f(x) = \begin{cases} ax+3, & \text{when } x>5 \\ 8, & \text{"} \quad x=5 \\ x^2+bx+1, & \text{"} \quad x<5 \end{cases}$

when $x=5$ then $f(5)=8$.

$\lim_{x\to5^-} f(x) = \lim_{x\to5^-} (x^2+bx+1) = 5^2+5b+1 = 5b+26$

$\lim_{x\to5^+} f(x) = \lim_{x\to5^+} (ax+3) = 5a+3$.

Since the function is continuous

$\therefore 5b+26 = 8 = 5a+3$

$\Rightarrow \therefore 5b+26=8$ and $5a+3=8$

$\Rightarrow 5b = -18$

$\therefore b = -\frac{18}{5}$

$\boxed{\begin{array}{l} \Rightarrow 5a\left(-\frac{18}{5}\right) = 8 \\ \Rightarrow 5a = 8+\frac{18}{5} = \frac{40+18}{5} = \frac{58}{5} \end{array}}$

$\Rightarrow 5a = 5$

$\therefore a = 1$.

$\therefore \left.\begin{array}{l} a=1 \\ b=-\frac{18}{5} \end{array}\right\}$ Ans.

(ii) when $x=1$ then $f(1)=3$

$L.H.L = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} (x^2-4x+b+3) = 1-4+b+3 = b$

$R.H.L = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} (ax+b) = a.1+b = a+b$

Since the function is continuous

$\therefore b = a+b = 3$

$\therefore b=3$ and $a+b=3$

$\Rightarrow a+3=3$

$\therefore a=0$

$\therefore \left.\begin{array}{l} a=0 \\ b=3 \end{array}\right\}$ Ans.

17. $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin^2 ax}{x^2} = \lim_{x\to 0} \left(\frac{\sin ax}{ax}\right)^2 \cdot a^2$

$= \lim_{x\to 0} a^2 \cdot 1^2$

$= a^2$

Again $f(0) = 1$.

If $a^2 = 1 \Rightarrow a = \pm 1$ then $\lim_{x\to 0} f(x) = f(0) = 1$.

$\therefore f(x)$ is continuous at $x = 0$.

If $a^2 \neq 1$ i.e $a \neq \pm 1$ then $\lim_{x\to 0} f(x) \neq f(0)$

Thus the function $f(x)$ is discontinuous at $x = 0$.

$\therefore$ The function $f(x)$ is discontinuous unless $a = \pm 1$

18. For continuity at $x = 1$ (6)

$L.H.L = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$

$R.H.L = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = 2-1 = 1$

$f(1) = 2-1 = 1$

Since $L.H.L = R.H.L = f(1)$

$\therefore f(x)$ is cont^s at $x = 1$.

For continuity at $x = 2$

$L.H.L = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2-x) = 2-2 = 0$

$R.H.L = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (2x - x^2) = 2\cdot 2 - 2^2 = 0$.

$f(2) = 2-2 = 0$.

Since $L.H.L = R.H.L = f(2)$.

So $f(x)$ is cont^s at $x = 2$.

For differentiability at $x = 1$

$L.H.D = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$

$$= \lim_{h\to 0^-} \frac{1+h-(2-1)}{h}$$

$$= \lim_{h\to 0} \frac{1+h-1}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

$$R.H.D = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{2-(1+h)-(2-1)}{h}$$

$$= \lim_{h\to 0^+} \frac{2-1-h-1}{h} = -1$$

Since $L.H.D \neq R.H.D.$

So $f(x)$ is not differentiable at $x=1$

For differentiability at $x=2$

$$\therefore L.H.D = \lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0^-} \frac{2-(2+h)-(2-2)}{h}$$

$$= \lim_{h\to 0^-} \frac{\cancel{2}-\cancel{2}-h-0}{h}$$

$$= \lim_{h\to 0^-} \frac{-h}{h} = -1.$$

$$R.H.D = \lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0^+} \frac{2(2+h)-(2+h)^2-(2-2)}{h}$$

$$= \lim_{h\to 0^+} \frac{\cancel{4}+2h-\cancel{4}-4h-h^2-0}{h}$$

$$= \lim_{h\to 0^+} \frac{-2h-h^2}{h}$$

$$= \lim_{h\to 0^-} \frac{-\cancel{h}(2+h)}{\cancel{h}} = -2+0 = -2$$

Since $L.H.D \neq R.H.D$

$\therefore f(x)$ is not differentiable at $x=2$.

12 Here $f(x) = |x-1| + 2|x-2|$

For $2 < x < \infty$, $f(x) = x - 1 + 2(x-2)$
$= x - 1 + 2x - 4$
$= 3x - 5$

For $1 \le x \le 2$, $f(x) = x - 1 + 2\{-(x-2)\}$
$= x - 1 - 2x + 4$
$= -x + 3$

For $-\infty < x < 1$, $f(x) = -(x-1) - 2(x-2)$
$= -x + 1 - 2x + 4$
$= -3x + 5$ 7

$$\therefore f(x) = \begin{cases} 3x - 5 & \text{when } x > 2 \\ -x + 3 & \text{"} \quad 1 \le x \le 2 \\ -3x + 5 & \text{"} \quad x < 1 \end{cases}$$

For continuity at $x = 1$

$L.H.L = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (-3x + 5) = -3 \cdot 1 + 5 = 2$

$R.H.L = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-x + 3) = -1 + 3 = 2$

$f(1) = -1 + 3 = 2$

Since $L.H.L = R.H.L = f(1)$

$\therefore f(x)$ is continuous at $x = 1$

For continuity at $x = 2$

$L.H.L = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (-x + 3) = -2 + 3 = 1$

$R.H.L = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 5) = 3 \cdot 2 - 5 = 1$

$f(2) = -2 + 3 = 1$

Since $L.H.L = R.H.L = f(2)$

$\therefore f(x)$ is cont^{s} at $x = 2$

For differentiability at $x=1$

$$L.H.D = \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0^-} \frac{-3(1+h)+5 - (-1+3)}{h}$$

$$= \lim_{h\to 0^-} \frac{-3-3h+5-2}{h}$$

$$= \lim_{h\to 0^-} \frac{-3h}{h} = -3$$

$$R.H.D = \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{-(1+h)+3 - (-1+3)}{h}$$

$$= \lim_{h\to 0^+} \frac{-1-h+3-2}{h}$$

$$= \lim_{h\to 0^+} \frac{-h}{h} = 1$$

Since $L.H.D \neq R.H.D.$

$\therefore$ $f(x)$ is not differentiable at $x=1.$

For differentiability at $x=2$

$$R.H.D = \lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0^+} \frac{3(2+h)-5 - (-2+3)}{h}$$

$$= \lim_{h\to 0^+} \frac{6+3h-5-1}{h}$$

$$= \lim_{h\to 0^+} \frac{3h}{h} = 3$$

$$R.H.D = \lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0^+} \frac{-(2+h)+3 - (-2+3)}{h}$$

$$= \lim_{h\to 0^-} \frac{-2-h+3-1}{h}$$

$$= \lim_{h\to 0^-} \frac{-h}{h} = -1$$

Since $L.H.D = R.H.D.$

$\therefore$ $f(x)$ is non-differentiable at $x=2$. Ans.

~~proved~~

20(iv) ~~For differentiability~~ Continuity at $x=a$

$$\lim_{x\to a} f(x) = \lim_{x\to a} (x-a)\sin\frac{1}{x-a}$$

now $\left|(x-a)\sin\frac{1}{x-a}\right| = |x-a|\left|\sin\frac{1}{x-a}\right|$

$\leq |x-a|\cdot 1$ since $\left|\sin\frac{1}{x-a}\right|\leq 1$

$= |x-a|$

$\therefore \lim_{x\to a}|f(x)| \leq \lim_{x\to a}|x-a| = 0.$

$\therefore \lim_{x\to a} f(x) = 0.$

$f(a) = 0$

Since $\lim_{x\to a} f(x) = f(a) = 0.$

$\therefore f(x)$ is cont$^{\underline{s}}$ at $x=a.$

For differentiability at $x=a$

(8)

$$L.H.D = \lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h\to 0^-}\frac{(a+h-a)\sin\frac{1}{a+h-a} - 0}{h}$$

$$= \lim_{h\to 0^-}\frac{\cancel{h}\sin\frac{1}{h}}{\cancel{h}}$$

$$= \lim_{h\to 0^-}\sin\frac{1}{h}$$

which does not exist.

$\therefore f(x)$ is not differentiable at $x=a.$

Given that the function $f(x) = \begin{cases}(x-a)^2\sin\frac{1}{x-a} & , x\neq a.\\ 0 & , x=a.\end{cases}$

$$L.H.D = \lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h\to 0^-}\frac{(a+h-a)^2\sin\frac{1}{a+h-a} - 0}{h}$$

$$= \lim_{h\to 0^-}\frac{h^2\sin\frac{1}{h}}{h}$$

$$= \lim_{h\to 0^-} h\sin\frac{1}{h}$$

now $\left|h\sin\frac{1}{h}\right| = |h|\left|\sin\frac{1}{h}\right| \leq |h|$

$\therefore \lim_{h\to 0^-}\left|h\sin\frac{1}{h}\right| \leq \lim_{h\to 0^-}|h| = 0.$

$\Rightarrow \left|\lim_{h\to 0^-} h\sin\frac{1}{h}\right| \leq 0$

$\Rightarrow \lim_{h\to 0^-} h\sin\frac{1}{h} = 0.$

$\therefore L.H.D = 0.$

$$R.H.D = \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0^+} \frac{(a+h-a)^2 \sin\frac{1}{a+h-a} - 0}{h}$$

$$= \lim_{h\to 0^+} \frac{h^2 \sin\frac{1}{h}}{h}$$

$$= \lim_{h\to 0^+} h \sin\frac{1}{h}.$$

Now $\left|h \sin\frac{1}{h}\right| \le |h|$

$\therefore \lim_{h\to 0^+} \left|h \sin\frac{1}{h}\right| \le \lim_{h\to 0^+} |h| = 0$

$\therefore \lim_{h\to 0^+} h \sin\frac{1}{h} = 0.$

$R.H.D = 0.$

Since $L.H.D = R.H.D$

So $f(x)$ is ~~diffe~~ differentiable at $x = a$.

For continuity at $x=0$.

$$L.H.L = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \{-\sqrt{|x|}\} = \lim_{x\to 0^-} \{-\sqrt{-(-x)}\}$$

$$= \lim_{x\to 0^-} \{-\sqrt{-x}\} = -\sqrt{0} = 0$$

$$R.H.L = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{|x|} = \lim_{x\to 0^+} \sqrt{x} = \sqrt{0} = 0$$

$f(0) = \sqrt{|0|} = 0.$

Since $L.H.L = R.H.L = f(0)$

So $f(x)$ is cont^d at $x=0$.

For differentiability at $x=0$

$$L.H.D = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0^-} \frac{-\sqrt{|h|} - \sqrt{|0|}}{h}$$

$$= \lim_{h\to 0^-} \frac{-\sqrt{|h|} - 0}{h}$$

$$= \lim_{h\to 0^-} -\frac{1}{\sqrt{h}} = \text{undefined.}$$

So $f(x)$ is not differentiable at $x=0$.

21. Since f is derivable for every x. So f must be derivable at $x=1$ and hence f must be continuous at $x=1$.

Continuity at $x=1$.

when $x=1$, then $f(1) = 1+3+a = a+4$

$L.H.L = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2+3x+a = 4+a.$

$R.H.L = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} (bx+2) = b+2.$

Since $L.H.L = R.H.L = f(1)$.

$\therefore a+4 = b+2 = a+4$

$\Rightarrow a+4 = b+2 \cdots$ (i)

$\Rightarrow a-b = -2 \longrightarrow$ (ii)

(3)

Differentiability at $x=1$

$$Lf'(1) = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^-} \frac{(1+h)^2+3(1+h)+a-(1+3+a)}{h}$$

$$= \lim_{h\to 0^-} \frac{1+2h+h^2+3+3h+a-4-a}{h}$$

$$= \lim_{h\to 0^-} \frac{h^2+5h}{h}$$

$$= \lim_{h\to 0^-} (h+5) = 5$$

$$Rf'(1) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{b(1+h)+2-(a+1+3)}{h}$$

$$= \lim_{h\to 0^+} \frac{b+bh+2-(a+4)}{h}$$

$$= \lim_{h\to 0^+} \frac{b+bh+2-(b+2)}{h} \quad \left[\text{From } (1)\right]$$

$$= \lim_{h\to 0^+} \frac{\cancel{b}+bh+\cancel{2}-\cancel{b}-\cancel{2}}{h}$$

$$= \lim_{h\to 0^+} \frac{b\cancel{h}}{\cancel{h}} = b.$$

Since f is derivable at $x=1$

$$\therefore Lf'(1) = Rf'(1)$$

$$\Rightarrow 5 = b \qquad \therefore b = 5$$

From (ii) we have

$$a-5 = -2$$

$$\Rightarrow a = 5-2 \qquad \therefore a = 3$$

$\therefore a=3,\ b=5.$ Ans.

22. Since f is derivable ~~for~~ every value of x, so f is derivable at $x=1$ and hence f is continuous at $x=1$.

$$L.H.L = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} bx^2 = b.1^2 = b$$

$$R.H.L = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (ax-6) = a.1-6 = a-6.$$

$$f(1) = b.1^2 = b$$

Since $f(x)$ is cont^s at $x=1$

$$\therefore L.H.L = R.H.L = f(1)$$

$$\Rightarrow b = a-6 = b$$

$$\therefore b = a-6 \quad \cdots (i)$$

$$\Rightarrow a-b-6 = 0 \quad \cdots (ii)$$

Again, $L.H.D = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h}$

$$= \lim_{h\to 0^-} \frac{b(1+h)^2 - b.1}{h}$$

$$= \lim_{h\to 0^-} \frac{\cancel{b}+2bh+bh^2-\cancel{b}}{h}$$

$$= \lim_{h\to 0^-} \frac{h(2b+bh)}{h} = \lim_{h\to 0^-} (2b+bh) = 2b$$

$$Rf'(1) = \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{a(1+h) - 6 - b\cdot 1^2}{h}$$

$$= \lim_{h\to 0^+} \frac{a + ah - 6 - b}{h}$$

$$= \lim_{h\to 0^+} \frac{a + ah - 6 - (a-6)}{h} \quad \text{[From (i)]}$$

$$= \lim_{h\to 0^+} \frac{\cancel{a} + ah - \cancel{6} - \cancel{a} + \cancel{6}}{h}$$

$$= a$$

(10)

Since $f(x)$ is derivable at $x=1$ so

L.H.D = R.H.D

$\Rightarrow \quad 2b = a$ .... (iii)

From (ii) and (iii) we have.

$$2b - b - 6 = 0$$

or $\quad b - 6 = 0 \quad \therefore b = 6.$

$$\therefore a = 2b = 2\cdot 6 = 12$$

$\therefore a = 12,\ b = 6.$ Ans.

23 (i) $\lim_{x\to 2} \frac{\sin \pi x}{x-2}$ putting $x = 2+h \quad \therefore x - 2 = h$

when $x \to 2 \quad \therefore h \to 0.$

$$\therefore \lim_{h\to 0} \frac{\sin \pi(2+h)}{h} = \lim_{h\to 0} \frac{\sin(2\pi + \pi h)}{h} = \lim_{h\to 0} \frac{\sin \pi h}{\pi h}\cdot \pi$$

$$= 1\cdot\pi$$

$$= \pi$$

Again when $x = 2$ then $f(x)$ is undefined.

$\therefore f(x)$ is not cont^{n} at $x = 2$

If the function f(x) can be made continuous at x = 2 then the function is defined by as following

$$f(x) = \begin{cases} \frac{\sin \pi x}{x-2} & \text{when } x \neq 2 \\ \pi & \text{"} \quad x = 2 \end{cases}$$

(ii) $L.H.L = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (15 - x^2) = 15 - 2^2 = 15 - 4 = 11$

$R.H.L = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x+5) = 2 \cdot 2 + 5 = 9$

Since L.H.L = R.H.L

∴ f(x) is not cont^n at x = 2

If the function f(x) can be made continuous at x = 2 then the function is defined by as following

$$f(x) = \begin{cases} 2x+7 & \text{when } x \geq 2 \\ 15 - x^2 & \text{"} \quad x < 2 \end{cases}$$

Ans.

iii) Answer: $f(x) = \begin{cases} \frac{x^2 - x - 2}{x-2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$