COMS BC3262: Introduction to Cryptography

Lecture 6: OWFs and MACs

February 16, 2026

Logistics

Office hours:

- **Eysa**: Mondays 3-5, Milstein 512
- **Mark**: Normally Tuesdays 6:30-8:30, but he is traveling these next two weeks.
    - *No office hours Tuesday, Feb 17 or Tuesday, Feb 24*
    - *Mark's next two office hours are tentatively set for Sunday via Zoom, time TBD*

PS2 is due Thursday

Each late day is 10% off, but we'll cap the late deduction at 30%

Today's Lecture

- One-Way Functions (OWFs)
- Message Authentication Codes (MACs)

One-Way Functions (OWFs)

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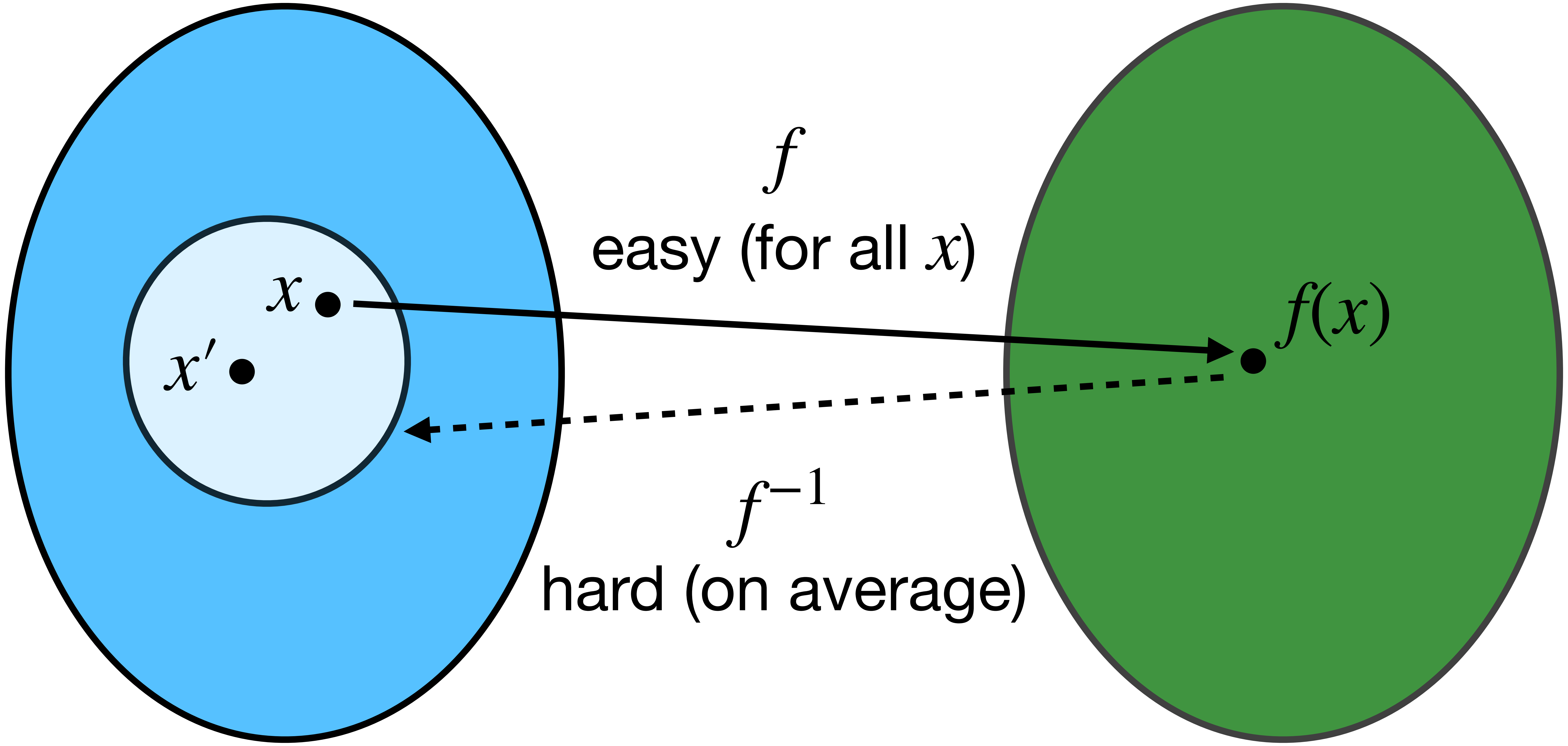
A function $f: \{0,1\}^* \rightarrow \{0,1\}^*$ that is easy to compute but hard to invert



One-Way Functions

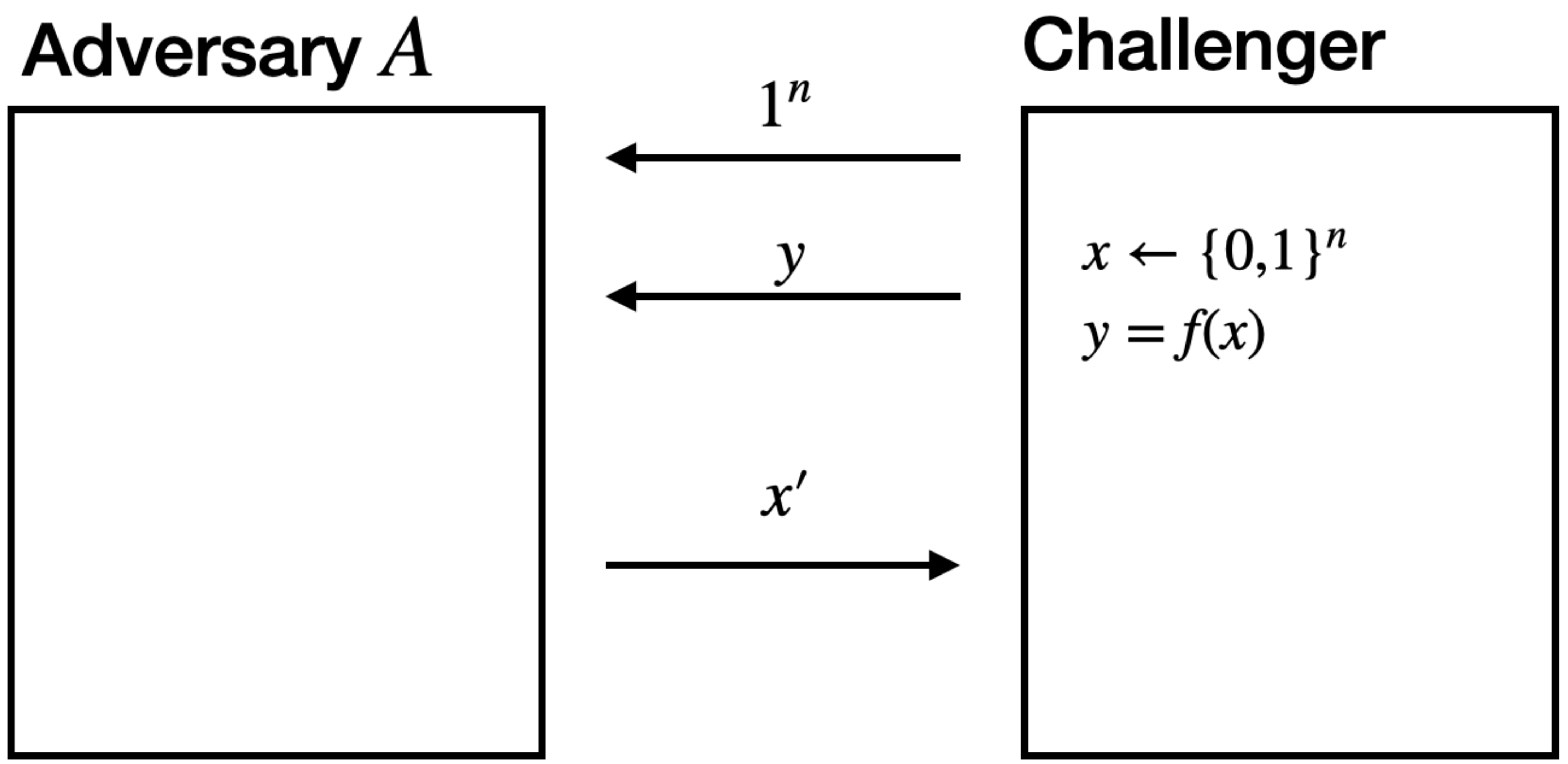
Given $f: \{0,1\}^* \rightarrow \{0,1\}^*$ and an adversary A, consider the experiment $\mathsf{Invert}_{f,A}(n)$:

Definition:

$f: \{0,1\}^* \rightarrow \{0,1\}^*$ is a **one-way function** if f can be computed in polynomial time, and for every PPT adversary A there exists a negligible function $\epsilon(\cdot)$ such that

$$\Pr[\mathsf{Invert}_{f,A}(n) = 1] \leq \epsilon(n)$$

where the probability is taken over the random coins used by A and by the experiment.



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and 0 otherwise

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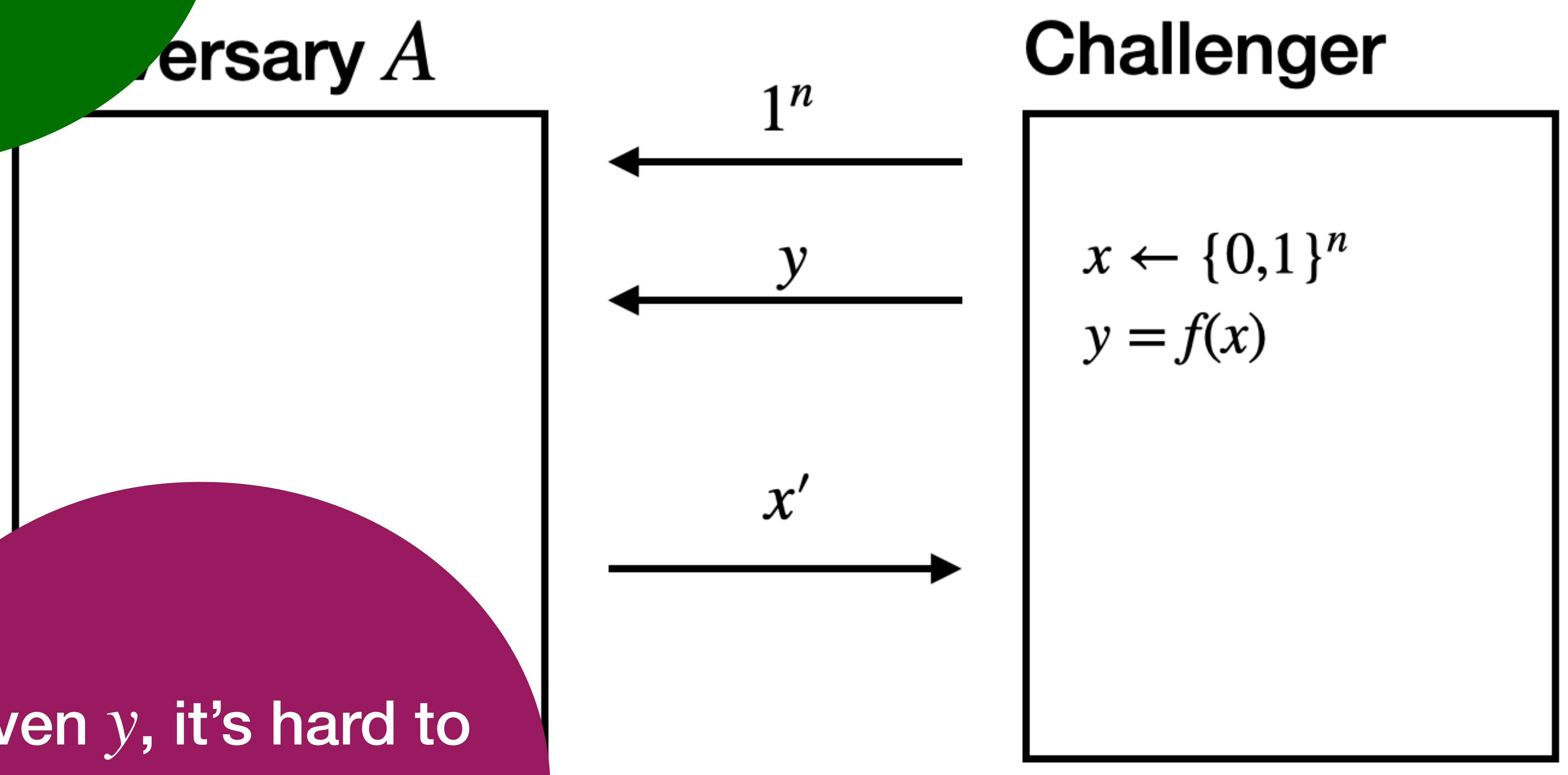


$\mathsf{Invert}_{f,A}(n) = 1$ if $f(x') = y$ and 0 otherwise

Some facts about OWFs

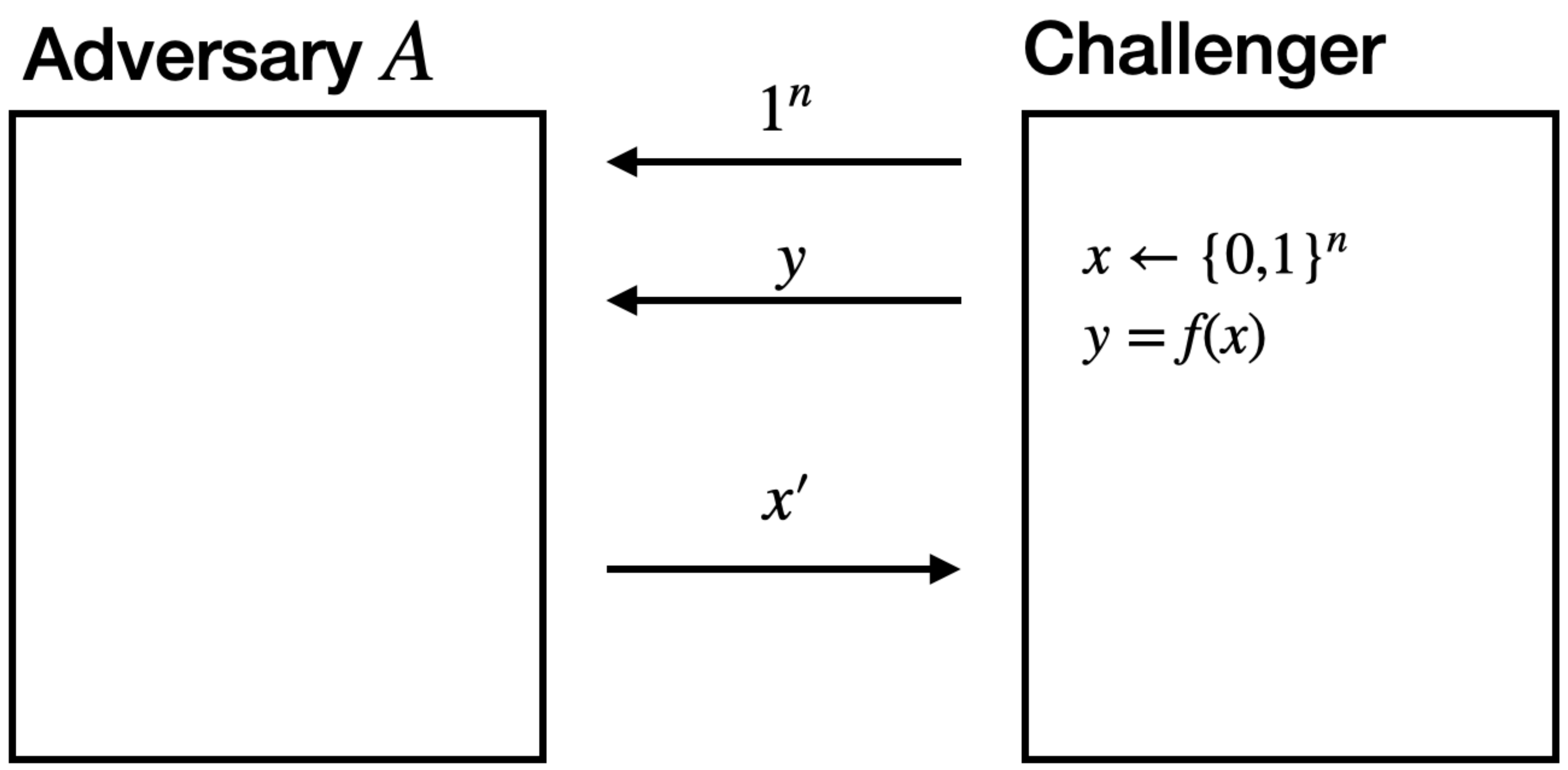
- A OWF does not need to hide all of its input!

- A OWF does not guarantee hardness of inverting *every* input

- If OWF exist, then $P \neq NP$

- OWF implies PRG

- If f is a OWF and a bijection, then f is a one-way permutation (OWP)



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Message Authenticity

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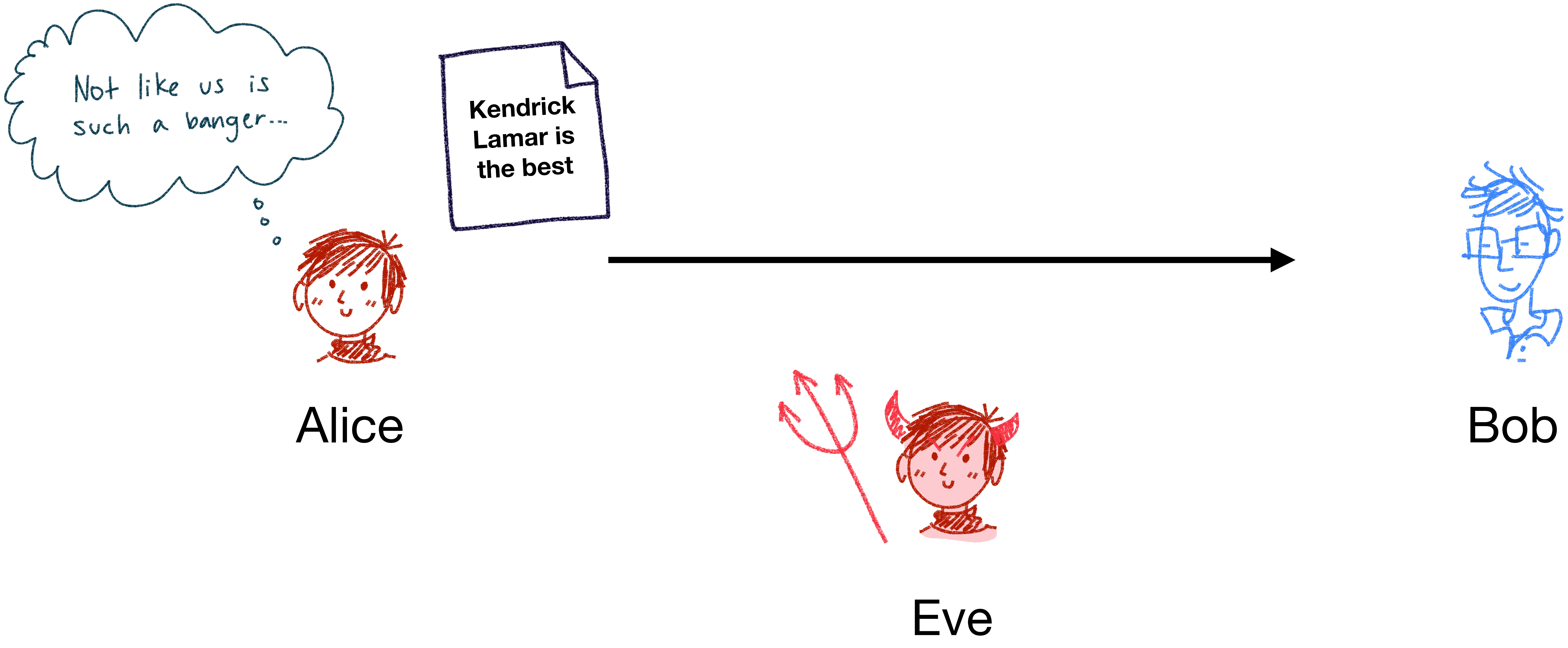

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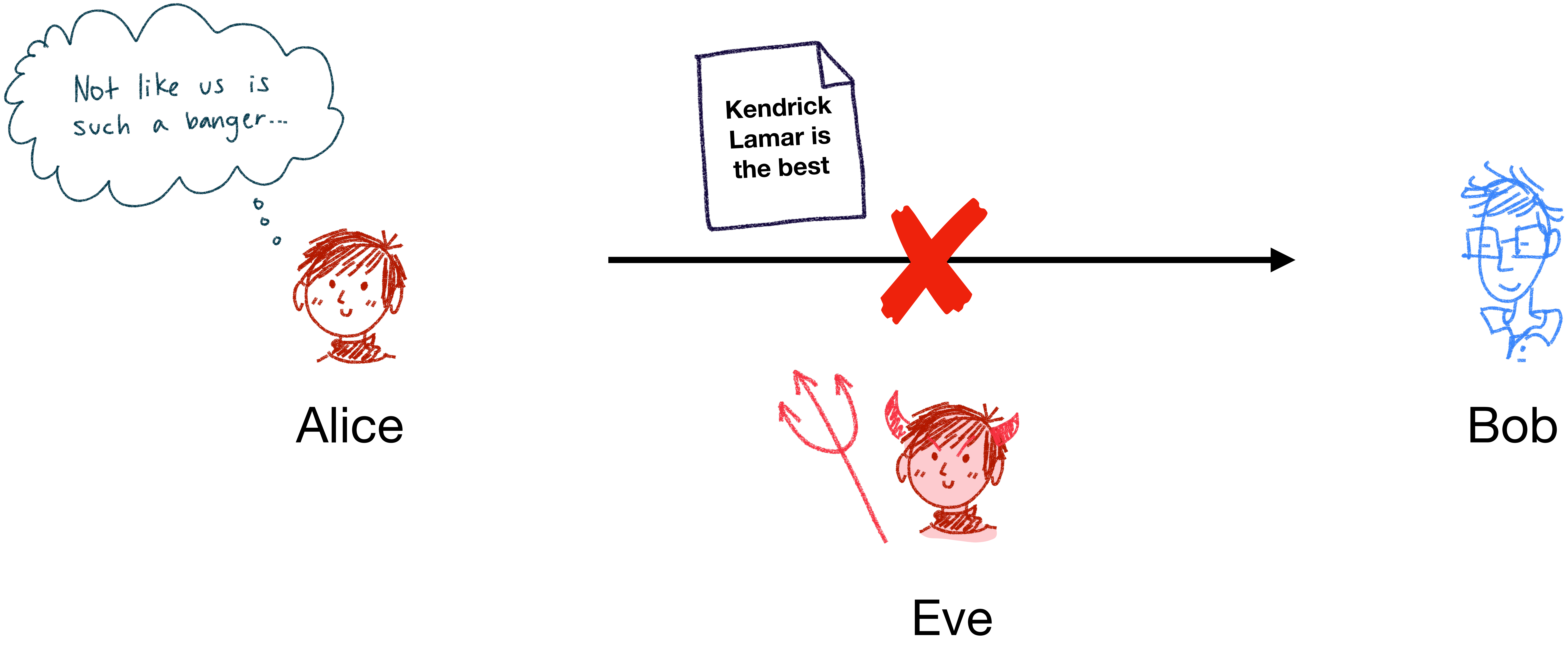

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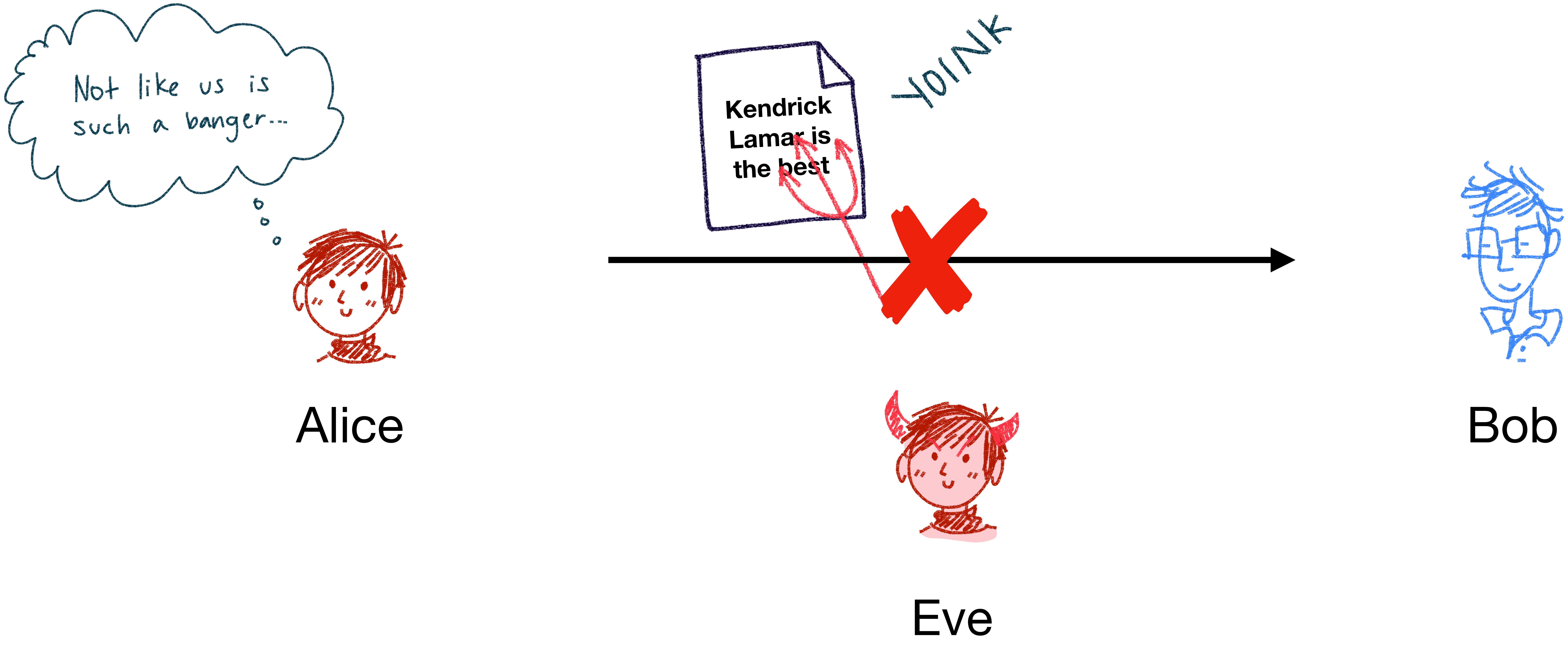

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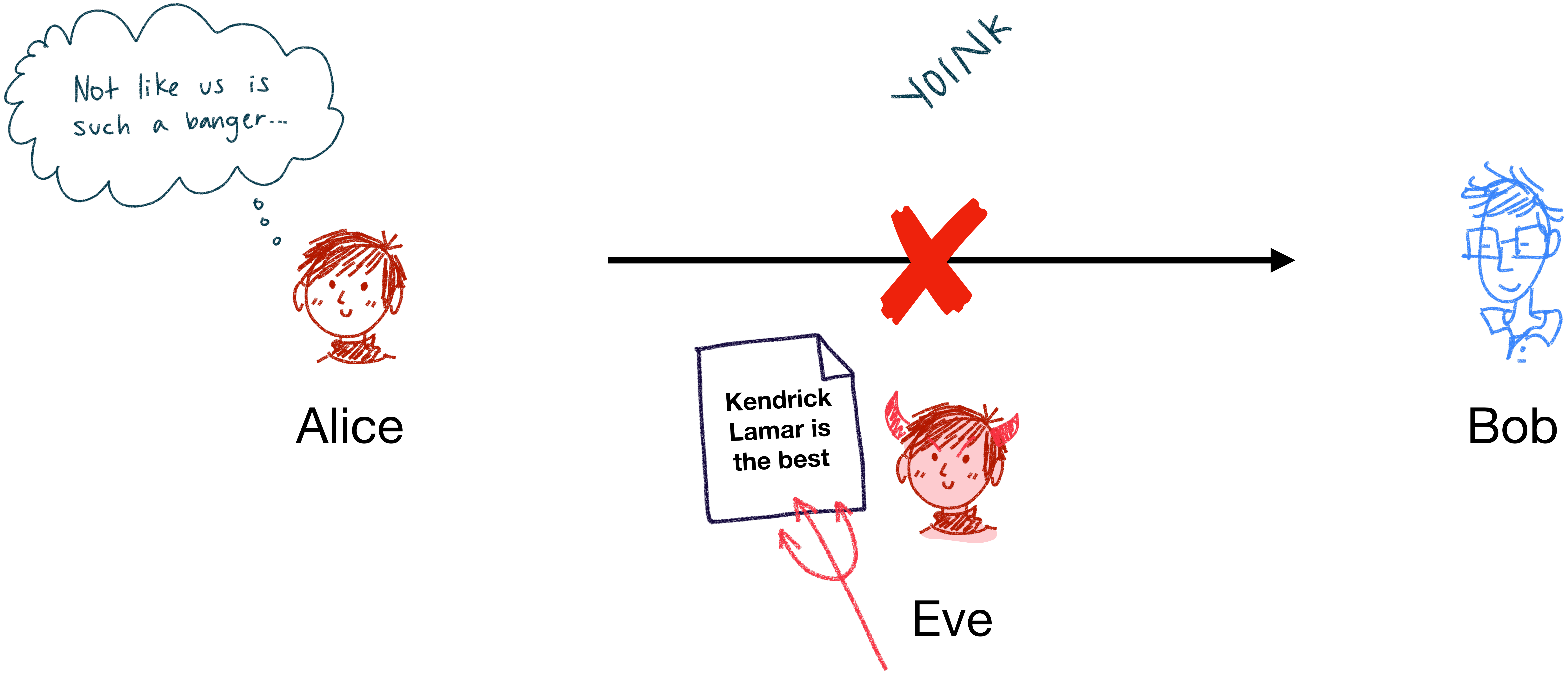

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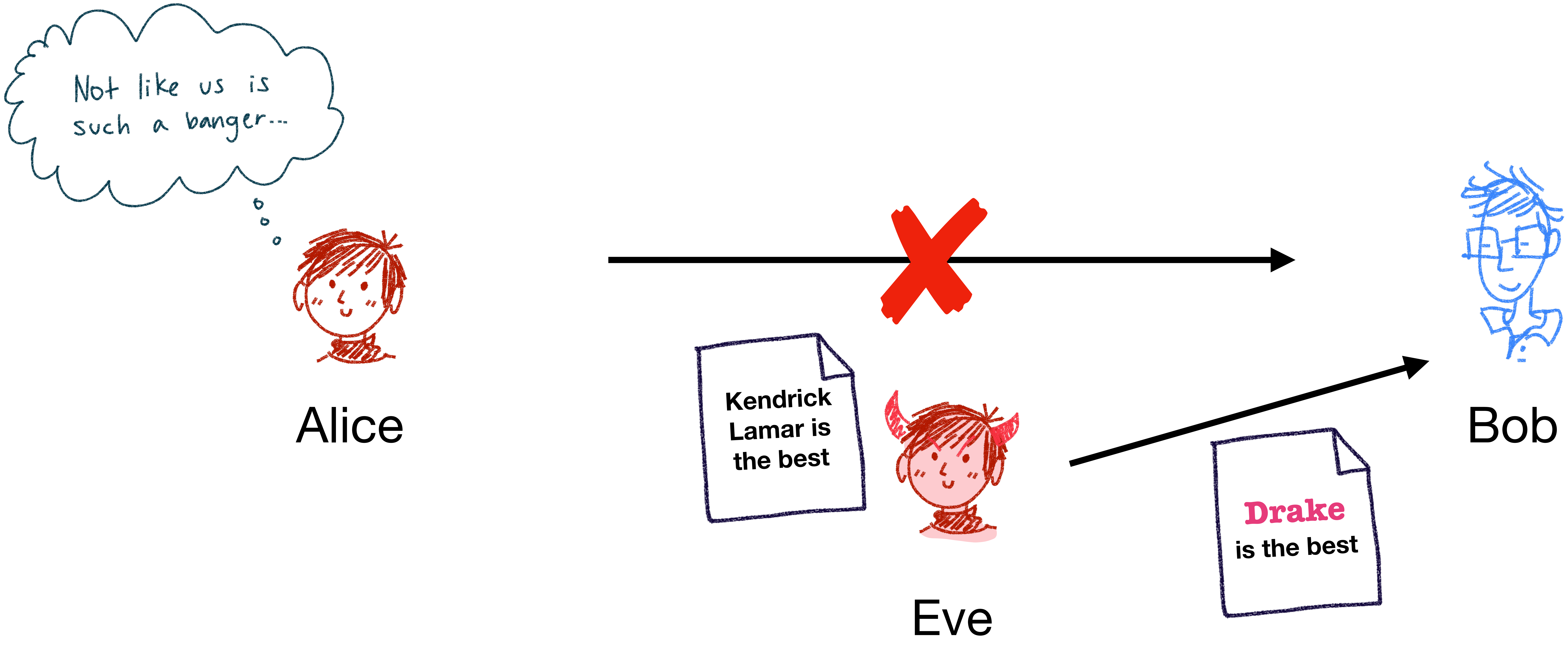
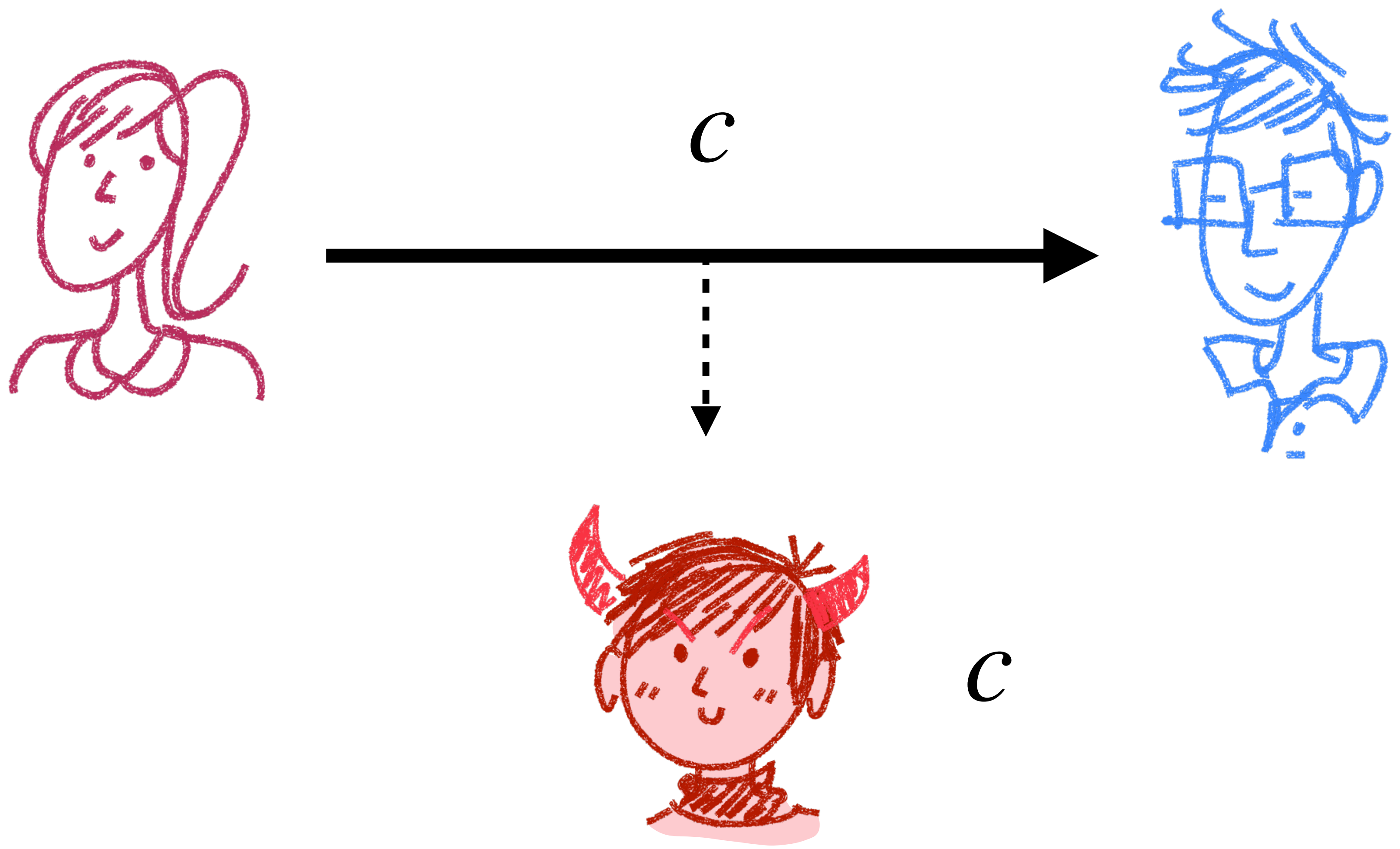


Encryption vs Authentication

Orthogonal aspects; in general, one does not guarantee the other

- **Encryption** focuses on data **secrecy**
  - Hiding the contents of the message from an adversary



- **Authentication** focuses on data **integrity**
  - Assuring a receiver that a message has not been modified

Secrecy vs Authentication

Consider the CPA-secure encryption scheme we saw in Lecture 4:

- Gen(1^n): Sample $k \leftarrow \{0,1\}^n$
- Enc(k, m): On input $m \in \{0,1\}^{\ell_{\text{in}}}$, sample $r \leftarrow \{0,1\}^{\ell_{\text{in}}}$ and output

$$c = (r, F_k(r) \oplus m)$$

- Dec(k, c): On input $c = (c_1, c_2)$, output $F_k(c_1) \oplus c_2$

How might an adversary generate ciphertexts that look like this but do not originate from Alice?

Secrecy vs Authentication

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- Replay attack: send the same ciphertext
  - Unavoidable for any scheme that can copy and resend, usually fixed on an application level with timestamps etc

Message Authentication Codes (MACs)

Syntax: Three algorithms $(\mathsf{Gen}, \mathsf{Mac}, \mathsf{Verify})$

- **Key generation** algorithm Gen takes input 1^n and outputs a key k
- **Tag generation** algorithm Mac takes a key k and a message $m \in \{0,1\}^*$ and outputs a tag $t \in \{0,1\}^*$
- **Verification** algorithm Verify takes a key k, a message m, a tag t, and outputs a bit b

m

$t \leftarrow \mathsf{Mac}_k(m)$

$$\xrightarrow{\quad (m, t) \quad}$$

$b = \mathsf{Verify}_k(m, t)$

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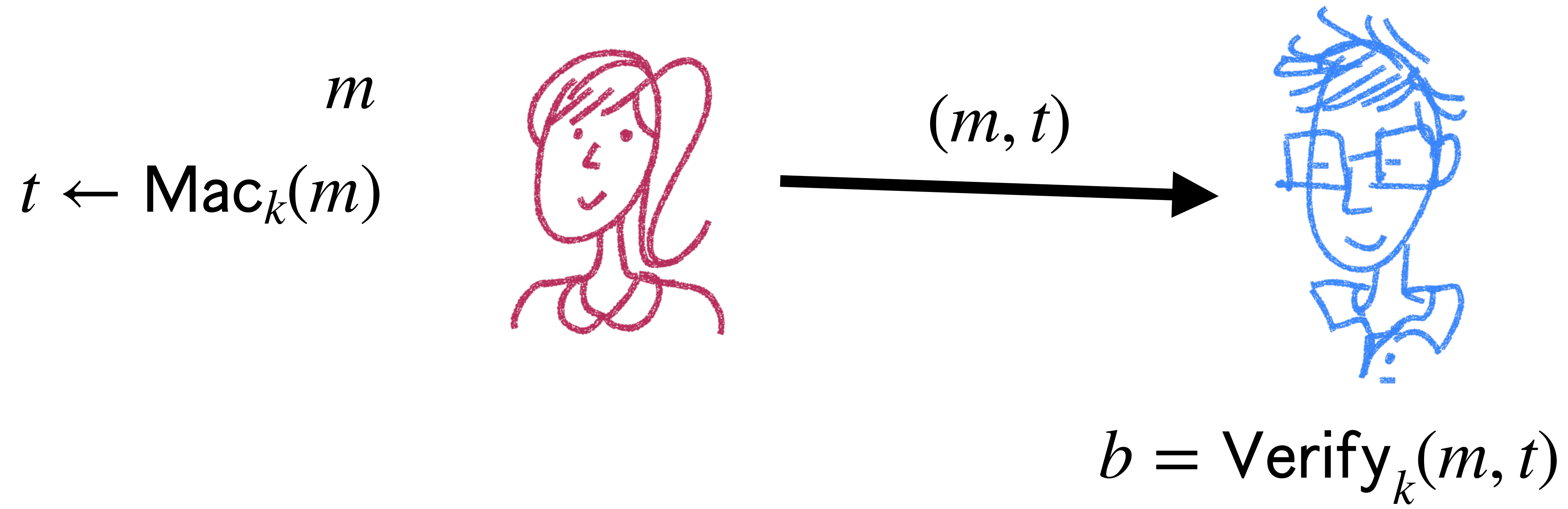
Correctness: $\forall n, \forall k$ output by Gen(1^n), $\forall m \in \{0,1\}^*$, $\forall t$ output by Mac$_k(m)$,

$$\text{Verify}_k(m, t) = 1$$



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Canonical Verification:
If Mac algorithm is deterministic, then Verify just checks if Mac$_k(m) = t$

$b =$ Verify$_k(m, t)$

How should we define security?

m

$t \leftarrow \mathsf{Mac}_k(m)$

Alice

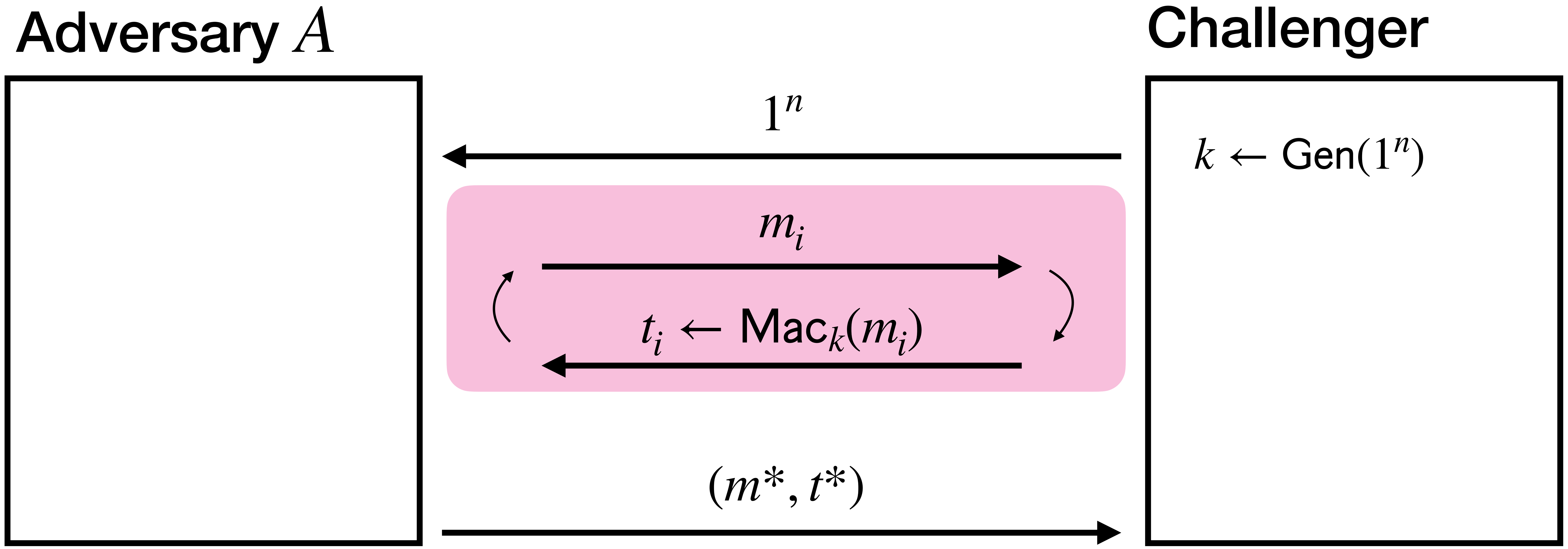
$b = \mathsf{Verify}_k(m, t)$

Bob

Eve

MAC Security

Let $\Pi = (\mathsf{Gen}, \mathsf{Mac}, \mathsf{Verify})$. We define $\mathsf{MacForge}_{\mathscr{A},\Pi}(n)$ as follows



We say the adversary succeeds ($\mathsf{MacForge}_{\mathscr{A},\Pi}(n) = 1$) if:

1. $\mathsf{Verify}_k(m^*, t^*) = 1$
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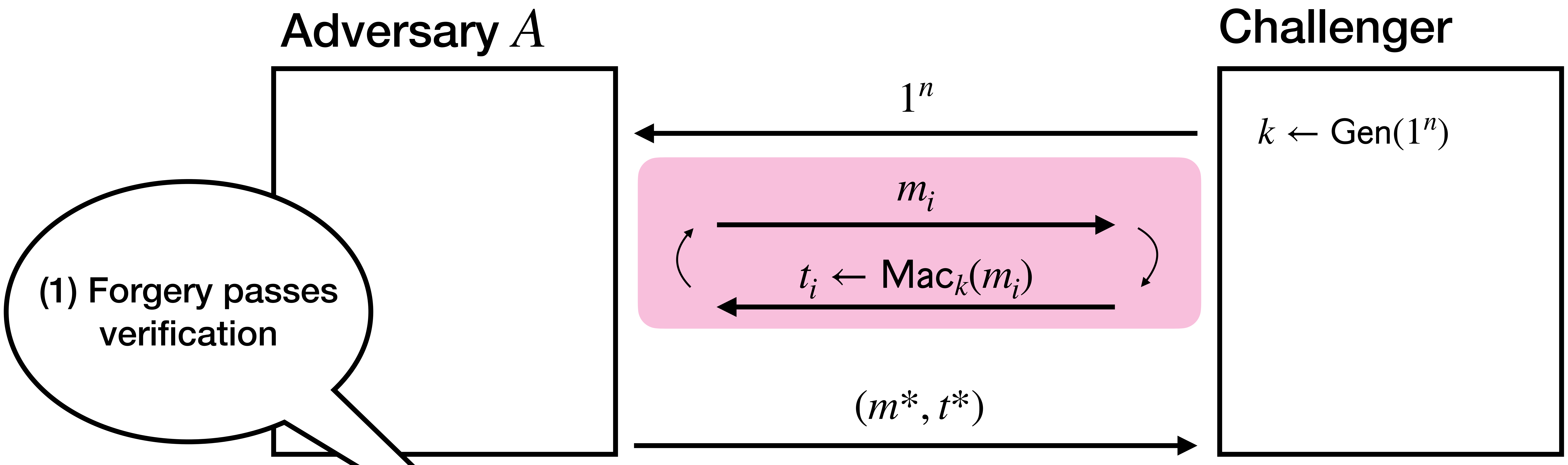

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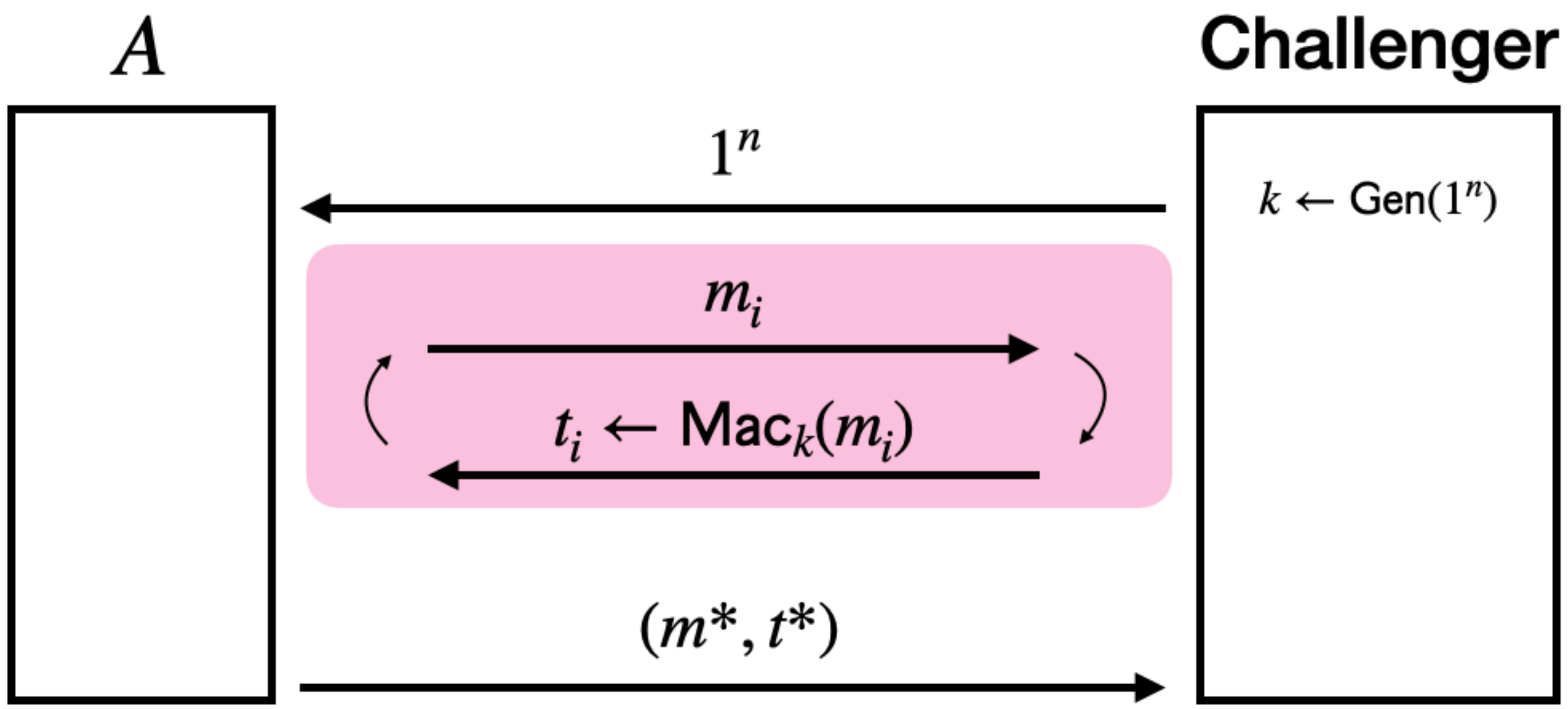
(2) Forgery is on a new message

MAC Security

Definition: A MAC $\Pi = (\mathsf{Gen}, \mathsf{Mac}, \mathsf{Verify})$ is existentially unforgeable under an adaptive chosen-message attack if for every PPT adversary A there exists a negligible function $\mathsf{negl}(\cdot)$ such that

$$\Pr[\mathsf{MacForge}_{\mathscr{A},\Pi}(n) = 1] \leq \mathsf{negl}(n)$$



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And is 0 otherwise

Strong MAC Security

Definition: A MAC $\Pi = (\text{Gen}, \text{Mac}, \text{Verify})$ is strongly secure if for every PPT adversary A there exists a negligible function $\text{negl}(\cdot)$ such that

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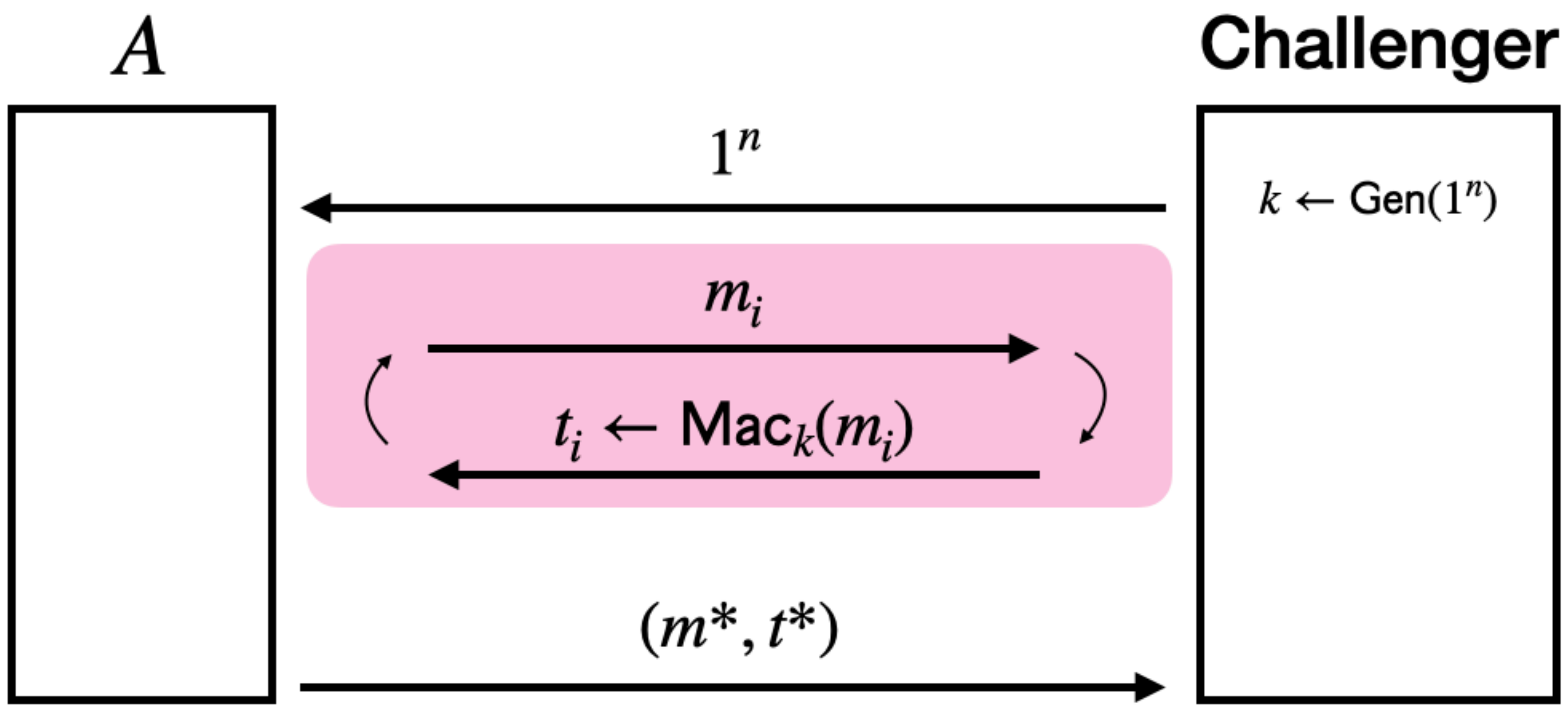

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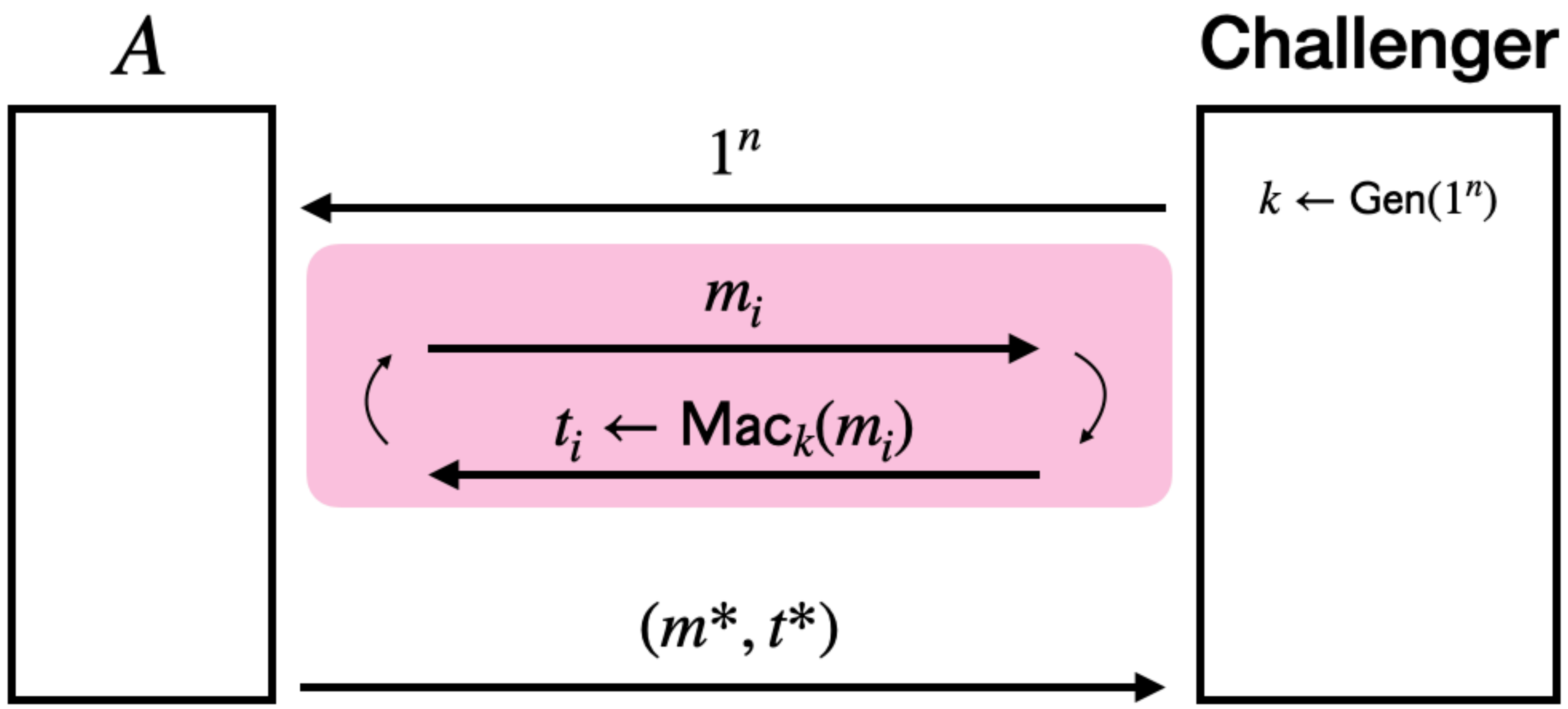

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Theorem: A secure MAC that uses canonical verification is a strong MAC

Fixed-Length MAC

A Fixed-Length MAC

Let $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^n$ be a PRF

We can construct a MAC as follows:

- **Key generation:** On input 1^n, output a randomly sampled $k \leftarrow \{0,1\}^n$
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Theorem: If F is a PRF, then the above MAC scheme is secure for messages of length n

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Proof idea:

- Given a forger for the MAC, construct a distinguisher for the PRF.

- The distinguisher has oracle access to a function which is either a PRF or a truly random function.

- Distinguisher guesses "PRF" if the forger is able to produce a valid forgery. Otherwise, distinguisher guesses "random"

Arbitrary-Length MACs

Authenticating Arbitrary-Length Messages

$$m \quad = \begin{array}{|c|c|c|c|c|c|c|c|} \hline m_1 & m_2 & & \ldots & & \ldots & & m_d \\ \hline \end{array}$$

Suppose we had a $(\mathsf{Gen}, \mathsf{Mac}, \mathsf{Verify})$ for fixed-length messages.

Can we construct a $(\widehat{\mathsf{Gen}}, \widehat{\mathsf{Mac}}, \widehat{\mathsf{Verify}})$ for arbitrary-length messages?

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Idea 1: Authenticate each block on its own

$$\hat{\mathsf{Mac}}_k(m_1 || m_2 || \dots || m_d) = \mathsf{Mac}_k(m_1) || \dots || \mathsf{Mac}_k(m_d)$$

Authenticating Arbitrary-Length Messages

$$m \quad = \quad \boxed{m_1} \boxed{m_2} \boxed{} \boxed{\ldots} \boxed{} \boxed{\ldots} \boxed{} \boxed{m_d}$$

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Can we construct a $(\hat{\mathsf{Gen}}, \hat{\mathsf{Mac}}, \hat{\mathsf{Verify}})$ for arbitrary-length messages?

Idea 2: Authenticate each block on its own with an index?

$$\hat{\mathsf{Mac}}_k(m_1 || m_2 || \ldots || m_d) = \mathsf{Mac}_k(1 || m_1) || \ldots || \mathsf{Mac}_k(d || m_d)$$

Next Time

- Today:
  - OWFs review
  - MACs
- Wednesday
  - Arbitrary-Length MACs
  - CCA-Security