Topology of Knots and Seifert Surfaces

Ayesha Arif (BSF1900982)
Supervisor: Dr. Arooj Adeel
University of Education, Lahore

Abstract

This project explores key topological concepts, including knots, Seifert surfaces, minimal genus Seifert surfaces, and the Kakimizu complex. These mathematical structures play a significant role in low-dimensional topology and are used to understand three-manifold structures. Through definitions, examples, and simplified explanations, this project aims to provide an accessible introduction to these complex topics.

Table of Contents

1 Knots

A **knot** is a closed, non-self-intersecting loop embedded in three-dimensional space. Mathematically, a knot is a smooth embedding of the circle S^1 into $\mathbb{R}^3$. Knots are studied up to **isotopy**, which means they can be continuously deformed without cutting or passing through themselves.

1.1 Examples of Knots

- **Unknot**: A simple closed loop without any crossings. Example: A standard rubber band.
- **Trefoil Knot**: The simplest non-trivial knot with three crossings. Example: Some shoelace knots.
- **Figure-eight Knot**: A more complex knot with four crossings. Example: Commonly used in rock climbing for securing ropes.
- **Borromean Rings**: A set of three intertwined loops where removing any one ring causes the structure to fall apart. Used in molecular chemistry.

2 Seifert Surfaces

A **Seifert surface** is an orientable, connected surface whose boundary is a given knot. These surfaces provide useful information about the complexity of knots and their topology.

2.1 Constructing Seifert Surfaces

Seifert surfaces can be obtained using **Seifert's algorithm**, which involves:

1. Orienting the knot projection.
2. Splitting crossings according to the orientation.
3. Connecting components with bands.

2.2 Examples of Seifert Surfaces

- The **unknot** has a Seifert surface that is a simple disk.
- The **trefoil knot** has a Seifert surface resembling a Möbius strip but with an extra twist.
- The **figure-eight knot** has a more complex Seifert surface, often requiring computational topology to analyze.

3 Minimal Genus Seifert Surfaces

The **genus** of a knot is defined as the smallest genus among all possible Seifert surfaces that bound the knot. The minimal genus Seifert surface provides the simplest surface associated with the knot.

3.1 Examples

- The **unknot** has a minimal genus of 0 because it bounds a disk.
- The **trefoil knot** has a minimal genus of 1.
- The **figure-eight knot** has a minimal genus of 1.
- More complex knots have higher genus values, often calculated using Alexander polynomials.

4 The Kakimizu Complex

The **Kakimizu complex** is a simplicial complex whose vertices represent isotopy classes of minimal genus Seifert surfaces of a given knot. A collection of vertices forms a simplex if the corresponding surfaces can be realized disjointly in the knot's exterior.

4.1 Key Properties

- The complex is **contractible** for many knot types.
- The structure of the Kakimizu complex depends on the knot genus.
- It helps in understanding minimal surfaces within 3-manifolds.
- Used in computational topology to classify knots algorithmically.

4.2 Examples

- The Kakimizu complex for the **unknot** is trivial, as there is only one minimal Seifert surface.
- For the **trefoil knot**, the Kakimizu complex consists of a 1-dimensional structure with two vertices connected by an edge.
- Higher-genus knots result in increasingly complex Kakimizu complexes.

5 Key Concepts Simplified

5.1 Why Are These Concepts Important?

- Knots appear in mathematical physics, quantum computing, and biology (e.g., DNA topology).
- Seifert surfaces help visualize and classify knots.
- The Kakimizu complex provides a higher-dimensional approach to studying knot theory.

5.2 Real-World Applications

- **DNA structure**: Understanding how DNA knots affect genetic processes.
- **Physics**: Knot theory is used in quantum field theory and fluid dynamics.
- **Material science**: Designing strong, flexible polymer structures using knot models.
- **Computer Science**: Knot invariants are used in cryptographic systems.
- **Chemistry**: Knot structures help in designing molecular bonds in synthetic chemistry.

6 Conclusion

The study of knots, Seifert surfaces, minimal genus surfaces, and the Kakimizu complex offers a rich framework for understanding topological structures. These concepts have broad applications in both pure and applied mathematics, making them an exciting area of study for mathematicians and scientists alike.

References

1. Seifert, H. "On the Construction of Knots and Their Surfaces." 1934.
2. Kakimizu, O. "Finding Minimal Genus Seifert Surfaces for Knots." 1992.
3. Thurston, W.P. "The Geometry and Topology of Three-Manifolds." 1978.
4. Various research papers on computational knot theory and its applications.