TABLE 14.7 Correlation Matrix on September 25, 2008, when EWMA Method Is Used

$$\begin{pmatrix} 1 & 0.611 & 0.629 & -0.113 \\ 0.611 & 1 & 0.971 & 0.409 \\ 0.629 & 0.971 & 1 & 0.342 \\ -0.113 & 0.409 & 0.342 & 1 \end{pmatrix}$$

Variable 1 is DJIA; variable 2 is FTSE 100; variable 3 is CAC 40; variable 4 is Nikkei 225.

14.4 HANDLING INTEREST RATES

It is out of the question to define a separate market variable for every single bond price or interest rate to which a company is exposed. Some simplifications are necessary when the model-building approach is used. One possibility is to assume that only parallel shifts in the yield curve occur. It is then necessary to define only one market variable: the size of the parallel shift. The changes in the value of a bond portfolio can then be calculated using the approximate duration relationship in equation (9.6)

$$\Delta P = -DP\Delta y$$

where P is the value of the portfolio, ΔP is the change in P in one day, D is the modified duration of the portfolio, and Δy is the parallel shift in one day. This approach gives a linear relationship between ΔP and Δy, but does not usually give enough accuracy because the relationship is not exact and it does not take account of nonparallel shifts in the yield curve.

The procedure usually followed is to choose as market variables the prices of zero-coupon bonds with standard maturities: 1 month, 3 months, 6 months, 1 year, 2 years, 5 years, 7 years, 10 years, and 30 years. For the purposes of calculating VaR, the cash flows from instruments in the portfolio are mapped into cash flows occurring on the standard maturity dates.

Consider a \$1 million position in a Treasury bond lasting 0.8 years that pays a coupon of 10% semiannually. A coupon is paid in 0.3 years and 0.8 years and the principal is paid in 0.8 years. This bond is, therefore, in the first instance, regarded as a \$50,000 position in 0.3-year zero-coupon bond plus a \$1,050,000 position in a 0.8-year zero-coupon bond. The position in the 0.3-year bond is then replaced by an equivalent position in three-month and six-month zero-coupon bonds, and the position in the 0.8-year bond is replaced by an equivalent position in six-month and one-year zero-coupon bonds. The result is that the position in the 0.8-year coupon-bearing bond is for VaR purposes regarded as a position in zero-coupon bonds having maturities of three months, six months, and one year. This procedure is known as *cash-flow mapping*.

Illustration of Cash-Flow Mapping

We now illustrate one approach to cash-flow mapping by continuing with the example we have just introduced. It should be emphasized that the procedure we use is just one of several that have been proposed.

TABLE 14.8 Data to Illustrate Cash Flow Mapping Procedure

	3-Month	6-Month	1-Year
Zero rate (% with ann. comp.)	5.50	6.00	7.00
Bond price vol (% per day)	0.06	0.10	0.20

Correlation between Daily Returns

	3-Month Bond	6-Month Bond	1-Year Bond
3-month bond	1.0	0.9	0.6
6-month bond	0.9	1.0	0.7
1-year bond	0.6	0.7	1.0

Consider first the \$1,050,000 that will be received in 0.8 years. We suppose that zero rates, daily bond price volatilities, and correlations between bond returns are as shown in Table 14.8. The first stage is to interpolate between the six-month rate of 6.0% and the one-year rate of 7.0% to obtain a 0.8-year rate of 6.6%. (Annual compounding is assumed for all rates.) The present value of the \$1,050,000 cash flow to be received in 0.8 years is

$$\frac{1{,}050{,}000}{1.066^{0.8}} = 997{,}662$$

We also interpolate between the 0.1% volatility for the six-month bond and the 0.2% volatility for the one-year bond to get a 0.16% volatility for the 0.8-year bond.

Suppose we allocate α of the present value to the six-month bond and $1 - \alpha$ of the present value to the one-year bond. Using equation (14.2) and matching variances, we obtain

$$0.0016^2 = 0.001^2\alpha^2 + 0.002^2(1-\alpha)^2 + 2 \times 0.7 \times 0.001 \times 0.002\alpha(1-\alpha)$$

This is a quadratic equation that can be solved in the usual way to give $\alpha = 0.320337$. This means that 32.0337% of the value should be allocated to a six-month zero-coupon bond and 67.9663% of the value should be allocated to a one-year zero-coupon bond. The 0.8-year bond worth \$997,662 is therefore replaced by a six-month bond worth

$$997{,}662 \times 0.320337 = \$319{,}589$$

and a one-year bond worth

$$997{,}662 \times 0.679663 = \$678{,}074$$

This cash-flow mapping scheme has the advantage that it preserves both the value and the variance of the cash flow. Also, it can be shown that the weights assigned to the two adjacent zero-coupon bonds are always positive.

For the \$50,000 cash flow received at 0.3 years, we can carry out similar calculations (see Problem 14.7). It turns out that the present value of the cash flow is

TABLE 14.9 The Cash-Flow Mapping Result

	\$50,000 Received in 0.3 Years	\$1,050,000 Received in 0.8 Years	Total
Position in 3-month bond (\$)	37,397		37,397
Position in 6-month bond (\$)	11,793	319,589	331,382
Position in 1-year bond (\$)		678,074	678,074

\$49,189. This can be mapped to a position worth \$37,397 in a three-month bond and a position worth \$11,793 in a six-month bond.

The results of the calculations are summarized in Table 14.9. The 0.8-year coupon-bearing bond is mapped to a position worth \$37,397 in a three-month bond, a position worth \$331,382 in a six-month bond, and a position worth \$678,074 in a one-year bond. Using the volatilities and correlations in Table 14.9, equation (14.2) gives the variance of the change in the price of the 0.8-year bond with $n = 3, \alpha_1 = 37{,}397, \alpha_2 = 331{,}382, \alpha_3 = 678{,}074; \sigma_1 = 0.0006, \sigma_2 = 0.001$, and $\sigma_3 = 0.002$; and $\rho_{12} = 0.9, \rho_{13} = 0.6, \rho_{23} = 0.7$. This variance is 2,628,518. The standard deviation of the change in the price of the bond is, therefore, $\sqrt{2{,}628{,}518} = 1{,}621.3$. Because we are assuming that the bond is the only instrument in the portfolio, the 10-day 99% VaR is

$$1621.3 \times \sqrt{10} \times 2.326 = 11{,}946$$

or about \$11,950.

Principal Components Analysis

As we explained in Section 9.8, a principal components analysis (PCA) can be used to reduce the number of deltas that are calculated for movements in a zero-coupon yield curve. A PCA can also be used (in conjunction with cash-flow mapping) to handle interest rates when VaR is calculated using the model-building approach. For any given portfolio that is dependent on interest rates, we can convert a set of delta exposures into a delta exposure to the first PCA factor, a delta exposure to the second PCA factor, and so on. The way this is done is explained in Section 9.8. Consider the data set in Table 14.10, which is the same as that in Table 9.9. In Section 9.8, the exposure to the first factor for this data is calculated as +0.05, and the exposure to

TABLE 14.10 Change in Portfolio Value for a One-Basis-Point Rate Move (\$ Millions)

3-Year Rate	4-Year Rate	5-Year Rate	7-Year Rate	10-Year Rate
+10	+4	−8	−7	+2

the second factor is calculated as −3.88. (The first two factors capture over 97% of the variation in interest rates.)

Suppose that f_1 and f_2 are the factor scores. The change in the portfolio value is approximately

$$\Delta P = 0.05f_1 - 3.88f_2$$

The factor scores in a PCA are uncorrelated. From Table 8.8 the standard deviations for the first two factors are 17.55 and 4.77. The standard deviation of ΔP is, therefore

$$\sqrt{0.05^2 \times 17.55^2 + 3.88^2 \times 4.77^2} = 18.52$$

The one-day 99% VaR is, therefore, $18.52 \times 2.326 = 43.08$. Note that the portfolio we are considering has very little exposure to the first factor and significant exposure to the second factor. Using only one factor would significantly understate VaR. (See Problem 14.9.) The duration-based method for handling interest rates would also significantly understate VaR in this case as it considers only parallel shifts in the yield curve.

14.5 APPLICATIONS OF THE LINEAR MODEL

The simplest application of the linear model is to a portfolio with no derivatives consisting of positions in stocks, bonds, foreign exchange, and commodities. In this case, the change in the value of the portfolio is linearly dependent on the percentage changes in the prices of the assets comprising the portfolio. Note that, for the purposes of VaR calculations, all asset prices are measured in the domestic currency. The market variables considered by a large bank in the United States are, therefore, likely to include the value of the Nikkei 225 index measured in dollars, the price of a 10-year sterling zero-coupon bond measured in dollars, and so on.

Examples of derivatives that can be handled by the linear model are forward contracts on foreign exchange and interest rate swaps. Suppose a forward foreign exchange contract matures at time T. It can be regarded as the exchange of a foreign zero-coupon bond maturing at time T for a domestic zero-coupon bond maturing at time T. For the purposes of calculating VaR, the forward contract is therefore treated as a long position in the foreign bond combined with a short position in the domestic bond. (As just mentioned, the foreign bond is valued in the domestic currency.) Each bond can be handled using a cash-flow mapping procedure, so that it is a linear combination of bonds with standard maturities.

Consider next an interest rate swap. This can be regarded as the exchange of a floating-rate bond for a fixed-rate bond. (See Appendix D.) The fixed-rate bond is a regular coupon-bearing bond. The floating-rate bond is worth par just after the next payment date. It can be regarded as a zero-coupon bond with a maturity date equal to the next payment date. The interest rate swap, therefore, reduces to a portfolio of long and short positions in bonds and can be handled using a cash-flow mapping procedure.

14.6 LINEAR MODEL AND OPTIONS

We now consider the performance of the linear model when there are options. Consider first a portfolio consisting of options on a single stock whose current price is S. Suppose that the delta of the position (calculated in the way described in Chapter 8) is δ.[7] Because δ is the rate of change of the value of the portfolio with S, it is approximately true that

$$\delta = \frac{\Delta P}{\Delta S}$$

or

$$\Delta P = \delta \Delta S \tag{14.4}$$

where ΔS is the dollar change in the stock price in one day and ΔP is, as usual, the dollar change in the portfolio value in one day. We define Δx as the return on the stock in one day so that:

$$\Delta x = \frac{\Delta S}{S}$$

It follows that an approximate relationship between ΔP and Δx is

$$\Delta P = S\delta\Delta x$$

When we have a position in several underlying market variables that includes options, we can derive an approximate linear relationship between ΔP and the Δx_i similarly. This relationship is

$$\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i \tag{14.5}$$

where S_i is the value of the ith market variable and δ_i is the delta of the portfolio with respect to the ith market variable. This is equation (14.1):

$$\Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i \tag{14.6}$$

with $\alpha_i = S_i \delta_i$. Equation (14.2) can, therefore, be used to calculate the standard deviation of ΔP.

---

[7] In Chapter 8, we denote the delta and gamma of a portfolio by Δ and Γ. In this section and the next one, we use the lower case Greek letters δ and γ to avoid overworking Δ.

EXAMPLE 14.1

A portfolio consists of options on Microsoft and AT&T. The options on Microsoft have a delta of 1,000, and the options on AT&T have a delta of 20,000. The Microsoft share price is \$120, and the AT&T share price is \$30. From equation (14.5) it is approximately true that

$$\Delta P = 120 \times 1{,}000 \times \Delta x_1 + 30 \times 20{,}000 \times \Delta x_2$$

or

$$\Delta P = 120{,}000\Delta x_1 + 600{,}000\Delta x_2$$

where Δx_1 and Δx_2 are the returns from Microsoft and AT&T in one day and ΔP is the resultant change in the value of the portfolio. (The portfolio is assumed to be equivalent to an investment of \$120,000 in Microsoft and \$600,000 in AT&T.) Assuming that the daily volatility of Microsoft is 2% and that of AT&T is 1%, and that the correlation between the daily changes is 0.3, the standard deviation of ΔP (in thousands of dollars) is

$$\sqrt{(120 \times 0.02)^2 + (600 \times 0.01)^2 + 2 \times 120 \times 0.02 \times 600 \times 0.01 \times 0.3} = 7.099$$

Because $N(-1.65) = 0.05$, the five-day 95% VaR is

$$1.65 \times \sqrt{5} \times 7{,}099 = \$26{,}193$$

Weakness of Model

When a portfolio includes options, the linear model is an approximation. It does not take account of the gamma of the portfolio. As discussed in Chapter 8, delta is defined as the rate of change of the portfolio value with respect to an underlying market variable and gamma is defined as the rate of change of the delta with respect to the market variable. Gamma measures the curvature of the relationship between the portfolio value and an underlying market variable.

Figure 14.1 shows the impact of a nonzero gamma on the probability distribution of the value of the portfolio. When gamma is positive, the probability distribution tends to be positively skewed; when gamma is negative, it tends to be negatively skewed. Figures 14.2 and 14.3 illustrate the reason for this result. Figure 14.2 shows the relationship between the value of a long call option and the price of the underlying asset. A long call is an example of an option position with positive gamma. The figure shows that, when the probability distribution for the price of the underlying asset at the end of one day is normal, the probability distribution for the option price is positively skewed.[8] Figure 14.3 shows the relationship between the value of a short

[8] The normal distribution is a good approximation of the lognormal distribution for short time periods.

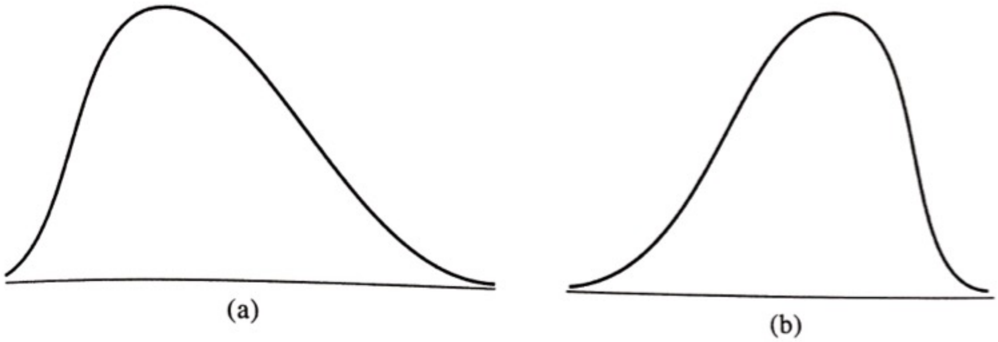

FIGURE 14.1 Probability Distribution for Value of Portfolio
(a) positive gamma, (b) negative gamma

call position and the price of the underlying asset. A short call position has a negative gamma. In this case, we see that a normal distribution for the price of the underlying asset at the end of one day gets mapped into a negatively skewed distribution for the value of the option position.

The VaR for a portfolio is critically dependent on the left tail of the probability distribution of the portfolio value. For example, when the confidence level used is 99%, the VaR is the value in the left tail below which only 1% of the distribution resides. As indicated in Figures 14.1a and 14.2, a positive gamma portfolio tends to have a less heavy left tail than the normal distribution. If the distribution is assumed to be normal, the calculated VaR will tend to be too high. Similarly, as indicated in Figures 14.1b and 14.3, a negative gamma portfolio tends to have a heavier left

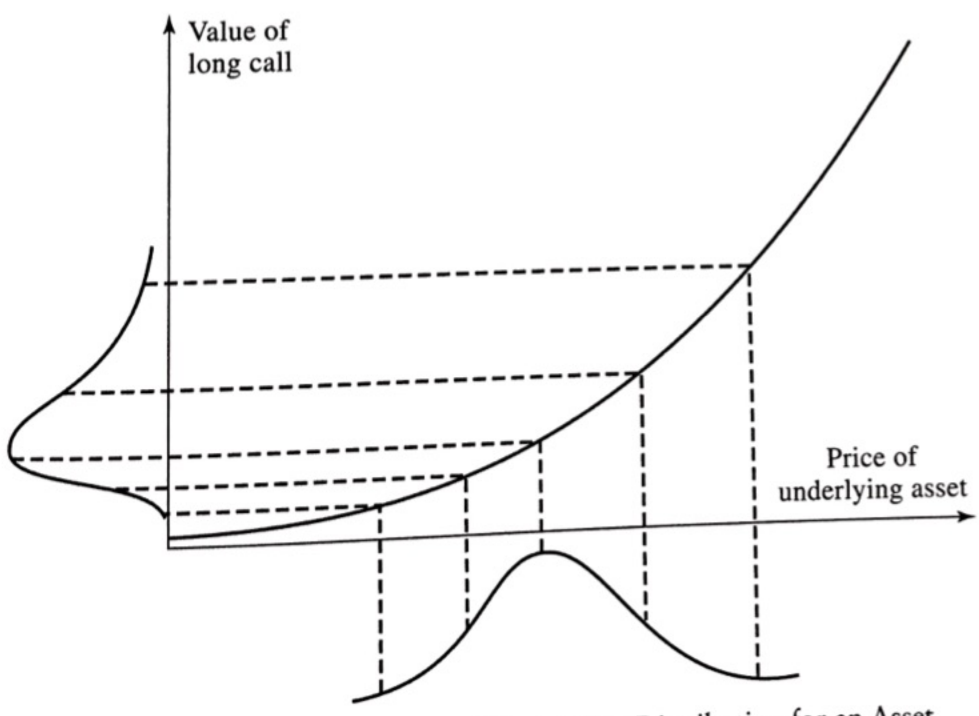

FIGURE 14.2 Translation of Normal Probability Distribution for an Asset into Probability Distribution for Value of a Long Call on the Asset

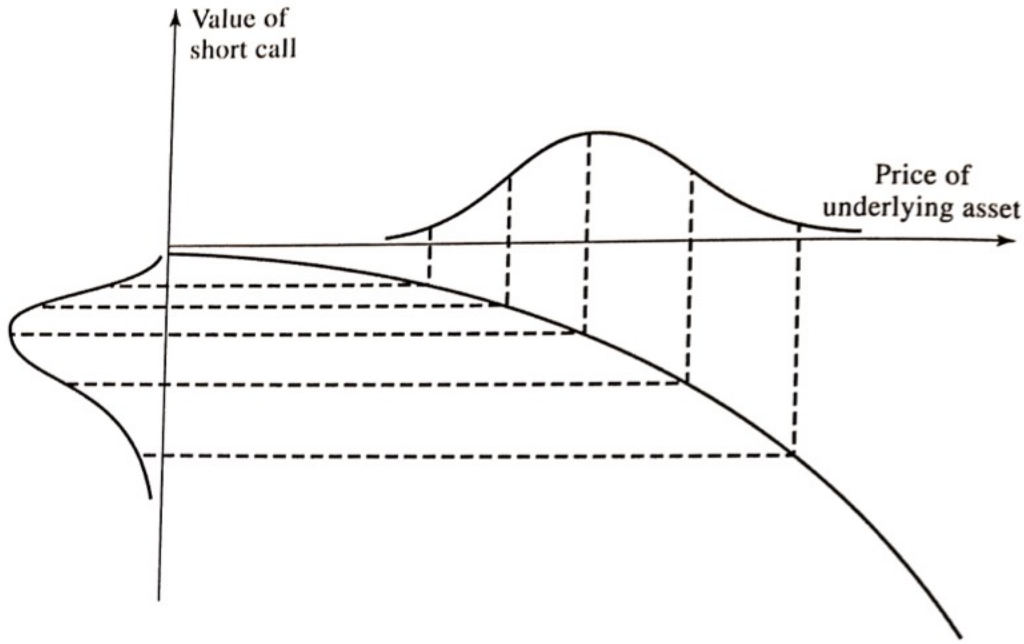

FIGURE 14.3 Translation of Normal Probability Distribution for an Asset into Probability Distribution for Value of a Short Call on the Asset

tail than the normal distribution. If the distribution is assumed to be normal, the calculated VaR will tend to be too low.

14.7 QUADRATIC MODEL

For a more accurate estimate of VaR than that given by the linear model, both delta and gamma measures can be used to relate ΔP to the Δx_i. Consider a portfolio dependent on a single asset whose price is S. Suppose δ and γ are the delta and gamma of the portfolio. As indicated in Chapter 8 and Appendix G, a Taylor Series expansion gives

$$\Delta P = \delta \Delta S + \frac{1}{2}\gamma(\Delta S)^2$$

as an improvement over the linear approximation $\Delta P = \delta \Delta S$.[9] Setting

$$\Delta x = \frac{\Delta S}{S}$$

[9] A fuller Taylor series expansion suggests the approximation

$$\Delta P = \Theta \Delta t + \delta \Delta S + \frac{1}{2}\gamma(\Delta S)^2$$

when terms of higher order than Δt are ignored. In practice, the $\Theta \Delta t$ term is so small that it is usually ignored.