The X-ray transform on Asymptotically Euclidean Spaces

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joint work with András Vasy,

Summary

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- Scattering calculus and our new calculus (6)
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X-Ray Transform on Compact Domains

X-Ray Problem

Consider a class $\mathcal{C}$ of parametrized curves γ on a domain Ω.
The X-ray transform I takes smooth functions f on Ω to functions $I(f)$ on $\mathcal{C}$ as follows:

$$I(f)(\gamma) = \int_\gamma f(\gamma(s))\, ds$$

Can f be determined if $I(f)$ is known?

The answer depends on Ω and $\mathcal{C}$. Let's explore the history of this problem.

Euclidean Case: Radon (1917)

Let's start with the example where $\mathcal{C}$ is the class of all Euclidean geodesics intersecting a bounded $\Omega \subset \mathbb{R}^2$. We can write each curve in this class using the formula

$$\gamma_{(v,y)}(s) = yv^{\perp} + sv$$

With this set-up, can the X-ray transform be inverted?

Answer: yes, using the Fourier transform to obtain

$$\mathcal{F}_{y\to\eta}(I(f))(\eta, v) = \int_{\mathbb{R}} e^{-iy\eta} f(yv^{\perp} + sv)\, ds\, dy$$
$$= \int_{\mathbb{R}^2} e^{-i\langle z, \eta v^{\perp}\rangle} f(z)\, dz = \hat{f}(\eta v^{\perp})$$

and then finishing with the Fourier inversion formula

Michel's Conjecture (1981)

This X-ray transform is related to several interesting nonlinear problems via linearization. For example:

Michel's Conjecture

In an Inventiones paper in 1981, Michel proposed a sufficient condition for determining the metric g of Ω if the distance function $\mathrm{dist}(y_1, y_2)$ is known for all $y_1, y_2 \in \partial\Omega$: that the metric g be simple, in other words

1. $\exp_p$ is a diffeomorphism for any $p \in \Omega$
2. Ω is strictly convex with respect to g

Results in the Compact Setting

One of the strongest results in the compact setting follows if one imposes a foliation condition on the region Ω:

Stefanov-Uhlmann-Vasy (2017)

If (M, g) is an n-dimensional Riemannian manifold with boundary, where $n \geq 3$, with strictly convex boundary and a convex foliation, then if there is another Riemannian metric $\hat{g}$ on M such that ∂M is still strictly convex with respect to $\hat{g}$, and if g and $\hat{g}$ have identical boundary distance functions, then they are the same up to a boundary-preserving diffeomorphism.

Related Linear Result:

The previous nonlinear rigidity result relied heavily on the following linear result:

Uhlmann-Vasy 2016

For compact Riemannian manifolds (M, g) with strictly convex boundary, the local geodesic X-ray transform is invertible on small enough collar neighborhoods of the boundary.

We will be pursuing linear results in the same vein but in a noncompact setting.

Noncompact Results

The next result is state-of-the-art and studies asymptotically Euclidean or conic Riemannian settings.

Guillarmou-Lassas-Tzou 2019

Under non-trapping and no conjugate point assumptions, the geodesic X-ray transform is injective.

We are going to work within similar setting but with vastly different tools, which don't require these non-trapping or conjugate point assumptions, instead relying on techniques similar to the work of Stefanov, Uhlmann, and Vasy cited above.

Noncompact Geometric Setting

One critical piece of work for us is

Melrose-Zworski (1995)

For any asymptotically Euclidean Riemannian space (M, g), its geodesic flow can be extended to a rescaled geodesic flow on the boundary.

Furthermore, this geodesic flow has analytic implications: it can be used to explicitly construct Poisson operators and scattering operators as Fourier Integral Operators (the main result).

Their insight into suitable parametrizations for geodesics on this space will be most useful to us here.

Scattering Blow-Up

The scattering blow-up can be understood as two successive blow-ups of Schwartz double space, first a b-blow up and then a 0-blow up, resulting in a manifold where the diagonal intersects the scattering front face, which forms corners as it intersects the b-front face.

The scattering blow-up corresponds to the following coordinate changes:

$$K_A(x,y,x',y') = \int a(x,y,\xi,\eta)e^{i((x-x')\xi+(y-y')\cdot\eta)}\,d\xi\,d\eta$$

$$= \int a(x,y,\xi,\eta)e^{i(\frac{x-x'}{x^2}(x^2\xi)+\frac{y-y'}{x}\cdot(x\eta))}\,d\xi d\eta$$

$$K_A^{sc}(x,y,X,Y) = \int a^{sc}(x,y,\tau,\mu)e^{iX\tau+Y\cdot\mu}\,d\tau\,d\mu$$

Scattering Symbols

Definition of Scattering Class Symbols

Smooth functions a^{sc} are scattering class symbols if

- $\left|\partial_\tau^\alpha \partial_\mu^\beta a^{sc}(x, y, \tau, \mu)\right| \leq C_{\alpha\beta} \langle \tau, \mu \rangle^{m-\alpha-|\beta|} \langle x^{-1} \rangle^\ell$
- Each $(x\partial_x)^{\alpha'} \partial_y^{\beta'} a^{sc}$ has the same regularity
- Principal symbols are a^{sc}'s leading order behavior in (τ, μ, x)

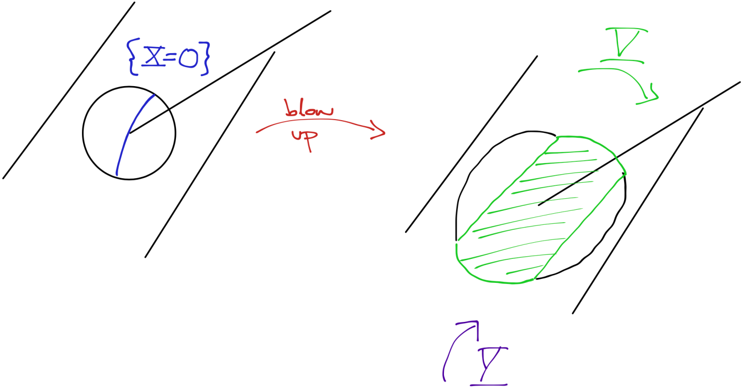
Symbolic Properties of the Scattering Calculus

Composition: $a \in \sigma_{sc}^{m,\ell}$, $b \in \sigma_{sc}^{m',\ell'} \implies A \circ B \in \Psi_{sc}^{m+m',\ell+\ell'}$
Mapping: A as above maps $H^{s,r} = \langle z \rangle^{-r} H^s \to H^{s-m,r-\ell}$
Ellipticity: $a^{sc}(x, y, \tau, \mu) \geq C \langle \tau, \mu \rangle^m x^{-\ell}$ for $\tau, \mu \gg 1$ or $x \ll 1$

Ellipticity of A establishes the existence of a parametrix $B \in \Psi_{sc}^{-m,-\ell}$ so that $I - A \circ B$'s error lies in $\Psi_{sc}^{-\infty,-\infty}$

One More Blow-Up

To understand the x-ray transform, we introduce a new algebra, called the 1-cusp algebra.



Defining Symbols

As $(x^2\partial_x, x\partial_y)$ and (X, Y) corresponded to (τ, μ) in the scattering case, $(x^3\partial_x, x\partial_y)$ and (V, Y) correspond to (β, μ) in this new class.

Definition of Class

Smooth functions $a^{new}(x, y, \beta, \mu)$ are symbols in this new class if

- $\left|\partial_\tau^a \partial_\beta^b a^{new}(x, y, \beta, \mu)\right| \leq C_{ab}\langle \beta, \mu\rangle^{m-a-|b|}\langle x^{-1}\rangle^{\ell}$ as before
- Each $(x\partial_x)^{a'}\partial_y^{b'} a^{new}$ has the same regularity.

This is the same definition as a scattering class symbol, but we turn it into an operator differently:

$$K_A^{new}(x, y, V, Y) = \int a^{new}(x, y, \beta, \mu) e^{iV\beta + Y\cdot\mu}\, d\beta\, d\mu$$

Coordinate Dependence

This new calculus, the 1-cusp calculus, depends on the choice of boundary-defining function x, specifically on the choice of equivalence class modulo $O(x^2)$. In the problem we consider, this boundary-defining function is constrained by the geometric conditions we place on the manifold

A Riemannian manifold is asymptotically Euclidean if its metric is a scattering metric, that is, it can be written in the following form:

$$\frac{dx^2}{x^4} + \frac{h(x,y)}{x^2} \quad \text{where } h(0,y) \text{ is nondegenerate on } T(\partial M)$$

and if additionally the cross-sections $\{x_0 = c\}$ are relatively close to spheres, for small values of c.

Ellipticity

Composition

If $A \in \Psi_{new}^{m,\ell}$ with symbol a and $B \in \Psi_{new}^{m',\ell'}$ with symbol b, then $A \circ B \in \Psi_{new}^{m+m',\ell+\ell'}$ with symbol ab.

Ellipticity

If $A \in \Psi_{new}^{m,\ell}$ with symbol a^{new},

$$a^{new}(x, y, \beta, \mu) \geq C\langle\beta, \mu\rangle^m x^{-\ell} \text{ for } \beta, \mu \gg 1 \text{ or } x \ll 1$$

then A is an elliptic and there is a parametrix $B \in \Psi_{new}^{-m,-\ell}$ with error in $\Psi_{new}^{-\infty,-\infty}$

As $\Psi_{new}^{-\infty,-\infty} \subseteq \Psi_{sc}^{-\infty,-\infty}$, this parametrix error is also compact, giving us Fredholm properties.

Returning to our friend the X-Ray Transform...

We defined the X-ray transform I earlier. As is typical, we study the normal operator I^*I, which maps functions on our space to functions on our space, bypassing the class of curves $\mathcal{C}$.

Theorem [Vasy-Z 2020]

A modified normal operator of the X-ray transform (on asymptotically Euclidean manifolds with no conjugate points on ∂X within distance $\pi/2$) is an elliptic operator (for sufficiently small x) of order $(-1, 0)$ in the new 1-cusp calculus.

Corollary

The normal operator I^*I, acting on exponentially quadratically decaying functions, has finite dimensional null space in a collar region of the boundary.

Our original X-ray normal transform is

$$K_{I^*I}(z,z') = \int_{s,\mu_0,\tau_0} \delta(\gamma_{z,\tau_0,\mu_0}(s) - z')\, ds\, d\mu_0\, d\tau_0$$

We convert this normal transform into coordinates (x, y, x', y'). Because we are working with asymptotically Euclidean space, the metric is a scattering metric, and so it is reasonable as a first guess to rewrite everything in terms of scattering coordinates.

$$K_{I^*I}(x,y,X,Y) = \int_{s,\mu_0,\tau_0} \delta(\gamma^x_{x,y,\tau_0,\mu_0}(s) - (x^2X + x))\delta(\gamma^y_{x,y,\tau_0,\mu_0}(s) - (xY + y))\, ds\, d\mu_0\, d\tau_0$$

Just to switch our work to oscillatory integrals, we move over to the Fourier side:

$$\sigma^{sc}_{I^*I}(x_0,y_0,\tau,\mu) = \int_{X,Y} e^{-iX\tau - iY\cdot\mu} \int_{s,\mu_0,\tau_0} \delta(x(s) - x_0 - x_0^2X)\delta(y(s) - y_0 - x_0Y)\, ds\, d\mu_0\, d\tau_0\, dX\, dY$$

$$\sigma^{sc}_{I^*I}(x_0,y_0,\tau,\mu) = \int_{s,\tau_0,\mu_0} e^{-i\frac{\tau}{x_0}\frac{x(s)-x_0}{x_0}} e^{-i\frac{\mu}{x_0}\cdot(y(s)-y_0)}\, ds\, d\mu_0\, d\tau_0$$

The unit bicharacterstics $\gamma_{(x_0,y_0,\tau_0,\mu_0)}(s)$ in scattering coordinates satisfy the following systems:

original coordinates | linearized Melrose-Zworski | linearized solution

$$\begin{cases} \partial_s x = x^2\tau \\ \partial_s y = x\mu \\ \partial_s \tau = -x|\mu|^2 \\ \partial_s \mu = x\tau\mu \end{cases} \qquad \begin{cases} \partial_r x = \frac{\tau x}{|\mu|} \\ \partial_r y = \hat{\mu} \\ \partial_r \tau = -|\mu| \\ \partial_r |\mu| = \tau \\ \partial_r \hat{\mu} = 0 \end{cases} \qquad \begin{cases} x = C\sin(r + r_0) \\ y = \hat{\mu} r + y_0 \\ \tau = \lambda\cos(r + r_0) \\ |\mu| = \lambda\sin(r + r_0) \\ \hat{\mu} = \hat{\mu}_0 \end{cases}$$

in full: $\partial_r y = \frac{1}{2} h'_\eta(y, \hat{\mu}) \qquad \partial_r \hat{\mu} = -\frac{1}{2} h_y(y, \hat{\mu})$

$$(y, \hat{\mu}) = \exp((r + r_0)H_{\frac{1}{2}h})(y_0, \mu_0).$$

Linearized bicharacteristics using Melrose-Zworski parametrization:

$$x = C\sin(r + r_0) \quad y = \hat{\mu} r + y_0$$
$$\tau = \lambda\cos(r + r_0) \quad |\mu| = \lambda\sin(r + r_0) \quad \hat{\mu} = \hat{\mu}_0$$

Finally, we introduce the coordinate $R = (r + r_0)x_0$. With this choice of coordinate, and parametrizing geodesic directions with spherical coordinates $(\xi_0 = \frac{\tau_0}{x_0}, \mu_0)$, our symbol so far becomes

$$\sigma^{sc}_{I^*I}(x_0, y_0, \tau, \mu) = \int_{s,\xi_0,\widehat{\tilde{\mu}_0}} e^{-i\left(\tau x_0\left[\frac{\xi_0}{|\tilde{\mu}_0|}\frac{\sin(x_0 R)}{x_0} + \frac{\cos(x_0 R)-1}{x_0^2}\right] + \mu\cdot\widehat{\tilde{\mu}_0} R\right)} ds\, d\widehat{\tilde{\mu}_0}\, d\xi_0$$

We then introduce two additional terms, both of which alter our operator from I^*I to A. First, we introduce a cut-off function $\phi(\xi_0)$ to reduce total information to geodesics which are near the boundary (where our asymptotically Euclidean assumption takes hold). Next, we also introduce an exponential weight $e^{\frac{c}{x'^2}-\frac{c}{x^2}}$ to place higher emphasis on geodesics which stay near infinity, not curving back away from the boundary.

Note that these choices correspond to considering the bracketed X-ray transform $\chi_\phi(\gamma)I(e^{c/\cdot^2}f(\cdot))(\gamma)$. These choices therefore shouldn't affect the null space of the operator, although they do restrict us to considering functions f on our space which have Gaussian decay at infinity.

Incorporating these changes, our modified symbol then takes the following form:

$$\sigma_A^{sc}(x_0, y_0, \tau, \mu) = \int_{R, \xi_0, \widehat{\tilde{\mu}_0}} \frac{1}{\tilde{x}|\tilde{\mu}|} \phi(\xi_0) e^{\frac{c}{\tilde{x}^2} - \frac{c}{x_0^2}} e^{-i\left(\tau x_0 \left[\frac{\tilde{\xi}_0}{|\tilde{\mu}_0|} \frac{\sin(x_0 R)}{x_0} + \frac{\cos(x_0 R) - 1}{x_0^2}\right] + \mu \cdot \widehat{\tilde{\mu}_0} R\right)} dR \, d\widehat{\tilde{\mu}_0} \, d\xi_0$$

In fact, we can now attempt to view this symbol in terms of the coordinates corresponding to our new calculus:

$$\sigma_A^{new}(x_0, y_0, \beta, \mu) = \int_{R, \xi_0, \widehat{\tilde{\mu}_0}} \frac{1}{\tilde{x}|\tilde{\mu}|} \phi(\xi_0) e^{(1 - f(x_0, R))/x_0^2} e^{-i\left(\beta \left[\frac{\tilde{\xi}_0}{|\tilde{\mu}_0|} \frac{\sin(x_0 R)}{x_0} + \frac{\cos(x_0 R) - 1}{x_0^2}\right] + \mu \cdot \widehat{\tilde{\mu}_0} R\right)} dR \, d\widehat{\tilde{\mu}_0} \, d\xi_0$$

Now we must verify that it really is a symbol in our new class.

Stationary Phase

$$I(\lambda) = \int_{\mathbb{R}^n} u(z) e^{i\lambda f(z)}\, dz$$

The function u(z) has compact support and within its support the only critical point of the phase function f is nondegenerate at 0. Then to leading asymptotic order in λ:

$$I(\lambda) \approx e^{i\lambda f(0)} \det\left(\lambda f''(0)/i2\pi\right)^{-1/2} u(0) \sim \lambda^{-n/2}$$

We will apply to the phase function

$$\Phi = \beta\left[\frac{\xi_0}{|\tilde{\mu}_0|}\frac{\sin(x_0 R)}{x_0} + \frac{\cos(x_0 R) - 1}{x_0^2}\right] + \mu \cdot \widehat{\tilde{\mu}_0} R$$

where β, μ will be the large parameters.

Overview of our Oscillatory Term

We focus on this new phase function.

$$\Phi = \beta \left[\frac{\xi_0}{|\tilde{\mu}_0|} \frac{\sin(x_0 R)}{x_0} + \frac{\cos(x_0 R) - 1}{x_0^2} \right] + \mu \cdot \widehat{\tilde{\mu}_0} R$$

Note that Φ is smooth in all variables shown (if we restrict to the support of the cut-off function).

Stationary phase will give us symbol estimates, using the critical set

$$C_\Phi = \left\{ R = 0,\ \ \beta \frac{\xi_0}{|\tilde{\mu}_0|} + \mu \cdot \widehat{\tilde{\mu}_0} = 0 \right\}$$

This critical set corresponds to bicharacteristics starting at the closest point to the origin and moving in orthogonal directions to the designated one.

Large β, μ: Correct calculus and ellipticity in one step

We get symbol estimates in several different regions. We first look at the region where depending on whether β dominates large (β, μ). Then we can apply stationary phase using that $\mathrm{Hess}_{R,\xi_0}(-i\Phi|_{C_\Phi}) = \beta^{-1}$ to observe that

$$|a^{new}(x, y, \beta, \mu)| \sim C\beta^{-1}$$

Similarly, when μ dominates large (β, μ), we can use $\mathrm{Hess}_{R,\widehat{\tilde{\mu}_0}}(-i\Phi|_{C_\Phi}) = |\mu|^{-1}$ to observe that

$$|a^{new}(x, y, \beta, \mu)| \sim C|\mu|^{-1}$$

These estimates also hold in these (β, μ) regions when we apply derivatives $\partial_\beta, \partial_\mu, x\partial_x, \partial_y$ to $a^{new}(x, y, \beta, \mu)$.

Finite β, μ

For larger values of β or μ, stationary phase told us that the main contribution occurs at $R = 0$, but for smaller values of β and μ we need to manually check that other regions (in terms of the bicharacteristic parameters) do not contribute.

We also need to check that a^{new} is a smooth, bounded function!

There are three different regions that we consider

Finite β, μ continued: ellipticity

In the small, β, μ region, we only have to show ellipticity for small x_0, which we can do by estimating the integral to show positivity at $x_0 = 0$.

The main technical difficulty for ellipticity is working with our cut-off function $\phi(\xi_0)$ which we don't want to integrate explicitly! So we estimate:

$$\phi(\xi_0) \approx e^{-M\xi_0^2/2}$$

and Taylor expansion in $(x_0 R)$ allows us to freeze many of the amplitude factors turning the verification of ellipticity into the computation of Gaussian integrals (in R and ξ_0) with complex argument, which can be manually checked.

Why this modified operator is reasonable

Because our modified normal operator is elliptic in this new calculus, we can construct a parametrix with compact error, giving us Fredholmness and that the modified normal operator has a finite dimensional nullspace supported in $x < \bar{x}$.

However, this modified operator varied only by introducing exponential weights and a cut-off ϕ which eliminated some information. In other words,

Theorem

The original geodesic X-ray normal operator and thus the X-ray transform itself, acting on functions with Gaussian decay, will have a finite dimensional nullspace supported in $x < \bar{x}$.

Thank you for your attention!