Computer Models for Physical Processes

Project presentation
on
“Finite Difference Method for Stationary 2D heat conduction problem”

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Contents

Introduction

The focus of this project is to determine the temperature distribution within the plate in a steady-state condition.

On a broader level the project can be split into the below 5 parts :

1. Physical Problem Formulation
2. Mathematical Formulation to establish an appropriate finite difference term
3. Investigation of the temperature field using MATLAB® implementation
4. Analytical validation using ANSYS® Fluent
5. Evaluation of results

Introduction

1. Physical Problem Formulation

Material parameters

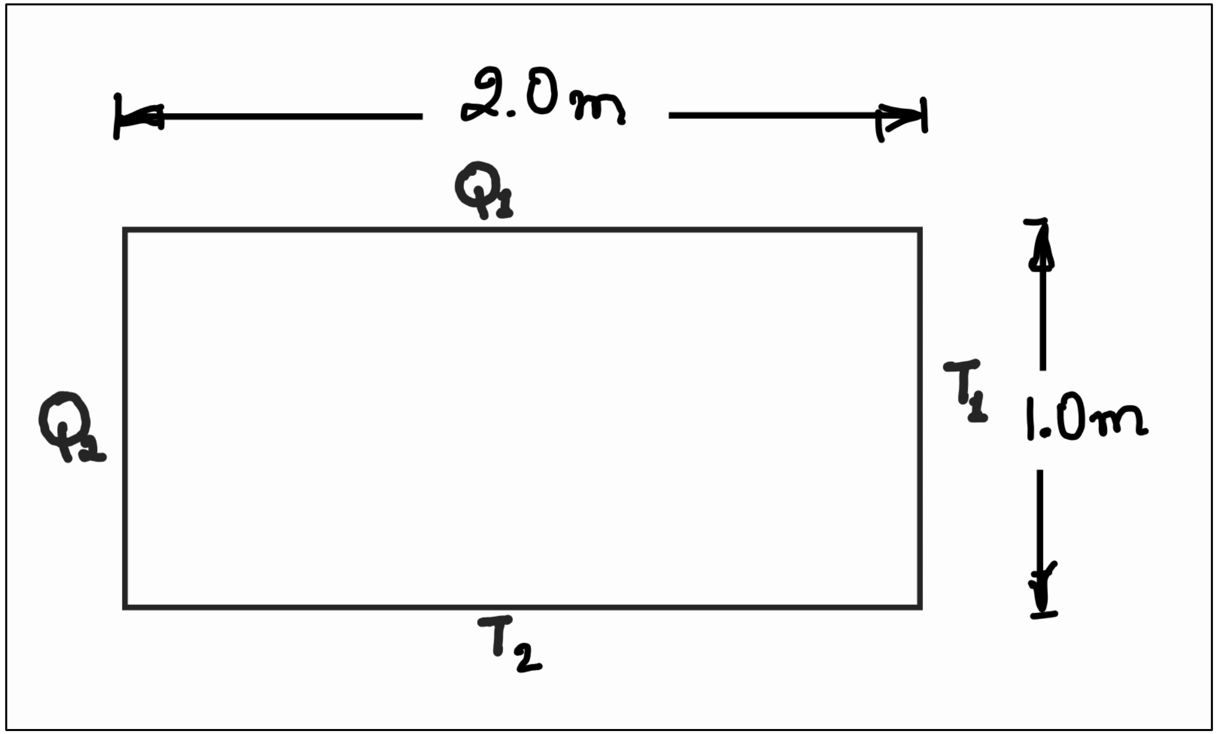
$heat\ conductivity\ c = 50\frac{W}{mK}$

$thickness\ of\ plate\ h\ = 0.15m$

Dimension = $2.0m\ \times\ 1.0m$

Boundary values:

$$T_1 = 50°\text{C}$$
$$T_2 = 10°\text{C}$$
$$Q_1 = -200\frac{W}{m^2}\ (heat\ output)$$
$$Q_2 = 0\frac{W}{m^2}$$



Mathematical Formulation to establish an appropriate finite difference term

General Equation for Heat Conduction Problem

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{Q}{K} = \rho \times C \times \frac{\partial T}{\partial t} \longrightarrow \text{Eq.1}$$

Assumptions and Simplifications

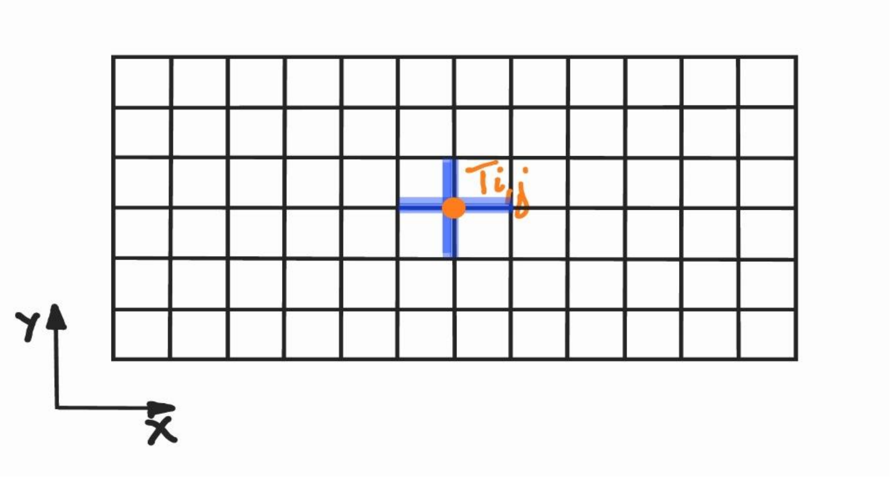
- 2D Steady-State Condition $\frac{\partial^2 T}{\partial z^2} \,\&\, \boldsymbol{\frac{\partial T}{\partial t} = 0}$
- Constant Thermal Conductivity $\boldsymbol{C}$
- No Internal Heat Generation $\boldsymbol{Q = 0}$

$$\boxed{\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0} \longrightarrow \text{Eq.2}$$

Mathematical Formulation to establish an appropriate finite difference term

Approximation of Derivatives using Finite Differences

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

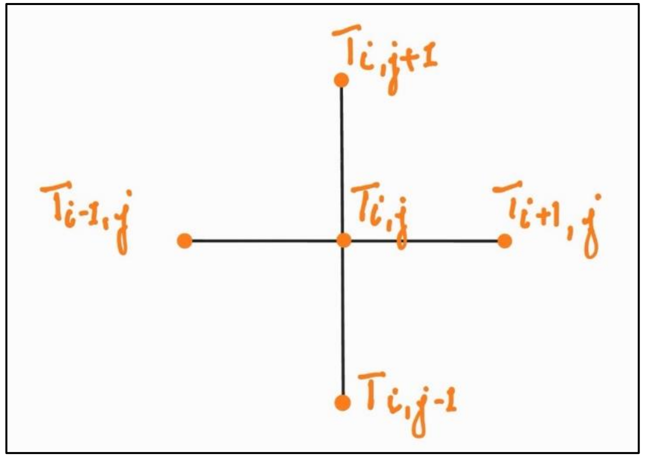


- **First-order Derivatives: Approximation of $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial y}$**
  - Central Difference (for $\frac{\partial T}{\partial x}$):

    $$\frac{\partial T}{\partial x} \approx \frac{(T_{i+1,j} - T_{i-1,j})}{2 \times \Delta x} \longrightarrow \text{Eq.3}$$

  - Central Difference (for $\frac{\partial T}{\partial y}$):

    $$\frac{\partial T}{\partial x} \approx \frac{(T_{i,j+1} - T_{i,j-1})}{2 \times \Delta x} \longrightarrow \text{Eq.4}$$



- **Second-order Derivatives: Approximation of $\frac{\partial^2 T}{\partial x^2}$ and $\frac{\partial^2 T}{\partial y^2}$**
  - Central Difference $(for\ \frac{\partial^2 T}{\partial x^2})$:

    $$\frac{\partial^2 T}{\partial x^2} \approx \frac{(T_{i+1,j} - 2T_{i,j} + T_{i-1,j})}{\Delta x^2} \longrightarrow \text{Eq.5}$$

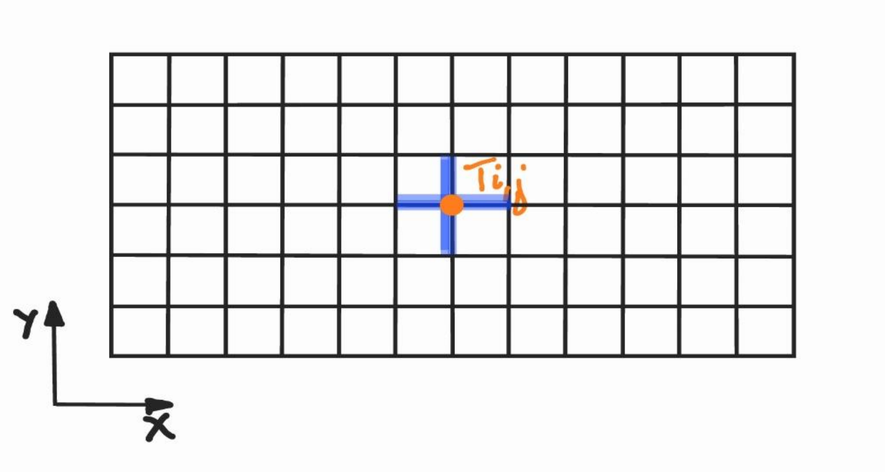
  - Central Difference $(for\ \frac{\partial^2 T}{\partial y^2})$:

    $$\frac{\partial^2 T}{\partial y^2} \approx \frac{(T_{i,j+1} - 2T_{i,j} + T_{i,j-1})}{\Delta y^2} \longrightarrow \text{Eq.6}$$

Mathematical Formulation to establish an appropriate finite difference term

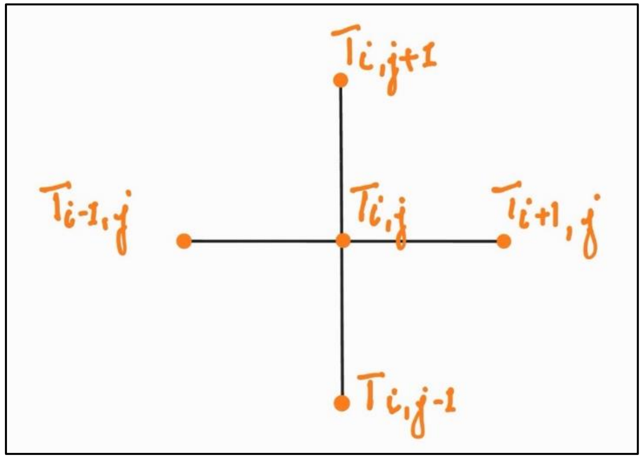
Approximation of Derivatives using Finite Differences

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$



For $\Delta x \neq \Delta y$, unequal grid spacing

$$T_{i,j} = \frac{(\Delta y^2) \times (T_{i+1,j} + T_{i-1,j}) + (\Delta x^2) \times (T_{i,j+1} + T_{i,j-1})}{2 \times (\Delta x^2 + \Delta y^2)} \longrightarrow \text{Eq.7}$$



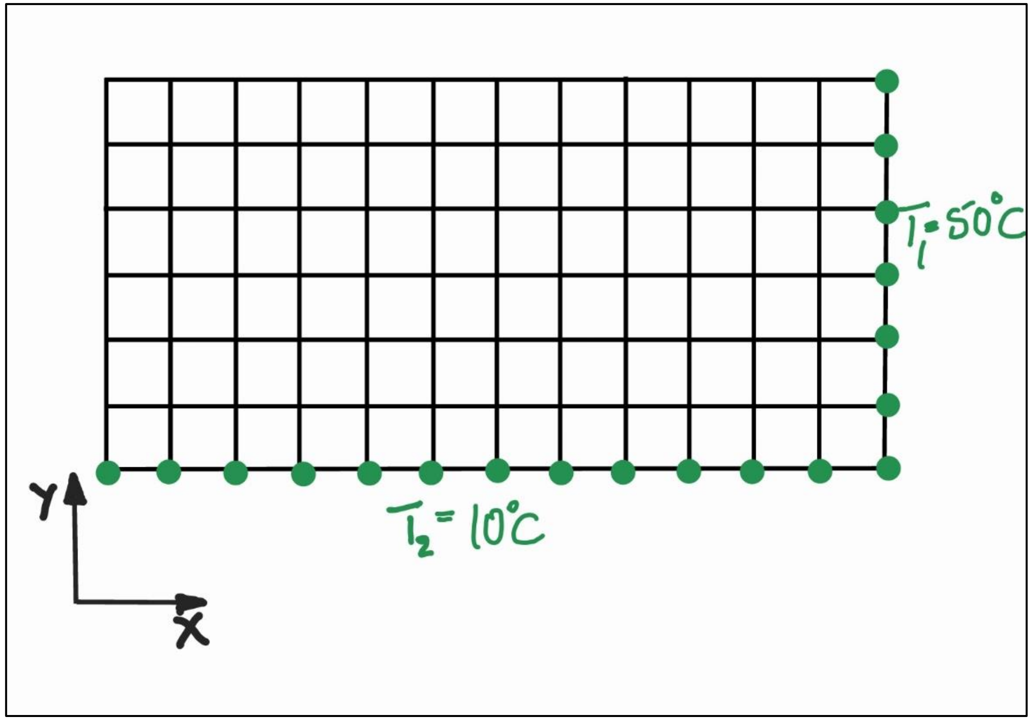
For $\Delta x = \Delta y$, for equal grid spacing

$$T_{i,j} = \frac{(T_{i+1,j}) + (T_{i-1,j}) + (T_{i,j+1}) + (T_{i,j-1})}{4} \longrightarrow \text{Eq.8}$$

Boundary Conditions

Temperature Boundary Conditions (T)

- $right\ side\ column\ of\ grid\ points = T_1$
- $bottom\ side\ row\ of\ grid\ points = T_2$



Boundary Conditions

- Heat Flux Boundary Conditions (Q)

➢ **For left side boundary condition (Q_2):**

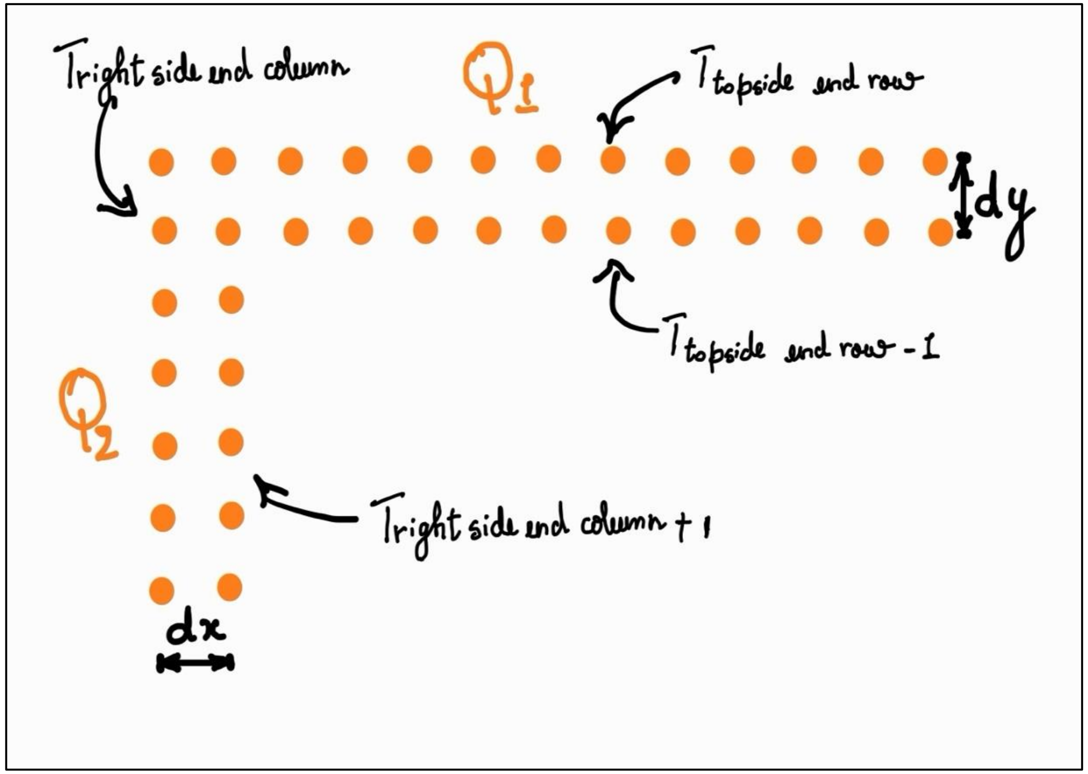
$$q = -C \times \frac{\partial T}{\partial x}$$

$$q = -C \times \frac{(T_{i,j} - T_{i+1,j})}{\Delta x} \longrightarrow \text{Eq.9}$$

➢ **For Top side boundary condition (Q_1)**

$$q = -C \times \frac{\partial T}{\partial y}$$

$$q = -C \times \frac{(T_{i,j} - T_{i,j-1})}{\Delta y} \longrightarrow \text{Eq.10}$$

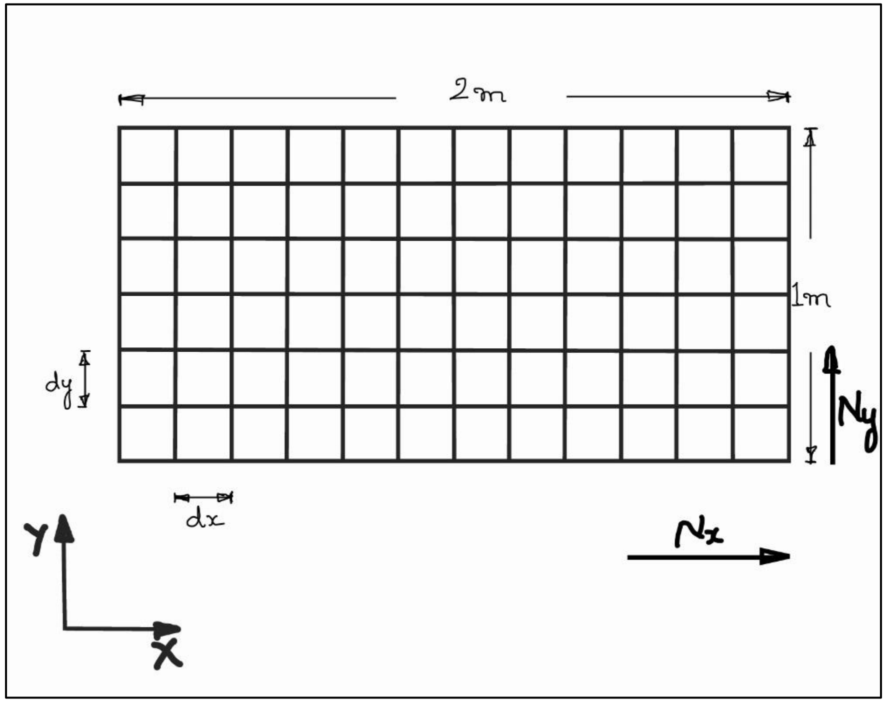


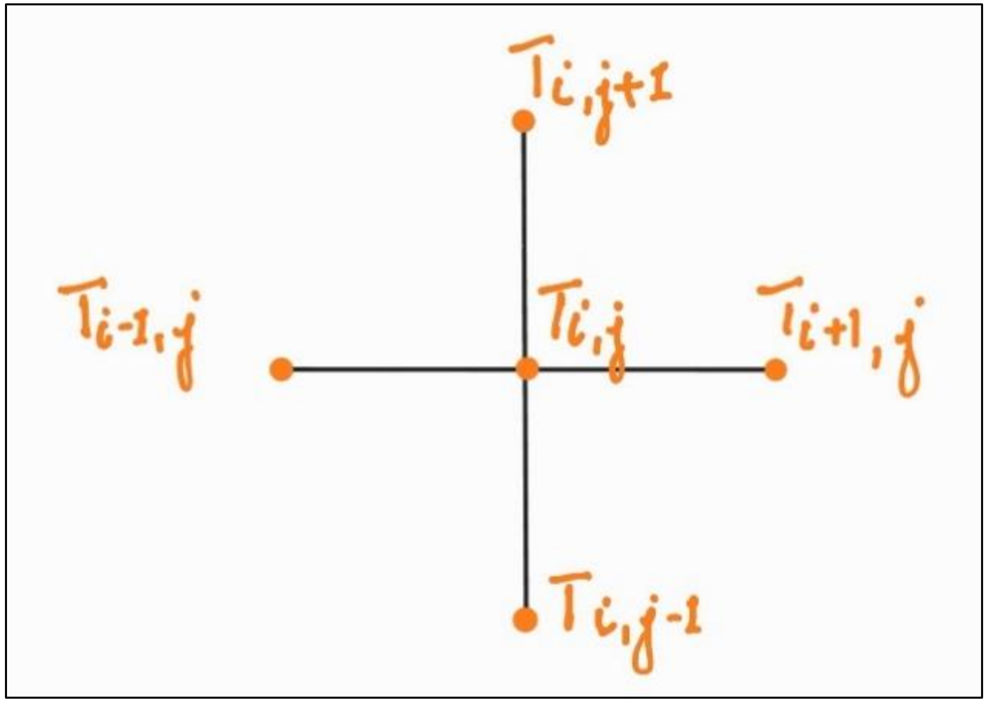
$$T_{left\ side\ end\ \ column} = T_{left\ side\ end\ column-1} - \frac{(Q_2 \times d_x)}{c}$$

$$T_{top\ side\ end\ \ row} = T_{top\ side\ end\ row-1} - \frac{(Q_1 \times d_y)}{c}$$

Investigation of the temperature field using MATLAB® implementation

Discretization of The 2D Plate

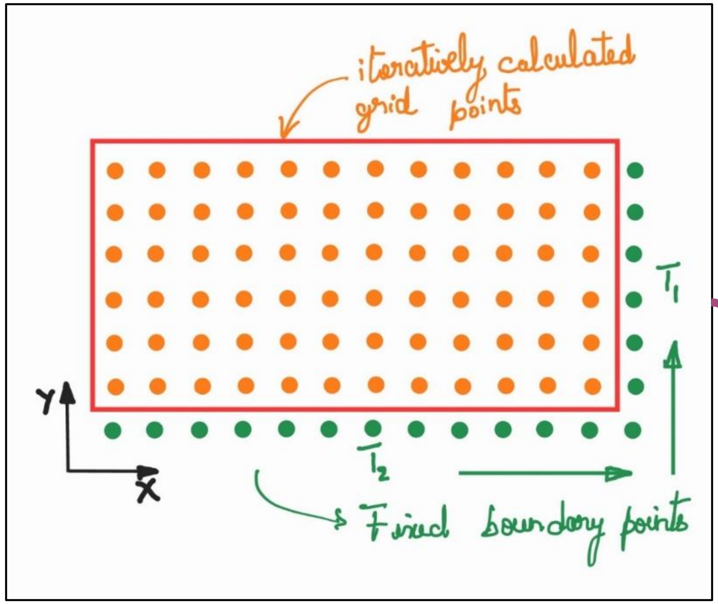


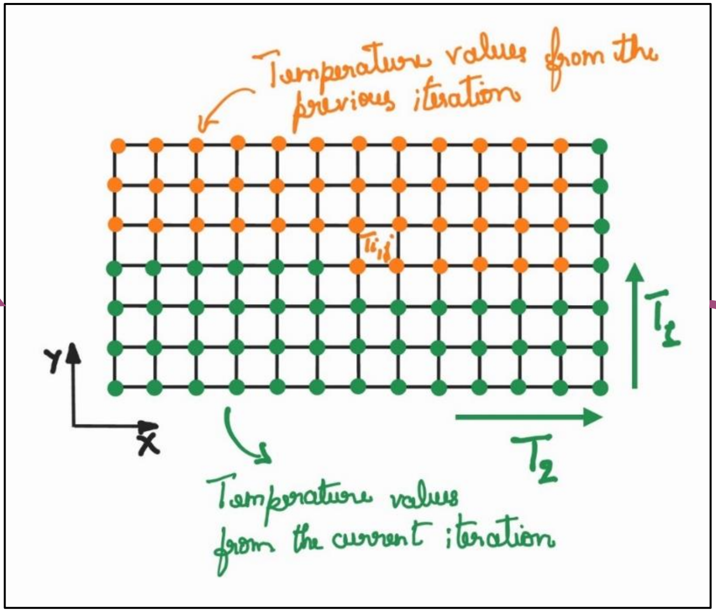


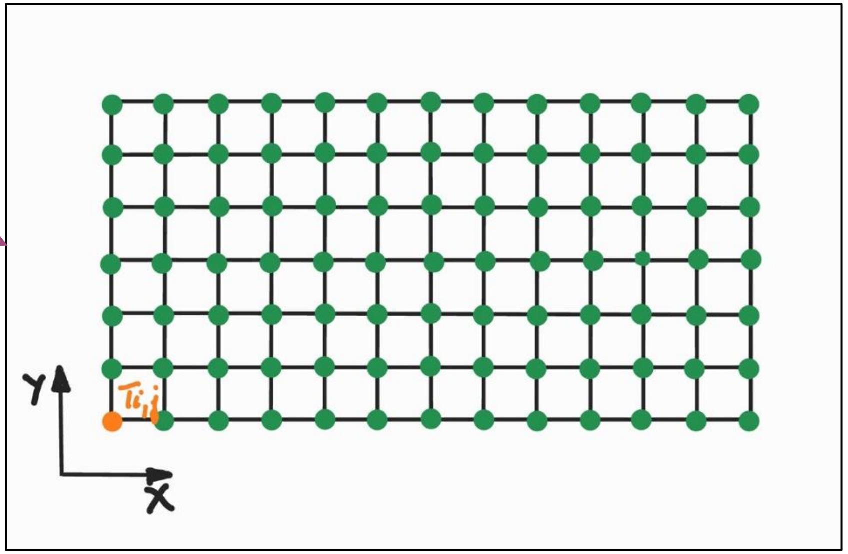
$$dx = \frac{L_x}{N_x - 1}$$

$$dy = \frac{L_y}{N_y - 1}$$

Investigation of the temperature field using MATLAB® implementation

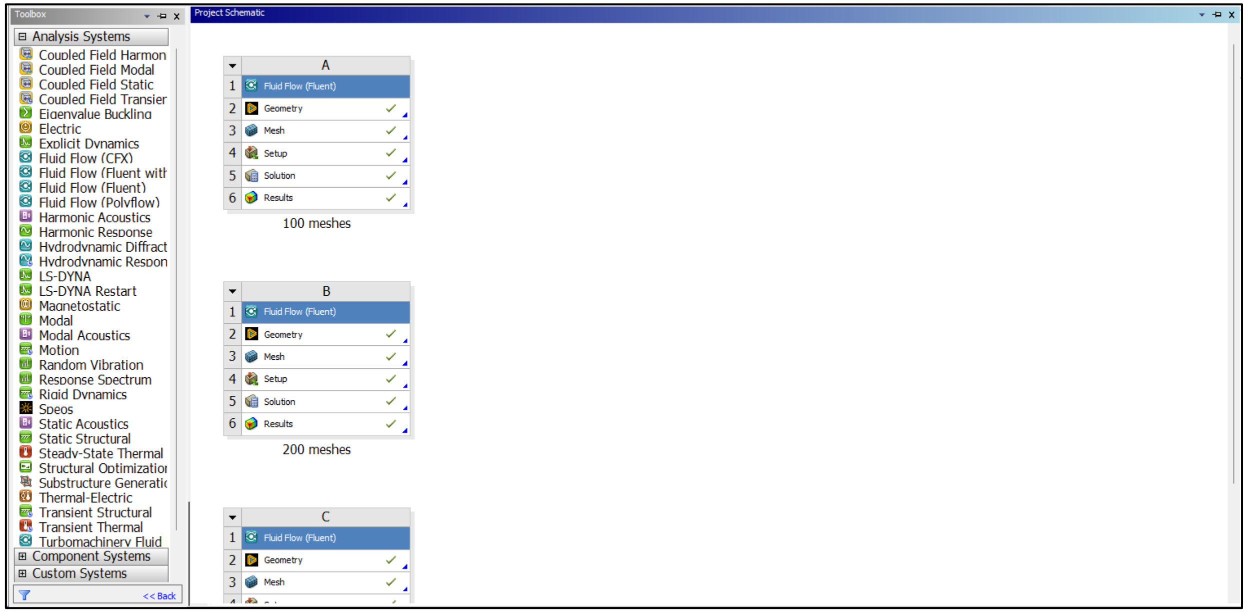






Analytical Validation

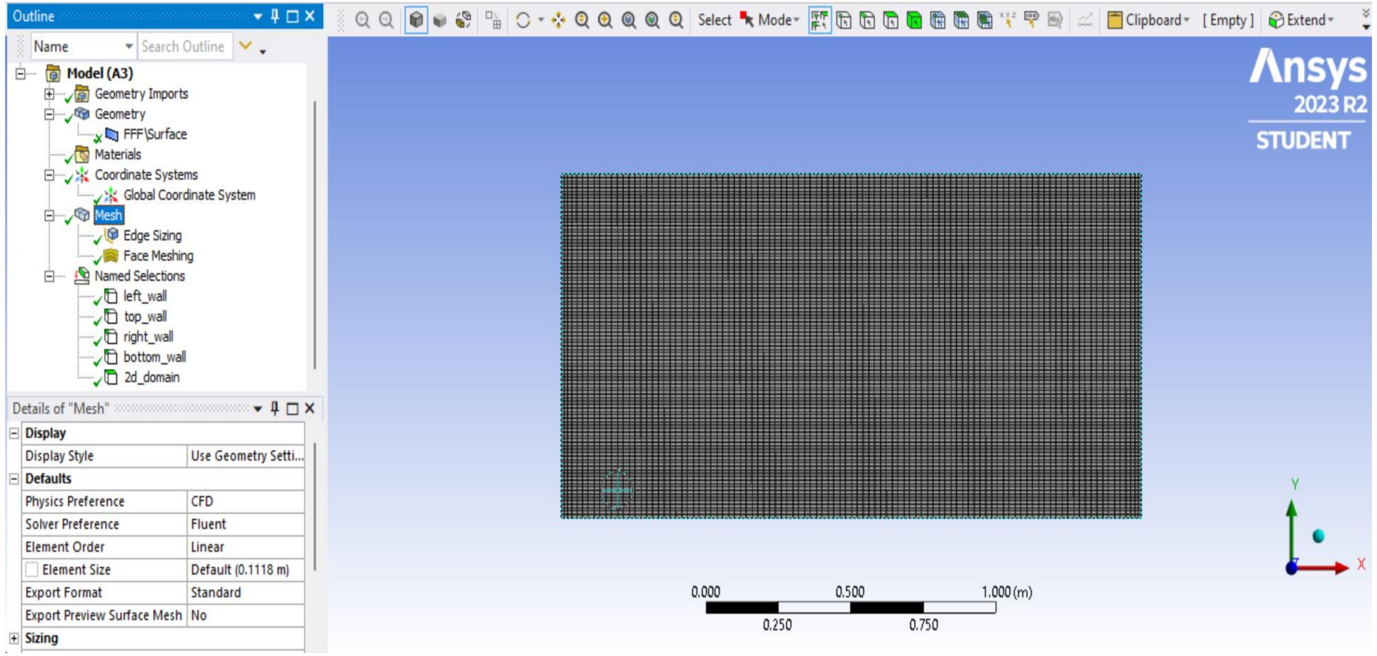
- ANSYS® Fluent is used to validated the MATLAB data
- The Fluid Flow (Fluent) in ANSYS® Workbench is used to do the required simulation



Analytical Validation

Mesh

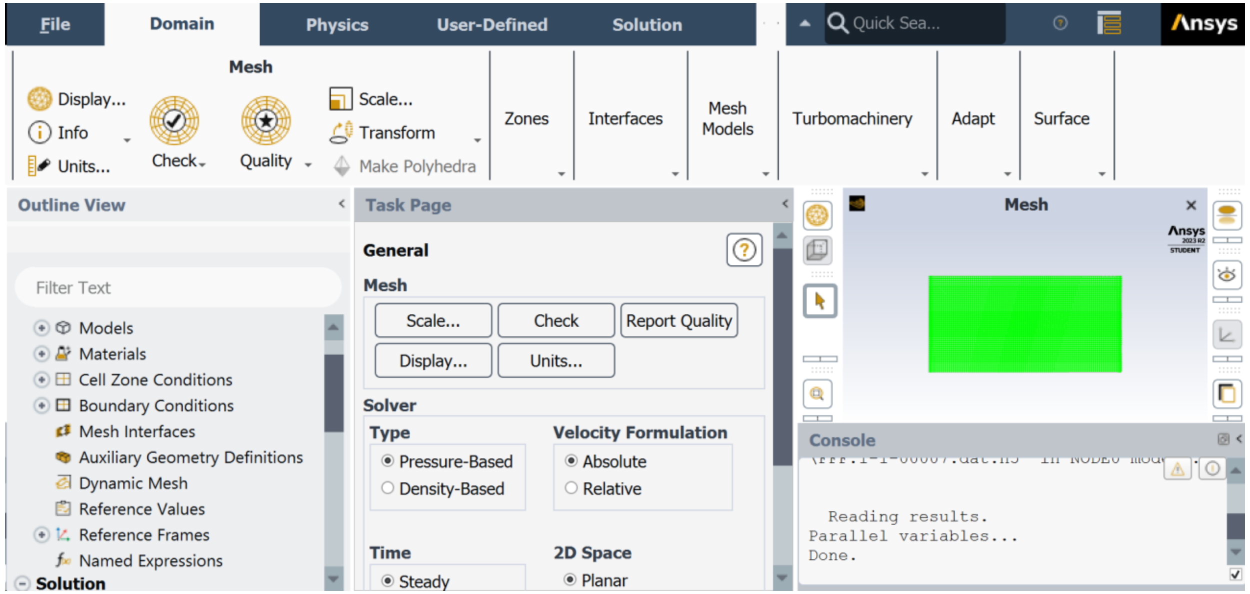
- In this section boundary walls are named and the 2-D domain
- Equidistance Quadrilateral meshes are drawn on the plate



Analytical Validation

Setup & Solution

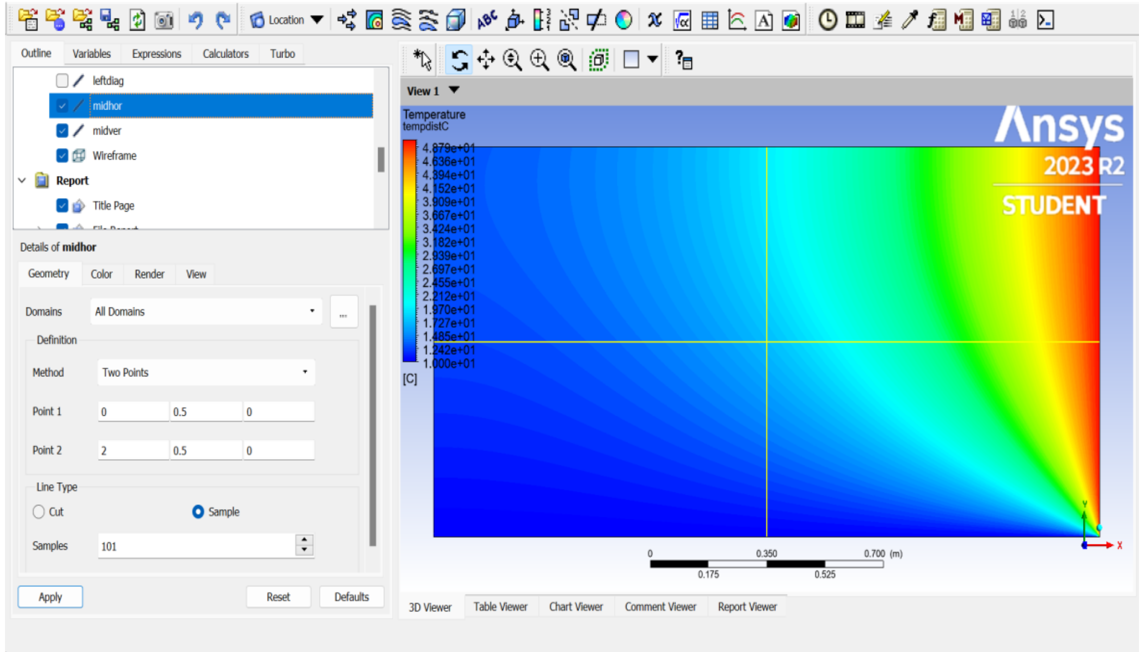
- In this section we select Energy equation for the simulation.
- The material of the plate is chosen, we kept it Aluminium because the material of the plate do not play any role in the calculation.
- We setup the boundary conditions provided in the problem statement before running the simulation for solution.

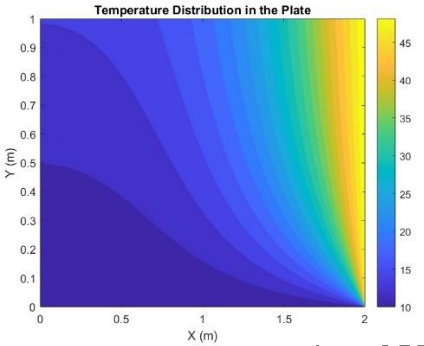


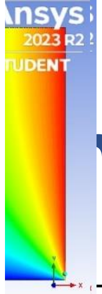
Analytical Validation

Result

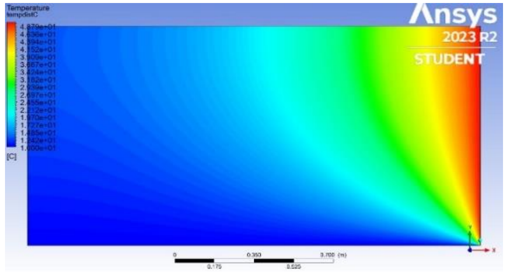
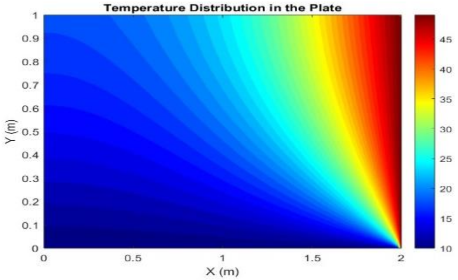
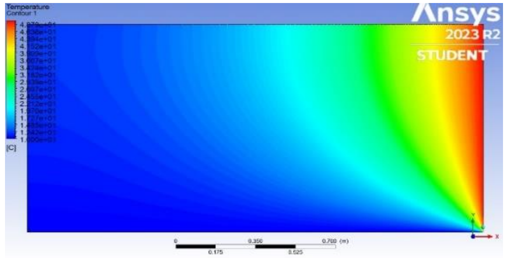
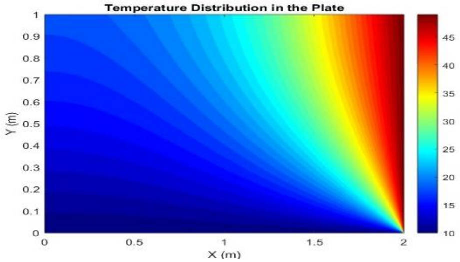
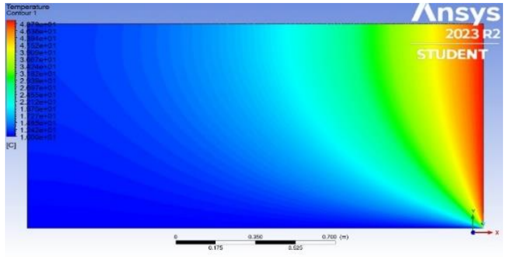
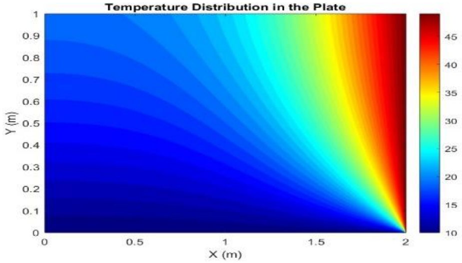
- In this section we extract the results from the solution.
- To compare the data, temperature data of horizontal mid-point line and vertical mid-point line of the plate is used.
- We divide the mid-point lines into grid points and extract the temperature value for each grid point. The grid points correspond to the number of meshes.







Comparison

No. of Mesh	ANSYS Plot	MATLAB Plot
100		
200		
300		

Conclusion

No. of Meshes	Horizontal midline		Vertical midline	
	Max difference	Average difference	Max difference	Average difference
100	2.51276381	1.832376204	4.3920025	2.147666243
200	2.07700129	1.726038221	4.19	2.052288557
300	0.55606093	0.237455986	0.94312382	0.225470157

References

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2. Powell, Adam. “Finite Difference Solution of the Heat Equation” dspace.mit.edu, 15 December 2023, www.dspace.mit.edu/bitstream/handle/1721.1/35256/22-00JSpring-2002/NR/rdonlyres/Nuclear-Engineering/22-00JIntroduction-to-Modeling-and-SimulationSpring2002/55114EA2-9B81-4FD8-90D5-5F64F21D23D0/0/lecture_16.pdf
3. dos Santos, Vânia Gonçalves de Brito, and Paula Tamires Gomes dos Anjos. "Finite Difference Method Applied in Two-Dimensional Heat Conduction Problem in the Permanent Regime in Rectangular Coordinates." *Advances in Pure Mathematics* 12.9 (2022): 505-518.

Thank You