Introduction to Financial Engineering

Week 45: Capital Asset Pricing Model

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Week 45

Mean-Variance Analysis

- Risk is quantified in terms of variance or standard deviation
- Not the only way, but widely used and fairly simple
- There is a trade-off between return and risk: Investors prefer to have higher returns for the same risk or lower risk for the same return
- In the absence of a risk-free asset, this leads to combinations of risk and return that takes a hyporbolic form (in a (standard deviation, mean)-space)

Two-fund seperation

- The (upper half) of the efficient frontier is where the investor wants to be
- It is where he gets the most return for the risk he's willing to take
- Or equivalently, it is where he gets the lowest risk for the return he wants
- All portfolios on the efficient frontier can be obtained as a weighted average of two other portfolios on the efficient frontier
- This can be proved using the expression of portfolio weights or verified numerically (see today's exercise 1)

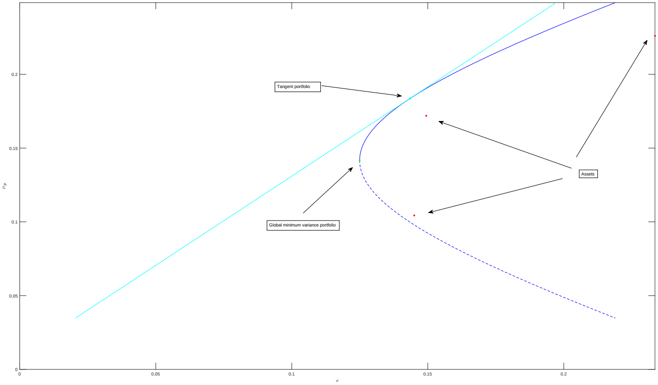
Adding a risk free asset

- Adding a risk-free asset expands the feasible set of portfolios
- The choice of optimal portfolios become a straight line in the (standard deviation, mean)-space
- The two-fund separation theorem now tells us that we only need to identify one other efficient portfolio to get all optimal portfolios
- For instance; the portfolio with all wealth invested in risky assets, i.e., the tangent portfolio
- All efficient portfolios can now be expressed as

$$\mu_P = \mu_0 + \frac{\mu_{tan} - \mu_0}{\sigma_{tan}}\sigma_P$$

- This can be obtained by investing $\frac{\mu_P - \mu_{tan}}{\mu_0 - \mu_{tan}}$ in the risk free asset and the rest in the tangent portfolio

Capital Market Line and Tangent Portfolio



Properties of the tangent portfolio

- The tangent portfolio is relevant to characterise optiomal portfolios
- The ratio of excess return for an asset to its' covariance with the tangent portfolio is constant:

$$\frac{\mu_i - \mu_0}{cov(r_i, r_{tan})} = \text{constant}$$

- Intuition: This is reward to marginal variance ratio. If they are not constant, then it's better to put more money in the asset with a high ratio and less money in the asset with a low ratio. This will increase the expected return, while lowering the variance. See numerical example 5.4
- Using the expression for the tangent portfolio derived earlier, it can also be proved that the above ratio is constant and equal to $\mathbf{1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e$

How expected return relates to the tangent portfolio

- The constant ratio condition specifically holds for the tangent portfolio itself:

$$\frac{\mu_i - \mu_0}{cov(r_i, r_{tan})} = \frac{\mu_{tan} - \mu_0}{cov(r_{tan}, r_{tan})}$$

- We can isolate μ_i to obtain:

$$\mu_i - \mu_0 = \frac{cov(r_i, r_{tan})}{var(r_{tan})}(\mu_{tan} - \mu_0)$$

- We denote $\frac{cov(r_i, r_{tan})}{var(r_{tan})}$ as the β of the asset. It is usually obtained as the slope of a regression of asset returns against the tangent portfolio
- β measures the covariance of an asset with the tangent portfolio (scaled with a constant)
- Plotting β against expected returns gives the Securities Market Line, a straigth line in the (β, expected return)-space

From the tangent portfolio to the market portfolio

- Theoretically (and in practice for a few assets) it is possible to find the tangent portfolio
- But in practice it is complex and inaccurate (and backward-looking)
- With a few sound theoretical assumptions, we can derive the tangency portfolio and thereby obtain a link between risk and expected return
- This approach is called the Capital Asset Pricing Model, CAPM
- Besides the assumptions from the mean-variance theory on frictionless markets and that investors only care about mean and variance, we assume that all investors have the same beliefs and information
- CAPM tells us that the tangent portfolio must be the market portfolio

What is the market portfolio

- The market portfolio is the portfolio, where the weights on each asset is the market value of that asset divided by the entire market value of risky assets
- Example 5.7:

Stock	Stock price	No of shares	Market value	w
HP	33	2 billion	66 billion	$66/268 = 0.25$
IBM	95	1.758 billion	167 billion	$167/268 = 0.62$
CPQ	20.25	1.7 billion	35 billion	$35/268 = 0.13$

- In practice, this would be an index like S&P 500 or similar

Why is the tangency portfolio the market portfolio

- Assume there are two investors in the market from the previous slide, Jack and Jill
- Mean-variance analysis states that both Jack and Jill will invest a proportion of their wealth in the tangent portfolio and the rest in the risk-free asset.
- Since they agree on the tangency portfolio, there must be a positive holding of each stock in the tangency portfolio
- Since they agree on the tangency portfolio, they must hold a portfolio where assets have the same relative weight
- This can only be obtained if the tangency portfolio is the same as the market portfolio
- This can be generalised to many assets and many investors

CAPM

- CAPM tells us that the tangency portfolio is the market portfolio
- The market portfolio can be proxied by a index like the S&P 500
- The relation between expected return of a portfolio/asset and risk is

$$\mu_P - \mu_0 = \beta_P(\mu_M - \mu_0)$$

  where β_P is obtained using the market portfolio
- The β of a portfolio is the portfolio-weighted average of the βs of individual assets

Some words on β

- CAPM tells us that we don't need to find the tangency portfolio
- The market portfolio can be used instead
- The market portfolio is practically impossible to obtain, but we use proxies like S&P 500
- After choosing the proxy, we need to estimate βs, risk-free return and market risk premium (or market expected return) to implement the CAPM
- βs are only estimated - they are not true values of β
- There are ways to get "better" βs – we will not look into these

Testing CAPM empirically

- Applying CAPM requires a market proxy
- If we are testing the relationship

$$\mu - \mu_0 = \beta(\mu_M - \mu_0)$$

  to see if there is a linear relationship between risk and stock βs, we can't distinguish between testing if CAPM is correct or if the proxy is in fact the tangency portfolio
- If data supports CAPM, a plot of β vs returns should give a straight line with the market risk premium as slope and the risk free rate as intercept
- If we further regress on other characteristics such as size, momentum etc., there should be no explanatory power if CAPM is true
- This is mostly not the case $\rightarrow$ many suggestions for altering the CAPM model and/or criticising the assumptions