Homework - 2

Problem - 1

$$f = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$$

$$\frac{\partial f}{\partial x_1} = 4x_1 - 4x_2 \quad , \quad \frac{\partial f}{\partial x_2} = -4x_1 + 3x_2 + 1$$

$$\frac{\partial^2 f}{\partial x_1^2} = 4 \quad , \quad \frac{\partial^2 f}{\partial x_2^2} = 3 \quad , \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = -4$$

$$4x_1 - 4x_2 = 0 \quad \Rightarrow x_1 = x_2$$

$$1 - 4x_1 + 3x_2 = 0$$
$$1 - 4x_1 + 3x_1 = 0 \quad \Rightarrow \boxed{x_1 = 1 = x_2}$$
$$1 - x_1 = 0$$

$$D(1,1) = 4 \times 3 - (-4)^2 = 12 - 16 = -4 < 0$$
$\Rightarrow (1,1)$ is a saddle point

Hessian Matrix $H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$

Eigen values of H

$$\det |H - \lambda I| = \begin{vmatrix} 4-\lambda & -4 \\ -4 & 3-\lambda \end{vmatrix}$$
$$= (4-\lambda)(3-\lambda) - (-4)(-4)$$
$$= 12 - 4\lambda - 3\lambda + \lambda^2 - 16$$
$$= \lambda^2 - 7\lambda - 4$$

$$\Rightarrow \lambda = \frac{7 \pm \sqrt{49 - (4 \times 1 \times -4)}}{2} = \frac{7 \pm \sqrt{65}}{2}$$

$\lambda_1 = 7.531$ and $\lambda_2 = -0.531$

Eigen vector of H

For λ_1 $\begin{bmatrix} \frac{-1+\sqrt{65}}{8} \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.8827 \\ 1 \end{bmatrix}$ | for λ_2 $\begin{bmatrix} -\frac{1+\sqrt{65}}{8} \\ 1 \end{bmatrix} \approx \begin{bmatrix} -1.327 \\ 1 \end{bmatrix}$

Hessian matrix H has one +ve and one -ve eigen values
$\Rightarrow$ H is a indefinite matrix

Taylor's ... sion at $[x_1, x_2] = [1, 1]$

$g_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad H_0 = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$ $\quad f_0 = 0.5$

$$f(x) = f_0 + g_0^T (x - x_0) + \frac{1}{2}(x - x_0)^T H (x - x_0)$$

$$f(x) = 0.5 + 0 + \frac{1}{2}[x_1 \; x_2] \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2}[x_1 \; x_2] \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = f(x) - 0.5 < 0$$

$$\frac{1}{2}(4x_1^2 - 8x_1x_2 + 3x_2^2) < 0$$

$$0.5(2x_1 - x_2)(2x_2 - 3x_2) < 0$$

$\Rightarrow$ $2x_1 - x_2 < 0$ AND $2x_2 - 3x_2 > 0$

OR

$2x_1 - x_2 > 0$ AND $2x_2 - 3x_2 < 0$

Problem 2

Let assume the point on the plane be (x_1, x_2, x_3)

distance b/w (x_1, x_2, x_3) and $(-1, 0, 1)$

$$= \sqrt{(x_1+1)^2 + x_2^2 + (x_3-1)^2}$$

we know if $\min(f(x)) \Rightarrow \min(\sqrt{f(x)})$

$\Rightarrow$ we have to minimize

$$\min\left[(x_1+1)^2 + x_2^2 + (x_3-1)^2\right] \quad -(1)$$

which is s.t

$$x_1 + 2x_2 + 3x_3 = 1$$

$$\Rightarrow x_1 = 1 - 2x_2 - 3x_3 \quad -(2)$$

put the value of x_1 in eqn (1)

$$\Rightarrow \min\left[(1 - 2x_2 - 3x_3 + 1)^2 + x_2^2 + (x_3-1)^2\right]$$

$$\Rightarrow f(x,y) = (2 - 2x - 3y)^2 + x^2 + (y-1)^2$$

Gradient $g = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix} = \begin{bmatrix} 10x + 12y - 8 \\ 20y + 12x - 14 \end{bmatrix}$

Hessian $H = \begin{bmatrix} \partial^2 f/\partial x^2 & \partial^2 f/\partial y \partial x \\ \partial^2 f/\partial x \partial y & \partial^2 f/\partial y^2 \end{bmatrix} = \begin{bmatrix} 10 & 12 \\ 12 & 20 \end{bmatrix}$

Hessian of this function is +ve

$\Rightarrow f(x,y)$ is a convex function

Problem 3

a) to proof $af(x) + bg(x)$ is convex s.t $a \geq 0$ and $b > 0$

For any point x_1 and x_2 which belong to $\mathcal{X}$

$\lambda x_1 + (1-\lambda) x_2$ is a convex function

If $\mathcal{X}$ is a convex set and 'λ' belongs to $[0, 1]$

$$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

Similarly

$$g(\lambda x_1 + (1-\lambda) x_2) \leq \lambda g(x_1) + (1-\lambda) g(x_2)$$

with a and b variable

$$a f(\lambda x_1 + (1-\lambda) x_2) + b g(\lambda x_1 + (1-\lambda) x_2 \leq a\lambda f(x_1) + a(1-\lambda) f(x_2) + b\lambda g(x_1) + b(1-\lambda) g(x_2)$$

$$a f(\lambda x_1 + (1-\lambda) x_2) + b g(\lambda x_1 + (1-\lambda) x_2 \leq \lambda [af(x_1) + bg(x_1)] + (1-\lambda) [af(x_2) + bg(x_2)]$$

As the above statement is a convex function

$\Rightarrow af(x) + bg(x)$ is a convex function

b) Condition for which $f(g(x))$ will be convex

Let $h(x) = f(g(x))$ dom $h = \{x \in \text{dom}\, g \mid g(x) \in \text{dom}\, f\}$

The second derivate of $h = f \circ g$

$$h''(x) = f''(g(x))\, g'(x)^2 + f'(g(x))\, g''(x)$$

Now if g is convex ($g'' \geq 0$)

f is convex and increasing ($f'' \geq 0$ and $f' \geq 0$)

$h'' \geq 0$

$\Rightarrow$ h is convex

Similarly

h is convex when

- f is convex and increasing, and g is convex
- f is convex and increasing, and g is concave

Problem 4

to show $f(x_1) \geq f(x_0) + g_{x_0}^T (x_1 - x_0)$ for a convex function

$f(x) : \mathcal{X} \to R$ and for $x_0, x_1 \in \mathcal{X}$

As f is a convex function, and x, y dom$f$.

Since dom f is convex

$\Rightarrow$ $0 < \lambda \leq 1$, $x + \lambda(y - x) \in \text{dom} f$

$\Rightarrow$ $f(x + \lambda(y-x)) \leq (1-\lambda) f(x) + \lambda f(x)$

Divide both sides by λ

we get,

$$f(y) \geq f(x) + \frac{f(x + d(y-x)) - f(x)}{d} \quad — ①$$

as $d \to 0$

let $z = \theta x + (1-\theta) y$

Applying ① twice yield

$$f(x) \geq f(z) + f'(z)(x-z) \quad , \quad f(y) \geq f(z) + f'(z)(y-z)$$

Multiply first inequality by θ and second by $1-\theta$

$$\theta f(x) \geq \theta f(z) + \theta f'(z)(x-z) \quad — ②$$

$$(1-\theta) f(y) \geq (1-\theta) f(z) + (1-\theta) f'(z)(y-z) \quad — ③$$

Add ② and ③, we get

$$\theta f(x) + (1-\theta) f(y) \geq f(z)$$

$$g(t) = f(dy + (1-d) x)$$

$$g'(t) = \nabla f(dy + (1-d) x)^T (y-x)$$

As f is conve which implies g is convex

$$\Rightarrow g(1) \geq g(0) + g'(0)$$

$$\Rightarrow f(y) \geq f(x) + \nabla f(x)^T (y-x)$$

$$\Rightarrow \boxed{f(x_1) \geq f(x_0) + g_{x_0}^T (x_1 - x_0)}$$

Problem 5

a)

For optimality function we have to minimize the error b/w the intensity level $a_k^T p$ and the target intensity I_t

Intensity level on k^{th} mirror $= a_k^T p$

where $p \to$ power output
$a \to$ distance b/w lamp and mirror

(I_k) Total Intensity $= \sum_{j=1}^{n} a_{kj} P_j$

$$\min \quad f(P_i)_{j=1}^{n} = \sum_{i=1}^{m} (I_k - I_t)^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{kj} P_j - I_t)^2$$

b)

$g = 2(a^T p - I)\, a$

$H = 2a\, a^T$

Here $H \geq 0$

We know
$d^T H d \geq 0$
Here $d \neq 0$
This mean that H is psd

$\Rightarrow$ The function is a convex problem c)

Yes there will be a unique solⁿ if the over-all power output is P^*

d) If only half of the lamp switched on then the functions should not have a unique solⁿ