Introduction to Financial Engineering

Week 39: Bonds

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Week 39

Time Value of Money

- First, using simple interest rates we saw that $V(0)$ grows to $V(t) = V(0)(1 + rt)$ after t years with r being the quoted interest rate expressed in annual terms
- If we instead use annual compounding $V(0)$ would grow to $V(t) = V(0)(1 + r)^t$ after t years
- Using more and more frequent compounding $V(0)$ would in the limit grow to $V(t) = V(0)e^{rt}$ after t years
- For the calculations done today and in most of this course (unless otherwise specified), we assume annual compunding and we don't restrict the time t to be an integer (recall last week's exercises)

Present Value of Future Cash Flows

- Finding the present value of a cash flow at time t is the reverse calculation. A cash flow C_t at time t has the following value at time 0:

$$\frac{C_t}{(1+r)^t}$$

- We refer to $\frac{1}{(1+r)^t}$ as the discount factor for time t
- The interest rate(s) or the yields are usually given (at uni; in the exercises or by a chosen required rate of return) or implied from a market using a set of comparable bonds
- Here, we are using government bonds to imply the yield curves

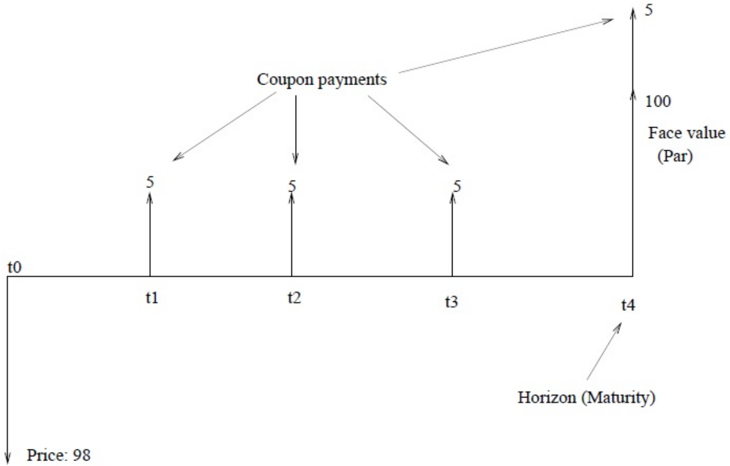
Types of bonds

The basic bonds are

- Zero Coupon Bond: No coupon, principal paid back at maturity
- Bond with coupons: Coupons paid (semi-)annually and principal paid back at maturity
- Annuity: Same payment every time
- (Other types, e.g., linear loans)

Often with bonds, the first thing to do is determine the cash flows from the bond's characteristics.

A bond with coupon has payments before maturity

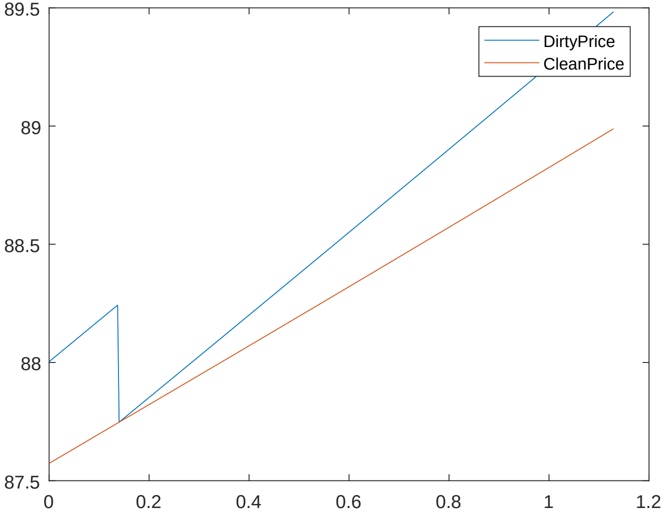


Most government bonds are of this type

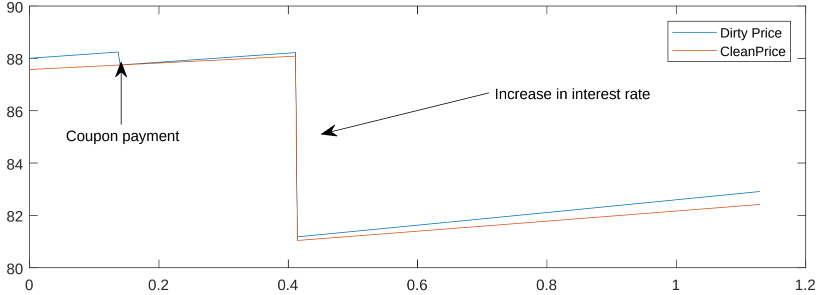
Quoted prices vs present value of cash flows

- There two types of bond prices
  - Clean price: The price the exchange quotes
  - Dirty price: The price you actually have to pay
  - Clean price = Dirty price - accrued interest
  - The accrued interest is calculated by the simple interest rule
- The dirty price is the present value of all future payments
- Remember: When setting up the cash flow and calculating yields (see later), the dirty price needs to be calculated from the clean price and the coupons
- Clean prices are more "clean". When they change, it's because of changes in the (interest rate) market
- The dirty price can change just because of time passing and coupons being paid

Clean vs. dirty prices at constant interest rate



Prices when interest rates increase



When clean prices move/jump, interest rates have changed. When dirty prices change, there is either a coupon payment or something happening with the interest rates.

Finding yields from bonds

- In the beginning, we think of bond prices as being determined by an interest rate and the bond's cash flow
- In reality, we should think of it opposite: "Which rate will this bond yield me if the market quotes this price"
- Or in other words "which rate should I use for discounting the cash flows to obtain the quoted price of the bond?"

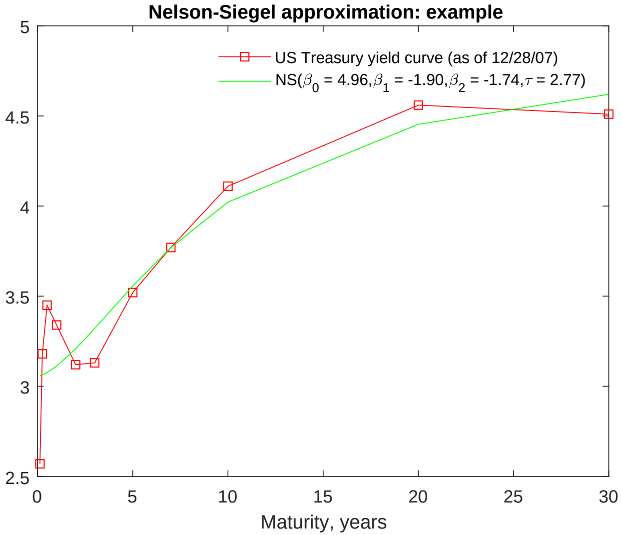
Yield calculation

- Denote the payment at time t_i from the bond by C_{t_i} for $i = 1, \ldots, N$, where $t_N = T$
- The yield to maturity solves the equation:

$$P(t,T) = \sum_{t_i = t_1}^{t_N} \frac{C_{t_i}}{(1 + y(t,T))^{t_i - t}}$$

- The yield $y(t,T)$ denotes the yield to maturity at time t for a bond maturing at time T
- If it's clear when t or T is, we might leave it out and just write y
- For this course, the dependence of T is more relevant than the dynamic dependence t

Nelson-Siegel term structure model



Interest term structure models

- The general idea for interest term structure models is to have a parametrized function for the interest rate curve
- Nelson-Siegel is the most widely used
- It has four parameters $\beta_0, \beta_1, \beta_2$ and τ:

$$y(T) = \beta_0 + \beta_1 \frac{1 - \exp\{-T/\tau\}}{T/\tau} + \beta_2 \left(\frac{1 - \exp\{-T/\tau\}}{T/\tau} - \exp\{-T/\tau\} \right)$$

- The parameters are found by applying routines in Matlab/R/similar programs
- Once we have the parameters, we have interest rates for all maturities

Interest rate risk

- The risk that the price/value of a security will change due to movements in the general level of the interest rates
- Obviously, the cash flow of the bond will (most likely) not change
- But the present value of the cash flow can have a different value
- Think of bonds issued by a government, who later wishes to repurchase the bonds instead of paying the coupons. If the interest rates have decreased, then prices of the bond has increased and there is a high cost for buying back the bonds

Price-rate relationship

- The relationship between rates (assume a constant interest rate r and $t = 0$) and the price of a bond is given by

$$P = \sum_{t_i=t_1}^{t_N} \frac{C_{t_i}}{(1+r)^{t_i}}$$

- If the interest rate shift up or down, how does the price change?
- Fill in the picture from the black board here:

Duration

- Think of the price of a bond as a function of r
- By Taylor-expansion around initial rate r_0,

$$P(r) - P(r_0) \approx P'(r_0)(r - r_0) + \tfrac{1}{2}P''(r_0)(r - r_0)^2$$

- So; we'd like to have a measure for how much the price moves when the interest rate changes:

$$P'(r) = -\frac{1}{1+r}\sum_{t_i=t_1}^{t_N} \frac{t_i C_{t_i}}{(1+r)^{t_i}}$$

- We call $D^{MOD} = -P'(r)/P$ the modified duration. It is the *percentage change* in price with respect to changes in the interest rate

Why modified duration?

- The term duration arises because the value

$$D^{MAC} = \sum_{t_i=t_1}^{t_N} t_i \frac{C_{t_i}}{(1+r)^{t_i}} / P$$

  can be seen as the time it takes to get the cash flows from the bond
- We get that $D^{MOD} = D^{MAC}/(1+r)$

Calculation of duration

- The input to duration calculations are
  - the cash flows of the bond
  - the price
  - the rate/yield
- The yield of the bond could be used as input for duration calculations
- Another option is to use the term structure of rates (see Nelson-Siegel and next slide)

Fisher-Weil duration

- Rather than assuming a flat interest rate r, the Fisher-Weil duration uses the zero coupon yield curve (or a similar proxy):

$$D^{FW} = -\frac{1}{P} \sum_{t_i=t_1}^{t_N} \frac{t_i C_{t_i}}{(1+r_{t_i})^{t_i+1}}$$

- Note that if we set r_{t_i} constant, we would get the modified duration
- The FW duration measures a *parallel shift* in the interest rates

Convexity adjustment

- Remember the relationship between price and rate. The first order approximation is not very good for larger changes in interest rates. Convexity is defined as the second order derivative relative to the price:

$$C = \frac{1}{P} \sum_{t_i = t_1}^{t_N} \frac{t_i(t_i + 1) C_{t_i}}{(1 + r_{t_i})^{t_i + 2}}$$

- Convexity adjustments will give a better approximation for changes in the bond price than just using the duration. Will duration over- or underestimate the new price of the bond?

For todays exercises

- Finish the exercise from last week if you haven't
- Look at the yields and think about which bonds to exclude if any (and perhaps why they are different)
- Fit the Nelson-Siegel term structure to the yields
- For each bond calculate duration and convexity – the definitions can be found in the slides

The next two week we will talk about

- characterising assets in terms of return and risk
- optimal portfolio choices

Readings:

- Chapter 3 until 3.4
- Lando & Poulsen Chapter 9.1