Astro 735: Cosmology
Lecture 7: The radiation era; geometry of the universe; the cosmological constant

Elena D'Onghia

1 The radiation era

We have seen that

$$a^4 \rho_{\rm rel} = \rho_{\rm rel,0}. \tag{1}$$

Compare this with the evolution of matter density with the scale factor,

$$a^3 \rho_m = \rho_{m,0}. \tag{2}$$

As the scale factor becomes smaller, $\rho_{\rm rel}$ increases more rapidly than ρ_m, which means that although matter dominates now, there must have been a time in the early universe as $a \to 0$ when relativistic particles were dominant. We can find out when this was by setting $\rho_{\rm rel} = \rho_{\rm rel,0}/a^4$ equal to $\rho_m = \rho_{m,0}a^3$. The value of the scale factor for matter-radiation equality is then

$$a_{r,m} = \frac{\Omega_{\rm rel,0}}{\Omega_{m,0}} = 8.25 \times 10^{-5} \Omega_{m,0}^{-1} h_{70}^{-2} = 3.05 \times 10^{-4}. \tag{3}$$

The corresponding redshift is

$$z_{r,m} = \frac{1}{a_{r,m}} - 1 = 1.21 \times 10^4 \Omega_{m,0} h_{70}^2 = 3270. \tag{4}$$

Since $aT = T_0$, the temperature at this time was

$$T_{r,m} = \frac{T_0}{a_{r,m}} = 3.31 \times 10^4 \Omega_{m,0} h_{70}^2 \text{ K} = 8910 \text{ K}. \tag{5}$$

The universe was **radiation dominated** until a redshift of $z = 3270$, at which time the temperature was $T = 8920$ K. After this, the universe was **matter dominated**.

We can also look at how the universe expands during the radiation era. Including the contributions of both matter and relativistic particles,

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi G}{3}(\rho_m + \rho_{\rm rel})\right] a^2 = -kc^2 \tag{6}$$

Writing ρ_m and $\rho_{\rm rel}$ in terms of their values today, we find

$$\left[\left(\frac{da}{dt}\right)^2 - \frac{8\pi G}{3}\left(\frac{\rho_{m,0}}{a} + \frac{\rho_{\rm rel,0}}{a^2}\right)\right] = -kc^2 \tag{7}$$

We can set $k = 0$ because the early universe was extremely close to flat. We can then show that the time of matter and radiation equality was $t_{r,m} = 5.5 \times 10^4$ yrs, and that when $a \ll a_{r,m}$,

$$a(t) = \left(\frac{16\pi G g_* a_{\rm rad}}{3c^2}\right)^{1/4} T_0 t^{1/2} \tag{8}$$

and

$$T(t) = \left(\frac{3c^3}{16\pi G g_* a_{\rm rad}} \right)^{1/4} t^{-1/2}. \tag{9}$$

This should be compared with the matter-dominated era, when $a \gg a_{r,m}$; in this case $a \propto t^{2/3}$, as we already showed for a flat universe containing only matter. So the universe expanded more slowly in the radiation era, because of the additional pressure of relativistic particles which slows the expansion.

2 The geometry of the universe

We've been talking about different scenarios for the evolution of the universe: open, flat or closed, depending on the density of mass and energy. This is very closely related to the geometry of the universe, which is described by general relativity. To understand the curvature of the universe, we need to consider some of the principles of non-Euclidean geometry.

2.1 Euclidean, Elliptic and Hyperbolic Geometries

In about 300 BC, Euclid worked out 5 postulates from which all the rules of geometry could be derived—these rules lay out the basic behaviors of straight lines, right angles, etc.

5th postulate: Given a line and a point not on the line, there is exactly one parallel line which passes through the point.

In the18th century, mathematicians realized it was possible to make fully consistent definitions of geometry using Euclid's first 4 postulates, but modifying the 5th. These correspond to **curved spaces**.

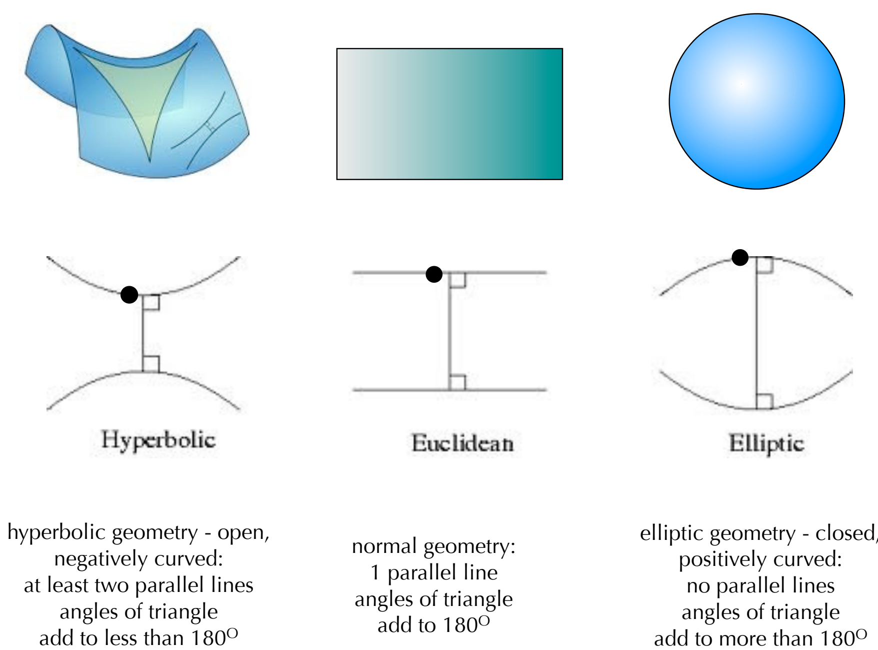

Figure 1: Hyberbolic, Euclidean and elliptic geometries.

These examples are curved 2-dimensional surfaces embedded in 3-d spaces—it's not too hard to understand the geometry of something when you can leave the space to look at it. But we can't do that with the universe—we can't pop off into a 4th spatial dimension to look at the curvature. Instead we have to try to understand it using measurements entirely within the space itself. These are called "inner properties."

The inner properties of a curved space are closely related to how distances are measured within that space.

- On a flat Euclidean plane, a circle has circumference $C = 2\pi r$.
- On a sphere of radius R, the radius of the circle $r = R\theta$, but the circumference is

$$C = 2\pi(R\sin\theta) \tag{10}$$
$$= 2\pi r\left(\frac{\sin\theta}{\theta}\right) \tag{11}$$
$$< 2\pi r \tag{12}$$

So, an observer on the surface of a sphere could measure the radius and circumference of a circle, and deduce that they were living in a positively curved world.

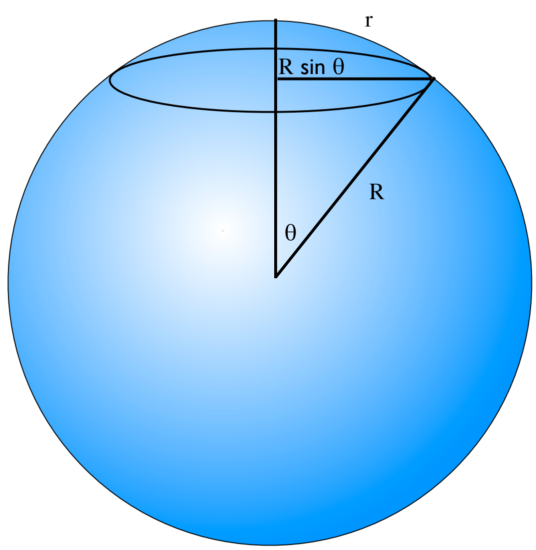

Figure 2: A circle on the surface of a sphere.

- Similarly, in a negatively curved space, $C > 2\pi r$.

3 Measuring distances and the Robertson-Walker metric

These differences in "big" distances must reflect differences in the infinitesimal distances from which they are made up:

$$L = \int dl \tag{13}$$

In Euclidean space, the distance between two points is

$$L^2 = x^2 + y^2 \tag{14}$$

or equivalently in polar coordinates,

$$L^2 = r^2. \tag{15}$$

We express infinitesimal separations similarly:

$$(dl)^2 = (dx)^2 + (dy)^2 \tag{16}$$

or in polar coordinates

$$(dl)^2 = (dr)^2 + (r\ d\phi)^2 \tag{17}$$

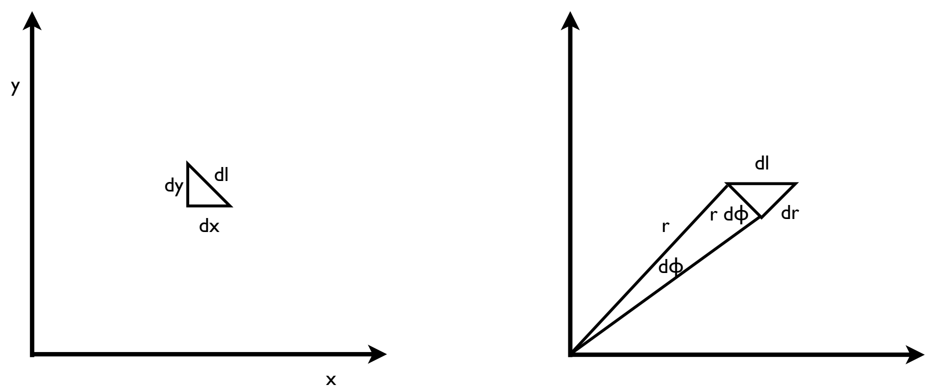

Figure 3: Infinitesimal distances.

We call the rule for how distances are measured in a space the **metric**, e.g. $dl^2 = dr^2 + r^2 d\phi^2$ is the metric for a flat Euclidean space.

We can define a different metric for a different geometry. E.g., the metric of a sphere of radius a is

$$(dl)^2 = \left(\frac{dr}{\sqrt{1 - r^2/a^2}}\right)^2 + (r d\phi)^2. \tag{18}$$

We set $K = 1/a^2$, where K is called the **curvature** of the surface, and generalize to 3-d (we're now using spherical coordinates instead of polar):

$$(dl)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (r\ d\theta)^2 + (r \sin\theta\ d\phi)^2 \tag{19}$$

K is now the curvature of the 3-d space. Note that as $K \to 0$ or $r \to 0$, the space acts Euclidean.

When considering the expanding universe, the curvature is a function of time. We therefore redefine

$$K(t) \equiv \frac{k}{a^2}, \tag{20}$$

where k is a time-independent constant. We can then rewrite the metric using k and substituting $r = a(t)\chi$, where $a(t)$ is the scale factor and χ is the comoving coordinate:

$$(dl)^2 = a^2(t)\left[\left(\frac{d\chi}{\sqrt{1 - k\chi^2}}\right)^2 + (\chi\ d\theta)^2 + (\chi \sin\theta\ d\phi)^2\right] \tag{21}$$

Finally, we are measuring distances in *spacetime*, not just space. By distance, we mean the proper distance between two spacetime events that occur simultaneously, according to an observer. The spacetime metric is

$$(ds)^2 = (c\,dt)^2 - (dl)^2. \tag{22}$$

Therefore, we have

$$\boxed{(ds)^2 = (c\,dt)^2 - a^2(t)\left[\left(\frac{d\chi}{\sqrt{1-k\chi^2}}\right)^2 + (\chi\,d\theta)^2 + (\chi\sin\theta\,d\phi)^2\right]} \tag{23}$$

This is the **Robertson-Walker metric**, and it's the general metric for any homogeneous, isotropic space (both positively and negatively curved, though we only showed it for the simplest, spherical case here).

Also note that this is the same k we've been talking about all along:

$$\left[\left(\frac{\dot{a}(t)}{a(t)}\right)^2 - \frac{8\pi}{3}G\rho\right]a^2(t) = -kc^2. \tag{24}$$

In our Newtonian derivation of this equation, it referred to the total energy of the universe, but when this equation is derived from general relativity, k is the curvature constant defined above.

4 The cosmological constant

Consider again the now familiar equation

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G\rho\right]a^2 = -kc^2. \tag{25}$$

We derived this equation from Newtonian physics, by considering the kinetic and potential energy of the universe, but this equation is also a solution to Einstein's field equations for an isotropic, homogeneous universe. In 1922 the Russian mathematician Aleksandr Friedmann solved the field equations and obtained this equation for a non-static universe. We've been calling it the Friedmann equation all along, but strictly speaking this refers to the equation as derived from general relativity. As we've seen, the constant k refers to the curvature of the universe.

Einstein developed his field equations before Hubble's discovery of the expanding universe, and he believed that the universe was static. In their original form, his field equations couldn't produce a static universe, so Einstein added an additional term (a constant of integration) in order to make the universe static. This term is the cosmological constant Λ, and with this addition the general solution to Einstein's field equations is

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G\rho - \frac{1}{3}\Lambda c^2\right]a^2 = -kc^2. \tag{26}$$

In our original Newtonian derivation, this would result from adding an additional potential energy term

$$U_\Lambda \equiv -\frac{1}{6}\Lambda mc^2r^2 \tag{27}$$

to our equation for the energy balance of the universe. The result of this new potential is a force

$$\mathbf{F_\Lambda} = -\frac{\partial U_\Lambda}{\partial r}\,\hat{\mathbf{r}} = \frac{1}{3}\Lambda mc^2r\,\hat{\mathbf{r}} \tag{28}$$

which is radially outward for $\Lambda > 0$: a repulsive force on the mass shell countering gravity, which allowed Einstein to balance the universe in an (unstable) equilibrium.

5 Effects of the cosmological constant

After Hubble's discovery of the expanding universe, Einstein called the inclusion of this term the "biggest blunder" of his life. However, recent results have indicated that the universe is actually dominated by some sort of energy which behaves like the cosmological constant. We call this **dark energy**, and we'll now look at its effect on the dynamics of the universe.

We write the Friedmann equation in a form that makes it clear that we're now dealing with a three-component universe of mass, relativistic particles and dark energy:

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G(\rho_m + \rho_{\rm rel}) - \frac{1}{3}\Lambda c^2\right]a^2 = -kc^2. \tag{29}$$

The fluid equation also comes from solving Einstein's field equations with the inclusion of the cosmological constant

$$\frac{d(a^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(a^3)}{dt} \tag{30}$$

where ρ and P are the density and pressure due to every component of the universe. (Λ does not appear in the fluid equation.)

As discussed for the two-component universe earlier, the Friedmann equation and the fluid equation can be combined to produce the acceleration equation:

$$\frac{d^2a}{dt^2} = \left[-\frac{4\pi G}{3}\left[\rho_m + \rho_{\rm rel} + \frac{3(P_m + P_{\rm rel})}{c^2}\right] + \frac{1}{3}\Lambda c^2\right]a \tag{31}$$

We now define the equivalent mass density of dark energy

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G} = \text{constant} = \rho_{\Lambda,0} \tag{32}$$

so that the Friedmann equation becomes

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G(\rho_m + \rho_{\rm rel} + \rho_\Lambda)\right]a^2 = -kc^2. \tag{33}$$

Note that because ρ_Λ remains constant as the universe expands, more and more dark energy must appear to fill the larger volume.

We can calculate the pressure due to dark energy from the fluid equation

$$\frac{d(a^3\rho_\Lambda)}{dt} = -\frac{P_\Lambda}{c^2}\frac{d(a^3)}{dt} \tag{34}$$

which is

$$3a^2\rho_\Lambda\frac{da}{dt} + a^3\frac{d\rho_\Lambda}{dt} = -\frac{P_\Lambda}{c^2}3a^2\frac{da}{dt} \tag{35}$$

Because ρ_Λ remains constant $d\rho_\Lambda/dt = 0$, so the second term is zero. Canceling $3a^2 da/dt$ from both sides, we find

$$P_\Lambda = -\rho_\Lambda c^2 \tag{36}$$

which is the equation of state for dark energy. In the general equation of state $P = w\rho c^2$, $w = -1$ for dark energy. The pressure due to the cosmological constant is *negative*, while the equivalent mass density is positive. We can now substitute expressions for ρ_Λ and P_Λ into the acceleration equation:

$$\frac{d^2a}{dt^2} = \left[-\frac{4\pi G}{3}\left[\rho_m + \rho_{\rm rel} + \rho_\Lambda + \frac{3(P_m + P_{\rm rel} + P_\Lambda)}{c^2}\right]\right] a \tag{37}$$

The Friedmann equation can also be written in terms of the Hubble constant and the density parameter Ω as

$$H^2[1 - (\Omega_m + \Omega_{\rm rel} + \Omega_\Lambda)]a^2 = -kc^2 \tag{38}$$

where

$$\Omega_m = \frac{\rho_m}{\rho_c} = \frac{8\pi G\rho_m}{3H^2} \tag{39}$$

$$\Omega_{\rm rel} = \frac{\rho_{\rm rel}}{\rho_c} = \frac{8\pi G\rho_{\rm rel}}{3H^2} \tag{40}$$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda c^2}{3H^2}. \tag{41}$$

We define the **total density parameter**

$$\Omega \equiv \Omega_m + \Omega_{\rm rel} + \Omega_\Lambda. \tag{42}$$

Note that Ω without a subscript refers to the total density parameter for all of the components of the model under consideration. The Friedmann equation is then

$$H^2(1 - \Omega)a^2 = -kc^2. \tag{43}$$

A flat universe with $k = 0$ requires $\Omega(t) = 1$.

The WMAP measurements of the cosmic microwave background give us values for the three components of the universe today:

$$\Omega_{m,0} = 0.27 \pm 0.04 \tag{44}$$

$$\Omega_{\rm rel,0} = 8.24 \times 10^{-5} \tag{45}$$

$$\Omega_{\Lambda,0} = 0.73 \pm 0.04 \tag{46}$$

Within our ability to measure it, the universe is flat and currently dominated by dark energy.

We can also define the **deceleration parameter** $q(t)$:

$$q(t) \equiv -\frac{a(t)[d^2a(t)/dt^2]}{[da(t)/dt]^2} \tag{47}$$

The name and the minus sign, which gives a positive value for a decelerating universe, reflect the once-common belief that the universe had to be decelerating. The deceleration parameter can also be written in terms of the density parameters for the different components of the universe:

$$q(t) = \frac{1}{2}\Omega_m(t) + \Omega_{\rm rel}(t) - \Omega_\Lambda(t). \tag{48}$$

With current values,

$$q_0 = -0.60, \tag{49}$$

telling us (because of the minus sign) that the universe is currently accelerating.

6 The Λ era

We've already discussed the dependence of the density of radiation and matter on the scale factor: $\rho_m \propto a^{-3}$ and $\rho_{\rm rel} \propto a^{-4}$. Because the radiation density decreases more quickly as the universe expands, the universe was dominated by radiation at early times but then became matter-dominated at a redshift of $z_{r,m} = 3270$. We now have another component to consider, ρ_Λ, which is *constant*. This means that at some point, as the matter density decreases as the universe expands, the universe will become dominated by the cosmological constant. As it turns out, this has already happened. The scale factor at the equality of matter and dark energy is

$$a_{m,\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} = 0.72, \tag{50}$$

which corresponds to a redshift $z_{m,\Lambda} = 0.39$.

We can also use the acceleration equation to find when the acceleration of the universe changed from negative to positive. The result is

$$a_{\rm accel} = \left(\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}\right)^{1/3} = 0.57, \tag{51}$$

corresponding to a redshift $z_{\rm accel} = 0.76$. So the universe began to accelerate *before* the equality of matter and dark energy; this is because dark energy has pressure as well as equivalent mass density which affects the dynamics of the universe, and (non-relativistic) matter does not. Note also that unlike all of the other cosmological times we've calculated, these are relatively recent. We routinely observe objects at these redshifts, and can use them to understand the dynamics of the universe.

Another consequence of the fact that $\rho_m \propto a^{-3}$ and $\rho_{\rm rel} \propto a^{-4}$ while ρ_Λ is constant is that in the early universe, the densities of matter and radiation were much higher than the dark energy density. This means that we can neglect Λ in the early universe, and all the results we've already derived for the early universe are still valid.