Putting Faith in Faithless Electors

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Abstract

The suitability of the Electoral College in determining the United States presidency has been the subject of much debate throughout history. The Electoral College suffers from the "faithless elector" problem, where an elector can cast their vote in opposition to the will of their state's popular vote. While the number of faithless electors has increased in recent elections, the impact of faithless electors on election outcomes has not been closely examined. In our paper, we present a model that incorporates the possibility that an elector votes for the party opposite of their state's popular vote winner, and we show under what conditions faithless electors can flip the electoral outcome. Furthermore, we theoretically and empirically quantify the level of faithlessness needed to make the electoral outcome converge to the popular outcome.

1 Introduction

1.1 Electoral College

The Electoral College is the process by which the United States elects the president. According to the Constitution, each state must appoint a quantity of electors equal to that state's congressional delegation (members of the House of Representatives plus two Senators), giving 535 electors to the states. In addition, Washington, D.C. is allocated 3 electors, resulting in 538 total electors [1]. In 48 of the 50 states, state laws mandate that the winner of the plurality of the statewide popular vote receive all of that state's electoral votes. In Maine and Nebraska, two electoral votes are assigned in this manner, while the remaining electoral votes are allocated based on the plurality of votes in each of their congressional districts [2]. Electors are selected state-by-state, as determined by the laws of each state. Electors are nominated by a party and pledge to vote for their party's candidate. A candidate must receive an absolute majority of electoral votes (currently 270) to win the presidency.

Under the Electoral College system, a president who has not won the popular vote has been elected five times, leading to criticisms that the Electoral College violates the democratic principle of "one person, one vote." A recent Pew Research Center poll found that 63% of U.S. adults favor a shift to a popular vote system, with 80% of Democrats supporting a change compared to only 42% of Republicans [5]. While a majority of Americans support moving away from the Electoral College, implementing a popular vote system lacks the bipartisan support needed to change the process.

1.2 Faithless Electors

Overwhelmingly, electors in the Electoral College have faithfully voted for their party's presidential and vice presidential nominees. However, on occasion, they deviate. Electors in the Electoral

College who cast a vote for someone other than their party's presidential and vice presidential nominees are called "faithless electors." To date, faithless electors have never changed the outcome of an election. Only one elector has ever voted for a member of an opposite party—in 1796, Samuel Miles voted for Democratic-Republican presidential candidate Thomas Jefferson, forgoing his pledge to vote for Federalist candidate John Adams [3].

More recently, the 2016 presidential election between Democratic nominee Hillary Clinton and Republican nominee Donald Trump contained seven faithless electors. The deviations resulted in a final tally of Trump 304, Clinton 227, Others 7, instead of the expected Trump 306 to Clinton 232 [3]. While the actions of the faithless electors did not affect the result of the 2016 election, it is possible that circumstances surrounding future elections may result in more deviations or exploitations of the possibility of faithless votes. Thus, it is important to model both the factors that can lead to elector deviations and the effect that faithless electors can have on the outcomes of elections, particularly for elections that are highly contested.

1.3 Our Results

With the concerns raised about the Electoral College system, the following question arises: is it possible to modify the current Electoral College such that the outcome of the electoral vote better aligns with the outcome of the popular vote? Our main result is an Electoral College model that incorporates the possibility of an elector turning faithless. We give a bound on the probability that an elector votes faithlessly and prove that this bound results in an equivalence between the Electoral College result and the popular vote result. Finally, we provide simulations on empirical data to quantify the level of faithlessness needed for equivalence.

2 Theoretical Model

2.1 Elector Apportionment

Let $\boldsymbol{n} = (n_0, \ldots, n_{50})$ represent the population of each state (and Washington, D.C.). Let $N = \sum_{i=0}^{50} n_i$ (i.e., the total population of the United States). Let $\boldsymbol{e} = (e_0, \ldots, e_{50})$ represent the number of electors in each state. Let $E = \sum_{i=0}^{50} e_i$ (i.e., the total number of electors in the Electoral College). Under our model, $e_i = \frac{E}{N} n_i$.

2.2 Probability of Faithlessness

Let $\boldsymbol{p} = (p_0, \ldots, p_{50})$ represent the probability that the electors in each state vote faithlessly. p_i increases as the margin of victory in state i decreases. That is, the more contested the statewide election is, the more likely an elector is to vote faithlessly. Let $\boldsymbol{d} = (d_0, \ldots, d_{50})$ represent the percent of Democratic votes in each state. Let $\boldsymbol{r} = (r_0, \ldots, r_{50})$ represent the percent of Republican votes in each state. We define the margin of victory in state i to be $m_i = |d_i - r_i|$ and the upper bound of the probability of faithlessness to be α. Under a *linear* probability of faithlessness, we let

$$p_i = -\alpha m_i + \alpha.$$

Under an *exponential* probability of faithlessness, we let

$$p_i = \frac{\alpha}{100^{m_i}}.$$

Note that if $d_i > r_i$, p_i is the probability that a Democratic elector will vote Republican, and if $r_i > d_i$, p_i is the probability that a Republican elector will vote Democrat.

2.3 Electoral Vote

Let D be the set of states where $d_i > r_i$, and let R be the set of states where $r_i > d_i$. Then, the total number of Democratic electoral votes is

$$\sum_{i \in D} e_i(1 - p_i) + \sum_{i \in R} e_i p_i,$$

and the total number of Republican electoral votes is

$$E - \left(\sum_{i \in D} e_i(1 - p_i) + \sum_{i \in R} e_i p_i \right).$$

We know that the Democrats win the electoral vote if

$$\left(\sum_{i \in D} e_i(1 - p_i) + \sum_{i \in R} e_i p_i \right) - \left(E - \left(\sum_{i \in D} e_i(1 - p_i) + \sum_{i \in R} e_i p_i \right) \right) > 0$$

$$2 \left(\sum_{i \in D} e_i(1 - p_i) + \sum_{i \in R} e_i p_i \right) - E > 0.$$

2.4 Popular Vote

Let $\boldsymbol{v} = (v_0, \ldots, v_{50})$ represent the number of voters in each state. We let $v_i = cn_i$, where $0 < c \leq 1$ (the number of voters in a state has to be some proportion of the state's population). The total number of Democratic popular votes is $\sum_{i=0}^{50} v_i d_i$, and the total number of Republican popular votes is $\sum_{i=0}^{50} v_i r_i$. We know that the Democrats win the popular vote if

$$\sum_{i=0}^{50} v_i d_i - \sum_{i=0}^{50} v_i r_i > 0$$

$$\sum_{i=0}^{50} v_i (d_i - r_i) > 0.$$

3 Impact of Faithless Electors

Theorem 1. *If a party wins the electoral vote without any faithless electors, that party maintains their electoral vote win with faithless electors who have a linear probability of faithlessness if* $\alpha < \frac{1}{4}\left(\frac{\sum_{i \in D} n_i - \sum_{i \in R} n_i}{\sum_{i \in D} n_i r_i - \sum_{i \in R} n_i d_i}\right)$ *and does not maintain their electoral vote win (i.e., the other party wins the electoral vote) if* $\alpha > \frac{1}{4}\left(\frac{\sum_{i \in D} n_i - \sum_{i \in R} n_i}{\sum_{i \in D} n_i r_i - \sum_{i \in R} n_i d_i}\right)$.

Proof. Without loss of generality, let us assume that the Democratic Party wins the electoral vote without any faithless electors. This means

$$\sum_{i \in D} e_i > \sum_{i \in R} e_i \implies \sum_{i \in D} \frac{E}{N} n_i > \sum_{i \in R} \frac{E}{N} n_i \implies \sum_{i \in D} n_i > \sum_{i \in R} n_i.$$

In order for the Democratic Party to win the electoral vote with faithless electors, we need

$$2\left(\sum_{i\in D} e_i(1-(-\alpha m_i+\alpha))+\sum_{i\in R} e_i(-\alpha m_i+\alpha)\right)-E>0.$$

We can rewrite this inequality as

$$\alpha\left(\sum_{i\in D} n_i-\sum_{i\in R} n_i\right)+\alpha\left(\sum_{i\in R} n_i(r_i-d_i)-\sum_{i\in D} n_i(d_i-r_i)\right)<\frac{1}{2}\left(\sum_{i\in D} n_i-\sum_{i\in R} n_i\right)$$

(full details are provided in the Appendix). Since we assume $\sum_{i\in D} n_i > \sum_{i\in R} n_i$, this means

$$\alpha\left(1-\frac{\sum_{i\in D} n_i(d_i-r_i)-\sum_{i\in R} n_i(r_i-d_i)}{\sum_{i\in D} n_i-\sum_{i\in R} n_i}\right)<\frac{1}{2},$$

which we can rewrite as

$$\alpha<\frac{1}{4}\left(\frac{\sum_{i\in D} n_i-\sum_{i\in R} n_i}{\sum_{i\in D} n_i r_i-\sum_{i\in R} n_i d_i}\right).$$

It directly follows that for the Republican Party to win the electoral vote with faithless electors

$$\alpha>\frac{1}{4}\left(\frac{\sum_{i\in D} n_i-\sum_{i\in R} n_i}{\sum_{i\in D} n_i r_i-\sum_{i\in R} n_i d_i}\right).$$

□

Example 1. *For simplicity's sake, let us have two states, state 1 and state 2. State 1's population is 51% of the total population, and state 2's population is 49% of the total population. In state 1, 49% of voters are Democratic and 51% are Republican. In state 2, 100% of voters are Democratic and 0% are Republican. Formally, this means we have the following (for each state i).*

i	n_i	e_i	d_i	r_i	m_i	p_i
1	$0.51T$	$0.51E$	0.49	0.51	0.02	0.98α
2	$0.49T$	$0.49E$	1	0	1	0

If there are no faithless electors, the Democrats win the popular vote and the Republicans win the electoral vote. However, intuitively, we can see that if state 1 has faithless electors, the Democrats could also win the electoral vote. In order for the Democrats to win the electoral vote with faithless electors, we need

$$2(e_2(1-p_2)+e_1p_1)-E>0$$

$$2(0.49E(1-0)+0.51E(0.98\alpha))-E>0$$

$$\alpha \gtrapprox 0.020.$$

Theorem 2. *Under an exponential probability of faithlessness, the Democratic Party wins the electoral vote if* $\alpha<\frac{1}{2}\left(\frac{\sum_{i\in D} n_i-\sum_{i\in R} n_i}{\sum_{i\in D}\frac{n_i}{100^{d_i-r_i}}-\sum_{i\in R}\frac{n_i}{100^{r_i-d_i}}}\right)$ *when* $\sum_{i\in D}\frac{n_i}{100^{d_i-r_i}}-\sum_{i\in R}\frac{n_i}{100^{r_i-d_i}}>0$ *and if* $\alpha>\frac{1}{2}\left(\frac{\sum_{i\in D} n_i-\sum_{i\in R} n_i}{\sum_{i\in D}\frac{n_i}{100^{d_i-r_i}}-\sum_{i\in R}\frac{n_i}{100^{r_i-d_i}}}\right)$ *otherwise.*

Proof. This follows a similar structure as the proof of Theorem 1. A full proof is provided in the Appendix. □

Example 2. *We assume the same setup as Example 1, except* $p_1 = \frac{\alpha}{100^{0.02}}$ *and* $p_2 = \frac{\alpha}{100}$ *now. In order for the Democrats to win the electoral vote with faithless electors, we need*

$$2(e_2(1 - p_2) + e_1 p_1) - E > 0$$
$$2\left(0.49E\left(1 - \frac{\alpha}{100}\right) + 0.51E\left(\frac{\alpha}{100^{0.02}}\right)\right) - E > 0$$
$$\alpha \gtrapprox 0.022.$$

Theorem 3. *Under a linear probability of faithlessness, an Electoral College with faithless electors (loosely) converges to the popular vote if* $\alpha < \frac{1}{2}$ *when* $\sum_{i \in R} n_i - \sum_{i \in D} n_i > 0$ *and* $\alpha > \frac{1}{2}$ *otherwise.*

Proof. Without loss of generality, let us assume that the Democratic Party wins the popular vote. This means

$$\sum_{i=0}^{50} v_i(d_i - r_i) > 0 \implies \sum_{i \in D} n_i(d_i - r_i) > \sum_{i \in R} n_i(r_i - d_i).$$

Now, we wish to find some bound on α such that the electoral vote result matches the popular vote result; that is, we want to find a bound on α such that the Democratic Party wins the electoral vote as well. This occurs if the following inequality holds:

$$\frac{N}{2} + \sum_{i \in R}(n_i \alpha(r_i - d_i) - n_i \alpha) < \sum_{i \in D}(n_i + n_i \alpha(d_i - r_i) - n_i \alpha).$$

We can rewrite this inequality as

$$\frac{1}{2\alpha}\left(\sum_{i \in R} n_i - \sum_{i \in D} n_i\right) + \left(\sum_{i \in D} n_i - \sum_{i \in R} n_i\right) < \left(\sum_{i \in D} n_i(d_i - r_i) - \sum_{i \in R} n_i(r_i - d_i)\right)$$

(full details are provided in the Appendix). For the popular vote to go to the Democratic Party, we need the right-hand side of the above inequality to be positive. We can guarantee this result conservatively by setting the left-hand side above to be positive, which gives us the inequality

$$\left(\sum_{i \in R} n_i - \sum_{i \in D} n_i\right) < \frac{1}{2\alpha}\left(\sum_{i \in R} n_i - \sum_{i \in D} n_i\right).$$

If $\sum_{i \in R} n_i - \sum_{i \in D} n_i > 0$, then $\alpha < \frac{1}{2}$. Otherwise, $\alpha > \frac{1}{2}$. $\square$

One point of interest from the above inequalities is that if the Democrats win the popular vote but the total population of states won by Democrats is lower than that of states won by Republicans, we have a lower bound imposed on α in order for the electoral and popular vote results to agree. Thus, this is another instance in which having faithless electors forces the Electoral College results to match that of the popular vote.

4 Empirical Implementation

In order to simulate our model on both constructed examples and previous presidential election data as a way to analyze the effect of changing α, we implemented it in a Python Jupyter Notebook[1], including code that parses 2019 U.S. population census data as a way to allocate electors for each state and previous election results (vote counts per party per state).

[1] See GitHub.

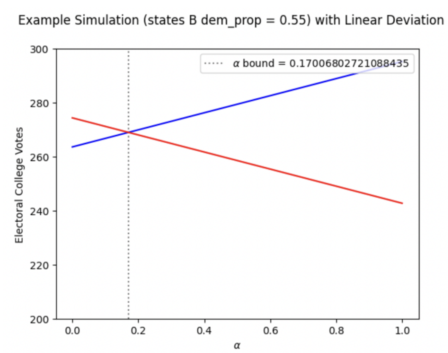
4.1 Constructed Examples

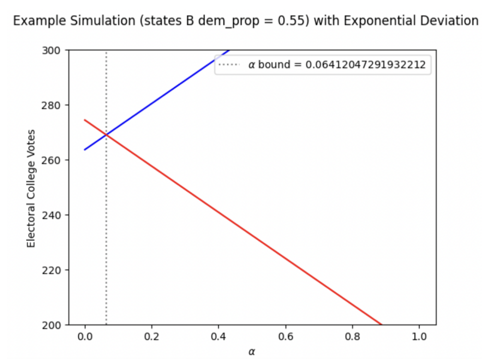
We set up the following example (very similar to our motivating example) as a way to analyze the effect of both modifying α and altering the popular vote proportions among states:

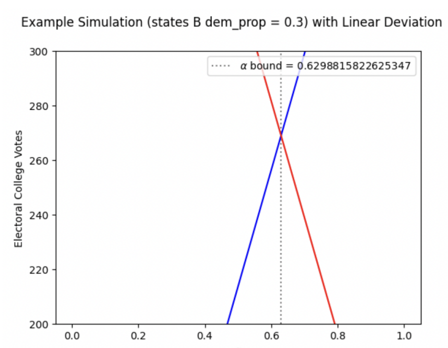
We have two sets of states in our example country: **A** and **B**. Set **A** contains 3 states and cumulatively makes up 51% of the population of our example country, while Set **B** contains 10 states and cumulatively makes up 49% of the population of our example country. The states in **A** have 49% of the population voting for Democrats and 51% voting for Republicans, while we varied the popular vote percentage allocated to each party for all states in **B** between 100% and 0%.

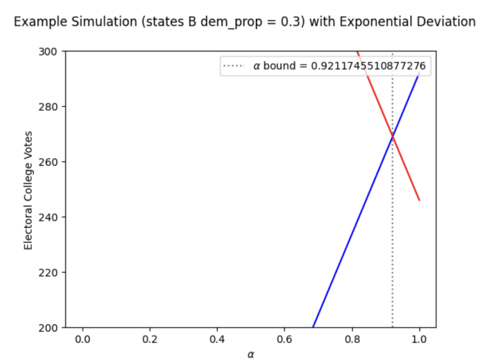
4.1.1 Linear vs. Exponential Deviation

We first looked at the difference between using a linear and exponential probability of faithlessness on the effect that α has on the electoral vote results. Plots demonstrating these effects for having 55% and 30% of the population voting for Democrats in states **B** are shown below.









In the plots, the red and blue lines represent how the number of electors change for the Republican and Democratic parties, respectively. Let $\alpha^\star$ be the alpha bound denoted by the vertical dotted lines in the plots, representing the value of α that would have flipped the electoral vote result from the initial point when there were no faithless electors. These plots give us a sense of the degree of faithlessness needed (when weighted according to p_i by the margin of victory in each state) for the party opposite of the "no faithlessness" winner to win the electoral vote.

In comparing linear and exponential deviation when the Democratic proportion in states **B** is 0.55, we see that the values for $\alpha^\star$ are relatively close, whereas when the Democratic proportion in states **B** is lowered to 0.3, the values for $\alpha^\star$ are much further apart. Intuitively, this makes sense: when the Democratic proportion is fairly close to that of the Republicans in all states, the values for p_i in both the linear and exponential deviation model will be somewhat similar. On the other hand, when the Democratic proportion is much lower in states $\mathbf{B}$, the p_i values in those states will be much more penalized under the exponential deviation model, sharply increasing the threshold $\alpha^\star$ needed to flip the electoral vote.

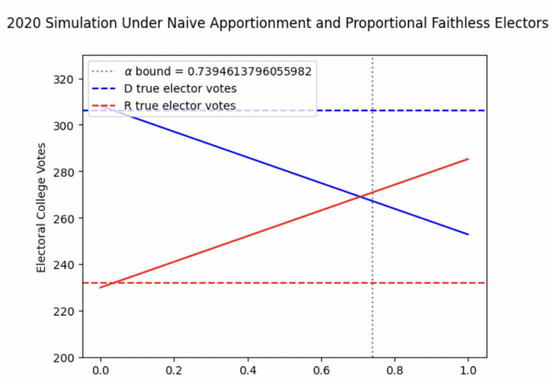
4.2 Past Presidential Election Simulations

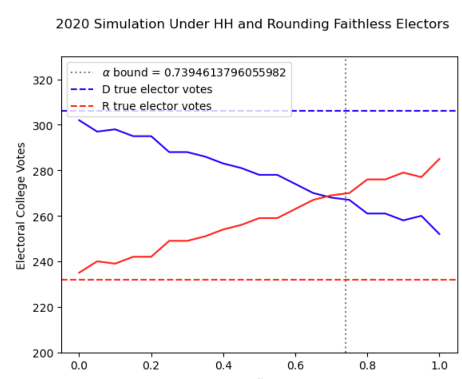
Next, we investigated the effect that α would have on historical election results. Specifically, we treated the electoral vote counts from past elections as fixed when $\alpha = 0$, then plotted how increasing levels of α changed the electoral vote results. Initially, we assume a naive apportionment of electors to states that is proportional to each state's population, and we assume a number of faithless electors exactly equal to the expected number of faithless electors for each state based on the faithlessness probability formula consistent with our model. Plots for the years 2020, 2016, and 2000 can be found in the Appendix.

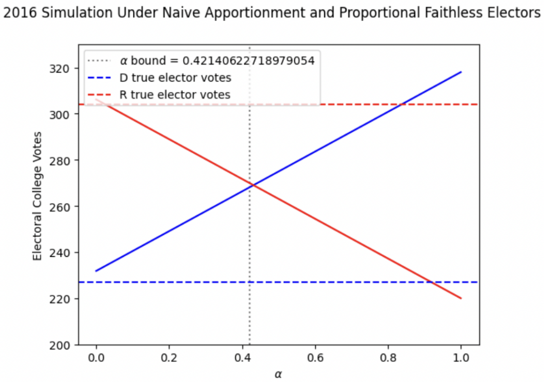
We informally call a high $\alpha^\star$ value as "high faithlessness" and vice versa. In the plots, the horizontal lines represent the *true* electoral vote count for each party in the election. Given that there are many assumptions our model makes that do not align with the true nature of the Electoral College today, we observe these horizontal lines to indicate the margin of victory between the parties only. In 2020, we see that Republicans would need a very high degree of faithlessness to win under our model, whereas Democrats would have needed an α of just over 0.4 in 2016. Interestingly, despite the very close election in 2000 where Republicans received just 271 votes to secure their win, the plot for 2000 indicates a fairly large degree of faithlessness needed in order for Democrats to win—however, it is hard to say what *would* have happened in these elections due to the many assumptions of our model. We note that the discrepancy between the crossover points and $\alpha^\star$ in the historical plots is due to the assumption, under our model, that some constant proportion c of the population in every state casts a vote, which we know is not true in practice.

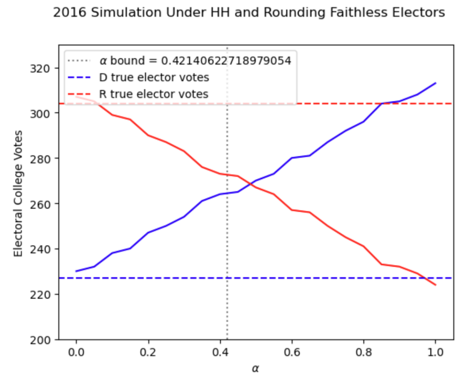
4.2.1 Our Apportionment vs. Huntington-Hill

We also investigated the impact of α on real-world elections under our naive apportionment of electors versus under a Huntington-Hill apportionment of electors with a rounded (to the nearest integer) number of faithless electors.









The graphs showing the Democratic and Republican vote counts over different values of α for each of these cases in 2016 and 2020 are shown above. We observe that both graphs have very similar trend lines in their slope and crossover points—the assumptions of naive apportionment and proportional faithless electors under our model do not seem to impact the trends seen in the plots all too much. These assumptions, however, were critical in our model so that we could mathematically derive a relatively clean value for $\alpha^{\star}$.

5 Discussion

Throughout American history, there have been 157 faithless electors. While some of these individuals deviated due to idiosyncratic reasons, others did so because they preferred the losing party's candidate. While our model considers the scenario that an elector switches their vote to the opposing party, the overwhelming majority of faithless electors have either voted for an alternative candidate in their pledged party or abstained from voting. However, the scenario of a faithless elector switching parties is not far-fetched, especially given that almost 40% of electors can vote faithlessly, without penalty [4]. Furthermore, given that faithlessness can (to a certain extent) drive convergence between electoral and popular vote results, permitting some level of faithlessness may better reflect the desires of the general population in elections.

Possible extensions include investigating more sophisticated models beyond our linear and exponential models for calculating the probability of faithlessness. Additionally, while our results depended on the simplified method of allocating electors based on state populations, proving convergence results for the Huntington-Hill apportionment method would give more realistic results. Lastly, future work could involve finding a tighter bound on the probability of faithlessness for convergence between electoral and popular vote results.

6 Acknowledgements

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References

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[3] Presidential elections. https://fairvote.org/resources/presidential-elections/, Feb 2023.

[4] Charles Tracy Drew Penrose. Supreme court resolves faithless elector dispute, still leaves electoral college problems on table, Sep 2022.

[5] Rebecca Salzer and Jocelyn Kiley. Majority of americans continue to favor moving away from electoral college, Jan 2023.

7 Appendix

7.1 Proof of Theorem 1

Under a linear probability of faithlessness, we have

$$\begin{aligned}
2\left(\sum_{i\in D} e_i(1-p_i)+\sum_{i\in R} e_i p_i\right)-E &= \left(\frac{2E}{N}\sum_{i\in D} n_i(1-p_i)\right)+\left(\frac{2E}{N}\sum_{i\in R} n_i p_i\right)-E \\
&= \left(\frac{2E}{N}\sum_{i\in D} n_i(1+\alpha(d_i-r_i)-\alpha)\right)+\left(\frac{2E}{N}\sum_{i\in R} n_i(\alpha(d_i-r_i)+\alpha)\right)-E.
\end{aligned}$$

In order for the Democrats to win the electoral vote under our model with faithless electors, we need

$$\begin{aligned}
0 &< \left(\frac{2E}{N}\sum_{i\in D} n_i(1+\alpha(d_i-r_i)-\alpha)\right)+\left(\frac{2E}{N}\sum_{i\in R} n_i(\alpha(d_i-r_i)+\alpha)\right)-E \\
E-\left(\frac{2E}{N}\sum_{i\in R} n_i(\alpha(d_i-r_i)+\alpha)\right) &< \left(\frac{2E}{N}\sum_{i\in D} n_i(1+\alpha(d_i-r_i)-\alpha)\right) \\
\frac{N}{2}-\sum_{i\in R} n_i(\alpha(d_i-r_i)+\alpha) &< \sum_{i\in D} n_i(1+\alpha(d_i-r_i)-\alpha) \\
\frac{N}{2}+\sum_{i\in R}(n_i\alpha(r_i-d_i)-n_i\alpha) &< \sum_{i\in D}(n_i+n_i\alpha(d_i-r_i)-n_i\alpha) \\
\frac{N}{2}+\sum_{i\in R} n_i\alpha(r_i-d_i)-\sum_{i\in R} n_i\alpha &< \sum_{i\in D} n_i+\sum_{i\in D} n_i\alpha(d_i-r_i)-\sum_{i\in D} n_i\alpha \\
\alpha\left(\sum_{i\in D} n_i-\sum_{i\in R} n_i\right)+\alpha\left(\sum_{i\in R} n_i(r_i-d_i)-\sum_{i\in D} n_i(d_i-r_i)\right) &< \frac{1}{2}\left(\sum_{i\in D} n_i-\sum_{i\in R} n_i\right).
\end{aligned}$$

If $\sum_{i\in D} n_i-\sum_{i\in R} n_i>0$ (that is, the Democrats would win the electoral vote without any faithless electors), we have

$$\begin{aligned}
\alpha\left(1-\frac{\sum_{i\in D} n_i(d_i-r_i)-\sum_{i\in R} n_i(r_i-d_i)}{\sum_{i\in D} n_i-\sum_{i\in R} n_i}\right) &< \frac{1}{2} \\
\alpha\left(\frac{\sum_{i\in D} 2n_i r_i-\sum_{i\in R} 2n_i d_i}{\sum_{i\in D} n_i-\sum_{i\in R} n_i}\right) &< \frac{1}{2} \\
\alpha &< \frac{1}{4}\left(\frac{\sum_{i\in D} n_i-\sum_{i\in R} n_i}{\sum_{i\in D} n_i r_i-\sum_{i\in R} n_i d_i}\right).
\end{aligned}$$

If $\sum_{i\in D} n_i-\sum_{i\in R} n_i<0$ (that is, the Republicans would win without any faithless electors), we have

$$\begin{aligned}
\alpha\left(1-\frac{\sum_{i\in D} n_i(d_i-r_i)-\sum_{i\in R} n_i(r_i-d_i)}{\sum_{i\in D} n_i-\sum_{i\in R} n_i}\right) &> \frac{1}{2} \\
\alpha\left(\frac{\sum_{i\in D} 2n_i r_i-\sum_{i\in R} 2n_i d_i}{\sum_{i\in D} n_i-\sum_{i\in R} n_i}\right) &> \frac{1}{2} \\
\alpha &> \frac{1}{4}\left(\frac{\sum_{i\in D} n_i-\sum_{i\in R} n_i}{\sum_{i\in D} n_i r_i-\sum_{i\in R} n_i d_i}\right).
\end{aligned}$$

7.2 Proof of Theorem 2

Under an exponential probability of faithlessness, we have

$$
\begin{aligned}
2\left(\sum_{i\in D} e_i(1-p_i)+\sum_{i\in R} e_ip_i\right)-E &= \left(\frac{2E}{N}\sum_{i\in D} n_i(1-p_i)\right)+\left(\frac{2E}{N}\sum_{i\in R} n_ip_i\right)-E\\
&= \left(\frac{2E}{N}\sum_{i\in D} n_i\left(1-\frac{\alpha}{100^{d_i-r_i}}\right)\right)+\left(\frac{2E}{N}\sum_{i\in R} n_i\left(\frac{\alpha}{100^{r_i-d_i}}\right)\right)-E.
\end{aligned}
$$

Let $\left(\frac{2E}{N}\sum_{i\in D} n_i\left(1-\frac{\alpha}{100^{d_i-r_i}}\right)\right)+\left(\frac{2E}{N}\sum_{i\in R} n_i\left(\frac{\alpha}{100^{r_i-d_i}}\right)\right)-E>0$. That is, we assume that the Democrats win the electoral vote under our model. Then, we have

$$
\begin{aligned}
0 &< \left(\frac{2E}{N}\sum_{i\in D} n_i\left(1-\frac{\alpha}{100^{d_i-r_i}}\right)\right)+\left(\frac{2E}{N}\sum_{i\in R} n_i\left(\frac{\alpha}{100^{r_i-d_i}}\right)\right)-E\\
E-\left(\frac{2E}{N}\sum_{i\in R} n_i\left(\frac{\alpha}{100^{r_i-d_i}}\right)\right) &< \left(\frac{2E}{N}\sum_{i\in D} n_i\left(1-\frac{\alpha}{100^{d_i-r_i}}\right)\right)\\
\frac{N}{2}-\sum_{i\in R} n_i\left(\frac{\alpha}{100^{r_i-d_i}}\right) &< \sum_{i\in D} n_i\left(1-\frac{\alpha}{100^{d_i-r_i}}\right)\\
\frac{N}{2}-\alpha\sum_{i\in R}\frac{n_i}{100^{r_i-d_i}} &< \sum_{i\in D} n_i-\alpha\sum_{i\in D}\frac{n_i}{100^{d_i-r_i}}\\
\alpha\left(\sum_{i\in D}\frac{n_i}{100^{d_i-r_i}}-\sum_{i\in R}\frac{n_i}{100^{r_i-d_i}}\right) &< \frac{1}{2}\left(\sum_{i\in D} n_i-\sum_{i\in R} n_i\right).
\end{aligned}
$$

If $\sum_{i\in D}\frac{n_i}{100^{d_i-r_i}}-\sum_{i\in R}\frac{n_i}{100^{r_i-d_i}}>0$ (that is, the sum of populations of states won by Democrats weighted by an exponential of the margins of victory is greater than the sum of populations of states won by Republicans weighted by an exponential of the margins of victory), we have

$$
\alpha<\frac{1}{2}\left(\frac{\sum_{i\in D} n_i-\sum_{i\in R} n_i}{\sum_{i\in D}\frac{n_i}{100^{d_i-r_i}}-\sum_{i\in R}\frac{n_i}{100^{r_i-d_i}}}\right).
$$

If $\sum_{i\in D}\frac{n_i}{100^{d_i-r_i}}-\sum_{i\in R}\frac{n_i}{100^{r_i-d_i}}<0$ (that is, the sum of populations of states won by Democrats weighted by an exponential of the margins of victory is less than the sum of populations of states won by Republicans weighted by an exponential of the margins of victory), we have

$$
\alpha>\frac{1}{2}\left(\frac{\sum_{i\in D} n_i-\sum_{i\in R} n_i}{\sum_{i\in D}\frac{n_i}{100^{d_i-r_i}}-\sum_{i\in R}\frac{n_i}{100^{r_i-d_i}}}\right).
$$

7.3 Proof of Theorem 3

If $2\left(\sum_{i\in D} e_i(1-p_i)+\sum_{i\in R} e_ip_i\right)-E$ and $\sum_{i=0}^{50} v_i(d_i-r_i)$ have the same sign, we know that the electoral vote converges to the popular vote. Let $\sum_{i=0}^{50} v_i(d_i-r_i)>0$. That is, without loss of generality, we assume that the Democrats win the popular vote. Then, we have

$$\sum_{i=0}^{50} v_i(d_i - r_i) > 0$$

$$c\left(\sum_{i\in D} n_i(d_i - r_i) + \sum_{i\in R} n_i(d_i - r_i)\right) > 0$$

$$\sum_{i\in D} n_i(d_i - r_i) > \sum_{i\in R} n_i(r_i - d_i).$$

Now, from the previous section, we know that if the Democrats win the electoral vote, the first inequality below holds. We progress with some arithmetic as follows.

$$\frac{N}{2} + \sum_{i\in R}(n_i\alpha(r_i - d_i) - n_i\alpha) < \sum_{i\in D}(n_i + n_i\alpha(d_i - r_i) - n_i\alpha)$$

$$\frac{N}{2} + \alpha\sum_{i\in R}(n_i(r_i - d_i)) - \alpha\sum_{i\in R} n_i < \sum_{i\in D} n_i + \sum_{i\in D} n_i\alpha(d_i - r_i) - \sum_{i\in D} n_i\alpha$$

$$\frac{N}{2} + \sum_{i\in D} n_i\alpha - \alpha\sum_{i\in R} n_i - \sum_{i\in D} n_i < \sum_{i\in D} n_i\alpha(d_i - r_i) - \alpha\sum_{i\in R}(n_i(r_i - d_i))$$

$$\frac{N}{2} + \sum_{i\in D} n_i\alpha - \sum_{i\in R} n_i\alpha - \sum_{i\in D} n_i < \alpha\left(\sum_{i\in D} n_i(d_i - r_i) - \sum_{i\in R} n_i(r_i - d_i)\right)$$

$$\frac{1}{2}\left(\sum_{i\in R} n_i - \sum_{i\in D} n_i\right) + \alpha\left(\sum_{i\in D} n_i - \sum_{i\in R} n_i\right) < \alpha\left(\sum_{i\in D} n_i(d_i - r_i) - \sum_{i\in R} n_i(r_i - d_i)\right)$$

$$\frac{1}{2\alpha}\left(\sum_{i\in R} n_i - \sum_{i\in D} n_i\right) + \left(\sum_{i\in D} n_i - \sum_{i\in R} n_i\right) < \left(\sum_{i\in D} n_i(d_i - r_i) - \sum_{i\in R} n_i(r_i - d_i)\right)$$

For the popular vote result to hold, we need $\sum_{i\in D} n_i(d_i - r_i) > \sum_{i\in R} n_i(r_i - d_i)$ to hold, meaning that the right-hand side of the above inequality must be positive. We can guarantee this result, then (very conservatively), by setting the left-hand side above to be positive, which guarantees that the right-hand side is positive and that the Democrats win the election. Thus, the proof for such a conservative bound on α is shown below, with also the assumption that $\sum_{i\in R} n_i - \sum_{i\in D} n_i > 0$.

$$0 < \frac{1}{2\alpha}\left(\sum_{i\in R} n_i - \sum_{i\in D} n_i\right) + \left(\sum_{i\in D} n_i - \sum_{i\in R} n_i\right)$$

$$\left(\sum_{i\in R} n_i - \sum_{i\in D} n_i\right) < \frac{1}{2\alpha}\left(\sum_{i\in R} n_i - \sum_{i\in D} n_i\right)$$

$$\alpha < \frac{1}{2}$$

If we assume $\sum_{i \in R} n_i - \sum_{i \in D} n_i < 0$, then the above process becomes the following.

$$0 < \frac{1}{2\alpha}\left(\sum_{i \in R} n_i - \sum_{i \in D} n_i\right) + \left(\sum_{i \in D} n_i - \sum_{i \in R} n_i\right)$$
$$\left(\sum_{i \in R} n_i - \sum_{i \in D} n_i\right) < \frac{1}{2\alpha}\left(\sum_{i \in R} n_i - \sum_{i \in D} n_i\right)$$
$$\alpha > \frac{1}{2}$$

Thus, the above inequalities show very loose bounds on α for the electoral and popular vote results to agree, given two different cases of assumptions.

7.4 Historical Presidential Election Plots

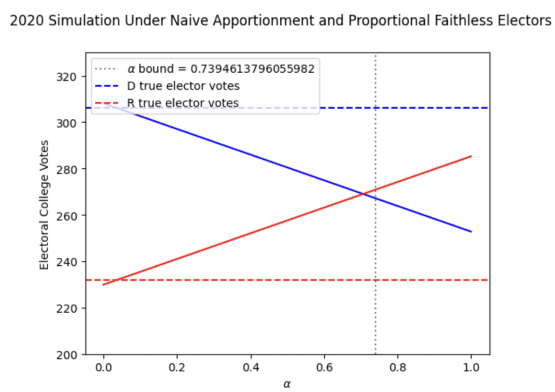

Figure 1: 2020 Presidential Election Simulation Under Our Model

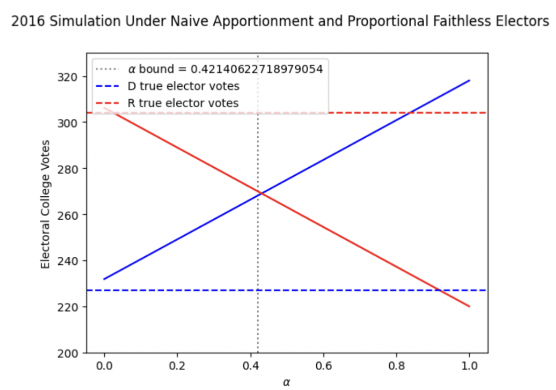

Figure 2: 2016 Presidential Election Simulation Under Our Model

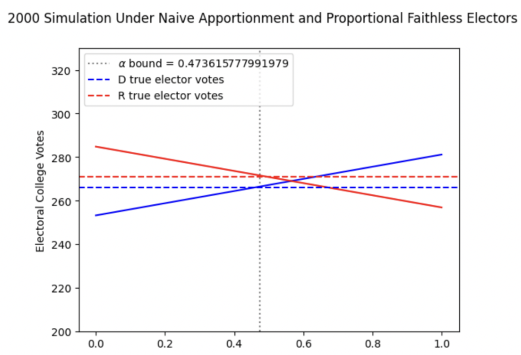

Figure 3: 2000 Presidential Election Simulation Under Our Model