Introduction to Financial Engineering

Week 40: Portfolio Choices

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Week 40

1 Making financial decisions
- Problem statement
- Combining assets to change the risk
- Extracting average and variance
- Other measures

2 Multiple assets

3 The maths of the efficient frontier

4 Illustrations

5 Wrap up

Making financial decisions

Financial assets are **risky assets**, meaning that the future cash flows (and value) are not known. Only the distribution of returns is known or assumed.

Overall problem

- Which assets to invest in
- How much to invest in each asset

Starting point: Characterizing assets in terms of

- expected return (average)
- variance / standard deviation

Small exercise: Combining risky assets

Market Condition	Return[a] Asset 1	Asset 2	Asset 3	Asset 5	Rainfall	Return[a] Asset 4
Good	15	16	1	16	Plentiful	16
Average	9	10	10	10	Average	10
Poor	3	4	19	4	Poor	4
Mean return	[redacted]					[redacted]
Variance						
Standard deviation						

[a]The alternative returns on each asset are assumed equally likely and, thus, each has a probability of $\frac{1}{3}$.

Figure: EGBG Table 4-3 page 47

- Compute expected returns, variance/standard deviation for each asset
- What happens if Asset 2 and Asset 3 are combined?
- What happens if Asset 2 and Asset 5 are combined?
- Is there enough information to say anything about the combination of Asset 2 and Asset 4?

Estimating expected returns and variance

- There is no way of knowing the true distribution of an asset's future returns
- This must be inferred from data (and potentially using sophisticated models)
- The expected return and variance / standard deviation of returns can be estimated from historical data
- Note: Remember the exercise on averaging from week 36 and 37

Other measures of a distribution

- The expected return and variance of returns is not a full description of a distribution
- Other measures to describe the dispersion (or more generally the distribution) are
  - deviations below the mean
  - the expectation of the squared deviation below the mean is called semivariance (this can be generalized to lower partial moments)
  - quantiles of the distribution (this is equivalent to the Value-at-Risk framework)
- If return distributions are symmetrical, then these other measures give the same ordering of portfolios as variance
- In portfolio literature, much theory is based on average and variance / standard deviation as adequate measures for choosing investments

1 Making financial decisions

2 Multiple assets
- Three asset example
- Combination of stock and bond portfolios
- Combination of two stock portfolios

3 The maths of the efficient frontier

4 Illustrations

5 Wrap up

Combining risky assets – stock market data

Month	IBM	Alcoa	GM	$\frac{1}{2}$ IBM + $\frac{1}{2}$ Alcoa	$\frac{1}{2}$ GM + $\frac{1}{2}$ Alcoa	$\frac{1}{2}$ GM + $\frac{1}{2}$ IBM
1	12.05	14.09	25.20	13.07	19.65	18.63
2	15.27	2.96	2.86	9.12	2.91	9.07
3	−4.12	7.19	5.45	1.54	6.32	0.67
4	1.57	24.39	4.56	12.98	14.48	3.07
5	3.16	0.06	3.72	1.61	1.89	3.44
6	−2.79	6.52	0.29	1.87	3.41	−1.25
7	−8.97	−8.75	5.38	−8.86	−1.69	−1.80
8	−1.18	2.82	−2.97	0.82	−0.08	−2.08
9	1.07	−13.97	1.52	−6.45	−6.23	1.30
10	12.75	−8.06	10.75	2.35	1.35	11.75
11	7.48	−0.70	3.79	3.39	1.55	5.64
12	−.94	8.80	1.32	3.93	5.06	0.19
$\bar{R}$	2.95	2.95	5.16	2.95	4.05	4.05
σ	7.15	10.06	6.83	6.32	6.69	6.02

Correlation Coefficient: IBM and Alcoa = 0.05;
GM and Alcoa = 0.22; IBM and GM = 0.48

Figure: EGBG Table 4-5 page 52

Combining risky assets – illustration in (σ, μ) diagram

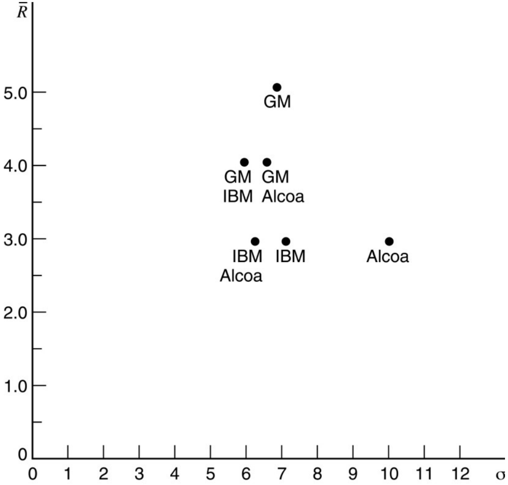

Figure: EGBG Figure 4-1 page 52

Investing in stocks and bonds

- Assume it is possible to invest in a stock portfolio and in a bond portfolio (each of these is one asset)
- Assume that the stock portfolio has an expected return of 12.5% and the standard deviation is 14.9%
- Assume that the bond portfolio has an expected return of 6% and the standard deviation is 4.8%
- The correlation between returns is 0.45
- What happens for different wealth allocations?

Combining stocks and bonds – portfolio data

Proportion Stocks	Proportion Bonds	Mean Return	Standard Deviation
1	0	12.5	14.90
0.9	0.1	11.85	13.63
0.8	0.2	11.2	12.38
0.7	0.3	10.55	11.15
0.6	0.4	9.9	9.95
0.5	0.5	9.25	8.80
0.4	0.6	8.6	7.70
0.3	0.7	7.95	6.69
0.2	0.8	7.3	5.82
0.1	0.9	6.65	5.16
0	1	6	4.80

Figure: EGBG Figure 4-11 page 62

Portfolios of stocks and bonds

Example (Expected return vs. standard deviation)

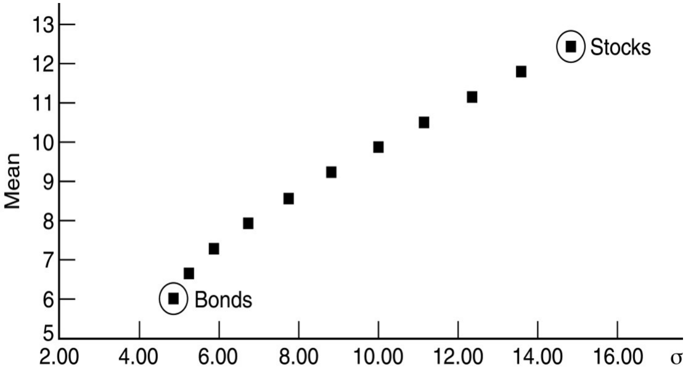

Figure: EGBG Figure 4-4 page 63

Investing in different stocks markets

- Assume it is possible to invest in a domestic stock portfolio and in a foreign stock portfolio (each of these is one asset)
- Assume that the domestic stock portfolio has an expected return of 12.5% and the standard deviation is 14.9%
- Assume that the foreign stock portfolio has an expected return of 10.5% and the standard deviation is 14.0%
- The correlation between returns is 0.33
- What happens for different wealth allocations?

Combining stocks – portfolio data

Proportion S&P	Proportion International	Mean Return	Standard Deviation
1	0	12.5	14.90
0.9	0.1	12.3	13.93
0.8	0.2	12.1	13.11
0.7	0.3	11.9	12.46
0.6	0.4	11.7	12.01
0.5	0.5	11.5	11.79
0.45	0.55	11.4	11.76
0.4	0.6	11.3	11.80
0.3	0.7	11.1	12.04
0.2	0.8	10.9	12.50
0.1	0.9	10.7	13.17
0	1	10.5	14.00

Figure: EGBG Figure 4-12 page 62

Portfolios of US and international stocks

Example (Expected return vs. standard deviation)

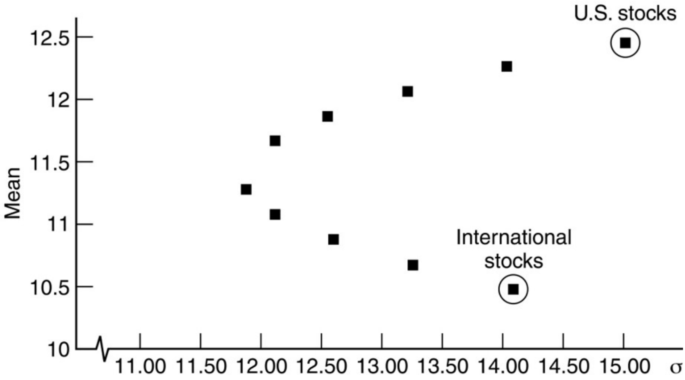

Figure: EGBG Figure 4-5 page 64

1 Making financial decisions

2 Multiple assets

3 The maths of the efficient frontier
- Matrix expressions
- Adding a risk free asset
- Two interest rates

4 Illustrations

5 Wrap up

Matrix expressions for expected return and variance

- Matrix multiplication makes it easy to formulate the expected return and variance of returns for a portfolio
- Assume that $\boldsymbol{m} = [\mu_1, \mu_2, \dots, \mu_n]$ is a one-row matrix of expected returns on different assets
- Assume that $\boldsymbol{w} = [w_1, w_2, \dots, w_n]$ are the fractions of the investors wealth invested in each asset
- The expected return on such a portfolio is $\mu_V = \boldsymbol{w}\boldsymbol{m}^T$
- The variance of returns for such a portfolio is $\sigma_V^2 = \boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T$, where $\boldsymbol{C}$ is the covariance matrix of the returns
- For two assets the variance is $\sigma_V^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2c_{12}$

Changing notation to Lando & Poulsen

- Assume that $\boldsymbol{\mu} = [\mu_1, \mu_2, \ldots, \mu_n]'$ is a vector of expected returns on different assets
- Assume that $\boldsymbol{w} = [w_1, w_2, \ldots, w_n]'$ are the fractions of the investors wealth invested in each asset
- Assume that $\boldsymbol{\Sigma}$ is the covariance matrix of the returns
- By definition, a covariance matrix is always positive semidefinite, but now it is assumed that it is **positive definite** and thus invertible
- Further, not all coordinates of $\boldsymbol{\mu}$ are equal
- For a given expected return μ_P, the objective is to find the portfolio with the lowest variance (or standard deviation)

Stating the optimization problem

- Consider the following problem

$$\min_{\boldsymbol{w}} \frac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}$$

- Under the following constraints:

$$\boldsymbol{w}'\boldsymbol{\mu} = \mu_P$$
$$\boldsymbol{w}'\boldsymbol{1} = 1$$

- Or in words, given a specific expected rate of return μ_P, what is the allocation of wealth into assets $1, \ldots, n$ that gives this expected return with the smallest variance

Setting up the Lagrange

$$\mathcal{L}(\boldsymbol{w}, \lambda_1, \lambda_2) = \frac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} - \lambda_1 \left(\boldsymbol{w}'\boldsymbol{\mu} - \mu_P\right) - \lambda_2 \left(\boldsymbol{w}'\mathbf{1} - 1\right)$$

FOCs for optimality are

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{w}} = \boldsymbol{\Sigma}\boldsymbol{w} - \lambda_1\boldsymbol{\mu} - \lambda_2\mathbf{1} = \mathbf{0} \qquad ([1])$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = \boldsymbol{w}'\boldsymbol{\mu} - \mu_P = 0 \qquad ([2])$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = \boldsymbol{w}'\mathbf{1} - 1 = 0 \qquad ([3])$$

Solving 1/3

First rearrange [1]:

$$\boldsymbol{\Sigma w} = \left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array}\right] \left[\begin{array}{c} \lambda_1 \\ \lambda_2 \end{array}\right] \Rightarrow$$

$$\boldsymbol{w} = \boldsymbol{\Sigma}^{-1} \left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array}\right] \left[\begin{array}{c} \lambda_1 \\ \lambda_2 \end{array}\right] \qquad ([4])$$

And then [2]-[3]:

$$\left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array}\right]' \boldsymbol{w} = \left[\begin{array}{c} \mu_P \\ 1 \end{array}\right] \qquad ([5])$$

Solving 2/3

Multiply [4] with $\begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}'$

$$\begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}' \boldsymbol{w} = \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}' \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \qquad ([6])$$

But [5] has the same left-hand side, meaning that

$$\begin{bmatrix} \mu_P \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}' \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}}_{A} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \qquad ([7])$$

If A is positive definite and then invertible, then [7] can be used to isolate λ_1 and λ_2 and this can be inserted in [4] to find $\boldsymbol{w}$.

Solving 3/3

A is invertible, because the coordinates of $\boldsymbol{\mu}$ are not all equal and because Σ and hence Σ^{-1} are invertible (trust this or study it yourself), so

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = A^{-1} \begin{bmatrix} \mu_P \\ 1 \end{bmatrix} \tag{[8]}$$

and finally the optimal portfolio weights for specific expected return μ_P is:

$$\hat{\boldsymbol{w}} = \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix} A^{-1} \begin{bmatrix} \mu_P \\ 1 \end{bmatrix} \tag{[9]}$$

Further, it follows that minimal portfolio variance for specific expected return μ_P is:

$$\begin{aligned} \sigma_P^2 &= \hat{\boldsymbol{w}}' \boldsymbol{\Sigma} \hat{\boldsymbol{w}} \\ &= \begin{bmatrix} \mu_P & 1 \end{bmatrix} A^{-1} \begin{bmatrix} \mu_P \\ 1 \end{bmatrix} \end{aligned} \tag{[10]}$$

Some words on A

A is defined

$$
\begin{aligned}
A &= \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}' \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1} \\ \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1} & \mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1} \end{bmatrix} := \begin{bmatrix} a & b \\ b & c \end{bmatrix}
\end{aligned}
$$

so the inverse of A is

$$
A^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}
$$

The minimum variance (or standard deviation) can then be expressed in terms of a, b and c:

$$
\sigma_P^2 = \frac{c\mu_P^2 - 2b\mu_P + a}{ac - b^2} \qquad \text{or} \qquad \sigma_P = \sqrt{\frac{c\mu_P^2 - 2b\mu_P + a}{ac - b^2}} \qquad ([11])
$$

Efficient frontier

Using the expression for σ_P as a function of μ_P, it's easy to find the portfolio with the smallest variance possible:

$$\frac{d\sigma_P^2}{d\mu_P} = \frac{2c\mu_P - 2b}{ac - b^2} = 0 \Rightarrow$$
$$\mu_{gmv} = b/c \quad \text{with} \quad \sigma_{gmv}^2 = 1/c$$

The portfolio weights can be expressed as

$$\hat{\boldsymbol{w}}_{gmw} = \frac{1}{c}\boldsymbol{\Sigma}^{-1}\mathbf{1}$$

In a (standard deviation, mean)-space or in a (variance, mean)-space, the **efficient frontier** or efficient portfolios is the upper half of the curve expressed by [11]. The efficient frontier will have expected specific returns greater than b/c and variances greater than $1/c$.

Inclusion of risk free asset

- Assume that a risk free asset exists with return μ_0
- Express returns as excess returns
  $\boldsymbol{\mu}^e = [\mu_1 - \mu_0, \mu_2 - \mu_0, \ldots, \mu_n - \mu_0]'$
- Assume that $\boldsymbol{w} = [w_1, w_2, \ldots, w_n]'$ are the fractions of the investors wealth invested in each risky asset
- Assume that $w_0 = 1 - \boldsymbol{w}'\mathbf{1}$ is invested in the risk free asset
- For a given expected excess return μ_P^e, the objective is to find the portfolio with the lowest variance (or standard deviation)

Stating the optimization problem

- Consider the following problem

$$\min_{\boldsymbol{w}} \tfrac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}$$

- Under the following constraints:

$$\boldsymbol{w}'\boldsymbol{\mu}^e = \mu_P^e$$

- Or in words, given a specific expected excess rate of return μ_P^e, what is the allocation of wealth into assets $1, \ldots, n$ that gives this expected excess return with the smallest variance

Setting up the Lagrange

$$\mathcal{L}(\boldsymbol{w}, \lambda) = \tfrac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} - \lambda_1 \left(\boldsymbol{w}'\boldsymbol{\mu}^e - \mu_P^e\right)$$

FOCs for optimality are

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{w}} = \boldsymbol{\Sigma}\boldsymbol{w} - \lambda\boldsymbol{\mu}^e = \mathbf{0} \quad ([12])$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \boldsymbol{w}'\boldsymbol{\mu}^e - \mu_P^e = 0 \quad ([13])$$

Solving 1/3

First rearrange [12]:

$$\begin{aligned} \boldsymbol{\Sigma w} &= \lambda \boldsymbol{\mu}^e \Rightarrow \\ \boldsymbol{w} &= \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e \lambda \end{aligned} \qquad ([14])$$

And then [13]:

$$(\boldsymbol{\mu}^e)' \boldsymbol{w} = \mu_P^e \qquad ([15])$$

Solving 2/3

Multiply [14] with $(\boldsymbol{\mu}^e)'$

$$(\boldsymbol{\mu}^e)'\boldsymbol{w} = (\boldsymbol{\mu}^e)'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e\lambda \qquad ([16])$$

But [15] has the same left-hand side, meaning that

$$\mu_P^e = (\boldsymbol{\mu}^e)'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e\lambda \qquad ([17])$$

Because Σ is positive definite and then invertible, then $(\boldsymbol{\mu}^e)'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e > 0$ and λ can be inserted in [14] to find $\boldsymbol{w}$.

Solving 3/3

So

$$\lambda = \frac{\mu_P^e}{(\boldsymbol{\mu}^e)' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \qquad ([18])$$

and the optimal portfolio weights for specific expected excess return μ_P^e is:

$$\boldsymbol{w} = \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e \frac{\mu_P^e}{(\boldsymbol{\mu}^e)' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \qquad ([19])$$

Further, it follows that minimal portfolio variance for specific expected excess return μ_P^e is:

$$\sigma_P^2 = \frac{(\mu_P^e)^2}{(\boldsymbol{\mu}^e)' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \qquad ([20])$$

Capital Market Line

The link between σ_P and of μ_P^e is:

$$\sigma_P = \frac{\mu_P^e}{\sqrt{(\boldsymbol{\mu}^e)' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e}}$$

or equivalently (this is called the Capital Market Line)

$$\mu_P = \sigma_P \sqrt{(\boldsymbol{\mu}^e)' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} + \mu_0$$

- It is reasonable to assume than $\mu_0 < \min\{\mu_1, \ldots, \mu_n\}$ (Why?)
- The CML touches the efficient frontier without the risk free asset in exactly one point (Why?)

Tangent Portfolio

The portfolio where everything is invested in risky assets is called the **tangent portfolio**. The excess return of the tangent portfolio is

$$\mu^e_{tan} = \frac{(\boldsymbol{\mu}^e)' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e}{\mathbf{1}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \qquad ([21])$$

with

$$\sigma_{tan} = \frac{\sqrt{(\boldsymbol{\mu}^e)' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e}}{\mathbf{1}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e}$$

One lending and one borrowing rate

If there is one borrowing rate and one lending rate, then the efficient portfolios will be

- The CML using the lending rate for $\sigma_P < \sigma^l_{tan}$
- The CML using the borrowing rate for $\sigma_P > \sigma^b_{tan}$
- The efficient frontier for the risky assets only $\sigma^l_{tan} < \sigma_P < \sigma^b_{tan}$

1 Making financial decisions

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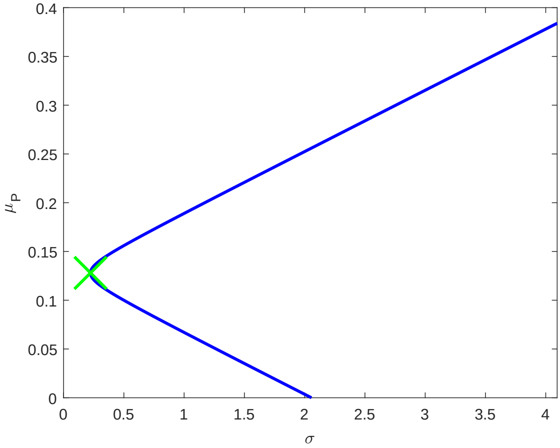
4 Illustrations
- Lando & Poulsen numbers
- Combination of stock and bond portfolios
- Combination of two stock portfolios

5 Wrap up

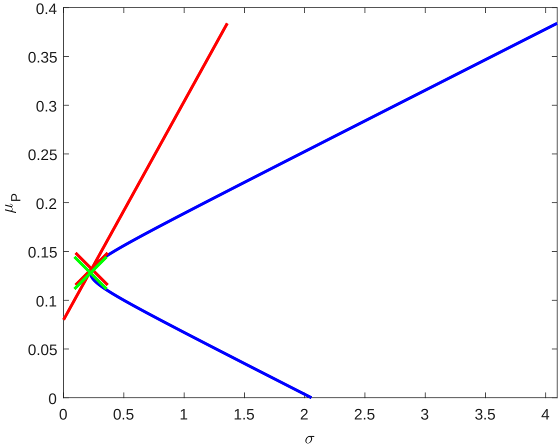
Risky Assets Only

Example (Minimum variance portfolios)



Including Risk Free Asset

Example (Minimum variance portfolios)

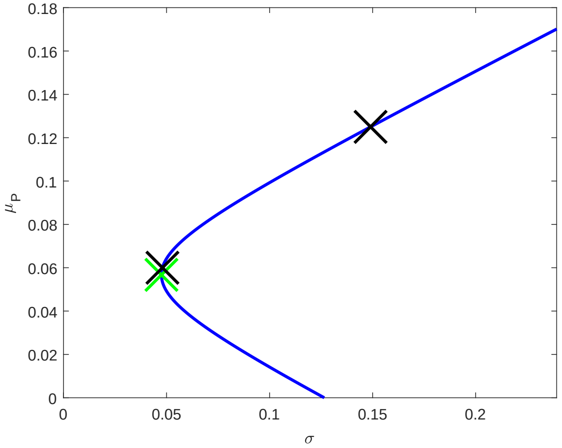


Investing in stocks and bonds

- Assume it is possible to invest in a stock portfolio and in a bond portfolio (each of these is one asset)
- Assume that the stock portfolio has an expected return of 12.5% and the standard deviation is 14.9%
- Assume that the bond portfolio has an expected return of 6% and the standard deviation is 4.8%
- The correlation between returns is 0.45
- What happens for different wealth allocations?

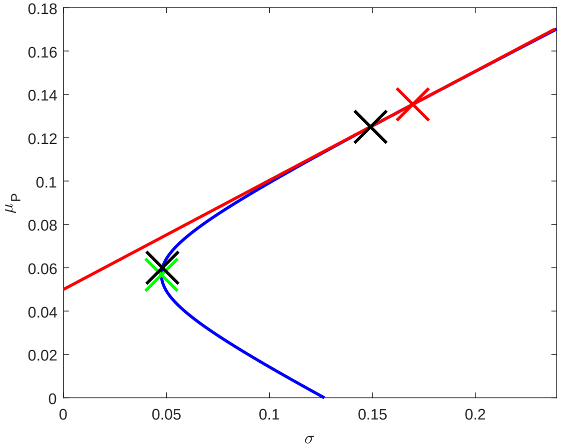
Portfolios of stocks and bonds

Example (Expected return vs. standard deviation)



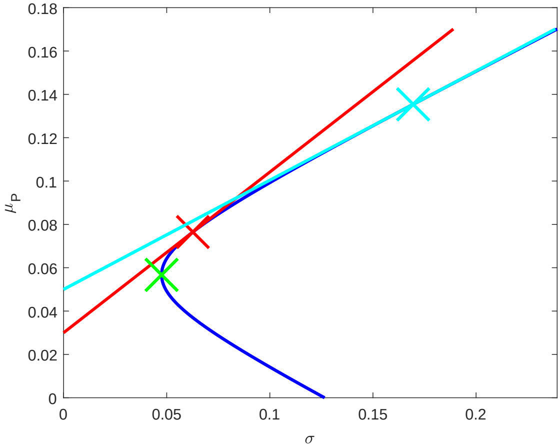
Portfolios of stocks and bonds and risk free asset

Example (Expected return vs. standard deviation)



Portfolios of stocks and bonds and two different rates

Example (Expected return vs. standard deviation)

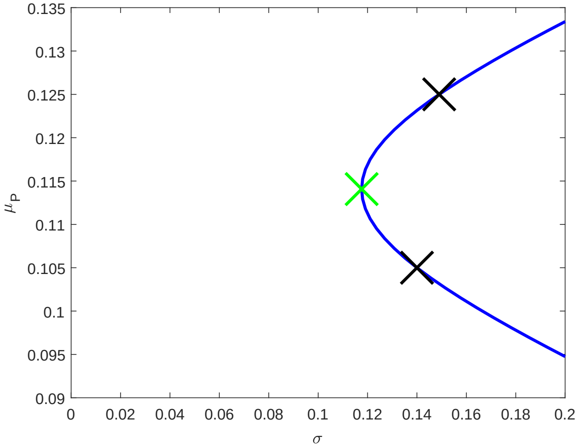


Investing in different stocks markets

- Assume it is possible to invest in a domestic stock portfolio and in a foreign stock portfolio (each of these is one asset)
- Assume that the domestic stock portfolio has an expected return of 12.5% and the standard deviation is 14.9%
- Assume that the foreign stock portfolio has an expected return of 10.5% and the standard deviation is 14%
- The correlation between returns is 0.33
- What happens for different wealth allocations?

Portfolios of stocks and bonds

Example (Expected return vs. standard deviation)



1 Making financial decisions

2 Multiple assets

3 The maths of the efficient frontier

4 Illustrations

5 Wrap up
- Literature
- Next week

Literature

- Textbook chapter 3 until 3.4
  - The two asset case
  - The examples with numbers
  - The mathematics presented
- Lando & Poulsen chapter 9
  - The mathematics presented

Next week

Next week, the following material is covered:

- The stuff we didn't manage to complete today
- Short selling constraints

Read the textbook, Lando & Poulsen and the slides again for preparation