A Recipe for Implementing The Delta-Normal Approach to Parametric VaR using Mapping

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Abstract

This note gives a recipe for how to implement the so-called Delta-Normal Approach to Parametric VaR for stocks, bonds, and financial derivatives like futures and options. First, in the introduction below, the overall idea of mapping is described. Then, in Section 2, a general recipe for how VaR can be calculated is given. In the following sections, some practical examples are given for specific types of assets, following the recipe in Section 2.

Since the recipe have to apply to all situations, it is quite general, and if you find the introduction a bit too abstract, you might go directly to the recipe and the examples.

1. Introduction

Assume that your overall financial wealth W at the date of the analysis t is equal to 5 million, $W_t = 5$ million. You are interested in knowing how the value of your wealth will change over the next time period.[1] If the value drops to W_{t+1} = 4.5 million, your gain is $W_{t+1} - W_t = \Delta W_{t+1} = -0.5$ million.[2]

The overall wealth might distributed on a large number I of assets $[x^1, x^2, \ldots, x^I]$. Asset 1 might be a stock with a market price $p^1 = 120$, and if you own $n^1 = 10$ stocks, your wealth in asset 1 is $x^1 = 1200$. What we are interested in here, is the changes in the values of the asset position, since the sum will give us the changes in the overall portfolio:

$$\Delta W = \Delta x^1 + \Delta x^2 + \cdots + \Delta x^I \tag{1}$$

It is not very convenient to work with changes in such absolute changes. For instance, you cannot get historical data in this format. It is much more convenient to deal with returns, $[r^1, r^2, \ldots r^I]$. They are independent of the size of the portfolio and historical observations can be found for almost all assets. We will very often use the following transformation from absolute changes to returns (here for asset 1):

[1] We will usually think of a time period as a day or a week.
[2] In order to simplify the notation, I will often not use the time subscript.

$$\begin{aligned}\Delta x^1 &= x^1 \frac{\Delta x^1}{x_t} \\ &= x^1 r^1.\end{aligned}$$

As in the example above, you could view a portfolio position $x^1 = 1200$ as 10 individual stocks multiplied by the price of 120: $x^1 = n^1 p^1$, but that would not change anything:

$$\begin{aligned}\Delta x^1 &= \Delta n^1 p^1 \\ &= n^1 \Delta p^1 \quad (\ n \text{ is fixed, only } p \text{ changes}) \\ &= n^1 p_t \frac{\Delta p^1}{p_t} \\ &= x^1 r^1.\end{aligned}$$

Hence, in any case, we can rewrite (1) to

(2) $$\Delta W = x^1 r^1 + x^2 r^2 + \cdots + x^I r^I$$

In portfolio theory, you would often divide Equation (2) with W to get an expression for the overall portfolio return, $r^W = \Delta W / W$:

$$\begin{aligned}\frac{\Delta W}{W} &= \frac{x^1}{W} r^1 + \frac{x^2}{W} r^2 + \cdots + \frac{x^I}{W} r^I \\ &\Downarrow \\ r^W &= \chi^1 r^1 + \chi^2 r^2 + \cdots + \chi^I r^I\end{aligned}$$

where the curly x's are portfolio weights that sums to one: $\sum \chi^i = 1$. This is a classical portfolio equation. If we want to calculate the variance of the portfolio return $\sigma^2(r)$, we "just" need to estimate the covariance of the returns, $r^1, r^2, \cdots$, and then do a few multiplications.

The problem in risk management is, that the number of assets I in the portfolio can be very high, and we would like to reduce the size of the problem by choosing a relative low number $N < I$ of key risk factors $RF = [r^a, r^b, \ldots, r^N]$ who influences the wealth changes ΔW via the exposure positions $[w^a, w^b, \ldots, w^N]$ in the different risk factors, such that we can replace Equation (2) with:

(3) $$\Delta W = w^a r^a + w^a r^a + \ldots w^N r^N$$

Typical choices of factors are "return of a stock index", "return of a zero-coupon bond with 1 year maturity", and so on.[3] Hence, if the stock index has a return equal to $r^a = 0.025$, or 2.5%, and the exposure to this risk factor is w^a = 1 million, the contribution from this stock index to ΔW is $w^a r^a$= 25,000.

Again, we would like to divide Equation (3) with W to get an expression for the portfolio return:

[3] In an exam situation, the these risk factors will usually be determined by the exam text.

$$
\begin{aligned}
\frac{\Delta W}{W} &= \frac{w^a}{W} r^a + \frac{w^a}{W} r^a + \dots \frac{w^N}{W} r^N \\
&\Downarrow \\
r^W &= \omega^a r^a + \omega^a r^a + \dots \omega^N r^N
\end{aligned} \tag{4}
$$

where the curly w's sum to one $\sum \omega^n = 1$. Once we have Equation (4), we can calculate portfolio return variance, standard deviations, VaR figures etc as before. We only need to determine the covariance matrix of the new returns instead.

The big question is, of course, how to come from the original assets in (2) to factors in (3). It is not straight forward to do, and the reason why this note was written. However, it's the same tricks you use each time, and this note will show you how to do, for a number of assets.

I think the best way to think about the problem, is to take one of the assets in the portfolio at a time, and start with an an equation like this one: (we use asset 1 as an example here)

$$
\Delta W = x^1 r^1 \tag{5}
$$

and then try to re-write the right hand side, until the return of the asset r^1 is replaced by the returns of some of the factors $r^a, r^b, \dots r^N$. The exposures of each risk factor from asset one, is then the expression in front of the return.

MAPPING EXAMPLE, STOCK TO INDEX: Let's consider a typical stock, say Pepsico Inc, and let's assume that Pepsi has a beta-value with the market return r^m equal to 0.75. Moreover let's assume that we own for 1 million worth of stocks in Pepsi. Finally, we will assume, as an approximation based on the CAPM, that $r^{pep} = \beta^{pep} r^m$. Then, let's elaborate on (5) to see if we can arrive at an equation with r^m instead of r^{pep}:

$$
\begin{aligned}
\Delta W &= x^{pep} r^{pep} \\
&= 1,000,000 r^{pep} \\
&= 1,000,000 \beta^{pep} r^m \quad \text{using the CAPM result} \\
&= 1,000,000 \cdot 0.75 r^m \\
&= \underbrace{750,000}_{w^m} r^m
\end{aligned}
$$

Hence, the stock position was mapped into the market risk factor r^m with an exposure equal to $w^m = 750,000$.

2. THE VaR RECIPE

Here comes the recipe. It is general and perhaps a bit ”long” for simple examples. In return, the recipe can be used for most situations.

1. CHOOSING RISK FACTORS: Choose a number of risk factors $[r^a, r^b, \ldots, r^N]$ which will describe the changes in your wealth ΔW according to Equation (3) via some exposures $[w^a, w^b, \ldots, w^N]$ which will be decided my the mapping step below.

2. THE COVARIANCE MATRIX: Calculate the covariance matrix S for the risk factors, usually based on some historical observations for the returns $r^a, r^b \cdots r^N$.

3. MAPPING: Map the asset positions in the portfolio $[x^1, x^2, \ldots, x^I]$ into the risk factors as *exposures* $[w^a, w^b, \ldots, w^N]$.

4. RELATIVE EXPOSURES: Calculate the *relative exposures* $\omega = [\omega^a, \omega^b, \ldots, \omega^N]$ simply by dividing each of the exposures of the risk factors $[w^a, w^b, \ldots, w^N]$ with the overall wealth W:

$$\omega = \begin{bmatrix} \omega^a \\ \omega^b \\ \vdots \\ \omega^N \end{bmatrix} = \begin{bmatrix} w^a/W \\ w^b/W \\ \vdots \\ w^N/W \end{bmatrix}$$

   Hence, if the exposure position w^a in the stock index is 1 million and the overall wealth is $W = 5$ million, the relative exposure is $\omega^a = 0.2$, or 20%.

5. PORTFOLIO RETURN VARIANCE: Based on Equation (4) we can calculate the variance of the overall wealth return r^W using the relative exposures ω, the covariance matrix S, and the matrix multiplication formula:

$$\begin{aligned} \sigma^2(r^W) &= \omega^\top \cdot S \cdot \omega \\ &= \begin{bmatrix} \omega^a \ldots \omega^N \end{bmatrix} \begin{bmatrix} \sigma^2(r^a) & \ldots & \mathrm{cov}(r^a, r^N) \\ \vdots & \ddots & \\ \mathrm{cov}(r^N, r^a) & & \sigma^2(r^N, r^N) \end{bmatrix} \begin{bmatrix} \omega^a \\ \vdots \\ \omega^N \end{bmatrix} \end{aligned} \tag{6}$$

6. VaR CALCULATION: From the the expression of variance of the portfolio return $\sigma^2(r^W)$, the standard deviation $\sigma(r^W)$, the *Value-at-Risk* (VaR), and the *Expected Tail Loss* (ELT) are easily calculated for the portfolio returns. If the figures are wanted in amounts, just multiply with W.

3. ONE STOCK

We will start out with the simplest of problems: The portfolio held is one type of stock, Pepsico Inc, and we will choose the Pepsi returns as the risk factor also. Hence, no real mapping. Assume that the overall wealth is W = \$ 5 million, the position in the stock is obviously also x^{pep} = \$ 5 million. (If we had the price p^{pep}, we would also be able to calculate the number of stocks we have n^{pep} using the relation $x^{pep} = n^{pep}p^{pep}$, but that is not important in the following.)

Let's try to follow the recipe from above using an American investor's perspective, i.e. all figures in US dollars:

1. CHOOSING RISK FACTORS: Only one risk factor $I = 1$, and it is r^{pep}.

2. THE COVARIANCE MATRIX: We only need the variance of r^{pep}. If we work with weekly observations, the standard deviation $\sigma(r^{pep})$ might be estimated to 0.02, or 2% from old return observations, and the variance σ^2 will be 0.02^2 = 0.0004. Hence, $S = 0.0004$.

3. MAPPING: Since the asset return and the risk factor are identical, the exposures are also the same: $x^{pep} = w^{pep} = 5,000,000$

4. RELATIVE EXPOSURES: Since we only have one stock in the portfolio, and since the asset is equal to the risk factor, both exposure and wealth is 5 million, and we get $\omega^{pep} = 5,000,000/5,000,000 = 1$. In other words, the returns on the portfolio and Pepsi are identical, and Equation (6) becomes.

$$
\begin{aligned}
r^W &= \omega^{pep} r^{pep} \\
&= r^{pep}
\end{aligned}
$$

5. PORTFOLIO RETURN VARIANCE: Again quite simple: $\omega S \omega = S = \sigma^2(r^{pep}) = 0.0004$

6. VAR CALCULATION: Portfolio standard deviation is $0.0004 = 0.02$, or 2%. The 5% VaR is $-1.64 \cdot 2\%$, and the ETL $2.06 \cdot 2\%$. With a total wealth of 5,000,000, the corresponding amounts for standard deviation, VaR, and ETL are: 100,000, -164,485, and 206,271.

4. Two stocks, no mapping

Assume that you invest your wealth W = 5,000,000 with $x^{pep} = 2,000,000$ in Pepsi and $x^{coc} = 3,000,000$ in Coca cola. Moreover, we will keep the two stocks as our risk factors. All figures in $.

1. Choosing Risk Factors: We will make no mappings in this example, and the risk factors are then:

$$RF = \begin{bmatrix} r^{pep} \\ r^{coc} \end{bmatrix}$$

2. The Covariance Matrix: Assume that the two stocks each have a weekly standard deviation equal to 0.02, or 2%, and a zero correlation/covariance. Hence,

$$S = \begin{bmatrix} 0.0004 & 0 \\ 0 & 0.0004 \end{bmatrix}$$

3. Mapping: No real mapping. Hence, $w = x$ and

$$w = \begin{bmatrix} w^{pep} \\ w^{coc} \end{bmatrix} = \begin{bmatrix} 2,000,000 \\ 3,000,000 \end{bmatrix}.$$

4. Relative Exposures: The relative exposures are

$$\omega = w/W = \begin{bmatrix} w^{pep}/W \\ w^{coc}/W \end{bmatrix} = \begin{bmatrix} \omega^{pep} \\ \omega^{coc} \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}.$$

5. Portfolio Return Variance: Using the standard formula we get $\sigma^2(r^W) = \omega^\top S\omega = 0.000144$.

6. VaR calculation: Standard procedure.

5. Stocks with Mapping

Assume same portfolio as in Section 4, but this time, the risk of the two stocks are mapped to an index. All calculations in $.

1. Choosing Risk Factors: The stocks are mapped to only one index:

$$RF = r^m.$$

2. The Covariance Matrix: The matrix is only one number. Assume that the weekly variance of the risk factor is estimated to $S = 0.000156$, or a standard deviation equal to 1.25%.

3. Mapping: Assume that the covariance between Pepsi and the index is 0.0001170 and for Coca Cola 0.0001950. Based on the CAPM and the formula $\beta^{pep} = \operatorname{cov}(r^{pep}, r^m)/\operatorname{var}(r^m)$, we get $\beta^{pep} = 0.75$ and $\beta^{coc} = 1.25$. Hence, $w^{pep} = x^{pep}\beta^{pep}$, and

$$\begin{aligned} w = w^m &= x^{pep}\beta^{pep} + x^{coc}\beta^{coc} \\ &= 2,000,000 \cdot 0.75 + 3,000,000 \cdot 1.25 \\ &= 1,500,000 + 3,750,000 \\ &= 5,250,000 \end{aligned}$$

4. Relative Exposures: There is only one relative exposure, as there is only one risk factor,

$$\omega = w/W = 5,250,000/5,000,000 = 1.05$$

5. Portfolio Return Variance: Using the standard formula we get $\sigma^2(r^W) = \omega^\top S\omega = 1.05 \cdot 0.000156 \cdot 1.05 = 0.000172$.

6. VaR calculation: Standard procedure.

6. Stocks and Zero Coupon

In this example, we assume a portfolio of 1,500,000 invested in Pepsi and 2,000,000 in Coca cola. In addition, $x^{5y} = 1,500,000$ is invested in a US five year zero coupon bond. All figures in $

1. Choosing Risk Factors: In this example, we will map the risk of the two stocks into the same index as in the example of Section 5. The zero coupon bond will not be mapped. We will use the bonds own return r^{5y} as risk factor. Hence, the risk factors are

$$RF = \begin{bmatrix} r^m \\ r^{5y} \end{bmatrix}.$$

2. The Covariance Matrix: The covariance S will be a 2×2 matrix estimated from historical observations for the risk factors.

$$S = \begin{bmatrix} \operatorname{var}(r^m) & \operatorname{cov}(r^m, r^{5y}) \\ \operatorname{cov}(r^{5y}, r^m) & \operatorname{var}(r^{5y}) \end{bmatrix}.$$

3. Mapping: The mapping of the two stocks is almost as before, except that the invested amounts x have changed:

$$\begin{aligned} w^m &= w^{pep} + w^{coc} = x^{pep}\beta^{pep} + x^{coc}\beta^{coc} \\ &= 1,500,000 \cdot 0.75 + 2,000,000 \cdot 1.25 \\ &= 3,625,000 \end{aligned}$$

As the bond is not mapped, we get $w^{5y} = x^{5y} = 1,500,000$. The two exposures are therefore

$$w = \begin{bmatrix} w^m \\ w^{5y} \end{bmatrix} = \begin{bmatrix} 3,625,000 \\ 1,500,000 \end{bmatrix}.$$

4. Relative Exposures:

$$\begin{aligned} \omega &= \begin{bmatrix} w^m/W \\ w^{5y}/W \end{bmatrix} = \begin{bmatrix} \omega^m \\ \omega^{5y} \end{bmatrix} \\ &= \begin{bmatrix} 0.725 \\ 0.3 \end{bmatrix}. \end{aligned}$$

Again, we do not have relative exposures summing to one.

5. Portfolio Return Variance: Standard procedure using the results above: $\sigma^2(r^W) = \omega^\top S\omega$.

6. VaR calculation: Standard procedure using the variance $\sigma^2(r^W)$ from above.

7. Stocks and Zero Coupon with Mapping

Same situation as in Section 6, but this time the zero coupon has maturity at 4.2 years, and should be mapped into zero coupon factors with time to maturity equal to 3 years and 5 years, such that the time to maturity is unchanged.

1. Choosing Risk Factors: Same risk factors as before, but this time with r^{3y} also:

$$RF = \begin{bmatrix} r^m \\ r^{3y} \\ r^{5y} \end{bmatrix}.$$

2. The Covariance Matrix: The covariance S will be a 3×3 matrix.

3. Mapping: Mapping of the stocks is as before. Mapping of the bond is done after finding the relative weights a and $1 - a$ according to (D = Duration/Time-to-Maturity):

$$D_{bond} = aD_{factor1} + (1 - a)D_{factor2}$$

and we get $a = 0.4$. Hence,

$$w = \begin{bmatrix} w^m \\ w^{3y} \\ w^{5y} \end{bmatrix} = \begin{bmatrix} 3,625,000 \\ a \cdot 1,500,000 \\ (1-a) \cdot 1,500,000 \end{bmatrix} = \begin{bmatrix} 3,625,000 \\ 600,000 \\ 900,000 \end{bmatrix}.$$

4. Relative Exposures: Standard procedure $\omega = w/W$

5. Portfolio Return Variance: Standard procedure $\sigma^2(r^W) = \omega^\top S\omega$.

6. VaR calculation: Standard procedure.

8. STOCKS AND A BOND

Same situation as in Section 6, but this time, instead of a zero coupon bond, the 1,500,000 are invested in a bond with the following payments:

t :	0.7	1.7	2.7	3.7
Payment:	4	4	4	104

To manage the risk of the bond, we must strip it into zero-coupon bonds, and for that, we need to find each payments present value. I.e., we need zero coupon interest rates for all the dates of payments. The zero coupon rates below for the different maturities are more or less randomly invented for the example.[4] The present value of the first payment is $4/(1.03)^{0.7} = 3.918$, etc. This gives a fair price of the bond equal to P^{bond} = 101.29. The present value "weight" of the first payment equal to 3.918/101.29 = 0.0387, etc.

t :	0.7	1.7	2.7	3.7
Zero coupon rate:	3%	3.3%	3.6%	4%
Present values:	3.918	3.785	3.636	89.95
Weights:	0.0387	0.0374	0.0359	0.8881

Hence, each time we invest one dollar in the bond, we get for 0.0387 dollars of a 0.7 years zero coupon bond etc. If we invest 1,500,000 dollars in the bond, this corresponds to the following zero coupon investments:

t :	0.7	1.7	2.7	3.7
Stripped values:	$58,022$	$56,055$	$53,841$	$133,2083$

1. CHOOSING RISK FACTORS: We choose the same risk factors as before and add the one year zero coupon rate r^{1y}:

$$RF = \begin{bmatrix} r^m \\ r^{1y} \\ r^{3y} \\ r^{5y} \end{bmatrix}.$$

2. THE COVARIANCE MATRIX: The covariance S will be a 4×4 matrix.

3. MAPPING: Note that the first payment of the bond takes place before one year has passed. Hence, we assign all risk of this payment to r^{1y}. Else, the mapping takes place as in the previous example, but this time four different payments contribute to the exposition as shown in Table 1.

   For example, the payment after 1.7 years with a present value of 56,055 is weighted with 0.65 or 65% into the 1 year zero coupon risk factor and with 0.35 into the 2 years zero coupon risk factor.

[4] We could use empirical rates, but don't.

TABLE 1

MAPPING FROM STRIPPED PAYMENTS TO RISK FACTORS

		0.7Y	1.7Y	2.7Y	3.7Y	Total
1Y zero	Weight:	1	0.65	0.15		
	Exposure:	58,022	36,435	8076		102,534
3Y zero	Weight:		0.35	0.85	0.65	
	Exposure:		19,619	45,764	865,854	931,237
5Y zero	Weight:				0.35	
	Exposure:				466,229	466,229

Adding all the contributions, gives an exposure equal to

$$w = \begin{bmatrix} w^m \\ w^{1y} \\ w^{3y} \\ w^{5y} \end{bmatrix} = \begin{bmatrix} 3,625,000 \\ 102,534 \\ 931,237 \\ 466,229 \end{bmatrix}.$$

4. RELATIVE EXPOSURES: Standard procedure $\omega = w/W$
5. PORTFOLIO RETURN VARIANCE: Standard procedure $\sigma^2(r^W) = \omega^\top S\omega$.
6. VAR CALCULATION: Standard procedure.

9. Stocks and Spot FX

Assume that you, as an American investor, invest your wealth W = 5,000,000 with $x^{pep} = 2,000,000$ in Pepsi, $x^{coc} = 1,000,000$ in Coca cola, and $x^{DKK} = 2,000,000$ in Danish kroner (DKK), i.e., spot currency.

1. CHOOSING RISK FACTORS: No mapping takes place here, so the three assets are also the risk factors:

$$RF = \begin{bmatrix} r^{pep} \\ r^{coc} \\ r^{DKK} \end{bmatrix}.$$

2. THE COVARIANCE MATRIX: The covariance S will be a 3×3 matrix. Note that the return standard deviation of a DKK investment from an American's point of view, is the same as the return standard deviation of a \$-investment from a Dane's point of view. Hence, if you use \$/DKK or DKK/\$ quotation for the FX position, should give the same result in this setup.

3. MAPPING: Nothing surprising here:

$$w = \begin{bmatrix} 2,000,000 \\ 1,000,000 \\ 2,000,000 \end{bmatrix}.$$

4. RELATIVE EXPOSURES: Standard procedure $\omega = w/W$

5. PORTFOLIO RETURN VARIANCE: Standard procedure $\sigma^2(r^W) = \omega^\top S \omega$.

6. VAR CALCULATION: Standard procedure.

10. Domestic and Foreign Stocks

Assume that you, as an American investor, invest your wealth W = 5,000,000 with $x^{pep} = 2,000,000$ in Pepsi, $x^{coc} = 1,000,000$ in Coca cola, and $x^{car} = 2,000,000$ in the Danish brewery Carlsberg listed in DKK.

1. Choosing Risk Factors: We will split the Carlsberg position into a stock position (quoted in DKK) and a FX spot position of DKK:

$$RF = \begin{bmatrix} r^{pep} \\ r^{coc} \\ r^{car} \\ r^{dkk} \end{bmatrix}.$$

2. The Covariance Matrix: The covariance S will be a 4×4 matrix.

3. Mapping: The Carlsberg position is mapped to both the stock and the FX-spot factors:

$$w = \begin{bmatrix} 2,000,000 \\ 1,000,000 \\ 2,000,000 \\ 2,000,000 \end{bmatrix}.$$

4. Relative Exposures: Standard procedure $\omega = w/W$

5. Portfolio Return Variance: Standard procedure $\sigma^2(r^W) = \omega^\top S\omega$.

6. VaR calculation: Standard procedure.

11. Stocks and Future with mapping

Assume that you invest your wealth W = 5,000,000 with $x^{pep} = 2,000,000$ in Pepsi, $x^{coc} = 3,000,000$ in Coca cola. In addition, assume that you sell $n^{fut} = 4,000$ future contracts written on the S&P 100 index, (of old habit, we will use m to denote the index). Assume that on the day of the analysis, the value of the index is $P^m = 950$, that the future contract has maturity equal to one year, that the relevant one-year interest rate is $r = 3.2\%$, and that the expected dividend payment of the stocks in the index is $d = 2\%$ yearly. In the following, we will assume that the future price is always determined by the arbitrage relationship.

$$F_t = \left(\frac{1+r}{1+d}\right)^{T-t} P_t^m$$

We will assume in the following, that r and d are constant. Hence,

$$\begin{aligned} F_t &= \frac{1+0.032}{1+0.02}\, P_t^m \\ &= 1.012\, P_t^m \end{aligned} \tag{7}$$

1. Choosing Risk Factors: In this exercise, we will map everything into the S&P 100 index. Hence, only one risk factor:

   $$RF = [\, r^m \,] .$$

2. The Covariance Matrix: The covariance S will be the variance of the index.

3. Mapping: Mapping of the two stocks is done like in Section 5. To map the future, we first note, from Equation (7) that one future contract is equal to an index exposure of 1.012 times the value of the index P^m, which is 950. Hence, one contract gives an exposure of 961.12. Hence, 4000 contracts sold gives an exposure of -3,844,706. Hence,

   $$\begin{aligned} w = w^m &= x^{pep}\beta^{pep} + x^{coc}\beta^{coc} - n^{fut}F_t \\ &= 1,500,000 + 3,125,000 - 3,844,706 \\ &= 805,489 \end{aligned}$$

4. Relative Exposures: Standard procedure $\omega = w/W$

5. Portfolio Return Variance: Standard procedure $\sigma^2(r^W) = \omega^\top S\omega$.

6. VaR calculation: Standard procedure.

12. Stocks and a Put Option with mapping

Assume that you invest your wealth W = 5,000,000 with $x^{pep} = 2,000,000$ in Pepsi, $x^{coc} = 2,500,000$ in Coca cola. In addition, assume that you invest $x^{put} = 500,000$ in at-the-money put options written on the underlying asset. The characteristics of the put option at the time of the analysis are the following:[5]

Price of underlying asset:	P^m	950
Exercise price:	K	950
Yearly interest rate:	r	3.2%
Expected yearly dividend yield:	d	2%
Time to maturity (years):	$T-t$	1
Yearly volatility:	σ	20.1%

This gives a Black-Scholes put price equal to $P^{put} = 68.72$ and, hence, $n^{put} = 7276.17$ contracts (we will ignore the problem with the non-integer number of contracts). Moreover, we will assume that the empirical price is determined by the Black-Scholes formula.

1. Choosing Risk Factors: In this exercise, we will map everything into the S&P 100 index. Hence, only one risk factor:

$$RF = [\, r^m \,] .$$

2. The Covariance Matrix: The covariance S will be the variance of the index.

3. Mapping: Mapping of the two stocks is done like in Section 5. To map the option, we use the fact, that the sensitivity of the option, with respect to the return of the underlying asset r^m, is approximately given by

$$\begin{aligned}\frac{\Delta n^{put} P^{put}}{\Delta r^m} &= n^{put} P^m \frac{\partial P^{put}}{\partial P^m} \\ &= n^{put} P^m \frac{N(-d_1) - 1}{(1+d)^{T-t}} \\ &= n^{put} P^m \cdot -0.5526 \\ &= -3,819,511.\end{aligned}$$

where $N(-d_1)$ refers back to the usual option pricing notation. The alternative expression, depending on the invested amount x^{put} instead of number of contracts n^{put} is:

$$\begin{aligned}\frac{\Delta P^{put}}{\Delta r^m} &= x^{put} \frac{P^m}{P^{put}} \cdot -0.5526 \\ &= -3,819,511.\end{aligned}$$

[5] To be consistent with previous notation, we use P^m as the price of the underlying asset. Normally, S is used instead in option pricing literature.

Combining this with the stock results previous the section, we get the following index exposure:

$$\begin{aligned} w = w^m &= x^{pep}\beta^{pep} + x^{coc}\beta^{coc} - n^{put}P^m \frac{\partial P^{put}}{\partial P^m} \\ &= 1,500,000 + 3,750,000 - 3,819,511. \\ &= 1,405,294 \end{aligned}$$

4. RELATIVE EXPOSURES: Standard procedure $\omega = w/W$

5. PORTFOLIO RETURN VARIANCE: Standard procedure $\sigma^2(r^W) = \omega^\top S\omega$.

6. VaR CALCULATION: Standard procedure.

13. Exercises

General information to Guiding solutions: All results are based on the empirical data of the supplied Excel workbook. Variance and covariance estimates are based on simple (equally-weighted) estimates using the entire two-year sample. All VaR and ETL figures are calculated on a 5% confidence level, unless otherwise indicated.

Exercise 1: One stock
Calculate the VaR amounts of the setup in Section 3 using a variance based on empirical data.

Exercise 2: Two Stocks
Calculate the VaR amounts of the setup in Section 4 using a covariance matrix S based on empirical data.

Exercise 3: Stocks with Mapping
Calculate the VaR amounts of the setup in Section 5 using a covariance matrix S based on empirical data. Note, this also implies different beta-values.

Exercise 4: Stocks with Mapping and Zero Coupon without Mapping
Calculate the VaR amounts of the setup in Section 6 using a covariance matrix S based on empirical data.

Exercise 5: Stocks and Zero Coupon with Mapping
Calculate the VaR amounts of the setup in Section 7 using a covariance matrix S based on empirical data.

Exercise 6: Stocks and a Bond
Calculate the VaR amounts of the setup in Section 8 using a covariance matrix S based on empirical data. Note, you can use the zero coupon interest rates of Section 8 to determine the present value of the individual payments (I.e., use the same present values as in Section 8).

Exercise 7: Stocks and Spot FX
Calculate the VaR amounts of the setup in Section 9 using a covariance matrix S based on empirical data.

Exercise 8: Domestic and Foreign Stocks
Calculate the VaR amounts of the setup in Section 10 using a covariance matrix S based on empirical data.

Exercise 9: Stocks and a Future
Calculate the VaR amounts of the setup in Section 11 using a covariance matrix S based on empirical data. Note, also the beta-values for the stocks should be re-estimated.

Exercise 10: Stocks and a Put Option
Calculate the VaR amounts of the setup in Section 11 using a covariance matrix S based on empirical data. Note, also the beta-values for the stocks should be re-estimated.

14. Guiding solutions

General information

All results are based on the empirical data of the supplied Excel workbook. Variance and covariance estimates are based on simple (equally-weighted) estimates using the entire two-year sample. All VaR and ETL figures are calculated on a 5% confidence level, unless otherwise indicated. The results have not been checked, so there might be errors. Mail me, if you think I'm wrong.

Guiding solution to Exercise 1:
VaR amount = -211,180. ETL amount = 264,828.

Guiding solution to Exercise 2:
VaR amount = -208,441. ETL amount = 261,393.

Guiding solution to Exercise 3:
VaR amount = -137,354. ETL amount = 172,248. The results are obviously underestimating the real risk, due to the ignorance of unsystematic risk of the two stocks in a non-diversified portfolio.

Guiding solution to Exercise 4:
VaR amount = -91,702. ETL amount = 114,998.

Guiding solution to Exercise 5:
VaR amount = -91,706. ETL amount = 115,004.

Guiding solution to Exercise 6:
VaR amount = -91,932. ETL amount = 115,286.

Guiding solution to Exercise 7:
VaR amount = -135,120. ETL amount = 169,445.

Guiding solution to Exercise 8:
VaR amount = -236,615. ETL amount = 296,725.

Guiding solution to Exercise 9:
VaR amount = -22,100. ETL amount = 27,714.

Guiding solution to Exercise 10:
VaR amount = -36,970. ETL amount = 46,362.