ECE 830
Problem Set 3

1. Consider problem of detecting a signal in noise with known variance.

$$\begin{aligned} H_0: \; x &\sim \mathcal{N}(0,1) \\ H_1: \; X &\sim \mathcal{N}(\theta,1) \end{aligned}$$

where $\theta > 0$ is unknown. Does a UMP test exist for this problem? Design a test for this problem and determine the distribution of the test under H_0.

2. Consider problem of detecting a signal in noise with known variance.

$$\begin{aligned} H_0: \; x &\sim \mathcal{N}(0,1) \\ H_1: \; X &\sim \mathcal{N}(\theta,1) \end{aligned}$$

where $\theta \neq 0$ is unknown. Does a UMP test exist for this problem? Design a test for this problem and determine the distribution of the test under H_0.

3. Consider problem of detecting a signal in noise with known variance.

$$\begin{aligned} H_0: \; X &\sim \mathcal{N}(\mathbf{0},\sigma^2\boldsymbol{I}) \\ H_1: \; X &\sim \mathcal{N}(\theta\boldsymbol{s},\sigma^2\boldsymbol{I}) \end{aligned}$$

where $\boldsymbol{s} \in \mathbb{R}^n$ and σ^2 are known, but both $\theta \in \mathbb{R}$ is unknown. Design a test for this problem and determine the distribution of the test under H_0. How would the test change if the distributions had a general covariance $\boldsymbol{\Sigma}$ instead of a diagonal covariance?

Homework Problems

4. Suppose we observe two iid samples from an exponential distribution of the form

$$p(x) = \begin{cases} \lambda\, e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

and consider the hypothesis testing problem

$$\begin{aligned} H_0 &: \;\; \lambda = \lambda_0 \\ H_1 &: \;\; \lambda > \lambda_0 \end{aligned}$$

for a known $\lambda_0 > 0$. Propose a one-dimensional test statistic and P_{FA} as a function of the threshold. Does a UMP test exist? Explain your answer.

5. Suppose we observe two iid samples from an Laplacian distribution of the form

$$p(x) = \frac{\lambda}{2}\, e^{-\lambda|x|}, \quad x \in \mathbb{R}$$

and consider the hypothesis testing problem

$$\begin{aligned} H_0 &: \;\; \lambda = \lambda_0 \\ H_1 &: \;\; \lambda > \lambda_0 \end{aligned}$$

for a known $\lambda_0 > 0$. Propose a one-dimensional test statistic and P_{FA} as a function of the threshold. Does a UMP test exist? Explain your answer.

6. Consider problem of detecting a signal in noise with unknown variance.

$$\begin{aligned} H_0 : \; X &\sim \mathcal{N}(\mathbf{0}, \sigma^2 \boldsymbol{I}) \\ H_1 : \; X &\sim \mathcal{N}(\theta \boldsymbol{s}, \sigma^2 \boldsymbol{I}) \end{aligned}$$

where $\boldsymbol{s} \in \mathbb{R}^n$ is known, but both $\theta \in \mathbb{R}$ and $\sigma^2 > 0$ are unknown. Design a test for this problem and show that its distribution under H_0 is F-distributed with 1 and $n-1$ degrees of freedom.

7. Let $e_i \in \mathbb{R}^n$ denote the vector with the value 1 in the i-th location and 0 in all others. Let $x \in \mathbb{R}^n$ be an observation and consider the following hypotheses.

$$\begin{aligned} H_0 : \; x &\sim \mathcal{N}(0, I) \\ H_1 : \; x &\sim \mathcal{N}(\theta e_i, I)\;,\;\; \theta \in \mathbb{R},\; i \in \{1, \ldots, n\} \end{aligned}$$

where I is the $n \times n$ identity matrix. Propose a testing procedure with false-alarm probability at most α; i.e., the probability of accepting H_1 when $x \sim \mathcal{N}(0, I)$ should be bounded by α.

8. Consider the simple binary testing problem

$$\begin{aligned} H_0 &: \;\; X_1, X_2, \ldots \overset{iid}{\sim} \exp(-x)\;,\;\; x \geq 0 \\ H_1 &: \;\; X_1, X_2, \ldots \overset{iid}{\sim} 0.5 \exp(-x/2)\;,\;\; x \geq 0 \end{aligned}$$

a. Compute the mean of each distribution and plot the probability densities in Matlab.

b. Design the non-sequential LRT and Sequential Probability Ratio Test (SPRT) to achieve $P_{FA} = 1 - P_D \leq 0.05$. For more background on the sequential LRT, please read `http://nowak.ece.wisc.edu/ece830/ece830_fall11_lecture9.pdf`.

c. Theoretically compare the number of samples required using the non-sequential LRT with the expected number of samples required by the SPRT.

d. Simulate the SPRT in Matlab with 100 independent trials. What is the average stopping time for the test and what is the standard deviation? How does this compare with the theoretical calculations above.