Solve the following recurrence:

$$T(n) = 2\sqrt{n} \cdot T(\lfloor\sqrt{n}\rfloor) + logn$$

Solution: $\Theta(n\ log\ n)$

We observe that we cannot apply directly the Master Theorem to the given recurrence. Towards manipulating the expression so that we can eventually apply the Master Theorem, we perform the following transformation on both the domain and range of the recurrence. Let $n = 2^m$ (i.e., $m = logn$), and $S(m) = T(2^m) = T(n)$. Then:

$$\begin{aligned} S(m) = T(2^m) &= 2 \cdot 2^{\frac{m}{2}} \cdot T(2^{\frac{m}{2}}) + m \\ &= 2 \cdot 2^{\frac{m}{2}} \cdot S(\frac{m}{2}) + m \end{aligned}$$

At this point we can still not apply the Master Theorem as the number of subproblems of size $\frac{m}{2}$ that arise is a function of m (in particular $2 \cdot 2^{\frac{m}{2}}$ problems arise). We can apply another transformation, however, so as to get an expresion where the number of subproblems is constant. To achieve this, multiply both sides with 2^{-m} and get:

$$2^{-m} \cdot S(m) = 2 \cdot 2^{\frac{-m}{2}} \cdot S(\frac{m}{2}) + m \cdot 2^{-m}$$

Then, define $R(m) = 2^{-m} \cdot S(m)$ and notice that:

$$\begin{aligned} R(m) = 2^{-m} \cdot S(m) &= 2 \cdot 2^{\frac{-m}{2}} \cdot S(\frac{m}{2}) + m \cdot 2^{-m} \Rightarrow \\ R(m) &= 2 \cdot R(\frac{m}{2}) + m \cdot 2^{-m} \qquad [Eq.1] \end{aligned}$$

This new recurrence, $R(m)$, can be solved using the general version of the Master's Theorem (which is also available on the CLRS book), and which specifies the following:

> Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence
>
> $$T(n) = aT(n/b) + f(n).$$
>
> Then, if $x = log_b a$, $T(n)$ can be bounded asymptotically as follows:

1: If $f(n) = O(n^{x-\epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^x)$.

2: If $f(n) = \Theta(n^x)$, then $T(n) = \Theta(n^x lgn)$.

3: If $f(n) = \Omega(n^{x+\epsilon})$ for some $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

So in the case of Equation 1, $f(n)$ is a function of m: $g(m) = m \cdot 2^{-m}$. Furthermore, $x = log_2(2) = 1$. We can show that $g(m) = O(n^{x-\epsilon})$, as $g(m)$ is linear in m times an exponentially diminishing factor of $\frac{1}{2^m}$, meaning we are in Case 1 of the Master's Theorem. This implies that $R(m) = \Theta(m^1) = \Theta(m)$ from the application of the general version of the Master Theorem.

To finish this problem, we transform it back into terms of n. Undoing the range transformation yields $S(m) = 2^m \cdot R(m) = \Theta(m \cdot 2^m)$. Undoing the domain transformation yields: $T(n) = T(2^m) = S(m) = \Theta(m \cdot 2^m) = \Theta(n\ log\ n)$.