9 Dynamic Programming

9.4 Option Pricing in Mathematical Finance (6 units)

This project is connected with material in the Stochastic Financial Models course. Students who are not taking that course but who wish to attempt the project will find the necessary definitions and background material in the references.

1 Black-Scholes model

A standard model used in option pricing is that the logarithm of the stock price follows a Brownian motion. Hence, if S_t is the stock price at time t, we assume that $\log(S_t/S_0)$ is normally distributed with mean μt and variance $\sigma^2 t$, where σ is the volatility, $\mu = \rho - \sigma^2/2$, and ρ is the continuously-compounded riskless interest rate.

The celebrated Black-Scholes formula gives the price of a call option (exercised only at expiry). The price of the option is

$$S_0\Phi\Big(\frac{\log(S_0/c) + (\rho + \sigma^2/2)t_0}{\sigma\sqrt{t_0}}\Big) - ce^{-\rho t_0}\Phi\Big(\frac{\log(S_0/c) + (\rho - \sigma^2/2)t_0}{\sigma\sqrt{t_0}}\Big) \tag{1}$$

where c is the strike price and t_0 is the expiry time. (See the Appendix for details of how to calculate Φ.)

Question 1 Write a routine to evaluate the Black-Scholes price (1). Compile a table of the price when $c = 40$, $S_0 = 52, 100$ or 107, $\sigma = 0.5$, $\rho = 0.035$, and $t_0 = 2$ or 3.

Question 2 How does the price vary with each of the parameters c, S_0, σ, ρ, t_0? Keep your explanations brief, but support them with solid mathematics where necessary.

2 Bernoulli approximation

The most widely-used method for approximating option prices which are based on the Black-Scholes model is to replace the Brownian motion by a discrete-time simple random walk. This approximation breaks up the interval $[0, t_0]$ into $[0, t_0/n, 2t_0/n, \ldots, (n-1)t_0/n, t_0]$ and assumes that between the times it_0/n and $(i+1)t_0/n$, $i = 0, \ldots, n-1$, the increment in the logarithm of the price is g or $-g$ with probability p or $1-p$ respectively, where g and p are chosen so that the increment has mean $\mu t_0/n$ and variance $\sigma^2 t_0/n$.

This approximation is primarily of interest for cases such as the American put where exact formulae are not available. For a European (or American) call option, the price obtained from the random walk will approximate the true price obtained from the Black-Scholes formula, and this can be a useful benchmark to judge the performance of the approximation.

One way to implement the approximation is to set

$$V_{i,j} = \big(pV_{i+1,j+1} + (1-p)V_{i+1,j}\big)e^{-\rho t_0/n} \quad \text{for } j = 0, \ldots, i \text{ and } i = n-1, \ldots, 0 \tag{2}$$

with boundary conditions

$$V_{n,j} = \Big(S_0 e^{(2j-n)g} - c\Big)^+ \quad \text{for } j = 0, \ldots, n. \tag{3}$$

Then $V_{0,0}$ is the approximate price.

Question 3 Calculate g and p as functions of the parameters.

Question 4 For the data in Question 1 and $n = 27$, compile a table of the approximate prices. How do they compare to the prices obtained from the Black-Scholes formula?

Question 5 What is the complexity of this algorithm, as a function of n?

Question 6 Consider an at-the-money case ($c = S_0$). Plot a graph of your approximation as a function of n. Indicate on your graph the true value obtained from the Black-Scholes formula. What do you notice? Explain this behaviour.

Question 7 Estimate the rate at which the error decreases as n increases. Explain every step and justify your answers.

3 American Put

Consider the case of an American put; now early exercise of the option may be optimal and no closed-form formula exists for the price.

Question 8 Modify your programs to approximate the price of the option by considering how equations (2) and (3) should be changed in this situation. Compile a table of the approximate price of the option for the same values of the parameters used in Question 1. Comment briefly on how the approximate price varies with n in this case.

4 Extrapolation

Suppose that f_n is the approximation to the option price, and we wish to find the limiting value of f_n as $n \to \infty$. One method of extrapolation assumes that f_n may be approximated by a polynomial in $1/n$:

$$f_n \approx g_0 + g_1 n^{-1} + g_2 n^{-2} + \cdots + g_s n^{-s}.$$

The limiting value of f_n is then approximated by g_0. One way to achieve this is as follows. Let $n_m = r^m n_0$ and calculate f_n at $n = n_0, \ldots, n_s$. Set

$$a_{m,0} = f_{n_m} \quad \text{for } m = 0, \ldots, s$$

and recursively calculate

$$a_{m,i} = a_{m,i-1} + \frac{a_{m,i-1} - a_{m-1,i-1}}{r^i - 1} \quad \text{for } m = i, \ldots, s \text{ and } i = 1, \ldots, s.$$

Then $a_{s,s}$ is taken as the approximation for g_0.

Question 9 Experiment with this extrapolation procedure for small values of r and s, say 2 to 4, for the at-the-money European call case studied above. How does this extrapolation compare in accuracy with just calculating f_n for a single suitably large value of n? Try to estimate the error analytically. Does your answer depend on whether n_0 is odd or even? If so, carefully explain why.

5 Binomial approximation

The approximation in Section 2 uses a Bernoulli (two-valued) distribution between time steps. The method may be refined by replacing the Bernoulli distribution between times it_0/n and $(i+1)t_0/n$ by, say, a binomial distribution taking $k+1$ equally-spaced values (for some $k \geqslant 1$) with mean and variance chosen to match those of the Brownian motion.

> **Question 10** Implement this refinement, and explain how you calculate p and g in this case. How do prices produced by the refined algorithm ($k > 1$) differ from those produced by the Bernoulli scheme ($k = 1$)? Does your answer depend on whether you are pricing the European call or American put option, and if so why? How does the computation time for this algorithm vary with n and k?

Appendix

An easy method to approximate the standard normal distribution function is as follows. For $x \geqslant 0$ set

$$1 - \Phi(x) = \frac{t}{2} \exp\Big(-\frac{x^2}{2} + \sum_{i=0}^{9} a_i t^i\Big)$$

where $t = (1 + x/\sqrt{8})^{-1}$ and

$$\begin{aligned}(a_0, \ldots, a_9) = (&-1.26551223, 1.00002368, 0.37409196, \\ &0.09678418, -0.18628806, 0.27886807, \\ &-1.13520398, 1.48851587, -0.82215223, 0.17087277).\end{aligned}$$

For $x < 0$ set $\Phi(x) = 1 - \Phi(-x)$.

References

[1] J. Hull, *Options, Futures and Other Derivative Securities*. Prentice-Hall, 1989.