Imperial College London

BSc/MSci EXAMINATION May 2017

This paper is also taken for the relevant Examination for the Associateship

COMPREHENSIVES PAPER II

For Third and Fourth Year Physics Students

Wednesday, 10 May 2017 10:00-13:10

You may attempt as many questions as you wish. Only the answers to the best EIGHT questions over the two papers will contribute to your mark.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the FIVE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in FIVE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. This question concerns the motion of a charged particle moving through a medium in which a charged sphere is embedded.

(i) A charged conducting sphere of radius a is embedded in a medium which produces an electrostatic potential outside the sphere of the form $V = V_s(a/r)^4$, where V_s is the potential of the sphere and r is the distance from its centre.

(a) Obtain an expression for the electric field outside the sphere.
[In spherical polar coordinates:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} . \quad]$$

A charged particle moves in this electric field. Assuming that no other forces are acting on it, explain why:

(b) its motion is confined to a plane, and

(c) its angular momentum is constant.

[5 marks]

(ii) We now assume that the charged particle (mass m) experiences an *attractive* force towards the sphere, and that initially it is so far from the sphere that its electrostatic potential energy is very small compared to its kinetic energy. Its initial speed is v_0 and it has an impact parameter of b (the impact parameter is the closest distance to the centre of the sphere the particle would reach if not deflected).

(a) Show that

$$\frac{1}{2}mv_r^2 + U^*(r) = \frac{1}{2}mv_0^2 ,$$

where v_r is the radial component of velocity,

$$U^*(r) = \frac{1}{2}mv_0^2\left(\frac{b}{r}\right)^2 - P\left(\frac{a}{r}\right)^4 ,$$

and P is the *magnitude* of the electrostatic potential energy the particle would have at the surface of the sphere.

(b) Show that for given values of P and v_0, $U^*(r)$ has a maximum outside the sphere if $b < b^*$, where

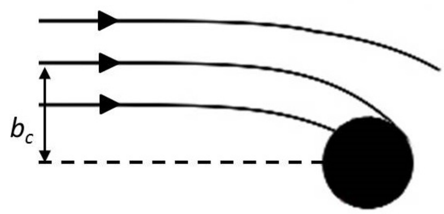
$$b^* = 2a\sqrt{\frac{P}{mv_0^2}} .$$

(c) Sketch separate graphs of $U^*(r)$ for the cases $b < b^*$ and $b > b^*$. Clearly indicate $r = a$ on both sketches.

[10 marks]

(iii) The following argument is sometimes suggested to determine if a given particle will hit the sphere. For given values of P and v_0 there is some critical impact parameter b_c: particles with $b < b_c$ do hit it, those with $b > b_c$ don't. The critical impact parameter corresponds to a particle which just grazes the sphere's surface (see diagram, below).

[This question continues on the next page . . .]



(a) Show that if this argument is valid then:

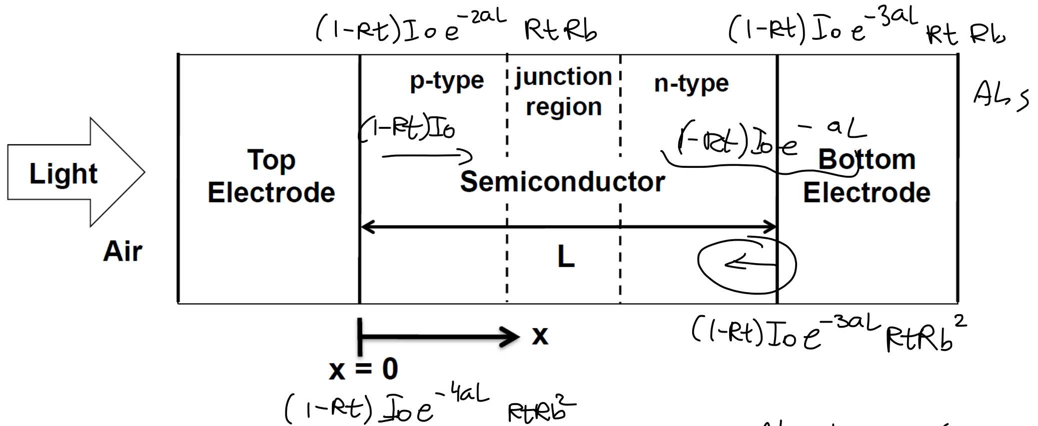
$$b_c = a\sqrt{1 + \frac{2P}{mv_0^2}}\ .$$

(b) Consider a particle whose initial kinetic energy is equal to $0.2P$. Using the properties of the function $U^*(r)$ discussed in part (ii), explain why the expression above for b_c is not valid for this particle.

[5 marks]

[Total 20 marks]

2. A semiconductor *pn* junction photodiode consists of a layer of silicon with equally thick *p*-type and *n*-type regions sandwiched between two metal electrodes. Assume that the top electrode does not absorb at all relevant wavelengths λ. Electrons and holes generated through the absorption of photons within the silicon layer diffuse to the junction region, resulting in a photocurrent between the electrodes. This question investigates using the photodiode to detect a relativistic light source.



(i) The intensity of light $\mathcal{I}$ varies with depth x according to:

$$\frac{d\mathcal{I}}{dx} = -\alpha(\lambda)\,\mathcal{I}$$

where $\alpha(\lambda)$ is the absorption coefficient.

Absorbed$_1$ = $(1-Rt)Io\,[1-e^{-aL}]$

Absorbed 2 = $\underbrace{(1-Rt)Io\,e^{-aL}Rb\,[1-e^{-aL}]}$

Absorbed 3 = $(1-Rt)Io\,e^{-2aL}RtRb\,[1-e^{-aL}]$

Abs 4 = $(1-Rt)Io\,e^{-3aL}RtRb^2\,[1-e^{-aL}]$

(a) If $\mathcal{I}_{SC}$ is the light incident on the semiconductor surface at $x = 0$, show that:

$$\mathcal{I} = \mathcal{I}_{SC} \exp[-\alpha(\lambda)\,x]$$

$(1-Rt)Io\,[1-e^{-aL}][1+Rbe^{-aL}+RtRbe^{-2aL}]$

[3 marks]

(b) Let $\mathcal{I}_0$ be the incident light intensity on the photodiode. Let the top and bottom electrode have a reflectance at all wavelengths of R_t and R_b respectively. By considering multiple internal reflections, show that the total light absorbed in the semiconductor layer is given by

$$\mathcal{I} = \underbrace{(1 - R_t)\,\mathcal{I}_0}_{\substack{x=0\\ \text{Incident}}}\,[1 - \exp(-\alpha(\lambda)L)]\left[\frac{1 + \overbrace{R_b \exp(-\alpha(\lambda)L)}^{\text{reflected from bottom}}}{1 - \underbrace{R_t R_b \exp(-2\alpha(\lambda)L)}_{\text{reflected from top after bottom}}}\right].$$

[4 marks]

(c) Assume the variation of $\alpha(\lambda)$ (in μm^{-1}) with wavelength λ (in μm) can be approximated by:

$$\alpha(\lambda) = \beta\,(1.112 - \lambda)^2$$

where β is a constant. If $\alpha(\lambda) = 0.06\,\mu\text{m}^{-1}$ when $\lambda = 0.78\,\mu$m, calculate the value of β. Let $L = 10\,\mu$m, $R_t = 0.05$ and $R_b = 0.9$. Find the fraction of incident light absorbed in the photodiode at a wavelength of $0.70\,\mu$m.

[2 marks]

(d) The photodiode has an internal quantum efficiency Φ (electrons and holes extracted per absorbed photon) of 90%. What can we conclude about the electron and hole diffusion lengths in silicon? [1 mark]

[This question continues on the next page . . .]

(ii) A relativistic space probe is launched which communicates with the Earth using a high-intensity laser with an emission wavelength at 0.70 μm and output power P. A large telescope focuses all of the light from the probe laser onto the photodiode.

(a) By considering the number of photons emitted by the laser per second, show that when the probe and the photodiode are at rest (relative velocity $u = 0$), the detected photodiode current i is given by:

$$i = (1 - R_t)\, e\Phi\, \frac{P\lambda}{hc}\, [1 - \exp(-\alpha(\lambda)L)] \left[\frac{1 + R_b \exp(-\alpha(\lambda)L)}{1 - R_t R_b \exp(-2\alpha(\lambda)L)} \right].$$

[3 marks]

(b) For a source moving away from the observer at velocity u, the frequency of light f is given by:

$$f = \sqrt{\frac{c-u}{c+u}}\, f_0$$

where f_0 is the frequency when $u = 0$. If the probe is travelling away from the Earth, find the change in the detected laser wavelength for $u = 0.3\,c$.

[1 mark]

(c) By carefully considering your answers to (ii)(a) and (b), find an equation relating the photodiode current i to the relativistic velocity u. Hence find the change in the detected photocurrent when the probe changes its velocity from $u = 0$ to $u = 0.3\,c$. [4 marks]

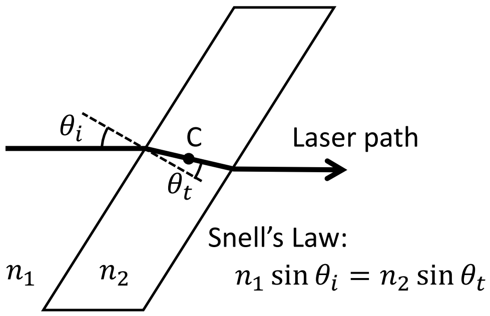
(d) Si has an energy gap $E_G = 1.12$ eV. Would this limit the range of detectable u? Would there be an advantage in using Ge ($E_G = 0.66$ eV) instead of Si? Justify your answer. [2 marks]

[Total 20 marks]

3. Forces are exerted at dielectric surfaces by photons as a result of their change in momentum as they are reflected or refracted. In particular, an increase in refractive index, n, when crossing a surface, slows photons down thereby increasing the magnitude of their wave-vector and thus their momentum according to $p = n\hbar k_0$.

This question considers the refractive forces exerted on a glass plate by the 10kJ energy light pulses typically found in laser fusion systems.

(i) The path of the laser pulses is shown as it passes through the centre, C, of a glass plate in cross-section in the figure below.



(a) By resolving the components of a single photon momentum as it is refracted at the interface, show that the impulse on the surface as a result of the refraction is always normal to the surface and derive an expression for its value. In what direction does the impulse act on the surface if the photon is moving from a lower refractive index to a higher one? Justify your answer.

[4 marks]

By orienting the plate at Brewster's angle, the reflection for p-polarised light is reduced to zero according to the Fresnel reflection coefficient:

$$R_p = \left| \frac{n_1 \cos\theta_t - n_2 \cos\theta_i}{n_1 \cos\theta_t + n_2 \cos\theta_i} \right|^2 = 0$$

and the incident and refracted angles are then also related by:

$$\tan\theta_i = \frac{n_2}{n_1} = \cot\theta_t$$

(b) Use these relations to show that the total net force, F, that would be exerted on the surface, for a p-polarised laser beam incident at Brewster's angle and of power P, containing many photons, may be written as:

$$F = \frac{P}{c} \frac{\left(n_1^2 - n_2^2\right)}{\sqrt{n_1^2 + n_2^2}}$$ [6 marks]

(ii) Equal but opposite forces generated on both entry and exit from the plate result in no net force on the plate but do result in a torque being exerted on the plate.

(a) Show that the laser exerts a torque T on the plate given by:

$$T = F\,d\,\frac{n_1}{n_2}$$

where d is the thickness of the plate. [2 marks]

(b) Calculate the angular impulse, $J = \int T\,dt$, that would be exerted on a 3.4cm thick glass amplifier plate with $n_2 = 1.5$ by a $U = \int P\,dt = 10\,\mathrm{kJ}$ laser pulse. Suppose the plate has a moment of inertia about the tilt axis through C of $1.7\ \mathrm{Kg\,m^2}$. Estimate the angular velocity of the plate if it were free to rotate after being hit by the laser pulse. [3 marks]

If the linear size of the plate is changed by a factor $\mathcal{L}$, but it retains the same shape and material used, then its volume and hence mass will scale with the cube of the linear dimension, i.e. by a factor $\mathcal{L}^3$.

(c) Show how the torque impulse and moment of inertia scale with laser pulse energy and plate size, and thereby compare the result from (ii)(b) to the angular velocity of a $3.4\,\mu$m thick plate of the same material and shape as the larger laser plate after it has been hit by a $1.0\,\mu$J laser pulse.

[3 marks]

(d) Radiation pressure due to refraction, reflection or absorption all result from changes in momentum of a magnitude similar to that of the incident photon. Comment on the practicality of using lasers for moving objects around using radiation pressure alone. [2 marks]

[Total 20 marks]

4. This question addresses the conservation of lepton and baryon numbers in particle physics, proton stability and its relation to the lifetime of human beings.

(i) Give the lepton and the baryon numbers for fundamental fermions (leptons and quarks) in the Standard Model (SM). Give two examples of decays forbidden in the SM, one violating the lepton number and one violating the baryon number. Draw the Feynman diagram for the leptonic muon decay, labelling all lines.

[3 marks]

(ii) The leptonic width of muon decay is given by $\Gamma(\mu^- \to e^- \bar{\nu}_e \nu_\mu) = G_F^2 m_\mu^5/(192\pi^3)$, where $G_F = 1.17 \times 10^{-5}\mathrm{GeV}^{-2}$, and the muon mass $m_\mu = 0.106\,\mathrm{GeV}$. Calculate the muon lifetime. You may assume that $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ is the only significant decay mode of the muon. [3 marks]

(iii) Explain why Γ must be proportional to m_μ^5 by use of dimensional grounds. Note that the amplitude of $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ decay is $\propto 1/M_W^2$, where M_W is the mass of the W boson. [3 marks]

(iv) Explain why protons are stable particles in the Standard Model. Most Grand Unified Theories explicitly break baryon number symmetry but require the conservation of the difference $B - L$ of baryon, B, and lepton, L, numbers. Explain why the decay $p \to e^+\pi^0$ is allowed in such theories. Draw the simplest Feynman diagram for this decay assuming that the decay is mediated by a new boson X, which can couple to both leptons and quarks. [3 marks]

(v) If the mass of the X boson is $M_X = 8 \times 10^{14}$ GeV, estimate the proton lifetime given that the muon lifetime is $\tau_\mu = 2.2\,\mu$s and the mass of W boson is 80 GeV. You may assume that the proton decays exclusively to the $e^+\pi^0$ final state and that the X boson has conventional weak couplings to the SM fermions.

[4 marks]

(vi) The best experimental limits on the proton lifetime are in the range $\tau_p > 10^{31} - 10^{33}$ years depending on the model. The proton lifetime can also be estimated from the requirement that the total radiation dose inside a human body does not exceed the annual dose delivered by cosmic ray muons, 30 mrad (1rad = 6.2×10^{10}MeV/kg). Calculate the value of the mean proton lifetime that would result in this dose, if protons underwent decay and their total mass energy (0.94 GeV) then appeared in the form of ionising radiation inside the human body. Comment on the significance of your result. You may assume the same number of protons and neutrons per 1 kg of the human body and that $m_p = m_n = 1.7 \times 10^{-27}$kg. [4 marks]

[Total 20 marks]

5. This question concerns a quantum double barrier and the phenomenon of resonant tunnelling, which is the basis of several devices such as a single electron transistor.

(i) A beam of electrons of wavevector k is incident from the left ($x \ll 0$) on a narrow barrier at $x = x_0$. The barrier is described by complex transmission and reflection coefficients t_1 and r_1, respectively, such that the wave function may be written as

$$\psi(x) = \begin{cases} e^{+ikx} + \left(r_1 e^{+i2kx_0}\right) e^{-ikx} & \text{for } x < x_0 \\ t_1 e^{+ikx} & \text{for } x > x_0 \end{cases}$$

where the factor e^{i2kx_0} accounts for the phase change as the wave travels from $x = 0$ to $x = x_0$ and back again.

(a) Write down the corresponding expression describing the wave function for a beam of electrons incident from the right.

(b) Write down the relationship between the complex numbers t_1 and r_1 that guarantees conservation of the number of electrons.

[3 marks]

(ii) A barrier is placed at $x = 0$ and a second identical barrier at $x = d$, such that

$$\psi(x) = \begin{cases} e^{+ikx} + r_2 e^{-ikx} & \text{for } x < 0 \\ a_+ e^{+ikx} + a_- e^{-ikx} & \text{for } 0 < x < d \\ t_2 e^{+ikx} & \text{for } x > d \end{cases}$$

(a) Solve for a_+ and a_- and hence show that the transmission coefficient of the double barrier system, t_2, may be written in the form

$$t_2 = t_1^2 \left[1 - |r_1|^2 e^{i2kd}\right]^{-1} .$$

(b) Hence show that for certain values of k the system is perfectly transmitting, $|t_2|^2 = 1$, and that these values correspond to the bound states of an infinite well.

[7 marks]

(iii) (a) Show that the value of k at which the transmitted intensity has half its maximum value ($|t_2|^2 = \frac{1}{2}$) is given by

$$|t_1|^4 = 4\,|r_1|^2 \sin^2(kd)$$

(b) Hence derive an approximate expression for the full width at half maximum of the peak in $|t_2|^2$, assuming that it is much narrower than the separation between the peaks.

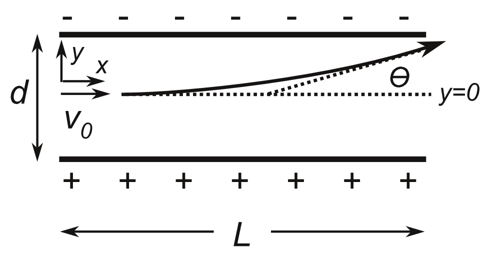
(c) Hence find the numerical value of δE ($= \frac{dE}{dk}\delta k$), the width of the lowest energy peak, where $|t_1|^2 = 0.1$ and $d = 1\,\mu$m.

[8 marks]

(iv) Although the above analysis tells us that an electron incident on the double barrier with a certain energy will certainly be transmitted, it does not tell us how long it takes to get through the double barrier. Use your knowledge of quantum physics to estimate the time taken. [2 marks]

[Total 20 marks]

6. (i) Electrostatic analysers are used to measure the kinetic energy W_0 of a charged particle. The simplest concept, sketched, consists of an ideal (no end effects) parallel plate capacitor of length L and separation d centred on the x-axis, charged to a voltage V. At $t = 0$ a particle of mass m and charge q moving non-relativistically at $\mathbf{v} = v_o\,\hat{\mathbf{x}}$ enters at the left end ($x = 0$).

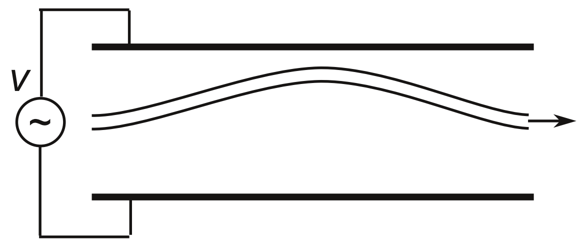


(a) By considering the particle's x and y equations of motion, or otherwise, calculate the angle of deflection θ when the particle exits at $x = L$. Show that this is a function of the particle's energy per charge, W_0/q, and the parameters of the analyser.

(b) Hence deduce that by measuring θ you cannot distinguish an alpha particle (doubly-ionised He atom (m = 4 amu)) with initial energy W_α from a proton with initial energy $W_\alpha/2$.

(c) Show also that if a proton and an alpha particle have the same initial speed v_0 then the alpha particle will have $\tan\theta_\alpha = (1/2)\tan\theta_p$. [7 marks]

(ii) A novel solution to extracting more information about the particle is to drive the capacitor voltage sinusoidally, so that $V = A\cos\omega t$ with A a constant, and to insert a passive sinusoidally-shaped narrow channel as sketched. The channel is non-conducting and non-polarising (its dielectric constant $\epsilon = 1$) so it does not disturb the uniformity of the field between the capacitor plates. It does, however, remove any particle that hits its walls. The path along the centre of the channel is $y = a\,[1 - \cos(2\pi x/L)]$.



(a) Particles enter the channel from the left along the x-axis at $t = 0$. Neglecting for the time being any induced magnetic field, show that particles which exit the channel at $x = L$ must not only have a specific energy per charge W/q but also a specific charge to mass ratio q/m. Give expressions for these quantities in terms of the parameters of the analyser. Thus this analyser IS capable of distinguishing protons and He^{++} ions.

(b) ESTIMATE without detailed derivation the magnitude of the induced magnetic field, and hence verify *a posteriori* that the magnetic force on a non-relativistic particle is negligible here.

[13 marks]

[Total 20 marks]

[Ampere's Law with displacement current is $\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t}$]

7. This question is about the change of entropy of a thermal bath, or heat reservoir, when viewed from both statistical physics and thermodynamic perspectives.

(i) We model a heat reservoir as made of a large number of particles, N, and we use the micro-canonical ensemble to describe its statistics. The reservoir is at equilibrium with, initially, an energy U_o. Its volume V is assumed to be constant through all parts of the question.

(a) Write an expression for the propability p_j that a microstate j of the system is occupied if the total number of microstates is Ω_o. [3 marks]

(b) Likewise, write an expression for the occupation number n_j of the microstate j. [3 marks]

(ii) An amount of energy Q is added to the reservoir by putting it in contact with a hot object. The energy of the reservoir after a new equilibrium has been reached is $U_1 = U_o + Q$.

(a) For a heat reservoir, we can assume that $Q \ll U_o$. Show that the change in entropy is then given by

$$\Delta S \approx \frac{Q}{T_o} \tag{1}$$

and that the total number of microstates Ω_1 in the final equilibrium state is:

$$\Omega_1 \approx \Omega_o exp\left(\frac{Q}{k_B T_o}\right), \tag{2}$$

where k_B is Boltzmann's constant and T_o is the temperature of the reservoir before heat has been added to it. [Hint: you may apply a Taylor expansion to $S_1 = S(U_o + Q, V)$ where $S = k_B \ln \Omega$ is Boltzmann's entropy (Ω being the total number of microstates in a given macrostate), and use the result $(\partial S/\partial U)_V = 1/T$ where T is temperature.] [4 marks]

(b) Compare the total number of microstates Ω_o and Ω_1 and interpret microscopically why the reservoir's entropy increases upon heating. [2 marks]

(iii) We now turn to a macroscopic interpretation of the result (1).

(a) Explain briefly why Clausius' statement of the second law can be written, for the transformation described in (ii), as

$$\Delta S > \int \frac{\delta Q}{T}, \tag{3}$$

in which δQ represents the incremental heat added/subtracted during the transformation. [3 marks]

(b) By writing $T = T_o + T'$ with $T' \ll T_o$ and using a Taylor expansion, show that application of eq. (3) to the heating process in (ii) yields

$$\Delta S > \frac{Q}{T_o} + \Delta S'. \tag{4}$$

Give an expression for the 2nd order term $\Delta S'$ in terms of δQ, T_o and T'. [3 marks]

[This question continues on the next page . . .]

(c) Give a physical interpretation of the inequality in equation (4) when compared to the result in equation (1). [2 marks]

[Total 20 marks]

8. This question concerns the motion of a satellite in orbit around a planet.

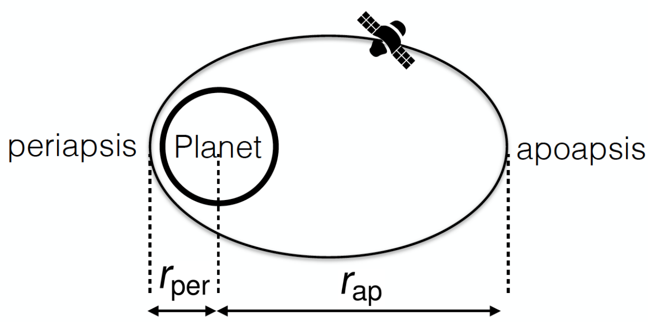
(i) Show that the period T of a satellite on a circular orbit of radius r around a planet with mass M is given by

$$T = 2\pi \sqrt{\frac{r^3}{GM}}$$

where G is the gravitational constant. [3 marks]

(ii) Find an expression for the total (kinetic+potential) energy of a satellite of mass m on a circular orbit as a function of M, m and r. [2 marks]

Following Kepler's third law, the above expression for T can also be applied to satellites on elliptical orbits, replacing r with the semi-major axis, a, of the orbit. The point on the orbit closest to the planet is the periapsis (r_{per}) and that furthest from the planet is the apoapsis (r_{ap}), as illustrated in the diagram below. For an orbit with eccentricity e we have $r_{\text{per}} = a(1 - e)$ and $r_{\text{ap}} = a(1 + e)$.



The ExoMars Trace Gas Orbiter (TGO) satellite arrived at Mars in October 2016 and entered into an initial elliptical orbit ($e = 0.926$) with a 94.3 hour period.

(iii) Calculate a, r_{per} and r_{ap} of the initial TGO orbit. [3 marks]

Beginning in spring of 2017, the orbit of the TGO is changed to one with a periapsis of 3520 km (130 km altitude above the Martian surface) and apoapsis of 95,000 km.

(iv) By considering the energy and angular momentum at periapsis and apoapsis, calculate the satellite speeds at periapsis, v_{per}, and apoapsis, v_{ap}. [6 marks]

This orbit results in so-called aerobraking, repeatedly "dipping" the spacecraft through the Martian atmosphere where close to periapsis it experiences an atmospheric drag force

$$F_{\text{drag}} = \frac{1}{2}\rho\, C_D\, A\, v^2$$

where $\rho \approx 2 \cdot 10^{-6}$ kg/m^3 is the atmosphere density near 130 km altitude, $C_D \approx 2.2$ is the satellite drag coefficient, $A \approx 35$ m^2 is the area of the spacecraft normal to its motion and v its speed. The purpose of aerobraking is to obtain a circular orbit.

[This question continues on the next page . . .]

(v) Calculate the approximate amount of kinetic energy lost by the satellite when it flies through the Martian atmosphere, assuming that it travels at roughly uniform speed at an almost constant altitude ($\sim$130 km) through the atmosphere over a horizontal distance of around 50 km during the dip. [2 marks]

(vi) After the first "dip" through the atmosphere, the satellite's orbital distance at apoapsis will have changed. Calculate the new value of r_{ap} after the "dip". [4 marks]

[Total 20 marks]

The mass of planet Mars is $M = 6.417 \cdot 10^{23}$ kg.
The radius of planet Mars is $R = 3390$ km.
The mass of the TGO satellite is $m = 3732$ kg.

9. (i) A planet is bathed in electromagnetic radiation emitted by a distant star and has no other source of energy. It rotates at a rate sufficient to achieve a uniform equilibrium temperature T_p at which the flux of thermal radiation it emits balances the flux of stellar radiation it absorbs. Assuming that the planet and the star are spheres, and behave as black bodies at temperatures T_p and T_s respectively, show that $T_p = \sqrt{r_s/(2d)}T_s$ where r_s is the radius of the star and d is the distance between the bodies ($r_s \ll d$). [5 marks]

(ii) The entropy flux J (power per unit area per unit temperature) of a beam of electromagnetic radiation is related to the energy flux F (power per unit area) by $dJ = (1/T)dF$ where T is the temperature associated with the radiation. Show that the flux of radiation entropy leaving a black body is $J = 4F/(3T)$.

[4 marks]

(iii) The spectral intensity $\mathcal{I}$ of the radiation emitted by a black body at frequency ν and temperature T is

$$\mathcal{I} = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/(k_B T)} - 1}.$$

(a) Show that

$$\frac{1}{T} = \frac{k_B}{h\nu}\left[\ln(1+y) - \ln y\right]$$

where $y = [c^2/(2h\nu^3)]\mathcal{I}$.

(b) Consider now the radiation entropy spectral intensity L associated with $\mathcal{I}$. Assuming $dL = (1/T)d\mathcal{I}$ (where the differentials are with respect to temperature) show that:

$$L = \frac{2k_B\nu^2}{c^2}\left[(1+y)\ln(1+y) - y\ln y\right]$$

.

[5 marks]

(iv) If the planet in part (i) is far from the star it is at a low temperature. Show that under these conditions: $L/\mathcal{I} \approx (1/T) + (k_B/h\nu)$. Comment (without mathematical derivation) on whether this is consistent with the result of part (ii). [6 marks]

[Useful integral: $\int \ln(ax+b)dx = \frac{1}{a}(ax+b)\ln(ax+b) - x$] [Total 20 marks]

10. Write short notes on FIVE of the following topics:

(i) Capacitance

(ii) Black holes

(iii) Einstein coefficients

(iv) Liquid drop model of the nucleus

(v) Pauli matrices

(vi) Bose-Einstein condensation

(vii) Bernoulli's equation

(viii) Relativistic energy

(ix) The critical point in thermodynamics

(x) Microscopic interpretation of entropy

[Total 20 marks]