Topic 3

PLANETARY AND SATELLITE MOTION

Content

- Kepler's laws of planetary motion
- Gravitational potential energy of a spherical shell
- Elliptical orbits and orbital transfers
- Concept of an effective radial potential

In the H3 Physics syllabus, building on the understanding of collisions and the significance of the centre of mass in equilibrium situations, candidates should understand and apply concepts related to non-relativistic dynamics viewed from different inertial frames.

Learning Outcomes

Candidates should be able to:

(a) show an understanding of Kepler's laws of planetary motion, and

(i) recall and apply Kepler's first law that the planets move in elliptical orbits with the Sun at one focus of the ellipse (knowledge of the eccentricity parameter is not required)

(ii) show an understanding of how Kepler's second law (that an imaginary line from the Sun to a moving planet sweeps out equal areas in equal intervals of time) is related to the conservation of angular momentum and apply this law to solve related problems

(iii) recall and apply Kepler's third law that the ratio of the square of a planet's period of revolution to the cube of the semi-major axis of its orbit around the Sun is a constant and that this constant is the same for all planets

(b) derive expressions for the gravitational potential energy of a point mass inside and outside a uniform spherical shell of mass, and relate these expressions to the justification for treating large spherical objects as point masses

(c) solve problems involving elliptical orbits and orbital transfers e.g. when a satellite fires its thrusters (knowledge of parabolic and hyperbolic trajectories is not required)

(d) derive the expression for the mechanical energy of a mass m interacting gravitationally with a large mass M, $E=\frac{1}{2}mv^2-\frac{GMm}{r}=\frac{1}{2}m\dot{r}^2+\frac{L^2}{2mr^2}-\frac{GMm}{r}$, where $\dot{r}$ is the component of the velocity of mass m in the radial direction and L is the angular momentum of mass m relative to mass M (where mass M is assumed to be stationary and $M \gg m$)

(e) discuss how the effective radial potential, $U_{eff}=\frac{L^2}{2mr^2}-\frac{GMm}{r}$, allows the determination of bound and unbound states, as well as turning points in the motion, and apply this to solve related problems.

1. Kepler's laws of planetary motion

1.1 KEPLER'S LAWS OF PLANETARY MOTION

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Kepler's 1st Law (elliptical orbits):

The orbit of a planet is an ellipse with the Sun at one of the two foci.

Kepler's 2nd Law (constant areal velocity):

A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.

Kepler's 3rd Law (relationship between orbital period and semi-major axis):

The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

We will examine these three laws in greater detail.

1.2 MECHANICS OF PLANETARY AND SATELLITE MOTION

The 2-body system of a satellite orbitting a more massive body (e.g., planet orbitting the Sun) can be simplified to a 1-body problem in which the satellite is observed from the point of view (frame of reference) of the larger central body, so that the motion of the central body is ignored. (see Annex A).

1.3 BASIC QUANTITIES IN ORBITAL MOTION

In the following discussion, we will use the Sun as the central body and a planet as the satellite. Of course, the conclusions reached can be applied to any other system in which one body orbits the other, e.g., artificial satellite orbitting the Earth.

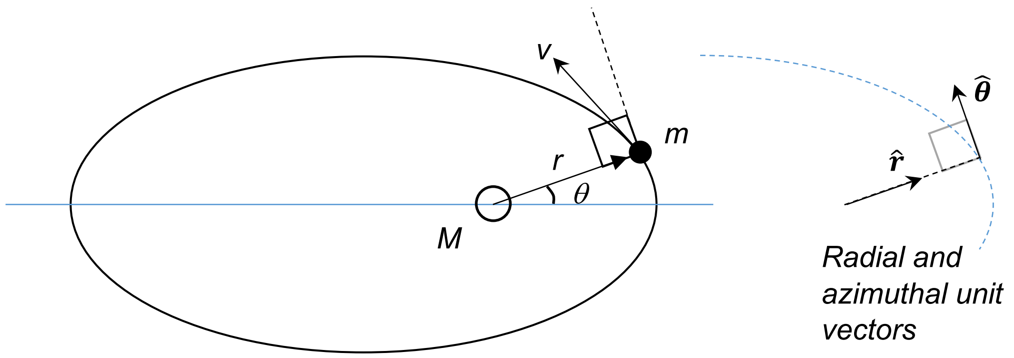

Fig. 1: Basic quantities in orbital motion

Let the mass of the Sun be *M*.

Let the mass of the planet be *m*.

Let the distance between the Sun and the planet be *r*.

Let the angle traversed by the radius in the orbital plane from an arbitrary direction be θ. This is usually called the azimuthal angle.

The radial direction is represented by the unit vector $\hat{\boldsymbol{r}}$.

The azimuthal direction is represented by the unit vector $\hat{\boldsymbol{\theta}}$. This is perpendicular to the radial direction $\hat{\boldsymbol{r}}$ in the anti-clockwise sense.

The velocity ***v*** of the planet is tangential to the elliptical path it follows as it orbits the Sun (Fig. 1). It is <u>not</u> perpendicular to the radius, unlike in circular motion.

The velocity ***v*** can be resolved into two components:

- The radial velocity component v_r represents the rate at which the radial distance *r* is changing with time, and so is written as:

$$v_r = \frac{dr}{dt} = \dot{r}$$

- The azimuthal velocity component v_θ represents the rate at which the azimuthal distance (measured in the direction perpendicular to the radius) is changing with time, and so is written as:

$$v_\theta = r\frac{d\theta}{dt} = r\dot{\theta}$$

Note that this becomes the familiar $v = r\omega$ when the orbit is perfectly circular (special case of an ellipse).

- Hence, the total velocity ***v*** can be written as a sum of these two velocity components:

$$\boldsymbol{v} = v_r\hat{\boldsymbol{r}} + v_\theta\hat{\boldsymbol{\theta}}$$

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + (r\dot{\theta})\hat{\boldsymbol{\theta}}$$

In terms of magnitude,

$$v^2 = \dot{r}^2 + \left(r\dot{\theta}\right)^2$$

1.4 CONSERVATION OF ANGULAR MOMENTUM & KEPLER'S 2ND LAW (CONSTANT AREAL VELOCITY)

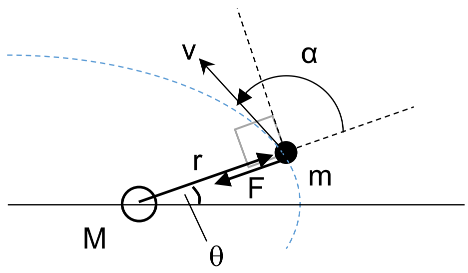

Fig. 2: Directions of vectors

The angular momentum of a planet of mass *m* orbitting the Sun is given by:

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p}$$

$$\therefore \boldsymbol{l} = \boldsymbol{r} \times m\boldsymbol{v}$$

$$\therefore \boldsymbol{l} = m(\boldsymbol{r} \times \boldsymbol{v})$$

The direction of the angular momentum ***l*** is therefore perpendicular to the plane containing vectors ***r*** and ***v***. In Fig. 1 and Fig. 2, the angular momentum vector ***l*** acts perpendicularly out of the page.

Because the gravitational force ***F*** exerted by the Sun on the planet acts along the line joining them, it does not exert any moment:

$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = rF\sin 180^o = \boldsymbol{0}$.

This means that the angular momentum ***l*** of the planet (mass *m*) is constant (in both magnitude and direction)*:

$$\boldsymbol{\tau} = \frac{d\boldsymbol{l}}{dt} = \boldsymbol{0}$$

⇒ ***l*** = constant

- The fact that the angular momentum ***l*** is constant in direction means that the planet's elliptical motion is confined to a single plane perpendicular to ***l***. This means that we can use polar coordinates in two dimensions and that the position of the planet can be specified by only two quantities *r* and θ.
- The fact that the angular momentum ***l*** is constant in magnitude is important because it leads to Kepler's 2nd Law (constant areal velocity):

l is constant

$$\Rightarrow \frac{d\boldsymbol{l}}{dt} = \boldsymbol{0}$$

But,

* This is analogous to Newton's Law applied to linear motion: If applied net force is zero, then $\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = \boldsymbol{0} \Rightarrow$ linear momentum ***p*** is constant.

$$\boldsymbol{l} = m(\boldsymbol{r} \times \boldsymbol{v})$$

Considering magnitudes,

$$l = mrv \sin \alpha$$

(See Fig. 2.)

$$l = mrv_\theta$$
$$l = mr(r\dot{\theta})$$
$$l = mr^2\dot{\theta}$$

So,

$$\frac{dl}{dt} = 0$$

leads to

$$\frac{d(mr^2\dot{\theta})}{dt} = 0$$
$$\frac{d(r^2\dot{\theta})}{dt} = 0$$
$$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0$$

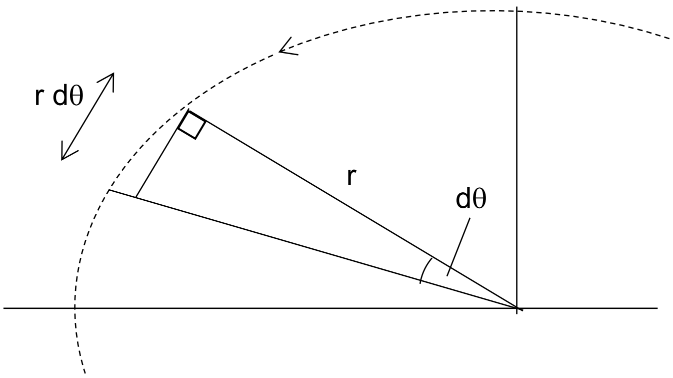

Fig. 3: The area *dA* swept out by the radius vector *r* is time *dt* is (1/2)*r(r* d*θ*).

The significance of this equation is seen if we consider the meaning of $\frac{1}{2}r^2\dot{\theta}$:

$$\frac{1}{2}r^2\dot{\theta} = \frac{1}{2}r\left(r\frac{d\theta}{dt}\right) = \frac{\frac{1}{2}r^2 d\theta}{dt} = \frac{dA}{dt}$$

, as

$$dA = \frac{1}{2}r^2 d\theta$$

is the area swept out by the radius vector ***r*** in time *dt* (See Fig. 3).
So, the earlier equation

$$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0$$
$$\Rightarrow \frac{d}{dt}\left(\frac{dA}{dt}\right) = 0$$

$$\Rightarrow \frac{dA}{dt} = \text{constant}$$

In words, this says that the area swept out by the radius vector per unit time (areal velocity) is constant, which is **Kepler's 2nd Law** **<u>(constant areal velocity):</u>**.

1.5 TOTAL ENERGY OF SYSTEM

Assuming that no resistive forces act on the 2-body system of a planet orbitting the Sun, the total energy of the system is conserved:

$$E = T + V$$

, where

E = total energy,

T = kinetic energy, and

V = gravitational potential energy between Sun and planet.

$$\therefore E = \frac{1}{2}mv^2 + V$$

But, from Section 1.3,

$$v^2 = \dot{r}^2 + \left(r\dot{\theta}\right)^2$$

$$\therefore E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V$$

Note that the gravitational potential energy *V* is given by:

$$V = -\frac{GMm}{r} = -\frac{k}{r}$$

, using the constant $k = GMm$ to simplify the expression.

From Section 1.4, the angular momentum *l* is

$$l = mr^2\dot{\theta}$$

$$\therefore \dot{\theta} = \frac{l}{mr^2}$$

Substituting this into the expression for *E* above gives

$$\therefore E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\left(\frac{l}{mr^2}\right)^2 + V$$

$$\therefore E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{GMm}{r} \qquad \text{(Total Energy of System)}$$

1.6 DERIVING KEPLER'S 1ST LAW (ELLIPTICAL ORBITS)

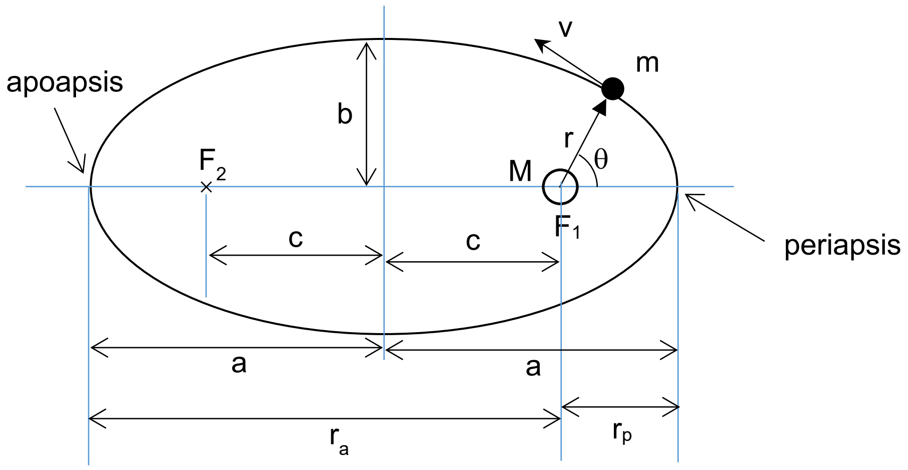

Fig. 4: Elliptical orbital trajectory. F_1 and F_2 are foci. The Sun is at F_1. The semi-major axis is distance *a*. The distance of each focus from the centre is distance *c*. The distance between the Sun and the periapsis is $r_p = a - c$.

Deriving the trajectory equation is <u>not in syllabus</u>, but it is always good to know basic concepts such as the meaning of terms like *semi-major axis* and *eccentricity*.

If you are interested, the derivation is available in Annex D.

The elliptical path is described by the equation (NOT in syllabus):

$$r = \frac{\frac{l^2}{mk}}{1 + \sqrt{1 + \frac{2El^2}{mk^2}} \cos\theta}$$

, where $k = GMm$.

This equation has the form of the equation of an ellipse in polar form:

$$r = \frac{p}{1 + e\cos\theta}$$

, thereby showing that the orbital path is elliptical.

1.7 RELATIONSHIP BETWEEN TOTAL ENERGY AND SEMI-MAJOR AXIS

Now that we are certain that the orbital trajectory of the planet is an ellipse, we can use this knowledge.

- As the orbit is elliptical in shape, the planet will have two turning points in its motion, which occur at the periapsis and apoapsis (the apsidal points).
- At the periapsis and apoapsis, the planet is instantaneously at the minimum and maximum distances respectively from the Sun, and so the radial velocity $\dot{r} = \frac{dr}{dt}$ is zero.

Using the expression for the total energy of the system from Section 0

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V$$

, and substituting $\dot{r} = 0$, we get:

$$E = 0 + \frac{l^2}{2mr^2} - \frac{k}{r}$$

Re-arranging, we get:

$$r^2 + \frac{k}{E}r - \frac{l^2}{2mE} = 0$$

The solutions to this quadratic equation are given by:

$$r = \frac{-\frac{k}{E} \pm \sqrt{\left(\frac{k}{E}\right)^2 + \frac{2l^2}{mE}}}{2}$$

The minimum distance r_p between the Sun and the orbit occurs at the periapsis (perihelion):

$$r_p = \frac{-\frac{k}{E} - \sqrt{\left(\frac{k}{E}\right)^2 + \frac{2l^2}{mE}}}{2}$$

The maximum distance r_a between the Sun and the orbit occurs at the apoapsis (aphelion):

$$r_a = \frac{-\frac{k}{E} + \sqrt{\left(\frac{k}{E}\right)^2 + \frac{2l^2}{mE}}}{2}$$

$$\therefore r_p + r_a = -\frac{k}{E}$$

But, from Fig. 4, $r_p + r_a = 2a$

Comparing the above two expressions for $r_p + r_a$, we get:

$$r_p + r_a = -\frac{k}{E} = 2a$$

Hence, the total energy *E* is related to the semi-major axis *a* via:

$$E = -\frac{k}{2a}$$

$$\boxed{E = -\frac{GMm}{2a}}$$

, as *k* = *GMm*.†

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† Notice that this expression is similar to the total energy of a satellite (mass *m*) with a perfectly circular orbital path,

$$E = -\frac{GMm}{2r}$$

1.8 VIS-VIVA EQUATION (ORBITAL ENERGY INVARIANCE LAW)

The total energy *E* is given by:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

But, we have derived the total energy in the previous section:

$$E = -\frac{GMm}{2a}$$

Hence,

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}$$

Re-arranging this, we can obtain the vis-viva equation, which is a useful equation that relates the magnitude of the planet's velocity *v* to its distance *r* from the Sun.

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

, where
v = speed of planet relative to Sun,
M = mass of central body (e.g., Sun),
r = radial distance of planet from Sun,
a = semi-major axis of elliptical orbit.

The vis-viva equation originates from the conservation of mechanical energy of the Sun-planet system; the total energy *E* is constant at every point in the trajectory.

1.9 KEPLER'S 3RD LAW (RELATIONSHIP BETWEEN ORBITAL PERIOD AND SEMI-MAJOR AXIS):

Kepler's 3rd Law states that:

The square of the orbital period of a planet's elliptical orbit around the Sun is proportional to the cube of the semi-major axis of its orbit.

$$\tau^2 \propto a^3$$

The full relationship is:

$$\tau^2 = \frac{4\pi^2 m}{k}a^3$$

, where $k = GMm$.
So, the relationship describing Kepler's 3rd Law is:

$$\tau^2 = \frac{4\pi^2}{GM}a^3$$

‡

‡ This expression looks similar to the period for a perfectly circular orbit (H2 syllabus):

$$\tau^2 = \frac{4\pi^2}{GM}r^3$$

2. Gravitational potential energy of a spherical shell

In the topic of Gravitation in the H2 syllabus, we calculated the gravitational potential ϕ and the gravitational field g outside of a central mass M (e.g., Earth, Sun) using the relations

$$\phi = -\frac{GM}{r} \qquad \text{and} \qquad \boldsymbol{g} = -\frac{GM}{r^2}\hat{\boldsymbol{r}}$$

, where r is measured from the centre of the central body, an assumption that we did not prove. This assumption is not a problem if the distance r is large compared to the radius R of the central body, as the planet or star can be treated as a point mass at such a large distance.

However, is this assumption correct if r is small compared to R? In the following, we shall prove that the above expression for the gravitational potential ϕ is valid for all points outside of a spherical shell (hollow sphere) or solid sphere.

2.1 Gravitational potential energy of a point mass located outside a spherical shell

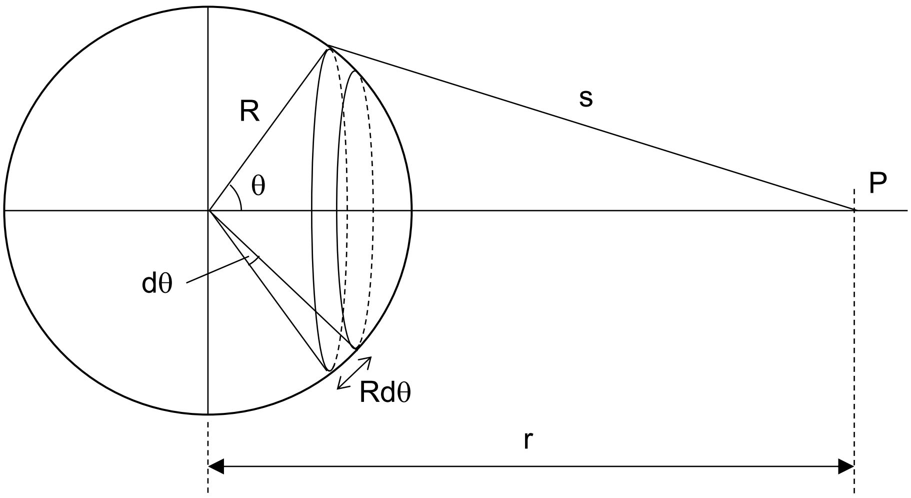

Fig. 5: Newton's method of calculating the gravitational potential a point P outside a spherical shell

We aim to calculate the gravitational potential ϕ at a point P outside of a spherical shell of radius R. The distance between P and the centre of the sphere is r. We will use the method used by Isaac Newton in his *Principia Mathematica*.

We cut the spherical shell into rings, each subtending an infinitesimal angle $d\theta$ at the centre. The width of the ring is then $R\,d\theta$. Each point on the ring is at a distance s from point P.

The gravitational potential due to the ring at P is given by

$$d\phi = -\frac{G\,dM}{s}$$

, where dM = mass of ring.

If σ = mass per unit area of the sphere, then

$$dM = \sigma \, dA$$

, where *dA* = area of ring.

$$\therefore dM = \sigma \times 2\pi R \sin\theta \times R \, d\theta$$

$$\therefore dM = \sigma \times 2\pi R^2 \sin\theta \, d\theta$$

But,

$$\sigma = \frac{M}{4\pi R^2}$$

, where *M* = total mass of spherical shell.

$$\therefore dM = \frac{M}{4\pi R^2} \times 2\pi R^2 \sin\theta \, d\theta$$

$$dM = \frac{M \sin\theta}{2} d\theta$$

$$\therefore d\phi = -\frac{G \, dM}{s}$$

$$d\phi = -\frac{GM \sin\theta}{2s} d\theta$$

Hence, the total gravitational potential at P due to the entire spherical shell is

$$\phi = \int d\phi = -GM \int_0^{\pi} \frac{\sin\theta}{2s} d\theta$$

Hence, we see that it will be difficult to perform the above integration with respect to θ as the integrand is a complex function. Newton's method is to use the relation between *s* and θ to modify the integral into one done with respect to *s*.

By the Cosine Rule,

$$s^2 = R^2 + r^2 - 2Rr\cos\theta$$

$$\therefore 2s\frac{ds}{d\theta} = 2Rr\sin\theta$$

, as *R* and *r* are constants.

$$\therefore \frac{\sin\theta}{2s} d\theta = \frac{1}{2Rr} ds$$

The integral thus becomes:

$$\phi = -GM \int_{r-R}^{r+R} \frac{1}{2Rr} ds$$

$$\phi = -\frac{GM}{2Rr} \int_{r-R}^{r+R} ds$$

$$\therefore \phi = -\frac{GM}{2Rr}[r + R - (r - R)]$$

Hence, we obtain the gravitational potential outside of a **spherical shell** (hollow sphere):

$$\therefore \phi = -\frac{GM}{r}$$

The gravitational potential <u>energy</u> of a point mass *m* placed at point P is then given by:

$$U = -\frac{GMm}{r}$$

The gravitational field *g* at the point outside of a **spherical shell** can be found via $g = -\frac{d\phi}{dr}$:

$$g = -\frac{GM}{r^2}$$

The gravitational potential due to a **solid sphere** can be obtained easily by treating it as a collection of an infinite number of spherical shells and adding together all of the separate potentials due to each shell. It is easily seen that this potential is given also by:

$$\phi = -\frac{GM}{r}$$

, as the sum of the masses of all the individual shells is equal to the total mass *M* of the entire solid sphere.

2.2 GRAVITATIONAL POTENTIAL ENERGY OF A POINT MASS LOCATED INSIDE A SPHERICAL SHELL

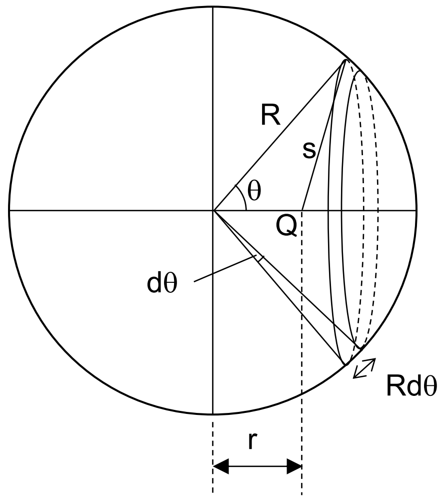

Fig. 6: Newton's method of calculating the gravitational potential a point Q inside a spherical shell

We can apply the same reasoning to find the gravitational potential ϕ at a point inside a spherical shell, such as Q in Fig. 6. Again, the spherical shell is treated as comprising many rings.

The same expression for ϕ is obtained:

$$\phi = -GM\int_0^{\pi} \frac{\sin\theta}{2s}\,d\theta$$

From Fig. 6, we see that the Cosine Rule still applies to *s*, *R* and *r* for point Q. Hence, we can again simplify the integral. However, the lower limit of the integration is now different:

$$\phi = -\frac{GM}{2Rr}\int_{R-r}^{R+r} ds$$

$$\therefore \phi = -\frac{GM}{2Rr}[R + r - (R - r)]$$

Hence, we obtain the gravitational potential inside a **spherical shell** (hollow sphere):

$$\phi = -\frac{GM}{R}$$

Hence, the gravitational potential ϕ inside a spherical shell is **constant**.

The gravitational potential energy of a point mass *m* placed at a point inside a spherical shell is then given by:

$$U = -\frac{GMm}{R}$$

As the gravitational potential ϕ is constant inside a spherical shell, the gravitational field *g* at the point inside of a **spherical shell** is therefore zero:

$$g = -\frac{d\phi}{dr} = 0$$

3. Elliptical orbits and orbital transfers

3.1 Hohmann Orbital transfer Maneuver

If a spacecraft needs to move from a planet orbiting the Sun to another planet, then the Hohmann transfer orbit is one method by which this can be achieved.

3.2 Change in velocity needed at the start of Hohmann orbital transfer maneuver

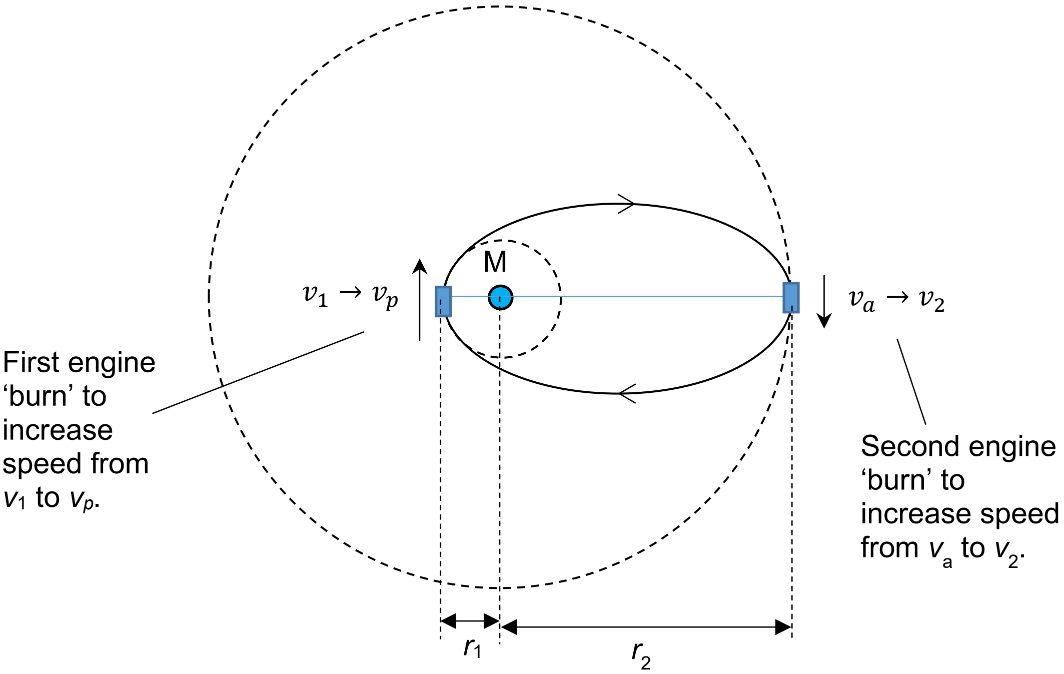

Fig. 7: Hohmann orbit transfer maneuver for increasing orbital radius from r_1 to r_2.

We assume that a spacecraft of mass *m* is initially orbitting a central body of mass *M* (which we call a 'planet' for convenience) along a circular path of radius r_1 and speed v_1, and that we want to move it to a higher orbit with a larger circular radius of r_2. The Hohmann transfer orbit is an elliptical orbit used to achieve this orbital transfer. The periapsis of the Hohmann orbit touches the initial circular orbit and the apoapsis touches the final circular orbit (assuming that the spacecraft is moving from a lower to a higher orbit).

From the H2 syllabus, we know that, for a circular orbit:

$$v_1 = \sqrt{\frac{GM}{r_1}}$$

To start the maneuver, the spacecraft needs to turn on its rocket engine so that its speed increases and it enters an elliptical path at the periapsis. This engine 'burn' is kept as short as possible so that the increase in speed can be taken to occur 'instantaneously' for the purpose of simplifying the calculations.

Let v_p = speed of spacecraft <u>after</u> the first engine 'burn' at the periapsis.

From the vis-viva equation (see Section 0), we know how the speed *v* of the spacecraft at any point in the elliptical Hohmann orbit is related to its distance *r* from the planet.

Hence, at the periapsis after the first engine 'burn',

$$v_p^2 = GM\left(\frac{2}{r_1} - \frac{1}{a}\right)$$

$$\therefore v_p = \sqrt{GM\left(\frac{2}{r_1} - \frac{1}{a}\right)}$$

, where *a* = semi-major axis of the elliptical Hohmann transfer orbit.

Hence, the increase in speed Δv_1 needed at the periapsis is:

$$\Delta v_1 = v_p - v_1$$

$$= \sqrt{GM\left(\frac{2}{r_1} - \frac{1}{a}\right)} - \sqrt{\frac{GM}{r_1}}$$

As $r_1 + r_2 = 2a$, we get:

$$\Delta v_1 = \sqrt{\frac{GM}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right)$$

3.3 Change in velocity needed at the end of Hohmann orbital transfer maneuver

Once the spacecraft reaches the position of the final circular orbit of radius r_2, the engine is turned on a second time to increase its speed again, so that it leaves the elliptical orbit and continue moving along this new circular orbit.

Again, from the H2 syllabus, we know that the speed v_2 of the spacecraft in the new circular orbit of radius r_2 is:

$$v_2 = \sqrt{\frac{GM}{r_2}}$$

Again, from the vis-viva equation, we can obtain the speed v_a of the spacecraft when it reaches the apoapsis of the elliptical orbit, before the second engine 'burn' to increase its speed to v_2:

$$v_a^2 = GM\left(\frac{2}{r_2} - \frac{1}{a}\right)$$

$$\therefore v_a = \sqrt{GM\left(\frac{2}{r_2} - \frac{1}{a}\right)}$$

Hence, the second increase in speed Δv_2 needed at the apoapsis is:

$$\Delta v_2 = v_2 - v_a$$

$$= \sqrt{\frac{GM}{r_2}} - \sqrt{GM\left(\frac{2}{r_2} - \frac{1}{a}\right)}$$

As $r_1 + r_2 = 2a$, we get:

$$\Delta v_2 = \sqrt{\frac{GM}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right)$$

3.4 Time taken for Hohmann orbital transfer maneuver

From Kepler's 3rd Law, we know the time taken τ_H for the spacecraft to complete one cycle of its elliptical orbit:

$$\tau_H^2 = \frac{4\pi^2}{GM}a^3 \Rightarrow \tau_H = \sqrt{\frac{4\pi^2}{GM}a^3}$$

The time taken to complete the Hohmann transfer orbit maneuver, from the periapsis to the apoapsis (or vice versa), is simply half of the full period:

$$t_H = \frac{1}{2}\tau_H$$

$$\therefore\ t_H = \frac{1}{2}\sqrt{\frac{4\pi^2}{GM}a^3}$$

As $r_1 + r_2 = 2a$, this becomes:

$$\therefore\ t_H = \pi\sqrt{\frac{(r_1 + r_2)^3}{8GM}}$$

4. Effective radial potential

4.1 DERIVATION

From Section 0, we obtained the total energy of the Sun-planet system:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V$$

, where

$\dot{r} = \frac{dr}{dt}$ = radial velocity = rate of change of radial distance between planet and Sun,

l = angular momentum, and

V = gravitational potential energy.

As the total energy *E* is constant in time (conserved), this means that

$$\frac{dE}{dt} = 0$$

Hence,

$$\therefore m\dot{r}\ddot{r} + \frac{d}{dt}\left(\frac{l^2}{2mr^2} + V\right) = 0$$

, where $\ddot{r} = \frac{d^2r}{dt^2}$.

Applying the chain rule of differentiation§ to the second term,

$$\therefore m\dot{r}\ddot{r} + \frac{d}{dr}\left(\frac{l^2}{2mr^2} + V\right)\frac{dr}{dt} = 0$$

$$\therefore m\dot{r}\ddot{r} + \frac{d}{dr}\left(\frac{l^2}{2mr^2} + V\right)\dot{r} = 0$$

As $\dot{r}$ is a common factor, it can be removed:

$$\therefore m\ddot{r} + \frac{d}{dr}\left(\frac{l^2}{2mr^2} + V\right) = 0$$

$$m\ddot{r} - \frac{l^2}{mr^3} + \frac{dV}{dr} = 0$$

$$m\ddot{r} = -\frac{dV}{dr} + \frac{l^2}{mr^3}$$

The left side of this equation represents the net force acting on the planet along the radius, i.e., the *effective radial force* ($F' = m\ddot{r}$).

§ Example of Chain Rule of differentiation: $\frac{dy}{dt} = \frac{dy}{dr} \times \frac{dr}{dt}$

The first term on the right side of the equation ($-\frac{dV}{dr}$) represents the gravitational force acting on the planet, as the potential energy

$$V = -\frac{k}{r} = -\frac{GMm}{r}$$

So, the gravitational force is

$$F = -\frac{dV}{dr} = -\frac{k}{r^2} = -\frac{GMm}{r^2}$$

The second term on the right side of the equation ($\frac{l^2}{mr^3}$) represents the centrifugal force experienced by the planet. This can be seen as follows:

$$\frac{l^2}{mr^3} = \frac{\left(mr^2\dot{\theta}\right)^2}{mr^3} = mr\dot{\theta}^2 = mr\omega^2$$

, where $mr\omega^2$ is recognized to be the centripetal force studied in the H2 syllabus.

Hence, the 2-dimensional motion of the planet can be seen as a 1-dimensional motion along radius (line joining the planet to the Sun), in which the planet experiences an *effective radial force* $F' = m\ddot{r}$ along the radius:

$$m\ddot{r} = -\frac{dV}{dr} + \frac{l^2}{mr^3}$$

$$F' = -\frac{GMm}{r^2} + \frac{l^2}{mr^3}$$

$$F' = F + \frac{l^2}{mr^3}$$

The above equation can be read as:

Effective radial force on planet = Gravitational force on planet + centrifugal force

The negative sign of the gravitational force $F\left(= -\frac{GMm}{r^2}\right)$ implies that it acts inwards, towards the Sun.

The positive sign of the centrifugal force $\left(+\frac{l^2}{mr^3}\right)$ implies that it acts outwards, away from the Sun. This term comes into consideration only because we are treating the motion as a 1-dimensional motion, and so are considering the system within the frame of reference of the rotating radius vector in which both Sun and planet are stationary, similar to the fictitious centrifugal force experienced by a person sitting in a train that is making a turn. In the external reference frame in which the Sun is stationary, no such centrifugal force exists (it is a fictitious force).

Corresponding to the effective radial force F' is an effective radial potential V':

$$F' = -\frac{dV'}{dr}$$

Hence, if

$$F' = F + \frac{l^2}{mr^3}$$

then

$$-\frac{dV'}{dr} = -\frac{dV}{dr} - \frac{d}{dr}\left(\frac{l^2}{2mr^2}\right)$$

$$\therefore -\frac{dV'}{dr} = -\frac{d}{dr}\left(V + \frac{l^2}{2mr^2}\right)$$

Hence, the effective radial potential is:

$$\therefore V' = V + \frac{l^2}{2mr^2}$$

This expression for the *effective radial potential* V' can likewise be obtained from the original expression for the total energy of the system:

$$E = \frac{1}{2}m\dot{r}^2 + \left(\frac{l^2}{2mr^2} + V\right)$$

If we consider this expression by pretending that the motion of the planet is 1-dimensional (along the radius) then the first term on the right side represents the kinetic energy of the planet along the radius (*radial kinetic energy*) while the two terms in the brackets represent the *effective radial potential* (energy). So, *E* can be written as:

$$E = \frac{1}{2}m\dot{r}^2 + V'$$

, where, again, the effective radial potential is:

$$V' = V + \frac{l^2}{2mr^2}$$

4.2 Types of orbits (Unbound, Bound, and Circular)

The significance of the above expression for the effective radial potential V' becomes clearer if we plot V', V and $\frac{l^2}{2mr^2}$ on the same set of axes (see Fig. 8).

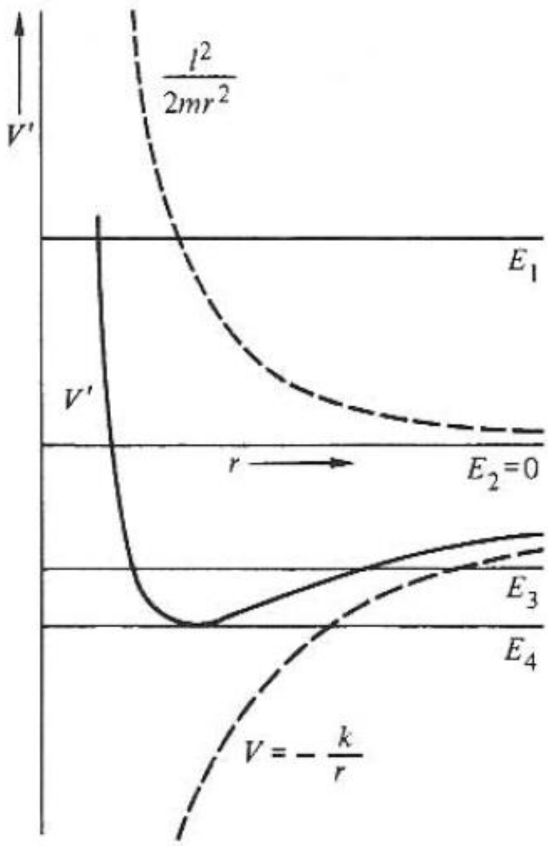

Fig. 8: Plot of effective radial potential (energy) against distance *r*.

From the expression for the total energy *E*:

$$E = \frac{1}{2}m\dot{r}^2 + V'$$

, we get an expression for the radial kinetic energy of the planet:

$$\frac{1}{2}m\dot{r}^2 = E - V'$$

As this kinetic energy cannot be negative, the only parts of the V' graph that are physically valid are those for which the total energy *E* is greater than V'. This allows us to categorize orbits according to their energy:

4.2.1 Total energy *E* > 0 (unbound orbit (hyperbola)):

For orbits with total energy *E* greater than zero[**], such as $E = E_1$ in Fig. 8, we see that almost the entire V' graph represents physically valid motion, except for the portion where the distance *r* between the Sun and the moving body is less than a minimum distance r_1.

This means that,

- at $r = r_1$, $E = V'$ and so the radial velocity $\dot{r} = 0$, and also that
- the moving body cannot come closer to the Sun than $r = r_1$. That is to say, $r = r_1$ is the point of minimum separation between the moving body and the Sun.

That is to say, the position at which $r = r_1$ is a turning point in the motion of the body.

In short, the moving body approaches the Sun from a distance ("infinity"), reaches a minimum distance r_1 from it, and then moves away from the Sun and never returns. In other words, this is not the typical elliptical orbital motion of a planet about the Sun, but is instead the hyperbolic trajectory followed by bodies such as a rock that sometimes pass through the Solar System. This type of orbit is termed *unbound* because the body is not captured by the Sun's gravity and never returns to it. The trajectory is described by the mathematical function of a hyperbola.

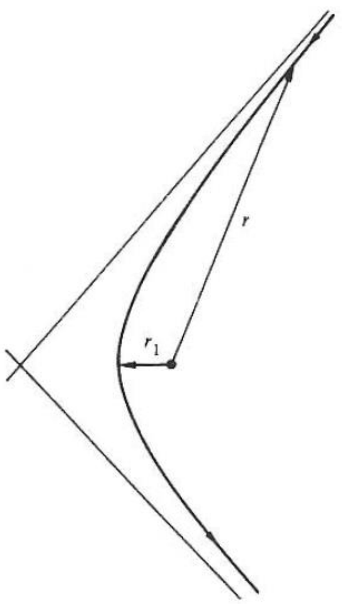

Fig. 9: Unbound orbit of an object with positive total energy.

4.2.2 Total energy *E* = 0 (unbound orbit (parabola)):

If the total energy *E* of the moving body is exactly zero ($E = E_2 = 0$ in Fig. 8), its motion is also *unbounded* and almost the same as that of of a body with $E > 0$, except that the shape of its trajectory is that of a parabola instead of a hyperbola. In this parabolic trajectory, the minimum distance between the moving body and the Sun is r_2 (see Fig. 8).

Such a body has just enough energy to escape the Sun's gravity.

[**] The potential energy to taken to be zero at infinity as usual.

4.2.3 Total energy *E* < 0 but <u>greater than</u> the minimum value for V' (bound elliptical orbit):

As before, for the motion to be physically meaningful, its radial kinetic energy must be positive ($\frac{1}{2}m\dot{r}^2 > 0$), and so the only parts of the V' graph which represent valid motion are those are below the graph for total energy *E*.

Hence, if *E* < 0 but above the minimum of the V' graph (e.g., $E = E_3$ in Fig. 8), then the motion of the orbitting planet (or other body) around the Sun is *bounded* and its distance *r* from the Sun must lie between r_p and r_a.

So,

$$r_p < r < r_a$$

, where

r_p = distance between the periapsis (perihelion) and the Sun, and

r_a = distance between the apoapsis (aphelion) and the Sun.

(see Fig. 4)

In short, the planet (or other body) will orbit the Sun (or other central body) in an elliptical trajectory, in which the position of minimum distance from the Sun (perihelion) occurs at $r = r_p$ and the position of maximum distance from the Sun (aphelion) occurs at $r = r_a$.

The planet does <u>not</u> have enough energy to escape the Sun's gravity, and so its motion is *bounded*.

4.2.4 Total energy *E* < 0 and <u>equal to</u> the minimum value for V' (bound circular orbit):

From Section 0, the radial kinetic energy of the planet:

$$\frac{1}{2}m\dot{r}^2 = E - V'$$

If the energy *E* of the system is exactly equal to the minimum value of V' (e.g., $E = E_4$ in Fig. 8), then the radial kinetic energy of the planet is

$$\frac{1}{2}m\dot{r}^2 = E - V' = 0$$

$$\therefore \dot{r} = \frac{dr}{dt} = 0$$

This means that the radius *r* (distance between the planet and the Sun) is constant in time for the entire orbit, which is possible only if the <u>orbit is perfectly circular in shape</u>.

Also, as this radius occurs at the minimum point in the V'- r graph (see Fig. 8),

$$\frac{dV'}{dr} = 0$$

But, from Section 0,

$$V' = V + \frac{l^2}{2mr^2}$$

$$\therefore \frac{dV'}{dr} = \frac{d}{dr}\left(V + \frac{l^2}{2mr^2}\right) = 0$$

$$\therefore \frac{dV}{dr} - \frac{l^2}{mr^3} = 0$$

$$\therefore -\frac{dV}{dr} = -\frac{l^2}{mr^3}$$

$$\therefore F = -\frac{l^2}{mr^3}$$

$$\therefore F = -mr\dot{\theta}^2 = -mr\omega^2$$

In other words, when the energy *E* is exactly equal to the minimum value of V', the <u>force of gravity is exactly equal in magnitude to the centripetal force needed for a perfectly circular trajectory</u>, as seen in the H2 syllabus.

This energy *E* is given by:

$$E = V' = V + \frac{l^2}{2mr^2}$$

From Section 0

$$l = mr^2\dot{\theta}$$

$$E = -\frac{k}{r} + \frac{(mr^2\dot{\theta})^2}{2mr^2}$$

$$E = -\frac{k}{r} + \frac{1}{2}m\left(r\dot{\theta}\right)^2$$

$$\therefore E = -\frac{GMm}{r} + \frac{1}{2}mv_\theta^2$$

, as $v_\theta = r\dot{\theta}$

We thus obtain the familiar total energy expression for circular motion (see H2 syllabus):

$$\therefore E = -\frac{GMm}{2r}$$

Annex

ANNEX A: REDUCTION OF A 2-BODY PROBLEM TO AN EQUIVALENT 1-BODY PROBLEM

We can reduce a 2-body problem like a planet orbitting the Sun into a 1-body problem.

Consider two masses m_1 and m_2 which exert attractive force forces (e.g., gravity) on each other.

The force exerted on m_1 by m_2 is = $\boldsymbol{F_{12}} = m_1\boldsymbol{a_1}$

The force exerted on m_2 by m_1 is = $\boldsymbol{F_{21}} = m_2\boldsymbol{a_2}$

As the two forces are oppositely directed, $\boldsymbol{F_{12}} = -\boldsymbol{F_{21}}$

$$\therefore m_1\boldsymbol{a_1} = -m_2\boldsymbol{a_2}$$

$\therefore$ the acceleration of m_2 is:

$$\boldsymbol{a_2} = -\frac{m_1}{m_2}\boldsymbol{a_1}$$

Hence, the acceleration of m_1 relative to m_2 is:

$$\boldsymbol{a_{12}} = \boldsymbol{a_1} - \boldsymbol{a_2}$$

$$\therefore \boldsymbol{a_{12}} = \boldsymbol{a_1} - \left(-\frac{m_1}{m_2}\boldsymbol{a_1}\right)$$

$$\therefore \boldsymbol{a_{12}} = \left(1 + \frac{m_1}{m_2}\right)\boldsymbol{a_1}$$

$$\therefore \boldsymbol{a_{12}} = \left(\frac{m_2 + m_1}{m_2}\right)\left(\frac{\boldsymbol{F_{12}}}{m_1}\right)$$

$$\therefore \boldsymbol{F_{12}} = \left(\frac{m_1 m_2}{m_1 + m_2}\right)\boldsymbol{a_{12}}$$

This means that, under the action of force $\boldsymbol{F_{12}}$, mass m_1 experiences an acceleration $\boldsymbol{a_{12}}$ relative to mass m_2. We can therefore analyze the motion of mass m_1 from the perspective of mass m_2 by treating m_1 as if it were a mass of $\left(\frac{m_1 m_2}{m_1+m_2}\right)$. In other words, in the frame of reference of mass m_2 , mass m_1 moves as if it has the mass $\left(\frac{m_1 m_2}{m_1+m_2}\right)$.

This equivalent mass is called the *reduced mass* μ:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

For example, if we analyze the motion of a planet (mass m) orbitting the Sun (mass M), we can take the Sun to be stationary and treat the planet as a point mass $\left(\frac{mM}{m+M}\right)$.

Annex B: Properties of an Ellipse

Basic properties of an Ellipse:

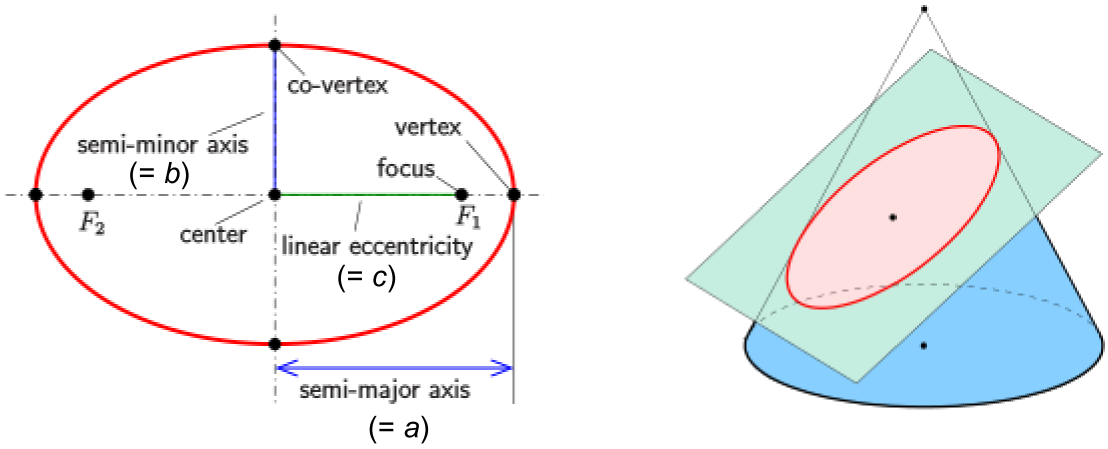

Fig. 10: Properties of an ellipse, which is obtained via a plane intersection through a cone.

- An **ellipse** is a curve in a plane surrounding two focal points (foci) such that the sum of the distances to the two focal points is constant for every point on the curve.
- A circle is a special type of an ellipse having both focal points at the same location (centre).
- The degree of elongation of an ellipse it is measured by a quantity called its *eccentricity e* which can take any value between 0 (for a circle) and a number less than 1 (Earth's orbit around the Sun has an eccentricity of 0.017 (almost circular), while that of Halley's Comet has an eccentricity of 0.97 (highly elongated)).
- An ellipse is a type of *conic section*, i.e., the plane curve at the intersection of a plane and a cone (see Fig. 10 right).
- The distance between the centre of the ellipse and its vertex is called the **semi-major axis** (symbol: *a*).
- The distance between the centre of the ellipse and its co-vertex is called the **semi-minor axis** (symbol: *b*).
- Equation of an ellipse in Cartesian coordinates:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Eccentricity:

$$e = \frac{c}{a}$$
$$b^2 + c^2 = a^2$$
$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

Semi-latus rectum:

$$p = \frac{b^2}{a}$$

Area of an ellipse,

$$A = \pi ab$$

Equation of an ellipse in polar coordinates:

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

, where θ is measured anti-clockwise from the line joining the focus to the vertex nearest to it.

$$r = \frac{p}{1 + e\cos\theta}$$

DERIVATION OF THE AREA *A* OF AN ELLIPSE

The equation of an ellipse in terms of Cartesian coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

To find the area unclosed, we integrate *y* with respect to *x*:

$$A = 2\int_{-a}^{a} y \, dx$$

, where the factor of 2 is needed as the curve for *y* exists above the below the *x*-axis.

$$A = \int_{-a}^{a} 2b\sqrt{1 - \frac{x^2}{a^2}} \, dx$$

$$A = \frac{b}{a}\int_{-a}^{a} 2\sqrt{a^2 - x^2} \, dx$$

Keeping in mind that the equation for a circle is $x^2 + y^2 = a^2$ ($\Rightarrow y = \pm\sqrt{a^2 - x^2}$), we that this integral is equal to the area of a circle, which is πa^2.

$$A = \frac{b}{a} \times \pi a^2$$

Hence, the area of an ellipse is obtained:

$$A = \pi ab$$

ANNEX C: DERIVATION OF RELATIONSHIP BETWEEN ORBITAL PERIOD AND SEMI-MAJOR AXIS (KEPLER'S 3RD LAW)

We start with the areal velocity obtained in Section 0:

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta}$$

This is the rate at which the radius sweeps out an area, i.e., it is the area swept out by the radius per unit time.

Hence, to find the area *A* of the ellipse, we integrate $\frac{dA}{dt}$ over one complete period τ as that is the time taken by the planet to go around the Sun exactly once.

$$A = \int_0^\tau \frac{dA}{dt}dt$$

$$\therefore A = \int_0^\tau \frac{1}{2}r^2\dot{\theta}\ dt$$

But, we know that the area *A* of an ellipse is

$$A = \pi ab$$

(see Section 0)

Equating these two expressions for *A*, we get:

$$\int_0^\tau \frac{1}{2}r^2\dot{\theta}\,dt = \pi ab$$

From Section 0, we also see that:

$$l = mr^2\dot{\theta}$$

$$r^2\dot{\theta} = \frac{l}{m}$$

So,

$$\int_0^\tau \frac{1}{2}\frac{l}{m}dt = \pi ab$$

$$\frac{l}{2m}\int_0^\tau dt = \pi ab$$

$$\frac{l}{2m}\tau = \pi ab$$

$$\frac{l^2}{4m^2}\tau^2 = \pi^2a^2b^2$$

We aim to eliminate *b*, which can be done by using the relation:

$$a^2 = b^2 + c^2$$

(See Fig. 10: The distance between a focus F_1 and the co-vertex is *b*.)

$$\frac{l^2}{4m^2}\tau^2 = \pi^2 a^2(a^2 - c^2)$$

c can also be eliminated by using the definition of the eccentricity *e*:

$$e = \frac{c}{a}$$

$$\therefore \frac{l^2}{4m^2}\tau^2 = \pi^2 a^2(a^2 - a^2 e^2)$$

$$\frac{l^2}{4m^2}\tau^2 = \pi^2 a^4(1 - e^2)$$

The eccentricity *e* had been obtained earlier in Section 0:

$$e = \sqrt{1 + \frac{2El^2}{mk^2}}$$

$$\therefore \frac{l^2}{4m^2}\tau^2 = \pi^2 a^4\left(1 - \left(1 + \frac{2El^2}{mk^2}\right)\right)$$

$$\therefore \frac{1}{4m^2}\tau^2 = -\pi^2 a^4\left(\frac{2E}{mk^2}\right)$$

From Section 0, we had obtained an expression for the total energy *E*:

$$E = -\frac{k}{2a}$$

$$\therefore \frac{1}{4m^2}\tau^2 = \pi^2 a^4\left(\frac{2}{mk^2} \times \frac{k}{2a}\right)$$

$$\therefore \frac{1}{4m}\tau^2 = \pi^2 a^3\left(\frac{1}{k}\right)$$

Hence, we get the relationship between the orbital period and the semi-major axis:

$$\tau^2 = \frac{4\pi^2 m}{k}a^3$$

As *k* = *GMm*, we get:

$$\tau^2 = \frac{4\pi^2}{GM}a^3$$

ANNEX D: DERIVATION OF ELLIPTICAL ORBITAL EQUATION (KEPLER'S 1ST LAW)

$$\therefore E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V \qquad \text{(Total Energy of System)}$$

Re-arranging this, we get

$$\frac{1}{2}m\dot{r}^2 = E - V - \frac{l^2}{2mr^2}$$

$$\therefore \dot{r}^2 = \frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)$$

$$\therefore \dot{r} = \sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}$$

$$\therefore \frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}$$

- Using this expression, we can calculate the <u>radial velocity</u> $\dot{r} = \frac{dr}{dt}$ for a separation *r* between Sun and planet if we know the total energy *E* and angular momentum *l*.
- We can also find the <u>total velocity</u> *v* using the basic expression for the total energy $E = \frac{1}{2}mv^2 + V$.
- As we know that $v^2 = \dot{r}^2 + \left(r\dot{\theta}\right)^2$ from Section 0, we can then use the total velocity *v* and the radial velocity $\dot{r}$ to find <u>azimuthal velocity</u> $r\dot{\theta} = r\omega = v_\theta$.

From the above, we obtained the radial velocity $\dot{r}$:

$$\therefore \dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}$$

We can manipulate this expression to derive the equation of the trajectory of the planet as it orbits the Sun in terms of the radius *r* and the angle θ measured from the direction of the point at which it is closest to the Sun (periapsis[††]). This angle θ is called the *true anomaly*. It is assumed that the planet orbits the Sun in the anti-clockwise direction from the periapsis (See Fig. 4).

We can use this to remove the variable *t* by applying the Chain Rule of differentiation to the expression for $\dot{r}$:

$$\frac{dr}{dt} = \frac{dr}{d\theta} \times \frac{d\theta}{dt}$$

But, in Section 0, the angular momentum *l* is seen to be:

$$l = mr^2\dot{\theta}$$

[††] As we are dealing with the Sun (Helios), the specific terms that can be used here are *perihelion* and *aphelion*.

$$\therefore l = mr^2 \frac{d\theta}{dt}$$

$$\therefore \frac{d\theta}{dt} = \frac{l}{mr^2}$$

Therefore,

$$\frac{dr}{dt} = \frac{dr}{d\theta} \times \frac{l}{mr^2}$$

Substituting this into the expression for $\dot{r}$, we get:

$$\frac{l}{mr^2}\frac{dr}{d\theta} = \sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}$$

Re-arranging:

$$\frac{d\theta}{dr} = \frac{l}{mr^2}\frac{1}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}$$

$$d\theta = \frac{l}{mr^2\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}dr$$

$$\int_{\theta_0}^{\theta} d\theta = \int_{r_0}^{r} \frac{l}{mr^2\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}dr$$

If we assume that lower limit of the integration is at the periapsis (point of minimum distance between Sun and planet), then

when $\theta_0 = 0^{\text{o}}$, $r_0 = r_p = a - c$

Upon performing this integration[‡‡], the equation of the trajectory of the planet around the Sun is obtained:

$$r = \frac{\frac{l^2}{mk}}{1 + \sqrt{1 + \frac{2El^2}{mk^2}}\cos\theta}$$

, where the constant *k* = *GMm* is found in the expression for gravitational potential energy:

$$V = -\frac{k}{r} = -\frac{GMm}{r}$$

Comparing the trajectory equation with the equation of an ellipse in polar form (see Section 0)

$$r = \frac{p}{1 + e\cos\theta}$$

[‡‡] Derivation of orbital trajectory equation is not required in the syllabus.

, we see that the trajectory of the planet around the Sun is indeed elliptical in shape, with the following parameters:
Semi-latus rectum *p*:

$$p = \frac{l^2}{mk}$$

Eccentricity *e*:

$$e = \sqrt{1 + \frac{2El^2}{mk^2}}$$

- Using the trajectory equation above, we are able to calculate the distance *r* between the Sun and the planet if the angular position θ (true anomaly) of the planet is known along with the total energy *E* and the angular momentum *l*.