Introduction to Financial Engineering

Week 43: Constraints and Estimation Methods

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Week 43

Notation

- Assume that $\boldsymbol{\mu} = [\mu_1, \mu_2, \ldots, \mu_n]'$ is a vector of expected returns on different assets
- Assume that $\boldsymbol{w} = [w_1, w_2, \ldots, w_n]'$ are the fractions of the investors wealth invested in each asset
- Assume that $\boldsymbol{\Sigma}$ is the covariance matrix of the returns
- By definition, a covariance matrix is always positive semidefinite, but now it is assumed that it is **positive definite** and thus invertible
- Further, not all coordinates of $\boldsymbol{\mu}$ are equal
- For a given expected return μ_P, the objective is to find the portfolio with the lowest variance (or standard deviation)

Optimal portfolios

For convenience, the matrix $\boldsymbol{A}$ was defined when the optimization problem was solved:

$$\begin{aligned} \boldsymbol{A} &= \left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array} \right]' \boldsymbol{\Sigma}^{-1} \left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array} \right] \\ &= \left[\begin{array}{cc} \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1} \\ \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1} & \mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1} \end{array} \right] := \left[\begin{array}{cc} a & b \\ b & c \end{array} \right] \end{aligned}$$

so the inverse of $\boldsymbol{A}$ is

$$\boldsymbol{A}^{-1} = \frac{1}{ac - b^2} \left[\begin{array}{cc} c & -b \\ -b & a \end{array} \right]$$

Optimal portfolios

The minimum variance (or standard deviation) for a given return μ_P can then be expressed in terms of a, b and c:

$$\sigma_P^2 = \frac{c\mu_P^2 - 2b\mu_P + a}{ac - b^2} \qquad \text{or} \qquad \sigma_P = \sqrt{\frac{c\mu_P^2 - 2b\mu_P + a}{ac - b^2}} \tag{[1]}$$

with corresponding portfolio weights:

$$\hat{\boldsymbol{w}} = \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix} \boldsymbol{A}^{-1} \begin{bmatrix} \mu_P \\ 1 \end{bmatrix} \tag{[2]}$$

Efficient frontier

Using the expression for σ_P as a function of μ_P, it's easy to find the portfolio with the smallest variance possible:

$$\frac{d\sigma_P^2}{d\mu_P} = \frac{2c\mu_P - 2b}{ac - b^2} = 0 \Rightarrow$$
$$\mu_{gmv} = b/c \quad \text{with} \quad \sigma_{gmv}^2 = 1/c$$

The portfolio weights can be expressed as

$$\hat{\boldsymbol{w}}_{gmw} = \frac{1}{c}\boldsymbol{\Sigma}^{-1}\mathbf{1}$$

In a (standard deviation, mean)-space or in a (variance, mean)-space, the **efficient frontier** or efficient portfolios is the upper half of the curve expressed by [1]. The efficient frontier will have expected specific returns greater than b/c and variances greater than $1/c$.

More notation

- Assume that a risk free asset exists with return μ_0
- Express returns as excess returns
  $\boldsymbol{\mu}^e = [\mu_1 - \mu_0, \mu_2 - \mu_0, \ldots, \mu_n - \mu_0]'$
- Assume that $\boldsymbol{w} = [w_1, w_2, \ldots, w_n]'$ are the fractions of the investors wealth invested in each risky asset
- Assume that $w_0 = 1 - \boldsymbol{w}'\mathbf{1}$ is invested in the risk free asset
- For a given expected excess return μ_P^e, the objective is to find the portfolio with the lowest variance (or standard deviation)

Capital Market Line

The link between σ_P and of μ_P^e is:

$$\sigma_P = \frac{\mu_P^e}{\sqrt{(\boldsymbol{\mu}^e)' \, \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e}}$$

or equivalently

$$\mu_P = \sigma_P \sqrt{(\boldsymbol{\mu}^e)' \, \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} + \mu_0 \qquad ([3])$$

with portfolio weights:

$$\boldsymbol{w} = \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e \frac{\mu_P^e}{(\boldsymbol{\mu}^e)' \, \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \qquad ([4])$$

Tangent Portfolio

The portfolio where everything is invested in risky assets is called the **tangent portfolio**. The excess return of the tangent portfolio is

$$\mu_{tan}^{e} = \frac{(\boldsymbol{\mu}^e)' \, \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e}{\mathbf{1}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \qquad ([5])$$

with

$$\sigma_{tan} = \frac{\sqrt{(\boldsymbol{\mu}^e)' \, \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e}}{\mathbf{1}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \qquad ([6])$$

The tangent portfolio touches the risky assets-only efficient frontier in exactly one point. The CML lies above the risky assets-only efficeint frontier.

Two risky assets case

- For two risky assets, both assets will be on the parabola of minimum-variance portfolios
- They will not necessarily be on the efficient frontier – the upper half curve. The position will depend on the assets (correlation, expected return, standard deviation)
- If short-selling is not allowed, the possible minimum-variance portfolios is on the parabola of minimum-variance portfolios between the two assets
- The efficient and feasible portfolios consist of the upper half curve for μ-values between $\max\{\min(\mu_1, \mu_2), \mu_{gmw}\}$ and $\max(\mu_1, \mu_2)$

Multiple risky assets

- For multiple risky assets, the assets will generally not be on the parabola of minimum-variance portfolios
- If short-selling is not allowed, the possible minimum-variance portfolios must be found by solving the optimization problem with the additional constraint that $w_i \geq 0$ for $i = 1, \ldots, n$
- This leads to portfolios with at least the same variance as without the additional constraint
- The highest feasible return is $\max(\mu_1, \mu_2, \ldots, \mu_n)$

Adding a risk free asset

- A risk free asset with possible different interest rates can also be incorporated in the optimization problem
- Risk less lending and no borrowing could for instance be incorporated by removing constraint $\mathbf{1}'\boldsymbol{w} = 1$ and instead constraining $w_0 \geq 0$ together with the previous condition of no short-selling of risky assets $w_i \geq 0$ for $i = 1, \ldots, n$
- Borrowing with no lending, or borrowing and lending at different rates etc. can also be incorporated by varying the constraints

Portfolios of particular interest

- The minimum variance portfolio: The portfolio with the lowest variance
- The maximum return portfolio: The portfolio with the highest expected return
- The tangent portfolio: This portfolio actually has the highest **Sharpe Ratio** $SR_P = \frac{\mu_P - \mu_0}{\sigma_P}$ among the risky asset only portfolios
- Choosing according to risk aversion: Finding the portfolio that (for instance) maximizes $\mu_P - \lambda\sigma_P^2$ for a given λ. If λ is high, the investor dislikes risk ("risk averse"). If λ is low, he favors return over risk.

Simplifying data input

To choose the optimal portfolio, analysts must assess

- the expected returns: N variables
- the risk (variance/standard deviation): N variables
- pairwise correlation between stocks: $(N-1)N/2$ variables
- for two assets, the total number of input is 2+2+1
- for three assets, the total number of input is 3+3+3
- for 100 assets, the total number of input is 100+100+4950

This must be simplified!

Single Index Model

The single index models says that the return on asset i can be explained by the market and a random factor. In other words, assume

$$R_i = \alpha_i + \beta_i R_m + e_i,$$

where:

- R_i is the return on asset i and R_m is the return on the market
- e_i is a random variable with mean zero
- σ_{ei} and σ_m are corresponding standard deviations
- β_i is the sensitivity that measures the change in R_i as a response to a change in R_m.
- α_i is expected return on asset i which is independent on the market

Single Index Model

Assume further that the residual risk is uncorrelated $cov(e_i, e_j) = 0$. The main implications of this setup are:

- $\bar{R}_i = \alpha_i + \beta_i \bar{R}_m$ is the expected return on asset i
- $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2$ is the variance of asset i
- $cov(R_i, R_j) = \sigma_{ij} = \beta_i \beta_j \sigma_m^2$
- This leads to a parameter reduction from $(3N + N^2)/2$ to $3N + 2$
- Parameters are easily obtained by linear regression
- Note: Using historical data to estimate parameters is (obviously) based on history and not necessarily a prediction of future performance.

Constructing portfolios

Consider the portfolio with weights X_i in asset i.

$$\begin{aligned}
\bar{R}_p &= \sum_{i=1}^{N} X_i \bar{R}_i \\
&= \sum_{i=1}^{N} X_i(\alpha_i + \beta_i \bar{R}_m) \\
&= \sum_{i=1}^{N} X_i \alpha_i + \sum_{i=1}^{N} X_i \beta_i \bar{R}_m \\
\sigma_p^2 &= \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2
\end{aligned}$$

Portfolio variance

Write $\alpha_p = \sum_{i=1}^{N} X_i \alpha_i$ and $\beta_p = \sum_{i=1}^{N} X_i \beta_i$. Then it's easily seen that

$$\bar{R}_p = \alpha_p + \beta_p \bar{R}_m$$
$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2$$

- When $N \to \infty$, the last term diminishes and the standard deviation of the portfolio approaches $\sigma_p = \beta_p \sigma_m$.
- Since the residual risk σ_{ei} can be eliminated by holding a large portfolio, β_i is often used as the measure for the risk of asset i.