7

The Correlation Structure of Security Returns

THE SINGLE-INDEX MODEL

In the first four chapters of this book we outlined the basics of modern portfolio theory. The core of the theory, as described in these chapters, is not new; in fact, it was presented as early as 1956 in Markowitz's pioneering article and subsequent book. The reader, noting that the theory is over 30 years old, might well ask what has happened since the theory was developed. Furthermore, if you had knowledge about the actual practices of financial institutions, you might well ask why the theory took so long to be used by financial institutions. The answers to both these questions are closely related. Most of the research on portfolio management in the last 30 years has concentrated on methods for *implementing* the basic theory. Many of the breakthroughs in implementation have been quite recent, and it is only with these new contributions that portfolio theory becomes readily applicable to the management of actual portfolios.

In the next three chapters we are concerned with the implementation of portfolio theory. Breakthroughs in implementation fall into two categories: The first concerns a simplification of the amount and type of input data needed to perform portfolio analysis. The second involves a simplification of the computational procedure needed to calculate optimal portfolios. As will soon become clear, these issues are interdependent. Furthermore, their resolution vastly simplifies portfolio analysis. This results in the ability to describe the problem and its solution in relatively simple terms—terms that have intuitive as well as analytical meaning, and terms to which practicing security analysts and portfolio managers can relate.

In this chapter we begin the problem of simplifying the inputs to the portfolio problem. We start with a discussion of the amount and type of information needed to solve a portfolio problem. We then discuss the oldest and most widely used simplification of the portfolio structure: the single-index model. The nature of the model as well as some estimating techniques are examined.

In Chapter 8 we discuss alternative simplified representations of the portfolio problem. In particular, we will be concerned with other ways to represent and predict the correlation structure between returns. Finally, in the last chapter dealing with implementation we will show how each of the techniques that have been developed to simplify the input to portfolio analysis can be used to reduce and simplify the calculations needed to find optimal portfolios.

Most of Chapters 7 and 8 will be concerned with simplifying and predicting the correlation structure of returns. Many of the single- and multi-index models discussed in these chapters were developed to aid in portfolio management. Lately, however, these models have been used for other purposes that are often viewed as being as important as their use in portfolio analysis. Although many of these other uses will be detailed later in the book, we briefly describe some of them at the end of this chapter and in Chapter 8.

THE INPUTS TO PORTFOLIO ANALYSIS

Let us return to a consideration of the portfolio problem. From earlier chapters we know that to define the efficient frontier we must be able to determine the expected return and standard deviation of return on a portfolio. We can write the expected return on any portfolio as

$$\bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i \tag{7.1}$$

while the standard deviation of return on any portfolio can be written as

$$\sigma_P = \left[\sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j \sigma_i \sigma_j \rho_{ij} \right]^{1/2} \tag{7.2}$$

These equations define the input data necessary to perform portfolio analysis. From Equation (7.1) we see that we need estimates of the expected return on each security that is a candidate for inclusion in our portfolio. From Equation (7.2) we see that we need estimates of the variance of each security, plus estimates of the correlation between each possible pair of securities for the stocks under consideration. The need for estimates of correlation coefficients differs both in magnitude and substance from the two previous requirements. Let's see why.

The principal job of the security analyst traditionally has been to estimate the future performance of stocks he or she follows. At a minimum this means producing estimates of expected returns on each stock he follows.[1]

With the increased attention that "risk" has received in recent years, more and more analysts are providing estimates of risk as well as return. The analyst who estimates the expected return of a stock should also be in a position to estimate the uncertainty of that return.

Correlations are an entirely different matter. Portfolio analysis calls for estimates of the pairwise correlation between all stocks that are candidates for inclusion in a portfolio. Most firms organize their analysts along traditional industry lines. One analyst might follow steel stocks or, perhaps in a smaller firm, all metal stocks. A second analyst might follow chemical stocks. But portfolio analysis calls for these analysts not only to estimate how a particular steel stock will move in relationship to another steel stock, but also how a particular steel stock will move in relationship to a particular chemical stock or drug stock. There is no nonoverlapping organizational structure that allows such estimates to be directly produced.

[1] Whether the analyst's estimates contain information or whether one is better off estimating returns from an equilibrium model (such as those to be presented in Chapters 13 and 14) is an open question. We have more to say about this later. However, the reader should note that portfolio selection models can help to answer this question.

The problem is made more complex by the number of estimates required. Most financial institutions follow between 150 and 250 stocks. To employ portfolio analysis, the institution needs estimates of between 150 and 250 expected returns and 150 and 250 variances. Let us see how many correlation coefficients it needs. If we let N stand for the number of stocks a firm follows, then it has to estimate ρ_{ij} for all pairs of securities i and j. The first index i can take on N values (one for each stock); the second can take on $(N - 1)$ values (remember $j \neq i$). This gives us $N(N - 1)$ correlation coefficients. However, since the correlation coefficient between stocks i and j is the same as that between stocks j and i, we have to estimate only $N(N - 1)/2$ correlations. The institution that follows between 150 and 250 stocks needs between 11,175 and 31,125 correlation coefficients. The sheer number of inputs is staggering.

It seems unlikely that analysts will be able to directly estimate correlation structures. Their ability to do so is severely limited by the nature of feasible organizational structure and the huge number of correlation coefficients that must be estimated. Recognition of thi has motivated the search for the development of models to describe and predict the corre lation structure between securities. In this chapter and in Chapter 8 we discuss some c these models and examine empirical tests of their performance.

The models developed for forecasting correlation structures fall into two categorie: index models and averaging techniques. The most widely used technique assumes that th co-movement between stocks is due to a single common influence or index. This model appropriately called the single-index model. The single-index model is used not only estimating the correlation matrix, but also in efficient market tests (discussed later) and equilibrium tests, where it is called a return-generating process. The rest of this chapter devoted to a discussion of the properties of this model.

SINGLE-INDEX MODELS: AN OVERVIEW

Casual observation of stock prices reveals that when the market goes up (as measured any of the widely available stock market indexes), most stocks tend to increase in pri and when the market goes down, most stocks tend to decrease in price. This suggests th one reason security returns might be correlated is because of a common response to ma ket changes, and a useful measure of this correlation might be obtained by relating t return on a stock to the return on a stock market index. The return on a stock can be wr ten as[2]

$$R_i = a_i + \beta_i R_m$$

where

a_i is the component of security i's return that is independent of the market's perfor ance—a random variable.

R_m is the rate of return on the market index—a random variable.

β_i is a constant that measures the expected change in R_i given a change in R_m.

This equation simply breaks the return on a stock into two components, that part du the market and that part independent of the market. β_i in the expression measures h

[2]The return on the index is identical, in concept, to the return on a common stock. It is the return the inv would earn if he or she held a portfolio with a composition identical to the index. Thus, to compute this re the dividends that would be received from holding the index should be calculated and combined with the changes on the index.

sensitive a stock's return is to the return on the market. A β_i of 2 means that a stock's return is expected to increase (decrease) by 2% when the market increases (decreases) by 1%. Similarly, a β of 0.5 indicates that a stock's return is expected to increase (decrease) by $\frac{1}{2}$ of 1% when the market increases (decreases) by 1%.[3]

The term a_i represents that component of return insensitive to (independent of) the return on the market. It is useful to break the term a_i into two components. Let α_i denote the expected value of a_i and let e_i represent the random (uncertain) element of a_i. Then

$$a_i = \alpha_i + e_i$$

where e_i has an expected value of zero. The equation for the return on a stock can now be written as

$$R_i = \alpha_i + \beta_i R_m + e_i \tag{7.3}$$

Once again, note that both e_i and R_m are random variables. They each have a probability distribution and a mean and standard deviation. Let us denote their standard deviations by σ_{ei} and σ_m, respectively. Up to this point we have made no simplifying assumptions. We have written return as the sum of several components but these components, when added together, must by definition be equal to total return.

It is convenient to have e_i uncorrelated with R_m. Formally, this means that

$$\operatorname{cov}(e_i R_m) = E\left[(e_i - 0)\left(R_m - \bar{R}_m\right)\right] = 0$$

If e_i is uncorrelated with R_m, it implies that how well Equation (7.3) describes the return on any security is independent of what the return on the market happens to be. Estimates of α_i, β_i, and σ_{ei}^2 are often obtained from time series-regression analysis.[4] Regression analysis is one technique that guarantees that e_i and R_m will be uncorrelated, at least over the period to which the equation has been fit. All of the characteristics of single-index models described to this point are definitions or can be made to hold by construction. There is one further characteristic of single-index models: it holds only by assumption. This assumption is the characteristic of single-index models that differentiates them from other models used to describe the covariance structure.

The key assumption of the single-index model is that e_i is independent of e_j for all values of i and j or, more formally, $E(e_i e_j) = 0$. This implies that the only reason stocks vary together, systematically, is because of a common co-movement with the market. There are no effects beyond the market (e.g., industry effects) that account for co-movement between securities. We will have more to say about this in our discussion of multi-index models in Chapter 8. However, at this time, note that, unlike the independence of e_i and R_m, there is nothing in the normal regression method used to estimate α_i, β_i, and σ_{ei}^2 that forces this to be true. It is a simplifying assumption that represents an approximation to reality. How well this model performs will depend, in part, on how good (or bad) this approximation is. Let us summarize the single-index model:

BASIC EQUATION

[3]We are illustrating the single-index model with a stock market index. It is not necessary that the index used be a stock market index. The selection of the appropriate index is an empirical rather than a theoretical question. However, anticipating the results of future chapters, the results should be better when a broad-based market-weighted index is used, such as the S&P 500 index or the New York Stock Exchange Index.

[4]This will be discussed in more detail later in the chapter.

$R_i = \alpha_i + \beta_i R_m + e_i$ for all stocks $i = 1, ..., N$

BY CONSTRUCTION

1. Mean of $e_i = E(e_i) = 0$ for all stocks $i = 1, ..., N$

BY ASSUMPTION

1. Index unrelated to unique return: $E[e_i(R_m - \bar{R}_m)] = 0$ for all stocks $i = 1, ..., N$
2. Securities only related through common response to market: $E(e_i e_j) = 0$ for all pairs of stocks $i = 1, ..., N$ and $j = 1, ..., N$ but $i \neq j$

BY DEFINITION

1. Variance of $e_i = E(e_i)^2 = \sigma_{ei}^2$ for all stocks $i = i, ..., N$
2. Variance of $R_m = E(R_m - \bar{R}_m)^2 = \sigma_m^2$

In the subsequent section we derive the expected return, standard deviation, and covarianc when the single-index model is used to represent the joint movement of securities. Th results are

1. The mean return, $\bar{R}_i = \alpha_i + \beta_i \bar{R}_m$.
2. The variance of a security's return, $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2$.
3. The covariance of returns between securities i and j, $\sigma_{ij} = \beta_i \beta_j \sigma_m^2$.

Note that the expected return has two components: a unique part α_i and a market-relate part $\beta_i \bar{R}_m$. Likewise, a security's variance has the same two parts, unique risk σ_{ei}^2 ar market-related risk $\beta_i^2 \sigma_m^2$. In contrast, the covariance depends only on market risk. This what we meant earlier when we said that the single-index model implied that the only re son securities move together is a common response to market movements. In this sectic of the text delineated by the solid line we derive these results. The reader uninterested the derivation can note the results and then skip to the end of the section.

The expected return on a security is

$$E(R_i) = E[\alpha_i + \beta_i R_m + e_i]$$

Since the expected value of the sum of random variables is the sum of the expected va ues, we have

$$E(R_i) = E(\alpha_i) + E(\beta_i R_m) + E(e_i)$$

α_i and β_i are constants and by construction the expected value of e_i is zero. Thus,

$$E(R_i) = \alpha_i + \beta_i \bar{R}_m \qquad \textbf{Result}$$

The variance of the return on any security is

$$\sigma_i^2 = E(R_i - \bar{R}_i)^2$$

Substituting for R_i and $\bar{R}_i$ from the expression above yields

$$\sigma_i^2 = E\left[(\alpha_i + \beta_i R_m + e_i) - (\alpha_i + \beta_i \bar{R}_m)\right]^2$$

Rearranging and noting that the α's cancel yields

$$\sigma_i^2 = E\left[\beta_i(R_m - \bar{R}_m) + e_i\right]^2$$

Squaring the terms in the brackets yields

$$\sigma_i^2 = \beta_i^2 E(R_m - \bar{R}_m)^2 + 2\beta_i E\left[e_i(R_m - \bar{R}_m)\right] + E(e_i)^2$$

Recall that by assumption (or in some cases by construction) $E[e_i(R_m - \bar{R}_m)] = 0$. Thus,

$$\sigma_i^2 = \beta_i^2 E(R_m - \bar{R}_m)^2 + E(e_i)^2$$

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2 \qquad \textbf{Result 2}$$

The covariance between any two securities can be written as

$$\sigma_{ij} = E\left[(R_i - \bar{R}_i)(R_j - \bar{R}_j)\right]$$

Substituting for R_i, $\bar{R}_i$, R_j, and $\bar{R}_j$ yields

$$\sigma_{ij} = E\Big\{\left[(\alpha_i + \beta_i R_m + e_i) - (\alpha_i + \beta_i \bar{R}_m)\right] \cdot \left[(\alpha_j + \beta_j R_m + e_j) - (\alpha_j + \beta_j \bar{R}_m)\right]\Big\}$$

Simplifying by canceling the α's and combining the terms involving β's yields

$$\sigma_{ij} = E\left[\left(\beta_i(R_m - \bar{R}_m) + e_i\right)\left(\beta_j(R_m - \bar{R}_m) + e_j\right)\right]$$

Carrying out the multiplication

$$\sigma_{ij} = \beta_i\beta_j E(R_m - \bar{R}_m)^2 + \beta_j E\left[e_i(R_m - \bar{R}_m)\right] + \beta_i E\left[e_j(R_m - \bar{R}_m)\right] + E(e_i e_j)$$

Since the last three terms are zero, by assumption

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2 \qquad \textbf{Result 3}$$

These results can be illustrated with a simple example. Consider the returns on a stock and a market index shown in the first two columns of Table 7.1. These returns are what an investor might have observed over the prior five months. Now consider the values for the single-index model shown in the remaining columns of the table. Column three just reproduced column one and is the return on the security. Accept for the moment that $\beta_i = 1.5$. The fifth column is just the second column times 1.5 or the market return times a Beta of 1.5. Where does e_i come from? Recall that the average value of e_i is zero. If the average value of e_i is zero, then the sum of e_i is also zero. The single-index model is an equality. The return over the 5 periods for the stock is 40; 30 of the 40 is market-related return, hence 10 must be non-market-related or unique. If e_i sums to zero, then for the single-index model to be an equality, α_i must sum to 10. Since α_i is a constant and there are 5 periods, α_i is $\frac{10}{5}$ or 2 per period.

Given the values for α_i and for $\beta_i R_m$, and since the single-index model is an equality, e_i

Table 7.1 Decomposition of Returns for the Single-Index Model

Month	1 Return on Stock	2 Return on Market	3 R_i		4 α_i		5 $\beta_i R_m$	6 $+ e_i$
1	10	4	10	=	2	+	6	+2
2	3	2	3	=	2	+	3	−2
3	15	8	15	=	2	+	12	+1
4	9	6	9	=	2	+	9	−2
5	3	0	3	=	2	+	0	+1
	40	20	40		10		30	0

is whatever is necessary to make the left- and right-hand sides the same. For example, in the first period the sum of α_i and $\beta_i R_m$ is 8. Since the return on the security in the first period is 10, e_i is + 2.

The reader should now understand where all the values of the single-index model come from except β_i. β_i divides return into market-related and unique return. When β_i is set equal to 1.5, the market return is independent of the residual return e_i. A lower value of e_i leaves some market return in e_i and the covariance of e_i with the market is positive. A β_i greater than 1.5 removes too much market return and results in a negative covariance between e_i and the market. Thus the value of β_i is unique and is the value that exactly separates market from unique return, making the covariance between R_m and e_i zero.

Before leaving the simple example, let's apply the formulas presented earlier. The mean return on the security is

$$\bar{R}_i = 40/5 = 8$$

using the formula from the single-index model.

$$\bar{R}_i = \alpha_i + \beta_i \bar{R}_m = 2 + 1.5(4) = 8$$

The variance of security i is calculating from the formula derived for the single-index model.

$$\begin{aligned}\sigma_i^2 &= \beta_i^2 \sigma_m^2 + \sigma_{ei}^2 \\ &= (1.5)^2(8) + 2.8 \\ &= 20.8\end{aligned}$$

Having explained the simple example, we can turn to the calculation of the expected return and variance of any portfolio if the single-index model holds. The expected return on any portfolio is given by

$$\bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i$$

Substituting for $\bar{R}_i$ we obtain

$$\bar{R}_P = \sum_{i=1}^{N} X_i \alpha_i + \sum_{i=1}^{N} X_i \beta_i \bar{R}_m \tag{7.4}$$

We know that the variance of a portfolio of stocks is given by

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j \sigma_{ij}$$

Substituting in the results stated above for σ_i^2 and σ_{ij}, we obtain

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2 \tag{7.7}$$

There are many alternative ways of estimating the parameters of the single-index model. From Equations (7.4) and (7.7) it is clear that expected return and risk can be estimated for any portfolio if we have an estimate of α_i for each stock, an estimate of β_i for each stock, an estimate of σ_{ei}^2 for each stock, and, finally, an estimate of both the expected return ($\bar{R}_m$) and variance (σ_m^2) for the market. This is a total of $3N + 2$ estimates. For an institution following between 150 and 250 stocks, the single-index model required between 452 and 752 estimates. Compare this with the 11,175–31,125 correlation estimates or 11,475–31,625 total estimates required when no simplifying structure was assumed. Furthermore, note that there is no requirement for direct estimates of the joint movement of securities, only estimates of the way each security moves with the market. A nonoverlapping organizational structure can produce all the required estimates.

The model can also be employed if analysts supply estimates of expected return for each stock, the variance of the return on each stock, the Beta (β_i) for each stock, and the variance of the market return.[5] This is $3N + 1$ estimates. This alternative set of estimates has the advantage that they are in more familiar terms.

We have discussed means and variances before. The only new variable is Beta. The Beta is simply a measure of the sensitivity of a stock to market movements.

Before we discuss alternative ways of estimating Betas, let us examine some of the characteristics of the single-index model.

CHARACTERISTICS OF THE SINGLE-INDEX MODEL

Define the Beta on a portfolio β_P as a weighted average of the individual β_is on each stock in the portfolio where the weights are the fraction of the portfolio invested in each stock. Then

$$\beta_P = \sum_{i=1}^{N} X_i \beta_i$$

Similarly define the Alpha on the portfolio α_P as

$$\alpha_P = \sum_{i=1}^{N} X_i \alpha_i$$

Then Equation (7.4) can be written as

$$\bar{R}_P = \alpha_P + \beta_P \bar{R}_m$$

If the portfolio P is taken to be the market portfolio (all stocks held in the same proportions as they were in constructing R_m), then the expected return on P must be $\bar{R}_m$. From the above equation the only values of β_P and α_P that guarantee $\bar{R}_P = \bar{R}_m$ for any choice of $\bar{R}_m$ is α_P equal to 0 and β_P equal to 1. Thus, the Beta on the market is 1 and stocks are thought of as being more or less risky than the market, according to whether their Beta is larger or smaller than 1.

Let us look further into the risk of an individual security. Equation (7.5) is

[5]The fact that these inputs are equivalent to those discussed earlier is easy to show. The expected returns can be used directly to estimate the expected return on a portfolio

$$\bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i$$

The estimates of the variance of return on a stock, the variance of the market, and the Beta on each stock can be used to derive estimates of its residual risk by noting that

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2$$

In addition, this structure is natural for those institutions that want analysts' estimates of means and variances and model estimates of correlations or covariances.

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2\beta_i^2\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{ei}^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} X_iX_j\beta_i\beta_j\sigma_m^2$$

In the double summation $i \neq j$, if $i = j$, then the terms would be $X_iX_i\beta_i^2\sigma_m^2$. But these are exactly the terms in the first summation. Thus, the variance on the portfolio can be written as

$$\sigma_P^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} X_iX_j\beta_i\beta_j\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{ei}^2$$

Or by rearranging terms

$$\sigma_P^2 = \left(\sum_{i=1}^{N} X_i\beta_i\right)\left(\sum_{j=1}^{N} X_j\beta_j\right)\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{ei}^2$$

Thus, the risk of the investor's portfolio could be represented as

$$\sigma_P^2 = \beta_P^2\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{ei}^2$$

Assume for a moment that an investor forms a portfolio by placing equal amounts of money into each of N stocks. The risk of this portfolio can be written as[6]

$$\sigma_P^2 = \beta_P^2\sigma_m^2 + \frac{1}{N}\left(\sum_{i=1}^{N}\frac{1}{N}\sigma_{ei}^2\right)$$

Look at the last term. This can be expressed as $1/N$ times the average residual risk in the portfolio. As the number of stocks in the portfolio increases, the importance of the average residual risk,

$$\sum_{i=1}^{N}\frac{\sigma_{ei}^2}{N}$$

diminishes drastically. In fact, as Table 7.2 shows, the residual risk falls so rapidly tha most of it is effectively eliminated on even moderately sized portfolios.[7]

The risk that is not eliminated as we hold larger and larger portfolios is the risk associ ated with the term β_P. If we assume that residual risk approaches zero, the risk of the port folio approaches

$$\sigma_P = \left[\beta_P^2\sigma_m^2\right]^{1/2} = \beta_P\sigma_m = \sigma_m\left[\sum_{i=1}^{N} X_i\beta_i\right]$$

Since σ_m is the same, regardless of which stock we examine, the measure of the contri bution of a security to the risk of a large portfolio is β_i.

The risk of an individual security is $\beta_i^2\sigma_m^2 + \sigma_{ei}^2$. Since the effect of σ_{ei}^2 on portfolio ris

[6]Examining the expression for the variance of portfolio P shows that the assumptions of the single-index mod are inconsistent with $\sigma_P^2 = \sigma_m^2$. However, the approximation is very close. See Fama [34] for a detailed discu sion of this issue.

[7]To the extent that the single-index model is not a perfect description of reality and residuals from the mark model are correlated across securities, residual risk does not fall this rapidly. However, for most portfolios t amount of positive correlation present in the residuals is quite small and residual risk declines rapidly as the nur ber of securities in the portfolio increases.

Table 7.2 Residual Risk and Portfolio Size

Number of Securities	Residual Risk (Variance) Expressed as a Percent of the Residual Risk Present in a One-Stock Portfolio with σ_{ei}^2 a Constant
1	100
2	50
3	33
4	25
5	20
10	10
20	5
100	1
1000	0.1

can be made to approach zero as the portfolio gets larger, it is common to refer to σ_{ei}^2 as diversifiable risk.[8] However, the effect of $\beta_i^2\sigma_m^2$ on portfolio risk does not diminish as N gets larger. Since σ_m^2 is a constant with respect to all securities, β_i is the measure of a security's nondiversifiable risk.[9] Since diversifiable risk can be eliminated by holding a large enough portfolio, β_i is often used as the measure of a security's risk.

ESTIMATING BETA

The use of the single-index model calls for estimates of the Beta of each stock that is a potential candidate for inclusion in a portfolio. Analysts could be asked to provide subjective estimates of Beta for a security or a portfolio. On the other hand, estimates of future Beta could be arrived at by estimating Beta from past data and using this historical Beta as an estimate of the future Beta. There is evidence that historical Betas provide useful information about future Betas. Furthermore, some interesting forecasting techniques have been developed to increase the information that can be extracted from historical data. Because of this, even the firm that wishes to use analysts' subjective estimates of future Betas should start with (supply analysts with) the best estimates of Beta available from historical data. The analyst can then concentrate on the examination of influences that are expected to change Beta in the future. In the rest of this chapter we examine some of the techniques that have been proposed for estimating Beta. These techniques can be classified as measuring historical Betas, correcting historical Betas for the tendency of historical Betas to be closer to the mean when estimated in a future period, and correcting historical estimates by incorporating fundamental firm data.

Estimating Historical Betas

In Equation (7.3) we represented the return on a stock as

$$R_i = \alpha_i + \beta_i R_m + e_i$$

This equation is expected to hold at each moment in time, although the values of α_i, β_i, or σ_{ei}^2 might differ over time. When looking at historical data, one cannot directly observe α_i, β_i, or σ_{ei}^2. Rather, one observes the past returns on the security and the market. If α_i, β_i,

[8]An alternative nomenclature calls this nonmarket or unsystematic risk.

[9]An alternative nomenclature calls this market risk or systematic risk.

and σ^2_{ei} are assumed to be constant through time, then the same equation is expected to hold at each point in time. In this case, a straightforward procedure exists for estimating α_i, β_i, and σ^2_{ei}.

Notice that Equation (7.3) is an equation of a straight line. If σ^2_{ei} were equal to zero, then we could estimate α_i and β_i with just two observations. However, the presence of the random variable e_i means that the actual return will form a scatter around the straight line. Figure 7.1 illustrates this pattern. The vertical axis is the return on security i and the horizontal axis is the return on the market. Each point on the diagram is the return on stock i over a particular time interval, for example, one month (t) plotted against the return on the market for the same time interval. The actual observed returns lie on and around the true relationship (shown as a solid line). The greater σ^2_{ei}, the greater the scatter around the line, and since we do not actually observe the line, the more uncertain we are about where it is. There are a number of ways of estimating where the line might be, given the observed scatter of points. Usually, we estimate the location of the line using regression analysis.

This procedure could be thought of as first plotting R_{it} versus R_{mt} to obtain a scatter of points such as that shown in Figure 7.1. Each point represents the return on a particular stock and the return on the market in one month. Additional points are obtained by plotting the two returns in successive months. The next step is to fit that straight line to the data that minimized the sum of the squared deviation from the line in the vertical (R_{it}) direction. The slope of this straight line would be our best estimate of Beta over the period to which the line was fit, and intercept would be our best estimate of Alpha (α_i).[10]

More formally, to estimate the Beta for a firm for the period from $t = 1$ to $t = 60$ via regression analysis use

$$\beta_i = \frac{\sigma_{im}}{\sigma^2_m} = \frac{\sum_{t=1}^{60}\left[\left(R_{it} - \overline{R}_{it}\right)\left(R_{mt} - \overline{R}_{mt}\right)\right]}{\sum_{t=1}^{60}\left(R_{mt} - \overline{R}_{mt}\right)^2}$$

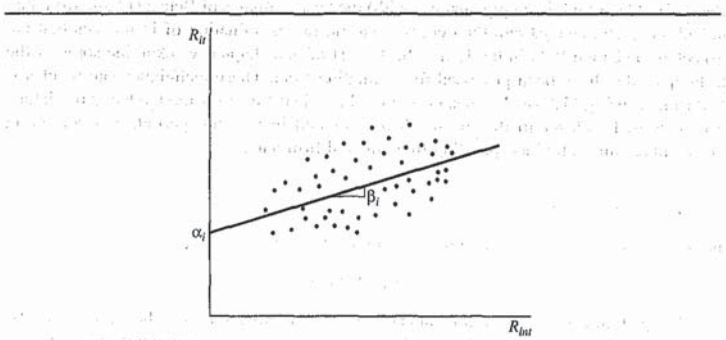

Figure 7.1

[10]If R_{it} and R_{mt} come from a bivariate normal distribution, the unbiased and most efficient estimates of α_i and are those that come from regressing R_{it} against R_{mt}, the procedure described above.

and to estimate Alpha use[11]

$$\alpha_i = \overline{R}_{it} - \beta_i \overline{R}_{mt}$$

To learn how this works on a simple example let us return to Table 7.1. We used the data in 7.1 to show how Beta interacted with returns. But now assume that all you observed was columns 1 and 2 or the return on the stock and the return on the market. To estimate Beta we need to estimate the covariance between the stock and the market. The average return on the stock was 40/5 = 8, whereas on the market it was 20/5 = 4. The Beta value for the stock is the covariance of the stock with the market divided by the variance of the market or

$$\beta_i = \frac{\left[\sum_{t=1}^{5}\left(R_i - \overline{R}_i\right)\left(R_m - \overline{R}_m\right)\right]\Big/5}{\sum_{t=1}^{5}\left(R_m - \overline{R}_m\right)^2/5}$$

The covariance is found as follows:

Month	Stock Return Minus Mean		Market Return Minus Mean		Value
1	(10 − 8)	×	(4 − 4)	=	0
2	(3 − 8)	×	(2 − 4)	=	10
3	(15 − 8)	×	(8 − 4)	=	28
4	(9 − 8)	×	(6 − 4)	=	2
5	(3 − 8)	×	(0 − 4)	=	20
				Total	60

The covariance is 60/5 = 12. The variance of the market return is the average of the sum of squared deviation from the mean

$$\sigma^2_m = \left[(4-4)^2 + (2-4)^2 + (8-4)^2 + (6-4)^2 + (0-4)^2\right]\Big/5 = 8$$

Thus Beta = 12/8 = 1.5

This value of Beta is identical to the number used in constructing Table 7.1.

Alpha can be computed by taking the difference between the average security return and Beta times the average return on the market.

$$\alpha_i = 8 - (1.5)(4) = 2$$

[11]Two other statistics of interest can be produced by this analysis. First, the size of σ^2_{ei} over the estimation period can be found by looking at the variance of the deviations of the actual return from that predicted by the model:

$$\sigma^2_{ei} = \frac{1}{60}\sum_{t=1}^{60}\left[R_{it} - (\alpha_i + \beta_i R_{mt})\right]^2$$

Remember that, in performing regression analysis, one often computes a coefficient of determination. The coefficient of determination is a measure of association between two variables. In this case, it would measure how much of the variation in the return on the individual stock is associated with variation in the return on the market. The coefficient of determination is simply the correlation coefficient squared, and the correlation coefficient is equal to

$$\rho_{im} = \frac{\sigma_{im}}{\sigma_i\sigma_m} = \frac{\beta_i\sigma^2_m}{\sigma_i\sigma_m} = \beta_i\frac{\sigma_m}{\sigma_i}$$

The values of α_i and β_i produced by regression analysis are estimates of the true α_i and β_i that exist for a stock. The estimates are subject to error. As such, the estimate of α_i and β_i may not be equal to the true α_i and β_i that existed in the period.[12] Furthermore, the process is complicated by the fact that α_i and β_i are not perfectly stationary over time. We would expect changes as the fundamental characteristics of the firm change. For example. β_i as a risk measure should be related to the capital structure of the firm and, thus, should change as the capital structure changes.

Despite error in measuring the true β_i and the possibility of real shifts in β_i over time, the most straightforward way to forecast β_i for a future period is to use an estimate of β_i obtained via regression analysis from a past period. Let us take a look at how well this works.

Accuracy of Historical Betas

The first logical step in looking at Betas is to see how much association there is between the Betas in one period and the Betas in an adjacent period. Both Blume [13] and Levy [61] have done extensive testing of the relationship between Betas over time. Let us look at some representative results from Blume's [13] study. Blume computed Betas using time series regressions on monthly data for nonoverlapping seven-year periods. He generated Betas on single stock portfolios, 2 stock portfolios, 4 stock portfolios, and so forth up to 50 stock portfolios and for each size portfolio examined how highly correlated the Betas from one period were with the Betas for a second period. Table 7.3 presents a typical result showing how highly correlated the Betas are for the period 7/54–6/61 and 7/61–6/68.

It is apparent from this table that, while Betas on very large portfolios contain a great deal of information about future Betas on these portfolios, Betas on individual securities contain much less information about the future Betas on securities. Why might observed Betas in one period differ from Betas in a second period? One reason is that the risk (Beta) of the security or portfolio might change. A second reason is that the Beta in each period is measured with a random error, and the larger the random error, the less predictive power Betas from one period will have for Betas in the next period.

Changes in security Betas will differ from security to security. Some will go up, some will go down. These changes will tend to cancel out in a portfolio, and we observe less change in the actual Beta on portfolios than on securities.

Table 7.3 Association of Betas Over Time

Number of Securities in the Portfolio	Correlation Coefficient	Coefficient of Determination
1	0.60	0.36
2	0.73	0.53
4	0.84	0.71
7	0.88	0.77
10	0.92	0.85
20	0.97	0.95
35	0.97	0.95
50	0.98	0.96

[12] In fact, the analysis will produce an estimate of the standard error in both α_i and β_i. This can be used to mal interval estimates of future Alphas and Betas under the assumption of stationarity.