Homework 2

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1a)

The pressure difference will be given by Bernoulli’s EQ:

$P_1 + \rho u_1{}^2 + g\rho h_1 \; = P_2 + \rho u_2{}^2 + g\rho h_2$

$P_2 - P_1 = \rho\left[\frac{u_2{}^2}{2} + gh_2\right]$

where

$\rho = \rho_{\text{air}} \; = 1.2\,\text{kg} \Big/ m^3$

$h = 50\; m$

$g = 10\; m \Big/ s^2$

$u = 10\; m\; / s$

then

$P' = 660\; \frac{\text{kg}}{m\;\; s^2} = 660\;\text{Pa}$

1b)

The pressure difference between upstream of the flow and at the top of the building will be

$P' = P_2 - P_1 = \frac{\rho}{2}\left[u_1{}^2 - u_2{}^2\right]$

we estimate $u_1 = \; u_2 + 1\; m\; / s = 9\; m\; / s$, then

$P' = \frac{1.2}{2}\,\text{kg} \Big/ m^3 \left((9\; m\; / s)^2 - (10\; m\; / s)^2\right)$

$P' = -11\;\text{Pa}$

1c)

We predict the normalized pressure in the pipe to be

$$\frac{(P_1 - P_2)\, d^3\, \rho}{v^2\, \rho^2\, L} = \frac{(P_1 - P_2)\, d^3}{v^2\, \rho\, L} = \frac{\left(660\; \frac{\text{kg}}{m\;\; s^2}\right) * (0.1\; m\;\;)^3}{\left(10^{-5}\; \frac{m\;\;\; ^2}{s}\right)^2 * 1.2\; \frac{\text{kg}}{m\;\;\; ^3} * 100\; m} = 5.5 \times 10^7$$

which is beyond the scope of the measured pressures in the diagram. We estimate the logarithmic slope of the relationship with finite differencing then can solve for Re

$\mathrm{Re} = 10^{\frac{\log(5.5\times 10^7)-1.5}{1.7}} = 4.7 \times 10^4$

with an estimated Re we are able to determine the flow speed:

$\mathrm{Re} = \frac{\mathrm{uL}}{v}$

$u = \frac{\mathrm{Re}\, v}{L} = \frac{4.6\times 10^4 * 2\times 10^{-5}\, m^2/s}{100\, m}$

$u = 9 \times 10^{-3}\, m/s$

2a)

The viscous shear is given by

$\tau = \frac{\partial v}{\partial r} \sim \frac{v}{r}$

$v = \tau * 2\pi 500\,\mathrm{km}$

$v \sim 300\, m/s$

2b)

The vorticity is $\omega = \nabla \times v = \frac{1}{r}\frac{\partial (r\, v)}{\partial r}$ in the k direction given the symmetry of the problem in cyclindrical coordinates.

$\phi = 1/r * \frac{\partial \mathrm{rv}}{\partial r} = \frac{v}{r} + \frac{\partial v}{\partial r}$

where $\frac{v}{r}$ is the curvature vorticity equal to the angular frequency Ω and $\frac{\partial v}{\partial r}$ is the shear vorticity. Assuming v does not vary hugely over r,

$\phi = \frac{2v}{r} = 2\Omega = \frac{2\times 300\, m/s}{5\times 10^5\, m} = 1.2 \times 10^{-3}\, s^{-1}$

2c)

The angular frequency is

$\Omega = \frac{v}{r}$

thus

$\phi = 2\Omega = 1.2 \times 10^{-3}\, s^{-1}$

3a)

We can estimate the scale height by calculating the difference in heights using the energy conservation from sea level to the desired height

$h_1 = \frac{c_V(\rho_0 T_0 - \rho_1 T_1) + P_0 - P_1}{\rho_1 g + c_V \rho_1 m_T}$

$$h_1 = \frac{P_0+P_1}{\rho_1\, g + c_V\, \rho_1\, m_T}$$
$$h_1 = \frac{101\,325\ \text{Pa} - 50\,000\ \text{Pa}}{1.2\, \frac{\text{kg}}{m^3} 9.8\, \frac{m}{s^2} + 700\, \frac{\text{kg}\, m^2}{s^2} * 1.2\, \frac{\text{kg}}{m^3} * 6500\, \frac{K}{m}}$$
$$h_2 = \frac{101\,325\ \text{Pa} - 100\,000\ \text{Pa}}{1.2\, \frac{\text{kg}}{m^3} 9.8\, \frac{m}{s^2} + 700\, \frac{\text{kg}\, m^2}{s^2} * 1.2\, \frac{\text{kg}}{m^3} * 6500\, \frac{K}{m}}$$

Leading to a scale height of

$$h_2 - h_1 = 2.9\ \text{km}$$

Analyzing
http://weather.unisys.com/upper_air/ua_cont.php?plot=thi&inv=0&t=cur
we find the thickness over WI is 5 km. This agrees mildly with our estimated scale height.

4a)

We estimate the 4km geostropic wind speed over Tateno Japan with

$$u_g = -\frac{1}{\rho f} \frac{\partial P}{\partial y}$$

We estimate $\Delta P = 500$ Pa and $\Delta y = 5 \times 10^5\ m$ using finite differencing, leading us to an estimate of u_g

$$\Omega = 10^{-5}\ \text{Hz}$$

$$u_g = -\frac{1}{1.2 * \frac{\text{kg}}{m^3} 2 * 10^{-5}\ s^{-1} * \text{Sin}[0.69]} * 10^{-3}\ \frac{\frac{\frac{\text{kg}\, m^2}{s^2}}{m^3}}{m}$$
$$u_g = -65\ m / s$$

Compared to the measured observations in winter (20 m/s) we are within approximation errors.

4b)

We can estimate the wind speed by relating the vertical acceleration to the temperature gradient

$$\frac{\partial u}{\partial z} = \frac{g}{f T} \frac{\partial T}{\partial x}$$
$$u = \frac{g}{f T} \frac{\partial T}{\partial x} z$$

where $\frac{\partial T}{\partial x} = \text{lapse rate} = \frac{1}{100} \frac{K}{\text{km}} = 10^{-5} \frac{K}{m}$, and $z = 5\ \text{km} = 5 \times 10^3\ m$, $f = 2 * 10^{-5}\ s^{-1} * \text{Sin}[0.75]$,

$$u = \frac{9.8\ m / s^2}{2 * 10^{-5}\ s^{-1} * \text{Sin}[0.75]\, 290\, K} 10^{-5} \frac{K}{m} 5 \times 10^3\ m$$
$$u = 123\ m / s$$

5a)

The coriolis and curvature terms in the horizontal momentum equations are

$$\hat{i}\ :\ \frac{\text{du}}{\text{dt}} = \frac{2\,u\,v\tan[\phi]}{z} + 2\,\Omega\,v\sin[\phi]$$
$$\hat{j}\ :\ \frac{\text{dv}}{\text{dt}} = -\frac{u^2\tan[\phi]}{z} - 2\,\Omega\,u\sin[\phi]$$

given the following known quantities

$\phi = 43.0667\text{ deg}$
$d = 100\text{ km} = 10^5\ m$
$z = 6\times 10^6\ m$
$u = -\frac{1000}{\sqrt{2}}\ \frac{m}{s}$
$v = -u$
$\Omega = 7\times 10^{-5}\text{ Hz}$

we can calculate the horizontal accelerations

$$\frac{\text{du}}{\text{dt}} = \frac{2*\frac{-1000}{2^{0.5}}\frac{m}{s}*\frac{1000}{2^{0.5}}\frac{m}{s}}{6.371\times 10^6\ m}\text{Tan}\left[43.0667\text{ deg} * \frac{\pi}{180\text{ deg}}\right] + 2*7\times 10^{-5}\ 1/s*\frac{1000}{2^{0.5}}\ \frac{m}{s}*\text{Sin}\left[43.0667\text{ deg} * \frac{\pi}{180\text{ deg}}\right]$$

$$\frac{\text{dv}}{\text{dt}} = \frac{\left(\frac{1000}{2^{0.5}}\frac{m}{s}\right)^2}{6.371\times 10^6\ m}\text{Tan}\left[43.0667\text{ deg} * \frac{\pi}{180\text{ deg}}\right] - 2*7\times 10^{-5}\ 1/s*\frac{-1000}{2^{0.5}}\ \frac{m}{s}*\text{Sin}\left[43.0667\text{ deg} * \frac{\pi}{180\text{ deg}}\right]$$

$$\frac{\text{du}}{\text{dt}} = -\frac{0.0791121\ m}{s^2}$$
$$\frac{\text{dv}}{\text{dt}} = \frac{0.140954\ m}{s^2}$$

To calculate the time traveled we use the displacement EQ

$$x = u\,t + a\,t^2$$
$$y' = x' = \frac{d}{\sqrt{2}}$$

Thus the time without deflection is

$$t = \frac{x'}{u} = \frac{d}{u} = 10^2\ s$$

With the curvature terms, we need to account for acceleration in the displacement of the missile:

$$\Delta x = x' - \left(u\,t + \frac{\text{du}}{\text{dt}}\,t^2\right) = -\frac{10^5}{2^{0.5}}\ m - \left(-\frac{1000}{2^{1/2}}\ \frac{m}{s}*10^2\ s - \frac{0.07911\ m}{s^2}*\left(10^2\ s\right)^2\right)$$
$$\Delta y = y' - \left(v\,t + \frac{\text{dv}}{\text{dt}}\,t^2\right) = \frac{10^5}{2^{0.5}}\ m - \left(\frac{1000}{2^{1/2}}\ \frac{m}{s}*10^2\ s + \frac{0.1409\ m}{s^2}*\left(10^2\ s\right)^2\right)$$
$$\Delta x = 800\ m$$
$$\Delta y = -1400\ m$$

6a)

The evolution of the mixing ratio will be determined by the continuity equation

$\frac{\partial n}{\partial t} + u \frac{\partial n}{\partial x} = S$

where S is the source, n is the number density of ozone, and u is the wind speed along the x direction.

$S = -\left(2 \times 10^{-4}\right) \text{ppbv} / s$
$u = \frac{20}{\sqrt{2}} \text{knots} = 0.03637\ m\ / s$
$n_{\text{madison},0} = 50 \text{ ppbv}$
$n_{\text{milwaukee},0} = 100 \text{ ppbv}$
$d = 125 \text{ km} = 1.25 \times 10^{5}\ m$
$\Delta\text{t} = 6 \text{ hours} = 21\,600\ s$

Assuming that the rate at which the number density of ozone changes over time is constant, we can solve for an evolving ozone density:

$n(t_1) = n(t_0) + \Delta\text{t}\left(S - u \frac{\Delta \text{n}}{\Delta \text{x}}\right)$
$n_{\text{madison}}\ (t = 6 \text{ hours}) = n_{\text{madison},0} + \Delta\text{t}\left(S - u \frac{n_{\text{milwaukee},0} - n_{\text{madison},0}}{d}\right)$
$n_{\text{madison}}\ (t = 6 \text{ hours}) = 54 \text{ ppbv}$

7a)

$\bar{P} = \bar{\rho}\, k\, \bar{T}$
$P = \bar{P} + P'$
$\rho = \bar{\rho} + \rho'$
$T = \bar{T} + T'$
$P = \left(\bar{P} + P'\right) k \left(\bar{T} + T'\right)$
$P = \bar{\rho}\, k\, \bar{T} + \rho'\, k\, \bar{T} + \bar{\rho}\, k\, T' + \rho'\, k\, T'$

but $\rho'\, k\, T'$ will be small

$P = \bar{P} + \rho'\, k\, \bar{T} + \bar{\rho}\, k\, T'$
$\frac{P' + \bar{P}}{\bar{P}} = 1 + \frac{\rho'\, k\, \bar{T}}{\bar{P}} + \frac{\bar{\rho}\, k\, T'}{\bar{P}}$
$\frac{P' + \bar{P}}{\bar{P}} = 1 + \frac{\rho'\, k\, \bar{T}}{\bar{\rho}\, k\, \bar{T}} + \frac{\bar{\rho}\, k\, T'}{\bar{\rho}\, k\, \bar{T}}$
$\frac{P'}{\bar{P}} = \frac{\rho'}{\bar{\rho}} + \frac{T'}{\bar{T}}$

7b)

We would like to $\frac{P'}{\bar{P}} << \frac{T'}{\bar{T}}$

Examining forecast maps for 250hPa and 200hPa maps, where $P' = 25 \text{ hPa}$ and $\bar{P} = 225 \text{ hPa}$, we find $T' = 40 \text{ deg } C$ and $\bar{T} = 20 \text{ deg } C$ at 11,000 m. So

$$\frac{P'}{\bar{P}} = \frac{25}{225} << \frac{40}{20} = \frac{T'}{\bar{T}}$$

which leads us to believe the Boussineq approximation holds on synoptic scales.

Below are the maps I used:
http://tempest.aos.wisc.edu/wxp_images/gfs104_12UTC/gblav_c250_h000.gif
http://tempest.aos.wisc.edu/wxp_images/gfs104_12UTC/gblav_c200_h000.gif