Revealed Delegation and Persuasion

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Abstract

To delegate future choices, *planners* can influence *doers* in various ways. In the intrapersonal case, a planner can motivate her future self by commitments, promises, deadlines etc. In the interpersonal case, doers can be selected and/or *persuaded* to change their beliefs. Even if delegations and persuasions are not observed directly, they can be still revealed through other choice primitives. We obtain this identification from planners' preferences over menus in a consumption space Z. All possible delegations are derived as an endogenous class of doers' rankings of Z, and can be further refined by expected utility when Z consists of lotteries or uncertain prospects. The latter case can be interpreted in terms of persuasions that affect the doer's beliefs without changing her risk attitude. In this interpretation, the set of possible beliefs is identified uniquely. A further refinement of this model captures a planner who knows *objective* probabilities and can use them in persuasion. Finally, we discuss how delegations can be revealed through choices in menus rather than among menus.

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1 Introduction

Sunstein and Ulmann-Margalit [32] distinguish two general environments where decisions need to be delegated. In the intrapersonal case, an agent must delegate her upcoming choices to her *future self* who may succumb to spontaneous temptations and exhibit less patience towards delayed rewards. Such conflicts can explain dynamic inconsistences and various *commitment* strategies (e.g. Strotz [31], Thaler and Shefrin [34], Gul and Pesendorfer [14]). Commitments can impose physical constraints on the feasible set, but also impose emotional or monetary penalties. For example, people may keep only healthy foods and drinks at home, use self-exclusions from casino gambling, make promises and vows, set deadlines, etc (see the review of Bryan, Karlan, and Nelson [4]).

In the interpersonal case, *planners* delegate decisions to *doers* who do not share the same physical identity with the planner. Instead, doers may be selected and/or *persuaded* to change their tastes and beliefs.[1] There is a vast literature (e.g. Laffont and Martimort [22]) on the principal-agent problem with monetary incentives. In a different vein, Kamenica and Gentzkow [18] show that planners can affect doers' beliefs and choices through suitable informative messages rather than material rewards.

In general, it can be hard to observe delegations and persuasions directly. Indeed, personal commitments are often mental and leave no exogenous trace. Interpersonal communications and messages can be hidden from outside observers as well. In this paper, we show how unobservable delegations and persuasions can be revealed through other choice primitives.

To illustrate, suppose that a planner must delegate an investment decision to a doer who charges 20% of all the upside. Let a be a safe investment that returns 5% interest with certainty. Let b and c be risky strategies with uncertain returns

[1] Thaler and Shefrin [34] propose the terms *planners* and *doers* in their early model of commitments and self-control. We use this terminology in interpersonal settings as well.

contingent on some event E,

$$b = \begin{cases} 15\% & \text{if } E \text{ occurs} \\ -5\% & \text{otherwise} \end{cases} \qquad c = \begin{cases} -5\% & \text{if } E \text{ occurs} \\ 10\% & \text{otherwise.} \end{cases}$$

The planner is risk neutral and has a belief $\mu(E) = 0.5$. Then her commitment ranking is $a \succ b \succ c$. The doer is also risk neutral and has a belief $\pi(E) = 0.3$. Suppose that he can be persuaded to change his belief to $\mu(E) = 0.5$. For instance, if μ is based on some compelling objective data, then the planner can disclose it to the doer. The doer's rankings with the beliefs π and μ are represented by $v(c) > v(a) > v(b)$ and $v'(b) > v'(a) > v'(c)$ respectively. Then the planner's preference over menus in $\{a, b, c\}$ should be

$$\{a\} \sim \{a, b\} \sim \{a, c\} \succ \{a, b, c\} \sim \{b, c\} \sim \{b\} \succ \{c\}. \tag{1}$$

Indeed, the choice a can be delegated in the menus $\{a, b\}$ and $\{a, c\}$ via beliefs π and μ respectively. However, the planner cannot delegate a in the menu $\{a, b, c\}$ because a is never best for the doer. This limitation motivates rankings $\{a, c\} \succ \{a, b, c\}$ and $\{b, c\} \succ \{a, b, c\}$ where ex ante commitments are valuable. More broadly, (1) should be unchanged even if the planner could impose any belief $\pi^*(E)$ on the doer because it would still be impossible to make a the best choice for the doer.

Our main results are formulated for preferences $\succeq$ over *menus*—finite subsets of a consumption space Z. Each menu $A \subset Z$ is interpreted as a planner's ex ante action that makes the set A feasible for doers' choices ex post. In this framework, the planner's *commitment utility* u is revealed directly through her choices over singleton menus. Suppose that there is a family Θ of rankings of Z that the planner can impose on doers via suitable delegations. Accordingly, she should evaluate each menu A via the best *delegable* alternative that can be chosen in A by some doer with a ranking $R \in \Theta$,

$$U(A) = \max\{u(a) : \ a \text{ maximizes some } R \in \Theta \text{ in } A\}. \tag{2}$$

Our first result (Theorem 1) characterizes (2) when Z is finite.[2] Here the maximal set Θ that agrees with representation (2) can be identified uniquely in terms of the preference $\succeq$. Moreover, it is without loss in generality to assume that all orders $R \in \Theta$ are total and hence, doers should be never indifferent between distinct options. For example, the ranking (1) is represented by (2) where $u(a) > u(b) > u(c)$ and $\Theta = \{R, R'\}$ consists of two total orders R and R' such that $cRaRb$ and $bR'aR'c$ respectively.

Next, if Z is a compact simplex of lotteries, then the commitment utility u and all rankings $R \in \Theta$ can be refined by the expected utility model. Theorem 2 characterizes this refinement and identifies the *minimal* version of Θ uniquely. For example, the model of *changing tastes* in Gul and Pesendorfer [13] (henceforth, GP) is a special case of our delegation model where the minimal Θ is a singleton.

Delegations can be reinterpreted as *persuasions* when Z consists of uncertain prospects, and all rankings $R \in \Theta$ comply with the *subjective* expected utility model. In this framework, the planner can influence the doer's subjective beliefs even if consumption tastes are invariant as in the classic adage: "De gustibus non est disputandum". Theorem 3 identifies a *unique* set $\mathcal{C}$ of all beliefs π that can be imposed on the doer via persuasion.

Another refinement of our model captures a distinction between objective and subjective probabilities. Imagine that the planner's probabilistic belief μ is *objective*, that is, determined by some symmetries across states or by reliable statistical frequencies. Disclosing such objective evidence should persuade the doer to have the same belief μ. This persuasion is optimal for the planner in some menus, but not necessarily in all menus because the two agents may have different consumption tastes and risk attitudes. Besides the inclusion $\mu \in \mathcal{C}$, Theorem 4 characterizes a representation where $\mathcal{C}$ consists of just two elements, μ and π, such that the planner's objective belief μ is the only persuasion that can change the doer's own subjective belief π. Unlike the approach of Gilboa et al. [12], the distinction be-

[2] The countable Z is also allowed, which is convenient if ex post actions determine consumption together with its timing in a discrete line $T = \{1, 2, \dots\}$.

tween objective and subjective probabilities is revealed through persuasion rather than through incompleteness of objective preferences.

The delegation model (2) can be written as $U(A) = u(\phi(A))$ where the filter

$$\phi(A) = \{a \in A : \ a \text{ maximizes some } R \in \Theta \text{ in } A\} \tag{3}$$

has the *multi-utility* structure of Aizerman and Malishevski [1] (henceforth, AM). When the domain Z is finite, AM establish the equivalence of (3) and Plott's [28] *path independence*. However, AM's result does not hold if all rankings $R \in \Theta$ must comply with expected utility. In this case, we use a stronger form of path independence to characterize the multi-utility representation (3) where the set Θ must be closed in a suitable topology (Theorem 5). Unlike the ordinal model, the set Θ is determined uniquely when the expected utility structure is assumed. Besides our delegation model, representation (3) can be useful in models of *heterogenous preferences* where choices are observed in menus of lotteries or uncertain prospects.

Finally, we discuss how delegations and persuasions can be revealed ex post through *intransitive* choices. For example, a car salesman may be able to influence a potential buyer's decision in favor of a midlevel trim b over some basic model a, and also in favor of an upscale trim c when b is the only other option. However, the salesman may fail to talk the customer into buying c when a is available because the price gap between a and c may be too steep to overcome by persuasion. The corresponding delegated choices should be intransitive:

$$b \text{ in } \{a, b\}, \quad c \text{ in } \{b, c\}, \quad a \text{ in } \{a, c\}. \tag{4}$$

Theorem 6 characterizes all rankings R_0 of the planner (e.g. the car salesman) that are consistent with a dataset $\mathcal{D}$ of delegated choices in our model. The dataset $\mathcal{D}$ need not be complete (i.e. choices can be observed only in a few menus), and the algorithm for checking the consistency between R_0 and $\mathcal{D}$ takes polynomial time $O(|\mathcal{D}|^3)$ in the size $|\mathcal{D}|$. We run this algorithm on a dataset of choices in menus of lotteries from Apesteguia and Ballester [3]. Even though the delegation model does not seem plausible in their context, we use this empirical exercise to illustrate how our model can be *rejected* by binary choice data. By contrast, binary choice data is consistent with an arbitrary commitment ranking in models of inattention (Masatlioglu, Nakajima, and Ozbay [25])) and rationalizations (Cherepanov, Feddersen, and Sandroni [5]).

1.1 Related Literature

For finite Z, our menu framework is the same as in models of subjective state spaces (Kreps [20]), changing tastes (GP [15]), and limited willpower (Masatlioglu, Nakajima and Ozdenoren [26], henceforth MNO). However, our representations and axioms are distinct from those models. For example, the ranking (1) violates Set Betweenness assumed by GP and MNO and Monotonicity assumed by Kreps. Given Set Betweenness, MNO construct an endogenous filter ϕ that consists of all maximal elements of some *interval order* I on the finite Z. This construction does not extend to lottery frameworks. If Set Betweenness is imposed in Theorem 2, then our delegation model reduces to GP's model of *changing tastes* where Θ is a singleton. See Theorem 8 for details.

The set Θ in our delegation model can be interpreted as a *subjective state space* that is aggregated in a non-additive and non-monotonic way via the commitment utility u. In the lottery framework, this aggregation violates continuity that is assumed by Dekel, Lipman, and Rustichini [8].

In the lottery framework, the functional form (2) is used also by Tang and Zhang [33] in simultaneously ongoing research project. They focus on the intrapersonal interpretation where the agent is *optimistic* ex ante about her ex post temptation ranking. Their interpretation does not suggest any persuasion applications. We start with a distinct motivation, use only finite menus, impose different axioms,[3] obtain characterizations with subjective beliefs (Theorems 3–4), and for ex post choices in menus (Theorem 5–6).

Special cases of the multi-utility representation (3) have been characterized

[3] In particular, we do not assume the indifference to randomization, and convex menus are not even finite.

by Heller [16] and Seidenfeld, Schervish, and Kadane [30], who impose unanimity constraints on Θ and require that all rankings $R \in \Theta$ should agree on some pair of deterministic outcomes. Besides the arbitrary unanimity assumption, some of their conditions (such as CARNI in Heller) are existential. We could not derive such conditions from preferences over menus and proved a more general representation result (Theorem 5) from scratch.

Our notions of delegations and persuasions have some similarities with the models of subjective learning (Dillenberger, Lleras, Sadowski, Takeoka [10]), costly contemplation (Ergin and Sarver [11]), and rational inattention (de Oliveira, Denti, Mihm, and Ozbek [7]). In all of these representations, the planner takes some actions that affect ex post tastes and/or beliefs. In our approach, the planner's influence is bound by the endogenous set Θ rather than by a combination of possible costly signals and Bayesian probabilistic laws.

Delegations can be related to models of choice functions where a suitable filter ϕ is used to accommodate sequential rationalization (Manzini and Mariotti [24], Cherepanov, Feddersen, and Sandroni [5]), inattention (Masatlioglu, Nakajima, and Ozbay [25]), choice overload (Lleras, Masatlioglu, Nakajima, and Ozbay [23]), and other cognitive limitations. Besides the difference in primitives, we assume that ϕ is path independent. Examples in Section 3 illustrate that ex post choices may reject all ex ante commitment rankings R_0 in the delegation model, but agree with all R_0 in the weaker specifications. Moreover, our refinements of path independence to choices among lotteries and uncertain prospects do not have any counterparts with competition and attention filters used by Masatlioglu et al. and Lleras et al.

2 Main Representation Results

Let $Z = \{a, b, c, \dots\}$ be a set of consumption alternatives. This set can be finite, countable, or a convex compact subset of a Euclidean space.

Let $\mathcal{M} = \{A, B, C \dots\}$ be the set of all *menus*—non-empty finite subsets of Z. Singletons $\{a\}$ are written as a.

Let $\mathcal{R}$ be the set of complete and transitive relations R on Z. Such relations are called *weak orders*. A weak order $R \in \mathcal{R}$ is called *total* if for all $a, b \in Z$, $aRbRa$ implies $a = b$. Let $\mathcal{T} \subset \mathcal{R}$ be the set of all total orders on Z.

For any order $R \in \mathcal{R}$, set $\Theta \subset \mathcal{R}$, function $u : Z \to \mathbb{R}$, and menu $A \in \mathcal{M}$, let

$$
\begin{aligned}
u(A) &= \max_{a \in A} u(a),\\
R(A) &= \{a \in A : aRb \text{ for all } b \in A\},\\
\Theta(A) &= \bigcup_{R \in \Theta} R(A),\\
(u \circ \Theta)(A) &= u(\Theta(A)).
\end{aligned}
$$

Thus $u \circ \Theta$ maps $\mathcal{M}$ to $\mathbb{R}$. By definition, $R(A) \subset A$ is not empty. If R is total, then for each $A \in \mathcal{M}$, $R(A) \in Z$ is a singleton. By convention, let $u(\emptyset) = 0$.

Interpret elements in Z as consumptions that may become feasible *ex post* (i.e. at some future time period). Interpret each menu $A \in \mathcal{M}$ as an action that, if taken *ex ante*, makes the set $A \subset Z$ feasible ex post.

Consider a *planner*[4] with a preference $\succeq$ over menus. Write its asymmetric and symmetric parts as $\succ$ and $\sim$ respectively. Assume

Axiom 1 (Order)**.** $\succeq$ *is complete and transitive.*

Let $R_0 \in \mathcal{R}$ be the restriction of $\succeq$ to the set of singleton menus $Z \subset \mathcal{M}$. The weak order R_0 is called the *commitment ranking*.

Imagine that the planner must delegate ex post choices to *doers*—her future selves or other individuals. In general, delegations are *hidden* and cannot be observed directly. For example, the planner can motivate her future self by mental commitments, promises, cues etc. In the interpersonal case, doers may be selected and/or persuaded to change their beliefs and tastes, but such communications can be private as well.

Say that an element $a \in A$ is *delegable* in a menu $A \in \mathcal{M}$ if a doer should be willing to choose a in A after a suitable delegation by the planner. For any

[4]Both planners and doers make decisions in our models, so we avoid the generic name of *decision maker* as confusing.

$A \in \mathcal{M}$, let $\phi(A) \subset A$ be the non-empty subset of all delegable alternatives a. This interpretation does not make the function $\phi : \mathcal{M} \to \mathcal{M}$ observable, but suggests some axioms for the primitive preference $\succeq$.

Assume that for any menu $A \in \mathcal{M}$,

$$A \sim d_A \quad \text{for some } d_A \in R_0(\phi(A)). \tag{5}$$

According to (5), the planner should evaluate each menu $A \in \mathcal{M}$ via her best delegable choice d_A in A. To do so, she must know the entire set $\phi(A)$ and incur no direct costs from delegating the choice d_A.

Consider two axioms for the preference $\succeq$.

Axiom 2 (Positive Set-Betweenness (PSB))**.** *For all $A, B \in \mathcal{M}$,*

$$A \succeq B \quad \Rightarrow \quad A \succeq A \cup B.$$

Take any $A, B \in \mathcal{M}$. Assume that

$$\text{if } a \in A \text{ is delegable in } A \cup B, \text{ then } a \text{ is delegable in } A. \tag{6}$$

Indeed, a doer who is willing to select a in $A \cup B$ should agree to choose a in A as well. By (5)-(6), if $d_{A \cup B} \in \phi(A)$, then $A \sim d_A \succeq d_{A \cup B} \sim A \cup B$. Similarly, if $d_{A \cup B} \in \phi(B)$, then $B \succeq A \cup B$. Thus PSB holds. Note that condition (6) for choice functions is known as the Chernoff property or Sen's property α. PSB originally appears in Dekel, Lipman, and Rustichini's [9] model of cumulative temptations.

Axiom 3 (Mandatory Delegation (MD))**.** *For all $A, B \in \mathcal{M}$ and $c \in Z$,*

$$c \succ c \cup A \quad \Rightarrow \quad c \cup A \cup B \sim A \cup B.$$

Take any $A, B \in \mathcal{M}$ and $c \in Z$ such that $c \succ c \cup A$. By (5), $c \notin \phi(c \cup A)$. By (6), $c \notin \phi(c \cup A \cup B)$ and hence, $\phi(c \cup A \cup B) \subset \phi(A \cup B)$. Moreover, the opposite inclusion $\phi(A \cup B) \subset \phi(c \cup A \cup B)$ is also plausible. Indeed, if a doer chooses a in $A \cup B$, then she should be still willing to choose a in $c \cup A \cup B$ because she cannot agree to choose the non-delegable c. As $\phi(c \cup A \cup B) = \phi(A \cup B)$, then by

(5), $c \cup A \cup B \sim A \cup B$. MD is weaker than the Independence of Irrationalizable Choices (IIC) in Tang and Wang [33]. Their condition requires that a menu A must be compared to each of its elements, which makes IIC practically irrefutable except for very small menus.

Theorem 1. *If Z is countable, then the following claims are equivalent.*

(i) $\succeq$ *satisfies Axioms 1–3,*

(ii) $\succeq$ *is represented by $U = u \circ \Theta$ for some $u : Z \to \mathbb{R}$ and $\Theta \subset \mathcal{R}$,*

(iii) $\succeq$ *is represented by $U = u \circ \Theta$ for some $u : Z \to \mathbb{R}$ and $\Theta \subset \mathcal{T}$.*

All proofs are in the appendix. The main steps are sketched in Section 2.4.

The functional form $U = u \circ \Theta$ has two components—the planner's *commitment utility* $u : Z \to \mathbb{R}$ and a family $\Theta \subset \mathcal{R}$ of weak orders. The function u can be any utility representation for R_0 and hence, is unique up to a strictly increasing transformation.

The planner as portrayed by the utility function $u \circ \Theta$ evaluates any menu A by maximizing u over the set $\Theta(A) \subset A$. This set consists of all maximal elements in the menu A for weak orders $R \in \Theta$ in the endogenous family $\Theta \subset \mathcal{R}$. The family Θ can be interpreted as a collection of all rankings that the planner can impose on doers via suitable delegations.

By Theorem 1, there is no loss in generality to assume that all orders $R \in \Theta$ in the family Θ are total. In this case, the set $R(A)$ is a singleton for each menu $A \in \mathcal{M}$ and each $R \in \Theta$. Thus doers can be modelled without any tie-breaking rules. However, the larger class $\mathcal{R}$ can be more convenient than $\mathcal{T}$ because $\mathcal{R}$ accommodates expected utility representations.

The set Θ in the representation $U = u \circ \Theta$ is not determined uniquely by the preference $\succeq$, but the largest of all such Θ can be found explicitly as[5]

$$\Theta^* = \{R \in \mathcal{R} : A \succeq a \text{ for all } A \in \mathcal{M} \text{ and } a \in R(A)\}.$$

[5]Assume that $\succeq$ has a utility representation $U = u \circ \Theta$. Then $\Theta \subset \Theta^*$ because any ranking $a \succ A$ where $a \in R(A)$ implies that $R \notin \Theta$. Therefore, for all $A \in \mathcal{M}$, $U(A) = u(\Theta(A)) \leq u(\Theta^*(A))$ because $\Theta \subset \Theta^*$ Conversely, $U(A) \geq u(\Theta^*(A))$ because $A \succeq a$ for all $a \in \Theta^*(A)$. Thus

This endogenous set consists of all weak orders R on Z that do not allow any strict rankings $a \succ A$ for any $A \in \mathcal{M}$ and $a \in R(A)$. Indeed, if $a \succ A$ and $a \in R(A)$, then a is not delegable and hence, R cannot be imposed on any doer in representation $U = u \circ \Theta$. Similarly,

$$\Theta^{**} = \Theta^* \cap \mathcal{T} = \{R \in \mathcal{T} : A \succeq R(A) \text{ for all } A \in \mathcal{M}\}$$

is the largest among all sets of total orders $\Theta \subset \mathcal{T}$ such that $u \circ \Theta$ represents $\succeq$.

To illustrate, let $Z = \{a, b, c\}$. Use the pair of brackets $\langle \ldots]$ to list total orders $R \in \mathcal{T}$. For instance, $\langle abc]$ denotes the total order $aRbRc$. Recall the ranking (1) from the introductory example,

$$\{a\} \sim \{a, b\} \sim \{a, c\} \succ \{a, b, c\} \sim \{b, c\} \sim \{b\} \succ \{c\}.$$

Then $\Theta^{**} = \Theta^* \cap \mathcal{T} = \{\langle bac], \langle bca], \langle cab], \langle cba]\}$. The strict ranking $a \succ \{a, b, c\}$ excludes two total orders $\langle abc]$ and $\langle acb]$ from Θ^{**}, and the comparison $A \succeq a$ holds for all other $A \in \mathcal{M}$ and $a \in A$. Check that $\succeq$ is represented by $U = u \circ \Theta^{**}$ where $u(a) = 2 > u(b) = 1 > u(c) = 0$ is the commitment utility. Thus $\succeq$ obeys Axioms 1–3. Note that $\succeq$ is also represented by $U = u \circ \Theta$ with $\Theta = \{\langle bac], \langle cab]\}$. This set contains just two rankings rather than four in Θ^{**}. The maximal set Θ^* has six rankings: it contains Θ^{**} and two weak orders that are represented by $v_1(b) > v_1(a) = v_1(c)$ and $v_2(c) > v_2(a) = v_2(b)$ respectively.

The functional form $U = u \circ \Theta$ has other interpretations besides the use of delegations. In particular, it can capture a preference for *conformity*. In social contexts, people may desire to conform to popular choices and to avoid shameful/rude ones. For example, an agent may avoid a social event if she dislikes the most popular activity there even if her favorite options are also present. In our model, the set $\Theta \subset \mathcal{R}$ can be interpreted as all rankings that produce popular choices in any menu A.

$U = u \circ \Theta^*$. As $\Theta \subset \Theta^*$, then Θ^* is the largest set among all $\Theta \subset \mathcal{R}$ such that $u \circ \Theta$ represents $\succeq$.

2.1 Delegations and Expected Utility

Say that Z is a *convex consumption domain* (CCD) if Z is a convex, compact subset of some Euclidean space. For example, Z can be the set of all probability distributions on some finite set of deterministic outcomes. In this case, elements of Z are called *lotteries*. CCDs can also consist of uncertain prospects.

Let Z be a CCD. For any $A, B \in \mathcal{M}$ and $\gamma \in [0,1]$, define a mixture

$$\gamma A + (1-\gamma)B = \{\gamma a + (1-\gamma)b : a \in A, b \in B\}.$$

Endow $\mathcal{M}$ with the Hausdorff metric h such that for all $A, B \in \mathcal{M}$,

$$h(A,B) = \max\left\{\max_{a\in A}\min_{b\in B}||a-b||, \max_{b\in B}\min_{a\in A}||a-b||\right\}.$$

Let $\mathcal{U}$ be the class of all surjective functions $v : Z \to [-1,1]$ such that for all $\gamma \in [0,1]$ and $a, b \in Z$,

$$v(\gamma a + (1-\gamma)b) = \gamma v(a) + (1-\gamma)v(b). \tag{7}$$

If Z is a simplex of lotteries, then $\mathcal{U}$ is the class of all von Neumann–Morgenstern expected utility functions that are normalized to have the range $[-1,1]$ on Z. Metrize $\mathcal{U}$ by the uniform metric $\rho(v,v') = \max_{a\in Z}|v(a) - v'(a)|$ for all $v, v' \in \mathcal{U}$.

For any function $v \in \mathcal{U}$, let R_v be the order that is represented by v. Call an order $R \in \mathcal{R}$ *separable* if $R = R_v$ for some $v \in \mathcal{U}$.

Let $\Theta_{\mathcal{U}}$ be the class of all separable orders. Note that $R \in \Theta_{\mathcal{U}}$ if and only if R satisfies the axioms of the mixture space theorem in Herstein and Milnor [17]. Moreover, each $R \in \Theta_{\mathcal{U}}$ has a unique representation $v \in \mathcal{U}$ because the range of v is normalized to $[-1,1]$. Thus there is a natural bijection between $\mathcal{U}$ and $\Theta_{\mathcal{U}}$.

Let $\succeq$ be a preference over the set $\mathcal{M}$ of all finite menus.[6] By convention, $u \circ \emptyset = 0$. Thus the degenerate preference $\succeq$ such that $a \sim b$ for all $a, b \in Z$ has a trivial representation $U = u \circ \Theta$ with any function $u : Z \to \mathbb{R}$ and $\Theta = \emptyset$. Assume hereafter that $\succeq$ is *non-degenerate* so that $a \succ b$ for some $a, b \in Z$.

[6]Our model can be extended for preferences over all closed subsets $C \subset Z$, but the axioms and proofs are more parsimonious when $\mathcal{M}$ consists of finite menus.

Axioms 1–3 can be motivated as is. If Z is not a singleton, then it is not countable, and Theorem 1 does not apply. To refine the general delegation model for CCDs, assume that the commitment ranking R_0 and all orders in the set Θ are separable.

Axiom 4 (Singleton Independence (SI)). *For all $a, b, c \in Z$,*

$$a \succeq b \quad \Leftrightarrow \quad \gamma a + (1-\gamma)c \succeq \gamma b + (1-\gamma)c.$$

This standard condition is necessary for R_0 to be separable (e.g. to have an expected utility representation over lotteries). By contrast, Independence is not assumed for mixtures of non-singleton menus because if a and c are delegable in A and C respectively, then $\gamma a + (1-\gamma)c$ need not be delegable in $\gamma A + (1-\gamma)C$.

Axiom 5 (Mixture Monotonicity (MM)). *For any $A \in \mathcal{M}$, $a, b \in A$, and $\gamma \in [0,1]$,*

$$A \cup \{\gamma a + (1-\gamma)b\} \succeq A.$$

To motivate MM, assume that the planner expects to delegate some choice d_A in a menu A to a doer who maximizes some separable ranking R. Then the same ranking should be maximized by d_A in the larger menu $B = A \cup \{\gamma a + (1-\gamma)b\}$ where $a, b \in A$ and $\gamma \in [0,1]$.

Next, impose a weak form of continuity on the preference $\succeq$.

Axiom 6 (Partial Continuity (PC)). *For any $A, B \in \mathcal{M}$ and $c \in Z$,*

(i) the set $\{B' \in \mathcal{M} : B' \succeq B\}$ is closed in $\mathcal{M}$,

(ii) the set $\{\gamma \in [0,1] : B \succeq \gamma A + (1-\gamma)c\}$ is closed in $[0,1]$.

This condition requires (i) the upper semi-continuity over all menus and (ii) mixture continuity for mixtures between any menu $A \in \mathcal{M}$ and any singleton $c \in Z$. Both properties are implied by the desired representation in Theorem 2 below. Note that $\succeq$ need not be lower semi-continuous or a fortiori, continuous. Lower semi-continuity is also dismissed by GP [13, Theorem 3] in their generalization of the costly self-control model.

Let $\mathcal{K}$ be the family of all non-empty closed sets $\mathcal{V} \subset \mathcal{U}$. For any set $\mathcal{V} \in \mathcal{K}$ and function $u \in \mathcal{U}$, let $\Theta_{\mathcal{V}} = \{R_v : v \in \mathcal{V}\}$ and

$$u \circ \mathcal{V} = u \circ \Theta_{\mathcal{V}}$$

so that for any menu $A \in \mathcal{M}$, the value $(u \circ \mathcal{V})(A)$ maximizes the function u across all maximizers of functions $v \in \mathcal{V}$ in A.

Let $\mathcal{V}_*$ be the intersection of all $\mathcal{V} \in \mathcal{K}$ such that $\succeq$ is represented by $U = u \circ \mathcal{V}$ for some $u \in \mathcal{U}$.

Theorem 2. *$\succeq$ satisfies Axioms 1–6 if and only if $\succeq$ is represented by*

$$U = u \circ \mathcal{V} \tag{8}$$

for some $u \in \mathcal{U}$ and $\mathcal{V} \in \mathcal{K}$. Moreover, $u \circ \mathcal{V} = u \circ \mathcal{V}_*$.

Suppose that $\succeq$ is represented by (8). As $\mathcal{V} \in \mathcal{K}$ is not empty, then the commitment utility $u \in \mathcal{U}$ is unique. The planner as portrayed by representation (8) evaluates each menu $A \in \mathcal{M}$ by maximizing u across all maximizers of the functions $v \in \mathcal{V}$ in the menu A. The corresponding family $\Theta_{\mathcal{V}}$ consists of all separable orders that can be maximized by doers ex post.

The set $\mathcal{V}$ in representation (8) is not unique, but its minimal version $\mathcal{V}_*$ is. Unlike the ordinal case, Theorem 2 implies equality $u \circ \mathcal{V} = u \circ \mathcal{V}_*$ and hence, guarantees that $\mathcal{V}_*$ is not empty. It is also closed as an intersection of closed sets.

For example, if $\mathcal{V} = \{v\}$ is a singleton, then $\mathcal{V}_* = \mathcal{V}$ because $\mathcal{V}$ does not have any proper non-empty subsets. The corresponding functional form $U = u \circ \{v\}$ is the model of *changing tastes* proposed by GP [13]. Here the function v can be interpreted as a *temptation* utility.

The maximal set $\mathcal{V}^*$ in (8) can be derived from the preference $\succeq$ similarly to the family Θ^* in the ordinal case,

$$\mathcal{V}^* = \{v \in \mathcal{U} : A \succeq a \quad \text{for all } A \in \mathcal{M} \text{ and } a \in R_v(A)\}\,.$$

It is typical for $\mathcal{V}^*$ to contain many redundant elements. For example, if $\mathcal{V} = \{v\}$, then $\mathcal{V}^*$ consists of all $w \in \mathcal{U}$ such that w represents the same ranking as $\gamma v + (1-\gamma)(-u)$ for some $\gamma \in [0,1]$.

The identifications of $\mathcal{V}_*$ and $\mathcal{V}^*$ are clarified further in Theorem 8 below.

2.2 Persuasion

Consider a form of persuasion where the DM can affect the doers' *beliefs* rather than their consumption tastes or risk attitudes.

Let $\Omega = \{1, 2, \ldots, |\Omega|\}$ be a finite state space. Let $\Delta = \{\mu, \ldots\}$ be the simplex of all *probability distributions* on Ω.

Let $X = \{x, y, \ldots\}$ be a CCD. Let $Z = X^\Omega$ be the set of all *acts*—functions $a : \Omega \to X$. Interpret each act $a \in Z$ as a physical action that delivers outcomes $a(n)$ contingent on the state $n \in \Omega$. Each outcome $y \in X$ is identified with the constant act that delivers y in each state $n \in \Omega$. As X is a convex compact subset of some Euclidean space L, then Z is a convex compact subset of the Euclidean space L^Ω. Thus Z is a CCD.

Let $\succeq$ be a non-degenerate preference over Z. When restricted to X, preferences are called risk attitudes.

Axioms 1–6 and Theorem 2 can be invoked as is because Z is a CCD. To distinguish persuasions from other delegation forms, impose a form of monotonicity on the rankings of planners and doers.

Axiom 7 (Monotonic Persuasions (MP))**.** *For all $a, b \in Z$ and $B \in \mathcal{M}$,*

(i) if $a(n) \succeq b(n)$ for all $n \in \Omega$, then $a \succeq b$,

(ii) if $b(n) \succ \{a(n), b(n)\}$ for all $n \in \Omega$, then $b \succ \{a, b\}$,

(iii) if $\{a(n), b(n)\} \succ b(n)$ for all $n \in \Omega$, then $\{a, b\} \cup B \sim a \cup B$.

Part (i) is monotonicity for the planner's commitment ranking R_0. Part (ii) imposes monotonicity on the doer's preferences. Assume that her ranking of X is unaffected by persuasion, as in the classic adage: "De gustibus non est disputandum". Then for any outcomes $x, y \in X$, the ranking $x \succ \{x, y\}$ reveals that the doer should strictly prefer y to x. Therefore, if $b(n) \succ \{a(n), b(n)\}$ for all $n \in \Omega$, then the doer cannot be persuaded to choose b over a because she strictly prefers

$a(n)$ to $b(n)$ in each state of the world. Part (iii) uses a similar argument when $\{a(n), b(n)\} \succ b(n)$ holds for all n. In this case, the doer should weakly prefer a to b because she is willing to choose $a(n)$ over $b(n)$ in each state n. By PSB, $a(n) \succ b(n)$ for all n. By monotonicity, $a \succeq b$. Thus the best delegable element d in any menu $a \cup B$ remains the best delegable element in $\{a, b\} \cup B$.

Let $\mathcal{U}_X = \{r, t, \dots\}$ be the set of all surjective functions $r : X \to [-1, 1]$ such that for all $x, y \in X$ and $\gamma \in [0, 1]$,

$$r(\gamma x + (1 - \gamma)y) = \gamma r(x) + (1 - \gamma)r(y).$$

For any $r \in \mathcal{U}_X$ and $\mu \in \Delta$, let $v_{r,\mu} \in \mathcal{U}$ be the subjective expected utility function such that for all $a \in Z$,

$$v_{r,\mu}(a) = \sum_{n \in \Omega} r(a(n))\mu(n).$$

This form of subjective expected utility is due to Anscombe and Aumann [2].

For any $t \in \mathcal{U}_X$ and non-empty closed set $\mathcal{C} \subset \Delta$, let

$$\mathcal{V}_{t,\mathcal{C}} = \{v_{t,\mu} : \mu \in \mathcal{C}\}$$

be the set of all subjective expected utility functions with the risk attitude t and some belief $\mu \in \mathcal{C}$. As t has the range $[-1, 1]$, then $v_{t,\mu}$ must have the same range as well.

Say that $\succeq$ is *regular* if $x \succ \{x, y\}$ and $\{x', y'\} \succ y'$ for some $x, y, x', y' \in X$. Regularity excludes situations where the risk attitudes of the planner and the doer are either the same or exactly opposite. These situations are excluded only in the uniqueness statement below.

Theorem 3. *$\succeq$ satisfies Axioms 1–7 if and only if $\succeq$ is represented by*

$$U = v_{r,\mu} \circ \mathcal{V}_{t,\mathcal{C}} \tag{9}$$

for some $r, t \in \mathcal{U}_X$, $\mu \in \Delta$, and non-empty, closed set $\mathcal{C} \subset \Delta$.

The components r, t, μ are unique. If $\succeq$ is regular, then $\mathcal{C}$ is unique, and

$$\mathcal{V}_{t,\mathcal{C}} = \mathcal{V}_*.$$

The planner as portrayed by representation (9) evaluates each menu $A \in \mathcal{M}$ by maximizing her subjective expected utility $v_{r,\mu}$ over all elements $a \in A$ that may be selected in A by the doer whose belief is manipulated in the set $\mathcal{C}$ by the planner. The doer's risk attitude t is assumed to be unchanged by such persuasions. In the regular case, all components $(r, \mu, t, \mathcal{C})$ are derived uniquely from $\succeq$.

When $\succeq$ is regular, the set $\mathcal{C}$ in (9) is identified uniquely, and $\mathcal{V}_{t,\mathcal{C}}$ is the minimal set $\mathcal{V}_*$ in the more general representation (8). Note that $\mathcal{C}$ is closed, but need not be convex. For example, let $\Omega = \{1, 2\}$,

$$\mathcal{C} = \{\mu \in \Delta : \mu(1) \in \{0.3, 0.7\}\}$$
$$\mathcal{C}' = \{\mu \in \Delta : \mu(1) \in [0.3, 0.7]\}.$$

Assume that $r \neq t$ and $r \neq -t$. Then representations (9) with tuples $(r, \mu, t, \mathcal{C})$ and $(r, \mu, t, \mathcal{C}')$ generate distinct preferences $\succeq$ and $\succeq'$, and the sets $\mathcal{C}$ and $\mathcal{C}'$ can be reconstructed uniquely from $\succeq$ and $\succeq'$ respectively.

2.3 Persuasion via Objective Probabilities

Imagine that the planner's probabilistic belief μ is *objective*, that is, determined by some symmetries across states or by reliable statistical frequencies. Disclosing such objective evidence should persuade the doer to have the same belief μ. This persuasion is optimal for the planner in some menus, but not necessarily in all menus because the risk attitudes r and t need not be the same.

For any payoffs $x, y \in X$, let

- $X(x, y) = \{\gamma x + (1 - \gamma)y : \gamma \in [0, 1]\}$ be the set of all mixtures of x and y,
- $Z(x, y) \subset Z$ be the set of all prospects $a : \Omega \to X(x, y)$,
- $\mathcal{M}(x, y) \subset \mathcal{M}$ be the set of all menus $A \subset Z(x, y)$.

For any act $a \in Z$ and probability measure $\mu \in \Delta$, let

$$a(\mu) = \sum_{n \in \Omega} \mu(n) a(n)$$

be the average payoff of a induced via the distribution μ. If X is a lottery space, then $a(\mu)$ is the reduction of the compound lottery that delivers the payoffs $a(n)$ with probabilities $\mu(n)$. Note that if $a \in Z(x, y)$, then $a(\mu) \in X(x, y)$.

Axiom 8 (Objective Monotonicity (OM))**.** *For all $x, y \in X$ and $A, B \in \mathcal{M}(x, y)$,*

$$\{x, y\} \succ y \quad \Rightarrow \quad A \cup B \succeq A.$$

Here the ranking $\{x, y\} \succ y$ implies that both the planner strictly prefers x to y and expects the doer to have the same ranking so that x is delegable in $\{x, y\}$. Take any two menus $A, B \in \mathcal{M}(x, y)$. Take $a^* \in A \cup B$ such that $a^* \succeq a$ for all $a \in A \cup B$. The rankings $a^* \succeq a$ are equivalent to $a^*(\mu) \succeq a(\mu)$. If the planner can persuade the doer to have the same belief μ, then she can persuade him to have the same preference $\succeq$ over prospects in $Z(x, y)$ because $a(\mu) \in X(x, y)$ for all $a \in Z(x, y)$, and the doer should view x as better than y. Given such persuasion, the rankings $a^*(\mu) \succeq a(\mu)$ should make a^* the best choice in A for the doer. Thus a^* is delegable in $A \cup B$ and hence, $A \cup B \sim a^* \succeq d_A \sim A$.

Assume further that the planner can reveal some objective probabilities only to the doer with a full disclosure of the entire distribution μ. This assumption is plausible if all objective probabilities are derived from the same source (e.g. proprietary data) that the doer can use on his own after gaining access to it.

Axiom 9 (Objective Disclosure (OD))**.** *For all menus $A, B, C \in \mathcal{M}$ such that $A \succeq B \succeq C$, either $A \cup B \succeq C$, or $A \cup C \succeq C$, or $B \cup C \succeq C$.*

Take any three menus $A, B, C \in \mathcal{M}$. Consider several possible cases. First, suppose that the planner discloses μ both for A and for B. Let d_A and d_B be the best delegable elements in A and B respectively. As $A \succeq B$, then $d_A \succeq d_B$, and d_A is delegable in $A \cup B$ when μ is disclosed. Thus $A \cup B \succeq d_A \succeq A \succeq C$. Similarly, OD holds if μ is disclosed for A and C or for B and C. Suppose next that d_A and d_B do not require any persuasion in A and in B. Then d_A is delegable in $A \cup B$ without any persuasion as well. Thus $A \cup B \succeq A \succeq C$. Similarly, OD should hold if μ is not disclosed for any two menus in the list A, B, C.

Theorem 4. *Axioms 1–8 hold if and only if $\succeq$ is represented by* (9) *where $\mu \in \mathcal{C}$.*

If $\succeq$ is regular, then Axioms 1–9 hold if and only if $\succeq$ is represented by (9) *where $\mathcal{C} = \{\mu, \pi\}$ for some $\pi \in \Delta$.*

This result relates the planner's belief μ to her anticipated persuasions. In the former case, μ is one of such persuasions. In the latter case, the planner's belief μ is the only persuasion that can change the doer's own subjective belief π. Here the distinction between μ and π can be interpreted as a difference between objective and subjective probabilities because the former are more convincing than the latter. Besides this distinction, the sharp binary structure $\mathcal{C} = \{\mu, \pi\}$ should be convenient for stylized principal-agent problems with persuasion.

2.4 Delegations and Path Independence

Utility representations in Theorems 1 and 2 are derived from endogenous filters that select delegable elements in menus.

Say that $\phi : \mathcal{M} \to \mathcal{M}$ is a *filter* if $\phi(A) \subset A$ for all $A \in \mathcal{M}$. By definition, $\phi(A)$ is not empty. Call $\phi : \mathcal{M} \to \mathcal{M}$ *path independent* if

$$\phi(A \cup B) = \phi(A \cup \phi(B)) \qquad \text{for all } A, B \in \mathcal{M}.$$

This property is formulated by Plott [28]. AM [1] show that if Z is finite and ϕ is path independent, then there is $\Theta \subset \mathcal{R}$ such that

$$\phi(A) = \Theta(A) \quad \text{for all } A \in \mathcal{M}. \tag{10}$$

Moreover, it is without loss in generality to take $\Theta \subset \mathcal{T}$.

Assume that Z is countable, and $\succeq$ satisfies Axioms 1–3. Take any $u : Z \to \mathbb{R}$ that represents $\succeq$ on Z. In the appendix, we construct a path independent filter ϕ such that $\succeq$ is represented by

$$U(A) = u(\phi(A)) \quad \text{for all } A \in \mathcal{M}. \tag{11}$$

The required set Θ is then derived from (10) via an extension of AM's result to countable sets.

Let Z be a CCD. To adapt the above identification strategy, consider a refinement of path independence. For any menu $B \in \mathcal{M}$, write its convex hull as $\text{conv}B$.

Call a filter $\phi : \mathcal{M} \to \mathcal{M}$ *path-and-mixture independent* (PMI) if ϕ satisfies

- **Convex $\boldsymbol{\alpha}$**: for all $A, B \in \mathcal{M}$, $A \subset \text{conv}(B)$ implies $A \cap \phi(B) \subset \phi(A)$,
- **Reduction**: for all $A, B \in \mathcal{M}$, if $\phi(B) \subset A \subset B$, then $\phi(A) \subset \phi(B)$,
- **Closed Graph**: the set $\{(a, A) \in Z \times \mathcal{M} : a \in \phi(A)\}$ is closed in $Z \times \mathcal{M}$,
- **Mixture Independence**: for all $A \in \mathcal{M}$, $c \in Z$ and $\gamma \in (0, 1)$,

$$\phi(\gamma A + (1 - \gamma)c) = \gamma\phi(A) + (1 - \gamma)c. \tag{12}$$

Obviously, Convex α implies Sen's α *property*: for all $A, B \in \mathcal{M}$,

$$A \subset B \quad \Rightarrow \quad A \cap \phi(B) \subset \phi(A).$$

Therefore, PMI implies path independence which is equivalent to the combination of Sen's α and Reduction.[7]

To prove Theorem 2, we refine AM's result to a novel characterization.

Theorem 5. *A filter $\phi : \mathcal{M} \to \mathcal{M}$ is PMI if and only if for all $A \in \mathcal{M}$,*

$$\phi(A) = \Theta_{\mathcal{V}}(A) \tag{13}$$

where $\mathcal{V} \in \mathcal{K}$ is closed. Moreover, the set $\mathcal{V}$ is unique.

Besides our delegation model, this result can be applied when the filter ϕ is directly observed as a choice function with some class $\mathcal{V}$ of heterogenous expected utilities. Heller [16] characterizes a special case of representation (13) where the set $\mathcal{V}$ must allow $a, b \in Z$ such that $v(a) > v(b)$ for all $v \in \mathcal{V}$. This unanimity constraint would be restrictive in our model where the set $\mathcal{V}$ describes the planner's

[7] Both conditions appear in Chernoff [6] and have various names in the literature. Reduction is called the Aizerman condition by Moulin [27] because of its prominence in AM's results.

anticipation of doers' possible rankings. It appears awkward to derive such a constraint from axioms on preferences $\succeq$. Moreover, one of Heller's main assumptions (CARNI) is existential and would be cumbersome for our arguments as well.

Suppose that $\succeq$ satisfies Axioms 1–6. These axioms are sufficient for R_0 to have a representation $u \in \mathcal{U}$. In the appendix, we construct a PMI filter ϕ such that $\succeq$ is represented by (11). Then Theorem 2 is derived immediately from Theorem 5.

3 Discussion

In empirical settings, choices are often observed in menus rather than between menus. Moreover, the collection of such observations is typically much smaller than the size of the entire domain $\mathcal{M}$, which grows exponentially with the number of consumption alternatives $|Z|$. Next, we apply the ordinal delegation model in such settings.

Say that $\mathcal{D} \subset Z \times \mathcal{M}$ is a *dataset* if $a \in A$ for every $(a, A) \in \mathcal{D}$. Any such pair $(a, A) \in \mathcal{D}$ means that a is observed to be chosen in A. Let $M = |\mathcal{D}|$ be the number of such observations.

A ranking $R_0 \in \mathcal{R}$ is called *acceptable* for a dataset $\mathcal{D}$ if there is a set $\Theta \subset \mathcal{R}$ such that

$$a \in R_0(\Theta(A)) \quad \text{for all } (a, A) \in \mathcal{D}. \tag{14}$$

Thus $R_0 \in \mathcal{R}$ is acceptable if all observations in $\mathcal{D}$ are consistent with some delegation model where a hypothetical planner with the ranking R_0 delegates choices in menus A by selecting doers in the set $\Theta \subset \mathcal{R}$.

Let P_0 be the asymmetric component of R_0. For any menu $B \in \mathcal{M}$, let

$$\mathcal{N}(B) = \{b \in B : bP_0a \quad \text{for some } (a, A) \in \mathcal{D} \text{ such that } A \subset B\}$$

be the set of all elements $b \in B$ that are revealed to be non-delegable in some $A \subset B$ and hence, in B itself.

Theorem 6. *A ranking $R_0 \in \mathcal{R}$ is acceptable for a dataset $\mathcal{D} \subset Z \times \mathcal{M}$ if and only if for all $(a, A) \in \mathcal{D}$ and a menu $B \supset A$,*

$$B \neq \mathcal{N}(B) \cup [A \setminus a]. \tag{15}$$

Moreover, for any $R_0 \in \mathcal{R}$, it takes polynomial time $O(M^3)$ to establish whether R_0 is acceptable for $\mathcal{D}$ or not.

This result provides a criterion for acceptability of any given $R_0 \in \mathcal{R}$. This criterion can be checked in polynomial time via an algorithm that we discuss in the proof of Theorem 6 in the appendix.

3.1 Examples

Before we discuss several examples of acceptability, it is insightful to relax the definition (14) to adapt the models of inattention in Masatlioglu, Nakajima, and Ozbay [25] and choice overload in Lleras, Masatlioglu, Nakajima, and Ozbay [23]. Say that $\phi : \mathcal{M} \to \mathcal{M}$ is

- a *competition filter* if it satisfies Sen's α, $a \in R_0(\phi(A))$ for all $(a, A) \in \mathcal{D}$,
- an *attention filter* if ϕ satisfies a strong form of Reduction[8]: for all $A, B \in \mathcal{M}$,

$$\phi(B) \subset A \subset B \quad \Rightarrow \quad \phi(A) = \phi(B). \tag{16}$$

Say that R_0 is *a-acceptable* for a dataset $\mathcal{D}$ if there is an attention filter $\phi : \mathcal{M} \to \mathcal{M}$ such that

$$a \in R_0(\phi(A)) \quad \text{for all } (a, A) \in \mathcal{D}. \tag{17}$$

Say that R_0 is *c-acceptable* for a dataset $\mathcal{D}$ if (17) holds for some competition filter $\phi : \mathcal{M} \to \mathcal{M}$.

These weaker notions of acceptability portray a decision maker with the ranking R_0 who pays attention only to elements in the filter ϕ.

[8]The attention property is implied by path independence.

Let $Z = \{a, b, c\}$, and consider a standard intransitivity cycle,

$$\mathcal{D}_1 = \{(a, \{a, b\}), (\{b, \{b, c\}\}), (\{c, \{a, c\}\})\}.$$

For this dataset $\mathcal{D}_1$, the total rankings $\langle acb]$, $\langle bac]$, and $\langle cba]$ are not acceptable. In particular, let $R_0 = \langle acb]$. Let $A = \{a, b\}$ and $B = \{a, b, c\}$. Then $\mathcal{N}(B) = \{a, c\}$ and $B = \mathcal{N}(B) \cup [A \setminus a]$ which contradicts (15).

By contrast, any ranking R_0 is both a-acceptable and c-acceptable for $\mathcal{D}_1$. In fact, one can show a more general claim.

Say that $\mathcal{D}$ is *binary* if for all $(a, A) \in \mathcal{D}$, the menu $A = \{a, b\}$ has size two, and $(b, A) \notin \mathcal{D}$. In other words, binary datasets describe observations of a single choice in some two-elements menus.

Proposition 7. *Any ranking $R_0 \in \mathcal{R}$ is both a-acceptable and c-acceptable for any binary dataset $\mathcal{D}$.*

Proof. For any observation $(a, \{a, b\}) \in \mathcal{D}$ such that bP_0a, let $\phi_a(\{a, b\}) = \{a\}$. For all other menus A, let $\phi_a(A) = A$. Then ϕ_a is an attention filter, and (17) holds.

For any observation $(a, \{a, b\}) \in \mathcal{D}$ such that bP_0a, let $\phi_c(\{a, b\}) = \{a\}$. For any observation $(a, \{a, b\}) \in \mathcal{D}$ such that aR_0b, let $\phi_c(\{a, b\}) = \{a, b\}$. For all other menus A, let $\phi_c(A)$ be the set of minimizers of the commitment utility u. Then u_c is a competition filter, and (17) holds. □

This proposition suggests roughly that our delegation model imposes a minimal structure on the filter ϕ such that the representation (17) has non-vacuous implications for choices in binary menus.

We illustrate this point with an experimental dataset found in Apesteguia and Ballester (2020) [3], and which is publicly available on the both authors' websites. The data includes the choices of 87 individuals from all 36 binary menus for the following nine equiprobable lotteries:

Each individual's choices from the 36 binary menus are observed. We take the expected value rankings of the lotteries as R_0. Only 27 out of the 87 individuals

Table 1: Lotteries

lottery	**payoffs**	**lottery**	**payoffs**	**lottery**	**payoffs**
l_1	(17)	l_2	$(50,0)$	l_3	$(40,5)$
l_4	$(30,10)$	l_5	$(20,15)$	l_6	$(50,12,0)$
l_7	$(40,12,5)$	l_8	$(30,12,10)$	l_9	$(20,12,15)$

conform to the delegation model. In contrast, all 87 individuals conform to the competition filter model and the attention filter model.

It is also surprising that the combination of competition and attention properties delivers path independence, but the separate use of these conditions in (17) makes the model vacuous for binary datasets.

Next, consider a more stringent intransitivity where $Z = \{a, b, c, d\}$, and

$$\mathcal{D} = \{(a, \{a, b, c\}), (b, \{b, c, d\}), (c, \{c, d, a\}), \{d, \{d, a, b\}\}.$$

For this dataset, all total orders become unacceptable. Indeed, if $R_0 = \langle abcd]$, then (15) is violated for $B = \{a, b, c, d\}$ and $A = \{b, c, d\}$. However, all rankings $R_0 \in \mathcal{R}$ are weakly acceptable for $\mathcal{D}_2$.

3.2 Identifications of $\mathcal{V}_*$ and $\mathcal{V}^*$

Suppose that Z is a CCD, and $\succeq$ is a non-degenerate preference that satisfies Axioms 1–6. Let $\mathcal{V}_*$ and $\mathcal{V}^*$ be respectively the intersection and the union of all sets $\mathcal{V} \in \mathcal{K}$ such that $\succeq$ is represented by $U = u \circ \mathcal{V}$ for some $u \in \mathcal{U}$. By definition, $\mathcal{V}_* \subset \mathcal{V} \subset \mathcal{V}^*$. Next we construct $\mathcal{V}_*$ and $\mathcal{V}^*$ more explicitly.

For any $v, v' \in \mathcal{U}$, write

$$\begin{aligned} [v, v'] &= \{w \in \mathcal{U} : R_{\alpha w} = R_{\gamma v + (1-\gamma) v'} \quad \text{for some } \alpha, \gamma \in [0, 1]\}, \\ (v, v'] &= [v, v'] \setminus \{v\}. \end{aligned}$$

By definition, if $v = -v'$, then $[v, v'] = \mathcal{U}$ because $0w = \frac{1}{2}v + \frac{1}{2}v'$ for any $w \in \mathcal{U}$. If $v \neq -v'$, then w belongs to the interval $[v, v']$ if and only if w represents the same order on Z as $\gamma v + (1-\gamma)v'$ for some $\gamma \in [0, 1]$.

Theorem 8. *Suppose that $\succeq$ is represented by* (8) *where $u \in \mathcal{U}$ and $\mathcal{V} \in \mathcal{K}$.*

(i) $\mathcal{V}^*$ is closed, and $\mathcal{V}^* = \bigcup_{v \in \mathcal{V}} [v, -u]$,

(ii) $\mathcal{V}_*$ is the closure of the set $\mathcal{V} \setminus \bigcup_{v \in \mathcal{V}} (v, -u]$,

(iii) $\mathcal{V}_*$ is a singleton if and only if for all $A, B \in \mathcal{M}$,

$$A \succeq B \quad \Rightarrow \quad A \succeq A \cup B \succeq B. \tag{18}$$

This result derives the maximal and minimal sets $\mathcal{V}^*$ and $\mathcal{V}_*$ from the set $\mathcal{V} \in \mathcal{K}$ in representation (8). For example, suppose that $\succeq$ is represented by (8) with a singleton set $\mathcal{V} = \{v\}$. Then $\mathcal{V}_* = \{v\}$, but the maximal set $\mathcal{V}^*$ consists of the entire interval $[v, -u]$. This model of changing tastes follows is a special case of delegations where *Set Betweenness* (18) holds.

For example, let Z be the set of all lotteries with three outcomes x, y, z. Define functions $u, v_1, v_2, v_3 \in \mathcal{U}$ by the matrix

$$\begin{array}{ccccc} & u & v_1 & v_2 & v_3 \\ x & 1 & 1 & 1 & 0 \\ y & 0 & 0.6 & 0.2 & 1 \\ z & -1 & -1 & -1 & -1. \end{array}$$

If $\mathcal{V} = \{v_1, v_2\}$, then $\mathcal{V}^* = [v_2, -u]$ and $\mathcal{V}_* = \{v_2\}$ because $v_1 = 3v_2 + 2(-u)$ and hence, $v_1 \in [v_2, -u]$. Here Set Betweenness holds. If $\mathcal{V} = \{v_1, v_2, v_3\}$, then $\mathcal{V}^* = [v_2, -u] \cup [v_3, -u]$ and $\mathcal{V}_* = \{v_2, v_3\}$ because $v_2 \notin (v_3, -u]$ and $v_3 \notin (v_2, -u]$. Here Set Betweenness does not hold.

In general, the inclusion $v' \in [v, -u]$ is equivalent to $v' = \alpha v + \beta(-u) + \varepsilon$ for some $\alpha, \beta \geq 0$ and $\varepsilon \in \mathbb{R}$. The existence of such weights $\alpha, \beta, \varepsilon$ can be established by the standard Gauss algorithm. Indeed, the set $\mathcal{U}$ can be embedded into a Euclidean space, and the triple $(\alpha, \beta, \varepsilon)$ is unique whenever it exists for v' and $v \neq -u$.

A APPENDIX: PROOFS

We start with several observations that hold for any consumption domain Z.

A filter $\phi : \mathcal{M} \to \mathcal{M}$ satisfies Reduction if and only if for all $B \in \mathcal{M}$ and $b, c \in B$,

$$b, c \notin \phi(B) \quad \Rightarrow \quad b \notin \phi(B \setminus c). \tag{19}$$

Reduction implies (19) when $A = B \setminus c$. Show that (19) implies Reduction. Take any $A, B \in \mathcal{M}$ such that $\phi(B) \subset A \subset B$. Show Reduction by induction with respect to the size of B. Suppose that Reduction holds if the size of B is k. If B is a singleton, then $A = B$. Let B have size $k+1$. Take any $c \in B \setminus A$. Then $c \notin \phi(B) \subset A$. By (19), $\phi(B \setminus c) \subset \phi(B)$. As $A \subset B \setminus c$, then by the inductive assumption, $\phi(A) \subset \phi(B \setminus c)$. Thus Reduction holds.

Let $\succeq$ be a preference over $\mathcal{M}$. Suppose that $\succeq$ satisfies Axioms 1–3.

For any $A, B \in \mathcal{M}$, write $A \geqq B$ if $a \succeq b$ for all $a \in A$ and $b \in B$. Write $A \gg B$ if $a \succ b$ for all $a \in A$ and $b \in B$. Show that

$$A \geqq (\gg) B \quad \Rightarrow \quad A \succeq (\succ) B. \tag{20}$$

Proceed by induction with respect to the size of A and B. Suppose that (20) is true if the size of A and B do not exceed $k \in \mathbb{N}$. Take any $a, b \in Z$ and assume that $A \cup a \geqq B \cup b$. By the inductive assumption, $A \succeq B$, $A \succeq b$, $a \succeq B$, and $a \succeq b$. By PSB, $A \succeq B \cup b$ and $a \succeq B \cup b$. If $A \cup a \succeq a$, then $A \cup a \succeq a \succeq B \cup b$. If $a \succ A \cup a$, then by MD, $A \cup a \sim A \succeq B \cup b$. Thus $A \cup a \succeq B \cup b$, and (20) holds by induction. If $A \gg B$, then there are $a, b \in Z$ such that $A \geqq a \succ b \geqq B$ and hence, $A \succ B$.

Say that a menu $A \in \mathcal{M}$ *cancels* an alternative $c \in Z$ if $c \succ c \cup B$ for some $B \subset A$. Show that if A cancels c, then

$$c \succ c \cup B \quad \text{for some } B \subset A \text{ such that } c \gg B. \tag{21}$$

Take a minimal subset $B \subset A$ that cancels c. Suppose that there is $a \in B$ such that $a \succeq c$. Then $a \succeq c \succ c \cup B$. By MD, $c \cup B \sim c \cup (B \setminus a)$. Thus $B \setminus a$ cancels c. By contradiction, $c \gg B$.

Let the commitment ranking R_0 have a utility representation $u : Z \to \mathbb{R}$. This function can be ordinal or have some additional structure, such as expected utility.

Lemma A.1. *$\succeq$ satisfies Axioms 1–3 if and only if there is a path independent filter $\phi : \mathcal{M} \to \mathcal{M}$ such that $\succeq$ is represented by*

$$U(A) = u(\phi(A)) \qquad \textit{for all } A \in \mathcal{M}. \tag{22}$$

Proof. Assume that $\succeq$ is represented by (22). Take any $A, B \in \mathcal{M}$ and $c \in Z$. Take $a \in R_0(\phi(A \cup B))$. By (22), $U(A \cup B) = u(a)$. By Sen's α, If $a \in A$, then $a \in \phi(A)$ and hence, $U(A) \geq u(a) = U(A \cup B)$. Similarly, if $a \in B$, then $u(B) \geq U(A \cup B)$. Turn to MD. Suppose that $c \succ c \cup A$. By (22), $c \notin \phi(c \cup A)$. By Reduction, $c \notin \phi(c \cup A \cup B)$ and hence, $\phi(c \cup A \cup B) = \phi(A \cup B)$. By (22), $c \cup A \cup B \sim A \cup B$.

Conversely, assume that $\succeq$ satisfies Axioms 1–3. For any $A \in \mathcal{M}$, let

$$\phi(A) = \{c \in A : c \cup A' \succeq c \quad \text{for all menus } A' \subset A\} \tag{23}$$

be the set of all $c \in A$ that are not cancelled by A. Show that ϕ is a path independent filter. Take any $A \in \mathcal{M}$. Take $c \in R_{-u}(A)$ that minimizes u in A. By (21), $c \in \phi(A)$ because there is no menu $B \subset A$ such that $c \gg B$. Thus $\phi(A)$ is not empty.

Sen's α follows from the definition (23) directly. Turn to Reduction. Take any $B \in \mathcal{M}$ and $b, c \in B$ such that $b, c \notin \phi(B)$. Then $b \succ b \succ B'$ for some $B' \subset B$ such that $b \gg B'$. If $c \notin B'$, then $B \setminus c$ cancels b and $b \notin \phi(B \setminus c)$. Suppose that $c \in B'$. Then $b \succ c$. Take $C \subset B$ such that $c \succ c \cup C$ and $c \gg C$. By PSB, $b \succ (b \cup B') \cup C$. By MD, $b \cup B' \cup C \sim b \cup (B' \setminus c) \cup C$. Thus b is canceled by $B \setminus c$.

Show that (22) holds. If A is a singleton, then (22) is trivial. Assume (22) for all menus $A \in \mathcal{M}$ of size $k \geq 1$. Take any $A \in \mathcal{M}$ of size $k + 1$. Fix $c \in R_0(A)$. If $c \in \phi(A)$, then $A \succeq c$. By (20), $c \succeq A$. Thus $U(A) = u(c) = \max_{a \in \phi(A)} u(a)$. Suppose that $c \notin \phi(A)$. Then $c \succ c \cup B$ for some $B \subset A$ such that $c \gg B$. By MD, $A \setminus c \sim c \cup (A \setminus c) = A$, and

$$U(A) = U(A \setminus c) = \max_{c \in \phi(A \setminus c)} u(c).$$

By (16), $\phi(A \setminus c) = \phi(A)$, and (22) holds. $\square$

A.1 Proof of Theorem 1

Suppose that Z is countable. Then Order guarantees that R_0 has a utility representation $u : \mathcal{M} \to \mathbb{R}$, and Lemma A.1 applies. Theorem 1 is derived as follows.

Fix any $\Theta \subset \mathcal{R}$. For any $A \in \mathcal{M}$, let

$$\phi_\Theta(A) = \Theta(A).$$

Consider an extension of AM's representation result.

Lemma A.2. *If Z is countable, then a filter $\phi : \mathcal{M} \to \mathcal{M}$ is path independent if and only if $\phi = \phi_\Theta$ for some $\Theta \subset \mathcal{T}$.*

Proof. If Z is finite, then this lemma is exactly AM's main result.

Suppose that Z is an infinite countable set. Any filter $\phi_\Theta : \mathcal{M} \to \mathcal{M}$ is still path independent. Take any $A, B \in \mathcal{M}$. If $A \subset B$ and $a \in \phi_\Theta(B) \cap A$, then $a \in R(B)$ for some $R \in \Theta$ and hence, $a \in R(A) \subset \phi_\Theta(A)$. Thus ϕ_Θ satisfies Sen's α. Suppose instead that $\phi_\Theta(B) \subset A \subset B$. Then for any $R \in \Theta$, $R(B) \subset A \subset B$ and hence, $R(B) \subset R(A)$. For any $a \in R(A)$, aRb for some $b \in R(B)$ and hence, $a \in R(B)$ as well. Thus ϕ_Θ satisfies Reduction and hence, path independence.

Conversely, let ϕ be path independent. Show that $\phi = \phi_\Theta$ for some $\Theta \subset \mathcal{T}$.

Assume that $Z = \{a_1, a_2, \dots\}$ is an infinite countable collection of disjoint elements a_i. For any $k = 1, 2, \dots$, let

- $Z_k = \{a_1, \dots, a_k\}$,
- $\mathcal{M}_k$ be the set of all finite menus $A \subset Z_k$,
- $\mathcal{T}_k$ be the set of all total orders on Z_k,
- $\Theta_k \subset \mathcal{T}_k$ be the set of all $R \in \mathcal{T}_k$ such that $R(A) \in \phi(A)$ for all $A \in \mathcal{M}_k$,
- for any $n \geq k$, let Θ_{kn} be the set of all restrictions of $R \in \Theta_n$ on Z_k.

By AM's Theorem, for all k, $n > k$, and $A \in \mathcal{M}_k$,

$$\phi(A) = \Theta_k(A) = \Theta_{kn}(A).$$

By definition, Θ_k is the maximal subset of $\mathcal{T}_k$ that represents ϕ on $\mathcal{M}_k$. Thus $\Theta_{kn} \subset \Theta_k$ for all $n > k$. For any $k < n < m$, the projection of Θ_m on Z_n is a subset of Θ_n. Thus

$$\Theta_k \subset \Theta_{k(k+1)} \subset \Theta_{k(k+2)} \supset \dots$$

As Θ_n are non-empty, and $\mathcal{T}_k$ is finite, then this nested sequence of orders has a non-zero intersection $\Theta_k^* = \Theta_{kn}$ for all sufficiently large n. By construction, each $R_k \in \Theta_k^*$ has an extension $R_{k+1} \in \Theta_{k+1}^*$. Thus it has an extension $R \in \mathcal{T}$ such that the projection of R_n on any Z_n belongs to Θ_n^*. Let Θ^* be the set of all such $R \in \mathcal{T}$. The projection Θ_k^* on Z_k equals Θ_k. For any $A \in \mathcal{M}$, take k such that $A \in \mathcal{M}_k$. Then $\phi(A) = \Theta_k(A) = \Theta_k^*(A) = \Theta^*(A)$. □

Theorem 1 is implied by the combination of Lemmas A.1–A.2.

The other proofs follows the sequence Theorem 5, 2, 8, 3, and 6.

A.2 Proof of Theorem 5

Let Z be a CCD—a convex compact domain subset of some Euclidean space L.

Let $S_0 = \{s \in L : ||s|| \leq 1\}$ and $S = \{s \in L : ||s|| = 1\}$ be respectively the unit ball and the unit sphere in L.

Fix a point $a_0 \in Z$ in the relative interior of Z. Write the affine hull of Z as

$$\text{aff } (Z) = a_0 + L_0$$

where $L_0 \subset L$ is a linear subspace of L.

Lemma A.3. *For any $v \in \mathcal{U}$, there is a non-zero vector $s_v \in L_0$ and a constant $q_v \in \mathbb{R}$ such that*

$$v(a) = s_v \cdot a + q_v \qquad \textit{for all } a \in Z. \tag{24}$$

Both s_v and q_v are unique.

If $Z = S_0$, then $s_v \in S$, $q_v = 0$, and for all $v, w \in \mathcal{U}$,

$$\rho(v, w) = ||s_v - s_w||.$$

Proof. The proof is standard. Take any $v \in \mathcal{U}$. Then v has a unique extension to an affine function v^* on $a_0 + L_0$, which is the affine hull of Z. Then the function $f(x) = v^*(a_0 + x) - v^*(a_0)$ is linear on L_0. Thus there is a unique vector $s_v \in L_0$ such that $f(x) = s_v \cdot x$ for all $x \in L_0$. Then

$$v(a) = v(a_0) + v^*(a - a_0) = v(a_0) + s_v \cdot (a - a_0) = s_v \cdot a + (v(a_0) - s_v \cdot a_0)$$

delivers representation (24). The uniqueness of q_v follows from the uniqueness of $s_v \in L_0$.

If $Z = S_0$, then $\max_{a,b \in Z}(s_v \cdot a - s_v \cdot b) = 2||s_v|| = 2$ because the range of v is normalized to be $[-1, 1]$. Thus $s_v \in S$. The normalization implies $q_v = 0$.

By the Cauchy-Schwartz inequality, for all $v, w \in \mathcal{U}$ and $a \in S_0$,

$$|v(a) - w(a)| = |(s_v - s_w) \cdot a| \leq ||s_v - s_w||||a||.$$

Then $\rho(v, w) = ||s_v - s_w||$ because $|v(a) - w(a)| = ||s_v - s_w||$ for $a = \frac{s_v - s_w}{||s_v - s_w||}$. $\square$

Take any PMI filter $\phi : \mathcal{M} \to \mathcal{M}$. Let

$$\mathcal{V} = \{v \in \mathcal{U} : R_v(A) \subset \phi(A) \quad \text{for all } A \in \mathcal{M}\}. \tag{25}$$

Then $\mathcal{V}$ is closed. Indeed, let $v_1, v_2, \dots \in \mathcal{V}$ converge to some $v \in \mathcal{U}$. Take any $A \in \mathcal{M}$ and $a \in R_v(A)$. Take $b^*, b_* \in Z$ such that $v(b^*) = 1$ and $v(b_*) = -1$. For any $m = 1, 2, \dots,$, let

$$b_m = \tfrac{1}{m} b^* + \tfrac{m-1}{m} a \quad \text{and} \quad B_m = \tfrac{1}{m} b_* + \tfrac{m-1}{m} A.$$

Then $v(b_m) \geq v(B_m) + \frac{2}{m}$. For all sufficiently large k, $\rho(v_k, v) < \frac{1}{m}$ and hence,

$$v_k(b_m \cup B_m) \geq v(b_m \cup B_m) - \tfrac{1}{m} = v(b_m) - \tfrac{1}{m} \geq v(B_m) + \tfrac{1}{m} \geq v_k(B_m).$$

Thus $b_m \in R_{v_k}(b_m \cup B_m)$. As $v_k \in \mathcal{V}$, then $b_m \in \phi(b_m \cup B_m)$. By Closed Graph, $a \in \phi(A)$. Thus $v \in \mathcal{V}$.

Next we show that $\mathcal{V}$ delivers the required representation: for all $A \in \mathcal{M}$.

$$\phi(A) = \Theta_{\mathcal{V}}(A) = \{a \in A : v(a) = v(A) \quad \text{for some } v \in \mathcal{V}\}.$$

The inclusion $\Theta_{\mathcal{V}}(A) \subset \phi(A)$ holds by definition of $\mathcal{V}$. It remains to show that the opposite inclusion

$$\phi(A) \subset \Theta_V(A). \tag{26}$$

To show this claim, assume first that $Z = S_0$ is the unit ball in L. After we show (26) for the unit ball Z, we will quickly derive it for any CCD Z in L.

Lemma A.3 establishes a natural bijection between the set $\mathcal{U}$ and the sphere S. The bijection $v \to s_v$ preserves the metric structure of $\mathcal{U}$. Note that $\mathcal{V} \in \mathcal{K}$ is compact because the space $\mathcal{U}$ is homeomorphic to the compact unit sphere in L.

Note that for all $v \in \mathcal{U}$ and $a, b \in Z$,

$$|v(a) - v(b)| \leq ||a - b|| \tag{27}$$

because $|v(a) - v(b)| = |s_v \cdot (a - b)| \leq ||s_v|| ||a - b|| = ||a - b||$.

Moreover, for all $v \in \mathcal{U}$ and $A, B \in \mathcal{M}$,

$$|v(A) - v(B)| \leq h(A, B). \tag{28}$$

Take any $A, B \in \mathcal{M}$. For any $a \in A$ such that $v(a) = v(A)$, there is $b \in B$ such that $||a - b|| \leq h(A, B)$ and hence, $v(B) \geq v(A) - h(A, B)$. Similarly, $v(A) \geq v(B) - h(A, B)$ and hence, (28) holds.

Define the *evaluation* function $e : \mathcal{U} \times \mathcal{M} \to \mathbb{R}$ for all $v \in \mathcal{U}$ and $A \in \mathcal{M}$ as

$$e(v, A) = v(A).$$

Lemma A.3 implies that e is continuous. Indeed, for all $v, v' \in \mathcal{U}$ and $A, A' \in \mathcal{M}$,

$$|e(v, A) - e(v', A')| = |v(A) - v'(A')| \leq |v(A) - v'(A)| + |v'(A) - v'(A')| \leq \rho(v, v') + h(A, A').$$

Thus e is continuous on $\mathcal{U} \times \mathcal{M}$.

For any menu $A \in \mathcal{M}$, define its *polar* as

$$\mathcal{P}(A) = \{v \in \mathcal{V} : v(A) \leq 0\}.$$

The polar $\mathcal{P}(A) \subset \mathcal{V}$ is closed because if $v_k \to v$, then $v_k(A) \to v(A)$ for all $A \in \mathcal{M}$.

For any $A \in \mathcal{M}$ and $a \in A$, define the polar of A at a as

$$\mathcal{P}(A, a) = \{v \in \mathcal{V} : v(A) \leq v(a)\}.$$

Then $\mathcal{P}(A, a) = \mathcal{P}\left(\frac{A+(-a)}{2}\right)$ is closed. Let $\mathcal{M}_0$ be the set of all menus $A \in \mathcal{M}$ that contain the zero vector $0 \in L$.

Lemma A.4. *For all $A, B \in \mathcal{M}_0$ such that $\mathcal{P}(A) \subset \mathcal{P}(B)$,*

$$0 \in \phi(A) \quad \Rightarrow \quad 0 \in \phi(B). \tag{29}$$

For all $A, B \in \mathcal{M}$, $a \in A$, and $b \in B$ such that $\mathcal{P}(A, a) \subset \mathcal{P}(B, b)$,

$$a \in \phi(A) \quad \Rightarrow \quad b \in \phi(B). \tag{30}$$

Proof. Take any $A, B \in \mathcal{M}_0$ such that $\mathcal{V}(A) \subset \mathcal{V}(B)$. Suppose that $0 \in \phi(A)$. Take any $b \in B$. Suppose that $\gamma b \notin \text{conv}(A)$ for all $\gamma \in [0, 1]$. Then the relative interior of the segment $\{\gamma b : \gamma \in [0, 1]\}$ is disjoint from the polyhedral set conv(A). A standard separation result (Rockafellar [29, Theorem 20.2]) implies that there is $s \in S$ such that $s \cdot b > 0 \geq s \cdot a$ for all $a \in \text{conv}(A)$. The function $v \in \mathcal{V}$ such that $s = s_v$ belongs to $\mathcal{P}(A)$, but does not belong to $\mathcal{P}(B)$. This contradiction implies that for each $b \in \mathcal{B}$, there is $\gamma \in (0, 1]$ such that $\gamma b \in \text{conv}(A)$. As B is finite and $0 \in A$, then there is $\gamma \in (0, 1]$ such that $\gamma B \subset \text{conv}(A)$. By Convex α, $\gamma B \subset \text{conv}(A)$ and $0 \in \phi(A)$ implies that $0 \in \phi(\gamma B)$. By Mixture Independence, $0 \in \phi(B)$.

The second claim (30) follows from (30). Take any $A, B \in \mathcal{M}$, $a \in A$, and $b \in B$ such that $\mathcal{P}(A, a) \subset \mathcal{P}(B, b)$. Let $A' = \frac{A+(-a)}{2}$ and $B = \frac{B+(-b)}{2}$. Then $\mathcal{P}(A, a) = \mathcal{P}(A')$ and $\mathcal{P}(B, b) = \mathcal{P}(B')$. Assume $a \in \phi(A)$. By Mixture Independence, $0 \in \phi(A')$. By (29), $0 \in \phi(B')$. By Mixture Independence, $b \in \phi(B)$. □

Suppose that the inclusion (26) does not hold. It means that there is $A \in \mathcal{M}$ and $a \in \phi(A)$ such that for each $v \in \mathcal{P}(A, a)$, there is $B \in \mathcal{M}$ and $b \in B \setminus \phi(B)$ such that $v \in \mathcal{P}(B, b)$. By Mixture Independence, the pair (A, a) can be replaced by $(0.5A + 0.5(-a), 0)$. Thus take $A \in \mathcal{M}_0$ and $a = 0$. Take $b^*, b_* \in Z$ such that $v(b^*) = 1$ and $v(b_*) = -1$. For any $m = 1, 2, \ldots,$, let

$$b_m = \tfrac{1}{m} b^* + \tfrac{m-1}{m} b \quad \text{and} \quad B_m = \tfrac{1}{m} b_* + \tfrac{m-1}{m} B.$$

By Closed Graph, there is m such that $b_m \notin \phi(b_m \cup B_m)$. Let $b_v = b_m$ and $B_v = b_m \cup B_m$. Then b_v is the unique maximizer of v in B_v, and hence, v belongs to the interior of $\mathcal{P}(B_v, b_v)$. Write this interior as $\mathcal{P}^o(B_v, b_v)$. Thus

$$\mathcal{P}(A) \subset \cup_{v \in \mathcal{P}(A)} \mathcal{P}^o(B_v, b_v).$$

As $\mathcal{P}(A)$ is compact, then there is a finite collection of open sets $\mathcal{U}_i = \mathcal{P}^o(B_i, b_i)$ such that $b_i \in B_i \setminus \phi(B_i)$, and

$$\mathcal{P}(A) \subset \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_k. \tag{31}$$

Then there is $\varepsilon > 0$ such that for each $v \in \mathcal{P}(A)$, there is i such that $\mathcal{U}_i$ contains the ε-neighborhood of v. Suppose not. Take a sequence $v_1', v_2', \dots \in \mathcal{V}$ such that the $1/m$ neighborhood of v_j' does not belong to any $\mathcal{U}_i$. As $\mathcal{V}$ is compact, take the sequence v_j' converging to some $v \in \mathcal{V}(A, a)$. Take $\mathcal{U}_i$ that contains v together with some ε-neighborhood. Then v_j' belongs to $\mathcal{U}_i$ together with $1/m$-neighborhood for sufficiently large m.

As the sphere S is compact, there is a finite subset $C \subset S$ such that for any $a \in S$, there is $c \in C$ such that $||c - a|| < \varepsilon/3$. Consider menus

$$A_m = \tfrac{1}{m} C \cup A.$$

Take any $c \in C$. The polar set $\mathcal{P}(C, c)$ belongs to the $\varepsilon/3$ neighborhood of $v_c \in \mathcal{U}$ such that $v_c(a) = c \cdot a$. Suppose that $\mathcal{P}(C, c) \cap \mathcal{P}(A)$ is not empty. Then $\mathcal{P}(C, c)$ lies within ε neighborhood of some $v \in \mathcal{P}(A)$. Thus $\mathcal{P}(C, c) \subset \mathcal{U}_i \subset \mathcal{P}(B_i, b_i)$ for some i. By (30), $c \notin \phi(C)$ and hence, $\frac{1}{m} c \notin \phi(A_m)$.

Suppose that $\mathcal{P}(C, c) \cap \mathcal{P}(A)$ is empty. Then for sufficiently large m, $\mathcal{P}(A_m, \frac{1}{m} c) = \emptyset$. Indeed, for each $v \in \mathcal{P}(C, c)$, $v(A) > 0$. By continuity, there is $\delta > 0$ such that $v(A) \geq \delta$ for all $v \in \mathcal{P}(C, c)$. Thus $v(\frac{1}{m} c) < v(A)$ for sufficiently large m. As $\mathcal{P}(A_m, \frac{1}{m} c) = \emptyset$, then by (30), $\frac{1}{m} c \notin \phi(A_m)$. By Convex α, $0 \in \phi(A)$ should imply $0 \in \phi(A_m)$ because $\phi(A_m) \subset A$. But 0 is in the interior of A_m and hence, $\mathcal{P}(A_m) = \emptyset$. By (30), this is a contradiction with any rejection $b_i \in B_i \setminus \phi(B_i)$.

Necessity of PMI. Take any $\mathcal{V} \in \mathcal{K}$. For all $A \in \mathcal{M}$, let

$$\phi(A) = \Theta_{\mathcal{V}}(A) = \{a \in A : v(a) = v(A) \quad \text{for some } v \in \mathcal{V}\}.$$

Obviously, each $\phi(A)$ is not empty.

Show that ϕ is a PMI filter. Take any menus $A, B \in \mathcal{M}$. Suppose that $A \subset \text{conv}(B)$. Then

$$b \in A \cap \phi(B) \;\Rightarrow\; v(b) = v(B) \text{ for some } v \in \mathcal{U} \;\Rightarrow$$
$$v(b) = v(\text{conv}(B)) \geq v(A) \;\Rightarrow\; v(b) = v(A) \;\Rightarrow\; b \in \phi(A).$$

Thus ϕ satisfies Convex α.

Suppose instead that $\phi(B) \subset A \subset B$. Take any $a \in \phi(A)$. Then $v(a) = v(A)$ for some $v \in \mathcal{V}$. Take any $b \in B$ such that $v(b) = v(B)$. As $b \in A$, then $v(a) \geq v(b) = v(B) \geq v(A)$. Thus $v(a) = v(B)$ and hence, $a \in \phi(B)$. Thus $\phi(A) \subset \phi(B)$, and Reduction holds.

Show Closed Graph. Let $G = \{(a, A) \in Z \times \mathcal{M} : a \in \phi(A)\}$. Take any converging sequence $(a_i, A_i) \in G$ such that $(a, A) = \lim_{i\to\infty}(a_i, A_i)$. As $a_i \in \phi(A_i)$, then $v_i(a_i) = v_i(A_i)$ for some $v_i \in \mathcal{V}$. As $\mathcal{V}$ is closed in compact $\mathcal{U}$, then $\mathcal{V}$ is compact. Then it is without loss in generality to select the sequence (a_i, A_i) so that v_i is a converging sequence in $\mathcal{U}$. Let $v = \lim_{i\to\infty} v_i$. As evaluation function is continuous, then

$$v(a) = \lim_{i\to\infty} v_i(a_i) = \lim_{i\to\infty} v_i(A_i) = v(A).$$

Thus $(a, A) \in G$.

Show Mixture Independence. Take any $A \in \mathcal{M}$, $c \in Z$, and $\gamma \in [0, 1]$. Then for any $v \in \mathcal{U}$, $\mathcal{R}_v(\gamma A + (1-\gamma)c) = \alpha R_v(A) + (1-\gamma)c$.

Extension to an arbitrary CCD.

Assume that the unit ball S_0 is a subset of Z. This assumption is without loss in generality for Theorem 5. Indeed, let a_0 be some point in the relative interior of Z. Then there is $\varepsilon > 0$ such that for all a in the affine hull of Z,

$$||a - a_0|| \leq \varepsilon \quad \Rightarrow \quad a \in Z.$$

Let $Z^* = \frac{1}{\varepsilon}Z - \frac{1}{\varepsilon}a_0$. Then Z^* contains the origin and hence, the affine hull of Z^* is a linear subspace $L^* \subset L$ with the metric induced from L. Moreover, Z^* contains the unit ball in L^*. The definition of Z^* provides a natural bijection between Z and Z^* that preserves the topology and mixtures. Thus if Theorem 5 is true for $Z^* \subset L^*$ it is also true for $Z \subset L$.

A.3 Proof of Theorem 2

Suppose that Z is a convex compact set in some Euclidean space L. By Lemma A.3, any function $u \in \mathcal{U}$ is continuous.

Lemma A.5. *$\succeq$ satisfies Axioms 1–6 if and only if there is a function $u \in \mathcal{U}$ and a PMI filter $\phi : \mathcal{M} \to \mathcal{M}$ such that*

$$U(A) = u(\phi(A)) \quad \textit{for all } A \in \mathcal{M}. \tag{32}$$

Proof. Take any $u \in \mathcal{U}$ and a PMI filter $\phi : \mathcal{M} \to \mathcal{M}$. Assume that $\succeq$ is represented by (32). Axioms 1–3 hold because ϕ is path independent. Singleton Independence holds because $u \in \mathcal{U}$. Show Mixture Monotonicity. Take any $A \in \mathcal{M}$, $a, b \in A$, and $\gamma \in [0, 1[$. By (32), $U(A) = u(a)$ for some $a \in \phi(A)$. By Convex α, $a \in \phi(A \cup (\gamma a + (1 - \gamma)b))$. Thus $U(A \cup (\gamma a + (1 - \gamma)b)) \geq U(A)$.

Show Partial Continuity. Suppose that $A_i \succeq B$ for some menu $B \in \mathcal{M}$ and sequence $A_1, A_2, \cdots \in \mathcal{M}$. By (32), for each i, there is $a_i \in \phi(A_i)$ such that $u(a_i) \succeq U(B)$. As Z is compact, it is without loss in generality to assume that a_i is converging to some $a \in Z$. By Closed Graph, $a \in \phi(A)$. As u is continuous on Z, then $u(a) \geq U(B)$. By (32), $U(A) \geq u(A) \geq U(B)$. The mixture continuity part holds because $U(\gamma A + (1 - \gamma)c) = \gamma U(A) + (1 - \gamma)c$ for all $A \in \mathcal{M}$, $c \in Z$, and $\gamma \in [0, 1]$.

Conversely, suppose that $\succeq$ satisfies Axioms 1–6. If $a \sim b$ for all $a, b \in Z$, then let $u = 0$. By Lemma A.1, $\succeq$ is represented by $U = 0$. The constant function has the form (32) for any path independent ϕ.

Assume that R_0 is not degenerate. Partial Continuity implies mixture continuity for the commitment ranking R_0. The mixture space theorem implies that R_0 has a unique representation $u \in \mathcal{U}$. By Lemma A.1, for any $A \in \mathcal{M}$, there is

$d_A \in A$ such that $A \sim d_A$. Let

$$U(A) = u(d_A).$$

Then U represents $\succeq$. Take any $A \in \mathcal{M}$ and $c \in Z$. Show that for all $\gamma \in [0, 1]$,

$$U(\gamma A + (1 - \gamma)c) = \gamma U(A) + (1 - \gamma)u(c). \tag{33}$$

By Partial Continuity, the function $w(\gamma) = U(\gamma A + (1-\gamma)c)$ is continuous on $[0, 1]$. By definition of U, the value $w(\gamma)$ belongs to the finite set

$$W_\gamma = \{\gamma u(a) + (1 - \gamma)u(c) : a \in A\}.$$

As $w(1) = U(A) = u(d_A)$, then the only continuous function w such that $w(\gamma) \in W_\gamma$ is $w(\gamma) = \gamma u(d_A) + (1 - \gamma)u(c)$.

For all $A \in \mathcal{M}$, let

$$\phi(A) = \{c \in A : c \cup A' \succeq c \quad \text{for all menus } A' \subset \text{conv}(A)\} \tag{34}$$

be the set of all $c \in A$ that are not cancelled by the convex hull $\text{conv}(A)$. Note that $\text{conv}(A)$ is an infinite set, but the menus $A' \subset \text{conv}(A)$ are still required to be finite.

Show that ϕ is a PMI filter. Take any $A \in \mathcal{M}$. Take $c \in R_{-u}(A)$ that minimizes u in A. As R_0 is separable, then $c \in R_{-u}(A')$ for any $A' \subset \text{conv}(A)$. By (21), c is not cancelled by any $A' \subset \text{conv}(A)$ because there is no menu $B \subset A'$ such that $c \gg B$. Thus $\phi(A)$ is not empty.

Convex α follows directly from definition (34). Mixture Independence is implied by (33).

Show Closed Graph. Take any pair $(a, A) \in Z \times \mathcal{M}$ such that $(a, A) = \lim_{n\to\infty}(a_n, A_n)$ for some $(a_n, A_n) \in Z \times \mathcal{M}$ such that $a_n \in \phi(A_n)$. Suppose that $a \notin \phi(A)$. Then $a \succ a \cup A'$ for some $A' \subset \text{conv}(A)$. Take $b \in Z$ such that $a \succ b \succ a \cup A'$. As R_0 is represented by a continuous $u \in \mathcal{U}$ and $\succeq$ satisfies Partial Continuity, then there is $\varepsilon > 0$ such that

- $u(a^*) > u(b)$ for all a^* in the ε-neighborhood of a, and

- $b \succ A^*$ for all A^* in the ε-neighborhood of $a \cup A'$.

Thus for all sufficiently large n, the pair (a_n, A_n) must have $A'_n \in \text{conv}(A_n)$ such that a_n and A'_n are in the ε-neighborhood of a and A' respectively. Thus $a_n \succ b \succ a_n \cup A'_n$, which contradicts $a_n \in \phi(A_n)$.

Turn to Reduction. Take any $B \in \mathcal{M}$ and $b, c \in B \setminus \phi(B)$. Show that $b \notin \phi(B \setminus c)$. By (21), there is $C \subset \text{conv}(B)$ such that $c \succ c \cup C$ and $c \gg C$. We claim that there is $C^* \subset \text{conv}(B \setminus c)$ such that $c \succ c \cup C^*$ and $c \gg C^*$. Indeed, any $a \in C \subset \text{conv}(B)$ can be written as a mixture $a = \gamma c + (1-\gamma)a'$ where $a' \in \text{conv}(B \setminus c)$. As $c \gg C$, then $c \succ a$ and $c \succ a'$. By PSB, $c \succ (c \cup C) \cup a'$. Let $C_1 = (C \setminus a) \cup a'$. By Mixture Monotonicity $(c \cup C) \cup a' \succeq c \cup C_1$. Note that C_1 equals C where a is replaced with $a' \in \text{conv}(B \setminus c)$. Repeating this procedure generates $C^* \subset \text{conv}(B \setminus c)$ such that $c \succ c \cup C^*$. Note that $c \gg C^*$.

As $b \notin \phi(B)$, then there is $B' \subset \text{conv}(B)$ such that $b \gg B'$. We claim that there is $B^* \subset \text{conv}(B \setminus c)$ such that $b \succ b \cup B^*$. Indeed, any $a \in B'$ can be written as a mixture $a = \gamma c + (1-\gamma)a'$ where $a' \in \text{conv}(B \setminus c)$. Assume that $a \notin \text{conv}(B \setminus c)$. Then $\gamma > 0$. Recall that $c \succ c \cup C^*$ and $c \gg C^*$. Let $C_1 = \gamma C^* + (1-\gamma)a'$. Then $a \gg C_1$, and by (33), $c \succ c \cup C^*$ implies $a \succ a \cup C_1$. By PSB, $b \succ (b \cup B') \cup C_1$. By MD, $(b \cup B') \cup C_1 \sim b \cup (B' \setminus a) \cup C_1$. Let $B_1 = (B' \setminus a) \cup C_1$. Then B_1 equals to B' where a is replaced by $C_1 \subset \text{conv}(B \setminus c)$. Repeating this procedure generates $B^* \subset \text{conv}(B \setminus c)$ such that $b \succ b \cup B^*$. Thus $b \notin \phi(B \setminus c)$, and Reduction holds. $\square$

The equivalence of Axioms 1–6 and representation (8) is implied by the combination of Lemma A.5 and Theorem 5. The equality $U = u \circ V = u \circ \mathcal{V}_*$ is established in the proof of Theorem 8 below.

A.4 Proof of Theorem 8

Let Z be a CCD. Fix a point $a_0 \in Z$ in the relative interior of Z.

Consider an application of the Farkas Lemma.

Lemma A.6. *For all $u, v \in \mathcal{U}$ and $\mathcal{V} \in \mathcal{K}$, either (I) or (II) is true:*

(I) there is $w \in \mathcal{V}$ such that $v \in [w, -u]$ and $w \in [u, v]$,

(II) there is a menu $B \in \mathcal{M}$ such that

$$a_0 = R_u(B) = R_v(B) \quad \textit{and} \quad a_0 \notin R_w(B) \quad \textit{for all } w \in \mathcal{V}. \tag{35}$$

Proof. Take any functions, $u, v, w \in \mathcal{U}$. Write them in the form (24) with vectors $s_u, s_v, s_w \in L_0$ respectively. Consider several cases.

Case 1. $v = -u$. Then $v \in [w, -u]$ and $w \in [u, v]$.

Case 2. $s_w = \alpha s_u + \beta s_v$ for some $\alpha, \beta \geq 0$. As $s_w \neq 0$, then $\alpha + \beta > 0$. Thus $w \in [u, v]$ because w and $\gamma u + (1-\gamma)v$ represent the same order when $\gamma = \frac{\alpha}{\alpha+\beta}$. Suppose that $\beta = 0$. Then $w = u$ and hence, $v \in [w, -u]$. Suppose that $\beta > 0$. Then $s_v = \frac{1}{\beta}s_w - \frac{\alpha}{\beta}s_u$. Thus $v \in [w, -u]$.

Case 3. $v \neq -u$ and $s_w \neq \alpha s_u + \beta s_v$ for any $\alpha, \beta \geq 0$. By the Farkas Lemma, there is $b_0 \in L_0$ such that $s_u \cdot b_0 > 0$, $s_v \cdot b_0 > 0$, and $s_w \cdot b_0 < 0$. Let $b = a_0 + b_0$. As a_0 is in the relative interior of Z, then it is without loss in generality to assume $b \in Z$. By construction,

$$u(a_0) > u(b) \quad \text{and} \quad v(a_0) > v(b) \quad \text{and} \quad w(b) > w(a_0). \tag{36}$$

Suppose that $w \in [u, v]$. Then w and $\gamma u + (1-\gamma)v$ represent the same ranking for some $\gamma \in [0, 1]$. Thus $s_w = \alpha s_u + \beta s_v$ for some $\alpha, \beta \geq 0$. This contradiction shows that $w \notin [u, v]$. Similarly, $v \notin [w, -u]$.

Take any $\mathcal{V} \in \mathcal{K}$. If Cases 1 and 2 hold for some $w \in \mathcal{V}$, then statement (I) is true. Suppose instead that for any $w \in \mathcal{V}$, Case 3 holds and hence, there is $b \in Z$ such that (36). As w is continuous, then $w'(b) > w'(a_0)$ holds for all $w' \in \mathcal{V}$ in some neighborhood of w. As $\mathcal{V}$ is compact, then there are finitely many menus $b_1, \ldots, b_k$ such that for any $w \in \mathcal{V}$, b can be selected as b_i. Let $B = \{a_0, b_1, \ldots, b_k\}$. Then (35) holds. $\square$

Suppose that $\succeq$ is represented by $U = u \circ \mathcal{V}$ for some $u \in \mathcal{U}$ and $\mathcal{V} \in \mathcal{K}$. Let

$$\mathcal{V}^* = \bigcup_{v \in \mathcal{V}} [v, -u].$$

Show that $\mathcal{V}^*$ is closed. Take any sequence of $v_1^*, v_2^*, \cdots \in \mathcal{V}^*$ such that $\lim_{i\to\infty} v_i^* = v^* \in \mathcal{U}$. Then there are $v_i \in \mathcal{V}$ and $\gamma_i \in [0,1]$ such that v_i^* and $\gamma_i v_i + (1-\gamma_i)(-u)$ represent the same separable order R_i. Both $\mathcal{V}$ and $[0,1]$ are compact sets. Thus it is without loss in generality to take both sequences $\{v_i\}$ and $\{\gamma_i\}$ converging to some $v \in \mathcal{V}$ and $\gamma \in [0,1]$ respectively. By continuity, v^* and $\gamma v + (1-\gamma)(-u)$ represent the same separable order on Z. Thus $v^* \in \mathcal{V}^*$.

Show that $\mathcal{V}^*$ contains any closed $\mathcal{V}'$ such that $\succeq$ is represented by $U' = u \circ \mathcal{V}'$. Suppose not. Then there is $v \in \mathcal{V}' \setminus \mathcal{V}^*$. By definition of $\mathcal{V}^*$, $v \notin [w, -u]$ for all $w \in \mathcal{V}^*$. By Lemma A.6, there is $B \in \mathcal{M}$ such that

$$a_0 = R_u(B) = R_v(B) \quad \text{and} \quad a_0 \notin R_w(B) \quad \text{for all } w \in \mathcal{V}^*.$$

Then $U'(B) \geq u(a_0)$ because a_0 is delegable via v, but $(u \circ \mathcal{V})(B) < u(a_0)$ because a_0 is not delegable via any $w \in \mathcal{V}^*$ and a fortiori, $w \in \mathcal{V}$. By contradiction, $\mathcal{V}' \subset \mathcal{V}^*$.

Show that $u \circ \mathcal{V} = u \circ \mathcal{V}^*$. As $\mathcal{V} \subset \mathcal{V}^*$, then $(u \circ \mathcal{V})(A) \leq (u \circ \mathcal{V}^*)(A)$ for all $A \in \mathcal{M}$. Take any $B \in \mathcal{M}$. Then $(u \circ \mathcal{V}^*)(B) = u(b)$ for some $b \in R_w(B)$ and $w \in \mathcal{V}^*$. There are $v \in \mathcal{V}$ and $\gamma \in [0,1]$ such that w and $\gamma v + (1-\gamma)(-u)$ represent the same order on R_i. Take any $a \in R_v(B)$. Then $v(a) \geq v(b)$ and $w(b) \geq w(a)$. Moreover, either $v(a) = v(b)$ or $u(a) > u(b)$ must hold. In the first case, b is delegable under $\mathcal{V}$ and hence, $(u \circ \mathcal{V})(B) \geq u(b) = (u \circ \mathcal{V}^*)(B)$. In the second case, a is delegable under $\mathcal{V}$ and hence, $(u \circ \mathcal{V})(B) \geq u(a) > u(b) = (u \circ \mathcal{V}^*)(B)$. Thus $u \circ \mathcal{V} = u \circ \mathcal{V}^*$.

Turn to (ii). Let $\mathcal{V}_*$ be the closure of the set $\mathcal{V} \setminus \bigcup_{v \in \mathcal{V}} (v, -u]$. Then $\mathcal{V}_* \subset \mathcal{V}$, and sets $\mathcal{V}$ and $\mathcal{V}_*$ generate the same $\mathcal{V}^*$. Thus

$$u \circ \mathcal{V} = u \circ \mathcal{V}^* = u \circ \mathcal{V}_*.$$

Show that $\mathcal{V}_*$ is a subset of any closed $\mathcal{V}'$ such that $\succeq$ is represented by $U' = u \circ \mathcal{V}'$. Suppose not. Then there is $v \in \mathcal{V}_* \setminus \mathcal{V}'$. The sets $\mathcal{V}_*$ and $\mathcal{V}'$ must generate the same $\mathcal{V}^*$. Thus $y_{\in}[v', -u]$ for some $v' \in \mathcal{V}'$. Take $v_* \in \mathcal{V}_*$ such that $v' \in [v_*, -u]$. Then $v \in [v_*, -u]$. By definition of $\mathcal{V}_*$, $v = v_*$. As $v' \in [v, -u]$ and $v \in [v', -u]$, then $v = v' \in \mathcal{V}'$. By contradiction, $\mathcal{V}_* \subset \mathcal{V}'$.

Finally, suppose that $\mathcal{V}_*$ is not a singleton. Take distinct $v, v' \in \mathcal{V}_*$. Write $\mathcal{V}_* = \mathcal{V}_1 \cup \mathcal{V}_2$ where $v \in \mathcal{V}_1 \setminus \mathcal{V}_2$ and $v' \in \mathcal{V}_2 \setminus \mathcal{V}_1$. Take $B, B' \in \mathcal{M}$ such that

- $a_0 = R_u(B) = R_v(B)$ and $a_0 \notin R_w(B)$ for all $w \in \mathcal{V}'$,
- $a_0 = R_u(B') = R_{v'}(B')$ and $a_0 \notin R_w(B')$ for all $w \in \mathcal{V}$.

Then $U(B) = U(B') = u(a_0)$, but $U(B \cup B') < u(a_0)$, which violates Set Betweenness.

A.5 Proof of Theorem 3

Let X be a CCD with $x_0 \in X$ in the relative interior. Then $a_0 = x_0$ is also in the relative interior of $Z = X^\Omega$.

Suppose that $\succeq$ is represented by

$$U = v_{r,\mu} \circ \mathcal{V}_{t,\mathcal{C}} \tag{37}$$

for some $r, t \in \mathcal{U}_X$, $\mu \in \Delta$ and non-empty, closed set $\mathcal{C} \subset \Delta$. By Theorem 2, $\succeq$ satisfies Axioms 1–6. Check MP. Take any acts $a, b \in Z$. Assume that $a(n) \succeq b(n)$ for all $n \in \Omega$. Then $r(a(n)) \geq r(b(n))$ for all n and hence, $v_{r,\mu}(a) \geq v_{r,\mu}(b)$. Assume that $\{a(n), b(n)\} \succ b(n)$ for all $n \in \Omega$. By PSB, $a(n) \succ b(n)$ and hence, $a \succ b$. By (37), $t(a(n)) \geq t(b(n))$ because $b(n)$ is delegable in $\{a(n), b(n)\}$. Take any $B \in \mathcal{M}$. If b is not delegable in $\{a, b\} \cup B$, then $\{a, b\} \cup B \sim a \cup B$. If b is delegable in $\{a, b\}$, then there is $\pi \in \mathcal{C}$ such that b maximizes $v_{t,\pi}$ in $\{a, b\} \cup B$. Then a maximizes $v_{t,\pi}$ as well. Thus $\{a, b\} \cup B \sim a \cup B$. Part (iii) is similar.

Conversely, suppose that $\succeq$ satisfies Axioms 1–7. By Theorems 2 and 8, $\succeq$ is represented by

$$U = u \circ \mathcal{V} = u \circ \mathcal{V}_*$$

for some $\mathcal{V} \in \mathcal{K}$ and $\mathcal{V}_*$ that is the closure of the set

$$\mathcal{V}' = \mathcal{V} \setminus \bigcup_{v \in \mathcal{V}} (v, -u].$$

The function $u \in \mathcal{U}$ is monotonic and hence, has a representation

$$u(a) = \sum_{n \in \Omega} \mu(n) r(a(n))$$

for some $r \in \mathcal{U}_X$ and $\mu \in \Delta$. As u has the range $[-1, 1]$, then r has the range $[-1, 1]$ as well.

We claim that for all $v, w \in \mathcal{V}'$, their restrictions t_v and t_w to X are the same. Suppose not. Then either $t_v \notin [r, t_w]$ or $t_w \notin [r, t_v]$. Let $t_v \notin [r, t_w]$. By (36), there is $y \in X$ such that

$$r(x_0) > r(y) \quad \text{and} \quad t_w(x_0) > t_w(y) \quad \text{and} \quad t_v(y) < t_v(x_0).$$

Let $\mathcal{W}_* = \{w_* \in \mathcal{V}_* : w_*(x_0) \geq w_*(y)\}$. Then $\mathcal{W}_*$ is closed and $v \notin W_*$. As $\mathcal{W}_* \subset \mathcal{V}_*$, then it does not overlap with $[u, v]$. By Lemma A.6, there is a menu $B \in \mathcal{M}$ such that

$$R_u(B) = R_v(B) = x_0 \quad \text{and} \quad x_0 \notin R_{w_*}(B) \quad \text{for all } w_* \in \mathcal{W}_*.$$

Then $x_0 \sim \{x_0, y\} \succ y$, but $B \succ y \cup B$ because $x_0 \notin R_{w_*}(y \cup B)$ for all $w_* \in \mathcal{W}_*$, and $y \succ x_0$ for all $w_* \in \mathcal{V}_* \setminus \mathcal{W}_*$. Thus

$$x_0 \cup B = B \succ y \cup B = \{x_0, y\} \cup B$$

which contradicts MP. Thus $t_v = t_w$.

Let $t \in \mathcal{U}_X$ be the restriction of all $v \in \mathcal{V}'$ to X. By continuity, t also equals the restriction of any $v \in \mathcal{V}_*$ because $\mathcal{V}_*$ is the closure of $\mathcal{V}'$.

Take any $v \in V'$. We claim that there is $\pi \in \Delta$ such that

$$v = v_{t,\pi}.$$

Consider two cases.

Case 1. $t = -r$. Then Take any two acts $a, b \in Z$ such that $t(a(n)) \geq t(b(n))$ for all $n \in \Omega$, but $v(b) < v(a)$. By continuity, a and b can be modified so that $t(a(n)) > t(b(n))$ for all $n \in \Omega$, but $v(b) < v(a)$. Then $\{b(n)\} \succ \{a(n), b(n)\}$ for all $n \in \Omega$, but $\{a, b\} \sim a$ because a is delegable in $\{a, b\}$ via v. This contradiction of MP implies that for all $a, b \in Z$,

$$t(a(n)) \geq t(b(n)) \quad \text{for all } n \in \Omega \quad \Rightarrow \quad v(a) \geq v(b).$$

Anscombe–Aumann's Theorem implies that $v = v_{t,\pi}$ for some $\pi \in \Delta$.

Case 2. $t \neq -r$. As $v \in \mathcal{U}$ satisfies the mixture linearity (7) on the entire Z, then the function v represents the same preference as

$$v'(a) = \sum_{n \in \Omega} \pi(n) t_n(a(n))$$

for some $\pi \in \Delta$ and $t_1, \ldots, t_n \in \mathcal{U}_X$. This state-dependent representation is equivalent to the one in Kreps [21, Proposition 7.5]. Let $\Omega_* = \{n \in \Omega : \pi(n) > 0\}$. Suppose that $t_n \in [u, t]$ for all $n \in \Omega_*$. As t is a mixture of t_n for $n \in \Omega_*$, then $t_n = t$ for all $t \in \Omega_+$. Then $v = v_{t,\pi}$ has the required form.

Suppose that $t_n \notin [u, t]$ for some $n \in \Omega_*$. As $t \neq -r$, then there is y such that

$$u(x_0) > u(y) \quad \text{and} \quad t(x_0) > t(y) \quad \text{and} \quad t_n(y) > t_n(x_0).$$

Take an act a such that $a(n) = y$ and $a(i) = x_0$ for all $i \in \Omega \backslash n$. Then $v(a) > v(x_0)$. By continuity, take a' such that $x_0 \succ a'(n)$ and $t(x_0) > t(a'(n)$ for all $n \in \Omega$, but $w(a') > w(x_0)$ for all w in some neighborhood of v in $\mathcal{V}_*$. By Lemma A.6, there is a menu $B \in \mathcal{M}$ such that

$$R_u(B) = R_v(B) = x_0 \quad \text{and} \quad x_0 \notin R_{w_*}(B) \quad \text{for all } w_* \in \mathcal{W}_*.$$

Then $B \succ a' \cup B$ even though $x_0 \in B$ is such that $\{x_0, a'(n)\} \succ \{a'(n)\}$ for all $n \in \Omega$. This contradiction of MP implies that $t_n \notin [u, t]$ cannot hold for any $n \in \Omega_*$.

Finally, any $v \in \mathcal{V} \setminus \mathcal{V}_*$ has a risk attitude t_v that is represented by a mixture of t and $-r$. If $t \neq r$ and $t \neq -r$, then v does not equal t. Therefore, $\mathcal{V} \setminus \mathcal{V}_*$ is empty.

A.6 Proof of Theorem 4

Suppose that $\succeq$ satisfies Axioms 1–7. By Theorem 3, $\succeq$ is represented by (9) with a tuple $(r, t, \mu, \mathcal{C})$.

Suppose first that $t = -r$. Then $v_{r,\mu} = -v_{t,\mu}$ and hence, (9) still holds if $\mathcal{C}$ is replaced by $\mathcal{C} \cup \{\mu\}$. Objective Monotonicity is vacuous because $\{x, y\} \succ y$ is impossible when $t = -r$. As $\succeq$ is not regular in this case, then both statements of Theorem 4 are true.

Suppose next that $t \neq -r$. Then there are $x, y \in X$ such that $r(x) > r(y)$ and $t(x) > t(y)$. By (9), $x \sim \{x, y\} \succ y$.

Show the first claim of Theorem 4. Let $\mu \in \mathcal{C}$. Show Objective Monotonicity. Take $A, B \in \mathcal{M}(x, y)$ and $a^* \in A \cup B$ such that $a^* \succeq a$ for all $a \in A \cup B$. Then $r(a^*(\mu)) \geq r(a(\mu))$ for all $a \in A \cup B$. The functions r and t represent the same ranking on $X(x, y)$. Thus $t(a^*(\mu)) \geq t(a(\mu))$ for all $a \in A \cup B$. As $\mu \in \mathcal{C}$, then $a^* \in \mathcal{V}_{t,\mathcal{C}}(A \cup B)$. Thus $A \cup B \succeq a^* \succeq A$.

Suppose instead that $\succeq$ satisfies OM, but $\mu \notin \mathcal{C}$. Let $z = \frac{x+y}{2}$. The standard separation argument implies that for each $\pi \in \mathcal{C}$, there is $a \in Z(x, y)$ such that $v_{r,\pi}(a) > r(z) > v_{r,\mu}(a)$. As $\mathcal{C}$ is compact, there are finitely many $a_1, \ldots, a_k \in Z(x, y)$ such that for each $\pi \in \mathcal{C}$, $v_{r,\pi}(a_i) > r(z) > v_{r,\mu}(a_i)$ for some i. Let $B = \{a_1, \ldots, a_k\}$. Then z is not delegable in $B \cup z$ and hence,

$$z \succ a_i \sim A \cup B$$

for some i. This ranking contradicts OM.

Turn to the second claim of Theorem 4. Suppose that $\succeq$ is regular. Then $t \neq -r$ and $t \neq r$.

Let $\mathcal{C} = \{\mu, \pi\}$ for some $\pi \in \Delta$. As $\mu \in \mathcal{C}$, then OM holds. Take any menus $A, B, C \in \mathcal{M}$ such that $A \succeq B \succeq C$. Then $A \sim d_A$, $B \sim d_B$, and $C \sim d_C$ where d_A, d_B, d_C maximize $v_{r,\mu}$ in $\mathcal{V}_{t,\mathcal{C}}(A)$, $\mathcal{V}_{t,\mathcal{C}}(B)$, $\mathcal{V}_{t,\mathcal{C}}(C)$ respectively. Then at least one of the following six cases must hold.

(i) d_A and d_B maximize $v_{t,\mu}$ in A and B respectively,

(ii) d_A and d_B maximize $v_{t,\pi}$ in A and B respectively,

(iii) d_A and d_C maximize $v_{t,\mu}$ in A and C respectively,

(iv) d_A and d_C maximize $v_{t,\pi}$ in A and C respectively,

(v) d_B and d_C maximize $v_{t,\mu}$ in B and C respectively,

(vi) d_B and d_C maximize $v_{t,\pi}$ in B and C respectively.

In the first two cases, either d_A or d_B is delegable in $A \cup B$. Thus $A \cup B \succeq A \succeq C$ or $A \cup B \succeq B \succeq C$ must hold. Similarly, $A \cup C \succeq C$ in the middle two cases, and $B \cup C \succeq C$ in the last two cases.

Suppose instead that $\succeq$ is regular, and satisfies OM and OD. Then $\mathcal{C}$ is unique. Suppose that $\mathcal{C}$ has at least three distinct elements, $\{\pi_1, \pi_2, \pi_3\} \subset \mathcal{C}$ where $\pi_i \neq \pi_j$ for $i \neq j$. Recall that $r(x) > r(y)$ and $t(x) > t(y)$. As $\succeq$ is not regular, then $r \neq t$ and hence, there are $x', y' \in X$ such that $r(x') > r(y')$ and $t(y') > t(x')$. Let $x'' = \gamma x + (1-\gamma)y'$ and $y'' = \gamma y + (1-\gamma)x'$ where $\gamma \in (0,1)$ satisfies

$$\frac{\gamma}{1-\gamma} = \frac{r(x') - r(y')}{r(x) - r(y)}.$$

Then $r(x'') = r(y'')$ and $t(x'') > t(y'')$. Let $y_* = \gamma y + (1-\gamma)y'$. Then $r(x'') > r(y_*)$ and $t(x'') > t(y_*) > t(y'')$. Consider the set $Z(x'', y_*)$. It is a mixture space, and the utility functions $v_1 = v_{t,\pi_1}$, $v_2 = v_{t,\pi_2}$, $v_3 = v_{t,\pi_3}$ represent distinct rankings on $Z(x'', y_*)$. Thus there are three acts $a_1, a_2, a_3 \in Z(x'', y_*)$ such that $v_i(a_i) > v_i(a_j)$ for all $i, j \in \{1, 2, 3\}$ such that $i \neq j$. (This claim follows from Lemma A.1 in Kopylov [19].) Wlog $x'' \succ a_i(n)$ for all $n \in \Omega$. Take $x_1, x_2, x_3 \in X(x'', y'')$ such that

$$v_i(a_i) > t(x_i) > v_i(a_j)$$

for all $i, j \in \{1, 2, 3\}$ such that $i \neq j$. Note that $r(x_i) = r(x'') = r(y'')$. By monotonicity, $x_i \succ a_j$ for all i, j. Let $A = \{x_1, a_2, a_3\}$, $B = \{x_2, a_1, a_3\}$, and $C = \{x_3, a_1, a_2\}$. Wlog $A \succeq B \succeq C$. Then x_1 is delegable in A via v_1. As $x_1 \succ a_2$ and $x_1 \succ a_3$, then $x_1 \sim A$. Similarly, $x_2 \sim B$ and $x_3 \sim C$. However, x_1 and x_2 are not delegable in the union $A \cup B = \{x_1, x_2, a_1, a_2, a_3\}$. Thus $A \cup B \sim a_i$ for some i, and $C \succ A \cup B$. Similarly, $C \succ A \cup C$ and $C \succ B \cup C$. This violation of OD implies that $\mathcal{C}$ has at most two distinct elements.

A.7 Proof of Theorem 6

Show the necessity of condition (15). First we take any $(a, A) \in \mathcal{D}$. Because R_0 is *acceptable*, $a \in R_0(\Theta(A))$, where $\Theta(A) = \cup_{R\in\Theta} R(A)$, $\Theta \subseteq \mathcal{R}$. By Theorem 1 we can take $\Theta \subseteq \mathcal{T}$. Thus, there is $R_1 \in \Theta$, such that $a = R_1(A)$. If $A = B$, clearly, $a \notin B \setminus a \cup \mathcal{N}(B)$. We claim that for any $B \in \mathcal{M}$ such that $A \subsetneq B$, we have $R_1(B) \notin A \setminus a$. Indeed, as $A \subsetneq B$, $R_1(B)R_1 a$ or $R_1(B) = a$. If $R_1(B) = a$, then clearly $R_1(B) \notin A \setminus a$. If $R_1(B)R_1 a$ and $R_1(B) \in A \setminus a$, then $aR_1R_1(B)R_1 a$. This contradicts the antisymmetric property of $R_1 \in \mathcal{T}$. Next, we show that $R_1(B) \notin \mathcal{N}(B)$. For the sake of contradiction, suppose $R_1(B) \in \mathcal{N}(B)$. By the definition of $\mathcal{N}(B)$, there is $(b', B') \in \mathcal{D}$, such that $\{R_1(B), b'\} \subseteq B' \subseteq B$ and $R_1(B)P_0 b'$. This implies that $R_1(B) \notin \phi(B')$. By the property α, $R_1(B) \notin \phi(B)$. This contradicts Lemma A.2, which states that $\phi(B) = \Theta(B) = \cup_{R\in\Theta}(R(B))$.

Show the sufficiency of condition (15). Let $|\mathcal{D}| = M, |Z| = N$. We construct a set of linear orders $\Theta \subseteq \mathcal{T}$ and show that it makes R_0 acceptable. To constitute this Θ, we construct an R_i using Algorithm 1 for each $(a_i, A_i), i \in \{1, ..., M\}$, and take $\Theta = \cup_{i=1}^{M} R_i$. For each R_i, the algorithm stops at some set S_i, where no menu in $\mathcal{M}$ is a subset of S_i. Although S_i is not ranked by R_i, we can extend R_i on S_i in basically any fashion, since the data puts no restriction on the ranking of elements in S_i. Although $R_i \in \mathcal{T}$ is a linear order for all $i = 1, ..., M$, R_0 can be a weak order $R_0 \in \mathcal{R}$. Suppose there are two observations for one menu: (a_i, A_i) and (a_j, A_i). Then $R_i, R_j \in \Theta$ such that $R_i(A_i) = a_i$ and $R_j(A_i) = a_j$ implies that $a_i, a_j \in \Theta(A)$, and thus $a_i, a_j \in R_0(\Theta(A))$ is possible.

We next show that $\Theta = \cup_{i=1}^{M} R_i$ makes R_0 acceptable. We proceed in two steps. First, we demonstrate that for any $(a_i, A_i) \in R_i(A_i)$, $a_i = R_i(A_i)$. Next, we show that for any $m, i = 1, ..., M$, $R_m(A_i) \notin \mathcal{N}(A_i)$. This implies that $a_i R_0 R_m(A_i)$ for all $R_m \in \Theta, m \neq i$. Therefore $a_i \in R_0(\Theta(A_i))$.

To show the two claims above, we introduce the following two notations for each $R_i, i = 1, ..., M$:

(1)Denote the elements in Z as $a_i^1, ..., a_i^N$, where a_i^k indicates that this element is ranked k^{th} by R_i.

Algorithm 1: Start from any observation (a_i, A_i); find a ranking $R_i \in \mathcal{T}$ such that $a_i = R_i(A_i)$ and $R_i(A_m) \notin \mathcal{N}(A_m)$ for all $A_m \in \mathcal{M}$.

Result: R_i

Take one observation (a_i, A_i). Initialize $R_i = [\,]$, $B = Z$, $j = 1$, $a_0 = [\,]$, $a = a_i$, $A = A_i$;

while *There exists* $A' \in \mathcal{M}$ *such that* $A' \subsetneq B$ **do**

  define $\mathcal{N}(B) = \{b \in B : bP_0a' \text{ for some } (a', A') \in \mathcal{D}, \text{ such that } A' \subseteq B\}$;

  if $B \setminus (\mathcal{N}(B) \cup A) \neq \emptyset$;

  then

    take any element in $B \setminus (\mathcal{N}(B) \cup A)$ and call it a_j;

    $B = B \setminus a_j$;

    R_i.append(a_j);

    $j = j + 1$;

  end

  Assign $a_j = a$;

  $B = B \setminus a_j$;

  R_i.append(a_j);

  $j = j + 1$;

  if *there exists* $A' \in \mathcal{M}$ *such that* $A' \subsetneq B$ **then**

    take any such A', and let $A = A'$, $a = a'$;

  end

end

Return$(R_i = [a_1 a_2 \ldots. a_j])$.

(2)Denote $B_i^j = Z \setminus \cup_{l=1}^j a_i^j$.

Show that for any $(a_i, A_i) \in \mathcal{D}$, $a_i = R_i(A_i)$. Indeed, suppose that $a_i = a_i^k$. If $k = 1$, then we are done. If $k > 1$, then by Algorithm 1, $a_i^j \in Z \setminus (\mathcal{N}(Z) \cup A_i)$ for $j = 1, ..., k-1$. Therefore $a_i^j \notin A_i \setminus a_i$, which implies $A_i \setminus a_i \subset B_i^k$. The construction of R_i implies $a_i^k \gg_i B_i^k$, and thus $a_i \gg_i A_i \setminus a_i$. Therefore, $a_i \in R_i(A_i)$.

Show that for any $m, i = 1, ..., M$, $R_m(A_i) \notin \mathcal{N}(A_i)$. Since $m, i = 1, ..., M$ are arbitrary, if we show that $R_i(A_m) \notin \mathcal{N}(A_m)$ then we can conclude that $R_m(A_i) \notin \mathcal{N}(A_i)$. For simplicity in notation, we show $R_i(A_m) \notin \mathcal{N}(A_m)$. Take (a_i, A_i), $i \in \{1, ..., M\}$. By Algorithm 1, R_i ranks all elements in $Z \setminus S_i$. Let $|Z \setminus S_i| = r$. Suppose $R_i(A_m) = a_i^k$. If $k = 1$, then $R_i(A_m) = a_i^1 \in Z \setminus (\mathcal{N}(Z) \cup [A_i \setminus a_i])$. Because $\mathcal{N}(A_m) \subseteq \mathcal{N}(Z)$ by definition of $\mathcal{N}(\cdot)$, $R_i(A_m) \notin \mathcal{N}(A_m)$. Suppose $2 \leq k \leq r$. Then, $a_i^1, .., a_i^{k-1} \notin A_m$, and therefore, $A_m \subseteq B_i^{k-1}$. Take $B \supseteq B_i^{k-1}$ to be the smallest B in the algorithm that includes B_i^{k-1}. Then, by construction, $a_i^k \in B \setminus (\mathcal{N}(B)) \cup [A_j \setminus a_j])$ for some $j \in \{1, .., M\}$. Therefore, $a_i^k \notin \mathcal{N}(B)$. As $A_m \subseteq B_i^{k-1} \subseteq B$, by definition of $\mathcal{N}(\cdot)$, $\mathcal{N}(A_m) \subseteq \mathcal{N}(B_i^{k-1}) \subseteq \mathcal{N}(B)$. Therefore, $a_i^k \notin \mathcal{N}(A_m)$. The case $r < k \leq N$ is not possible. Indeed, a_i^k is the first element in A_m ranked by R_i. $k > r$ means that A_m is not ranked by R_i, and thus, $A_m \subseteq S_i$. This contradicts the fact that S_i has no subsets in $\mathcal{M}$. We thus have proved that $R_i(A_m) \notin \mathcal{N}(A_m)$ for any $i \in \{1, ..., M\}$.

Check time complexity. Algorithm 1 needs to be run on each observation, each repetition takes $O(M^2)$ time. Thus the total time complexity is $O(M^3)$.

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