4) LRT : $\dfrac{\prod_{i=1}^{n} \lambda_1 e^{-\lambda_1 x_i}}{\prod_{i=1}^{n} \lambda_0 e^{-\lambda_0 x_i}} \underset{H_0}{\overset{H_1}{\gtrless}} t$

$\sum_{i=1}^{n} -\lambda_1 x_i \log \lambda_1 + \lambda_0 x_i \log \lambda_0 \underset{H_0}{\overset{H_1}{\gtrless}} \log \tau$

$\sum_{i=1}^{n} x_i [\lambda_0 \log \lambda_0 - \lambda_1 \log \lambda_1] \underset{H_0}{\overset{H_1}{\gtrless}} \log \tau$

$\lambda_1 > \lambda_0$

$\sum_{i=1}^{n} x_i \underset{H_0}{\overset{H_1}{\lessgtr}} \log \tau [\lambda_0 \log \lambda_0 - \lambda_1 \log \lambda_1]^{-1}$

Our test statistic is $t(x_1, x_2, \dots, x_n) := \sum_{i=1}^{n} x_i$

we define the threshold $\gamma = \log \tau [\lambda_0 \log \lambda_0 - \lambda_1 \log \lambda_1]^{-1}$

$P_{FP} = \int_{\gamma}^{\infty} \lambda e^{-\lambda x} dx$

$= -e^{-\lambda x} \Big|_{\gamma}^{\infty}$

$\boxed{P_{FP} = e^{-\lambda \gamma}}$

a UMP does exist because for larger λ_1, where $\lambda_1 > \lambda_0$ γ will become a smaller negative #, thus P_{FP} will decrease with increasing λ_1

5) LRT: $$\frac{\prod_{i=1}^{n} \frac{\lambda_1}{2} e^{-\lambda_1 |x_i|}}{\prod_{i=1}^{n} \frac{\lambda_0}{2} e^{-\lambda_0 |x_i|}} \underset{H_0}{\overset{H_1}{\gtrless}} \tau$$

$$\sum_{i=1}^{n} -\lambda_1 |x_i| \log \lambda_1 + \lambda_0 |x_i| \log \lambda_0 \underset{H_0}{\overset{H_1}{\gtrless}} \log \tau$$

$$\sum_{i=1}^{n} |x_i| [\lambda_0 \log \lambda_0 - \lambda_1 \log \lambda_1] \underset{H_0}{\overset{H_1}{\gtrless}} \log \tau$$

$$\sum_{i=1}^{n} |x_i| \underset{H_0}{\overset{H_1}{\lessgtr}} \log \tau [\lambda_0 \log \lambda_0 - \lambda_1 \log \lambda_1]^{-1}$$

$$t := \sum_{i=1}^{n} |x_i|$$

$$P_{FP} = \int_{-\infty}^{-\gamma} \frac{\lambda}{2} e^{\lambda x} dx + \int_{\gamma}^{\infty} \frac{\lambda}{2} e^{-\lambda x} dx$$

$$= \frac{1}{2} e^{\lambda x} \Big|_{-\infty}^{-\gamma} - \frac{1}{2} e^{-\lambda x} \Big|_{\gamma}^{\infty}$$

$$= \frac{1}{2} e^{-\lambda \gamma} + \frac{1}{2} e^{-\lambda \gamma}$$

$$\boxed{P_{FP} = e^{-\lambda \gamma}}$$

There is a UMP test given reasoning explained in 4

6) $$\hat{\Lambda}(x) = \frac{\max_{\theta, \sigma_1^2} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-(x-\theta s)^2/2\sigma_1^2}}{\max_{\sigma_0^2} \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-x^2/2\sigma^2}} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

$$= \frac{\max_{\theta\, \sigma_1^2} -(x-\theta s)^2/2\sigma_1^2 - \frac{1}{2}\log 2\pi\sigma_1^2}{\max_{\sigma_0^2} -x^2/2\sigma_0^2 - \frac{1}{2}\log 2\pi\sigma_0^2} \underset{H_0}{\overset{H_1}{\gtrless}} \log\gamma$$

$$= \frac{\max_{\theta, \sigma_1^2} -\frac{1}{2\sigma_1^2}\left[x^Tx - 2\theta^T s^T x + \theta^T s^T s\theta\right] - \frac{1}{2}\log 2\pi\sigma_1^2}{\max_{\sigma_0^2} -\frac{1}{2\sigma_0^2} x^2 - \frac{1}{2}\log 2\pi\sigma_0^2}$$

$\max_{\theta}$: $$\frac{\partial}{\partial\theta}\left[-\frac{1}{2\sigma_1^2}\left[x^Tx - 2\theta^T s^T x + \theta^T s^T s\,\theta\right] - \frac{1}{2}\log 2\pi\sigma_1^2\right] = 0$$

$$= -\frac{1}{2\sigma_1^2}\left[-2s^Tx + 2s^Ts\theta\right] = 0$$

$$\boxed{\hat{\theta} = \frac{s^Tx}{s^Ts}}$$

$\max_{\sigma_1^2}$: $$\frac{\partial}{\partial\sigma_1^2} = \frac{1}{4\sigma_1^4}\left[x^Tx - 2\theta^T s^T x + \theta^T s^T s\theta\right] - \frac{1}{2}\frac{1}{2\pi\sigma_1^2} = 0$$

$$\hat{\sigma}_1^2 = \left[x^Tx - 2\theta^T s^T x + \theta^T s^T s\,\theta\right]$$

$$\boxed{\hat{\sigma}_1^2 = (x - \hat{\theta}s)^2}$$

$\max_{\sigma_0^2}$: $$\frac{\partial}{\partial\sigma_0^2}\left[-\frac{1}{2\sigma_0^2}x^Tx - \frac{1}{2}\log 2\pi\sigma_0^2\right] = 0$$

$$= \frac{1}{4\sigma^4}x^Tx - \frac{1}{2}\frac{1}{2\pi\sigma^2} = 0$$

$$\boxed{\hat{\sigma}_0^2 = x^Tx}$$

b) $\log \hat{\lambda}(x) = \log \dfrac{\frac{1}{\sqrt{2\pi \hat{\sigma}_1^2}} e^{-(x-\hat{\theta})^2/2\hat{\sigma}_1^2}}{\frac{1}{\sqrt{2\pi\hat{\sigma}_0^2}} e^{-x^2/2\hat{\sigma}_0^2}} \underset{H_0}{\overset{H_1}{\gtrless}} \log \gamma$

$= \left[-(x-\hat{\theta}s)^T(x-\hat{\theta}s)/2\hat{\sigma}_1^2 - \frac{1}{2}\log[2\pi\hat{\sigma}_1^2] \right] -$
$\left[-x^T x/2\hat{\sigma}_0^2 - \frac{1}{2}\log[2\pi\hat{\sigma}_0^2] \right]$

$= \left[[x^T x + 2\hat{\theta}^T s^T x - \hat{\theta}^T s^T s \hat{\theta}]/2\hat{\sigma}_1^2 - 2\log\hat{\sigma}_1 + \frac{1}{2}\log 2\pi \right] -$
$\left[-x^T x/2\hat{\sigma}_0^2 - \log \hat{\sigma}_0^2 + \frac{1}{2}\log 2\pi \right]$

$= \left[-\frac{1}{2} - \log \hat{\sigma}_1^2 \right] - \left[-\frac{1}{2} - \log \hat{\sigma}_0^2 \right]$

$= \log \hat{\sigma}_0^2 - \log \hat{\sigma}_1^2$

$= \log \dfrac{x^2}{(x-\hat{\theta}s)^2}$

$= \log \dfrac{x^2}{\left(x - \frac{s^T x}{s^T s} s\right)^2} \underset{H_0}{\overset{H_1}{\gtrless}} \log \gamma$

$\hat{\lambda}(x) = e^{\left(\frac{x}{x - \frac{s^T x}{s^T s} s}\right)^2} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$

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7)

$$\sum x_i \sim N(0, n)$$

$$P_{FP} = \alpha = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi n}} e^{-x^2/2n} dx \qquad y = \frac{x}{\sqrt{n}} \quad dy = \frac{dx}{\sqrt{n}}$$

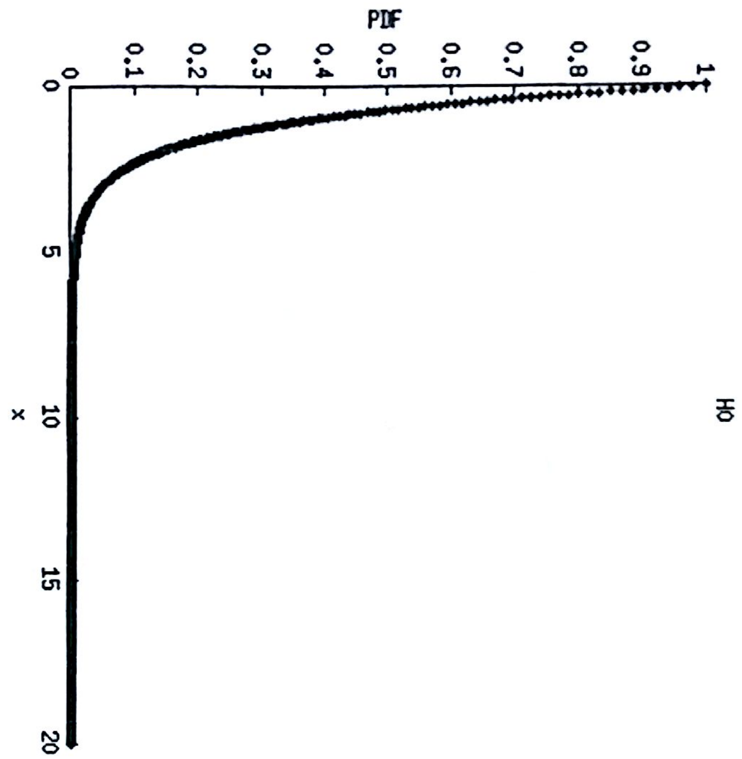
$$= \int_{\frac{\gamma}{\sqrt{n}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy$$

$$= Q\left(\frac{\gamma}{\sqrt{n}}\right)$$

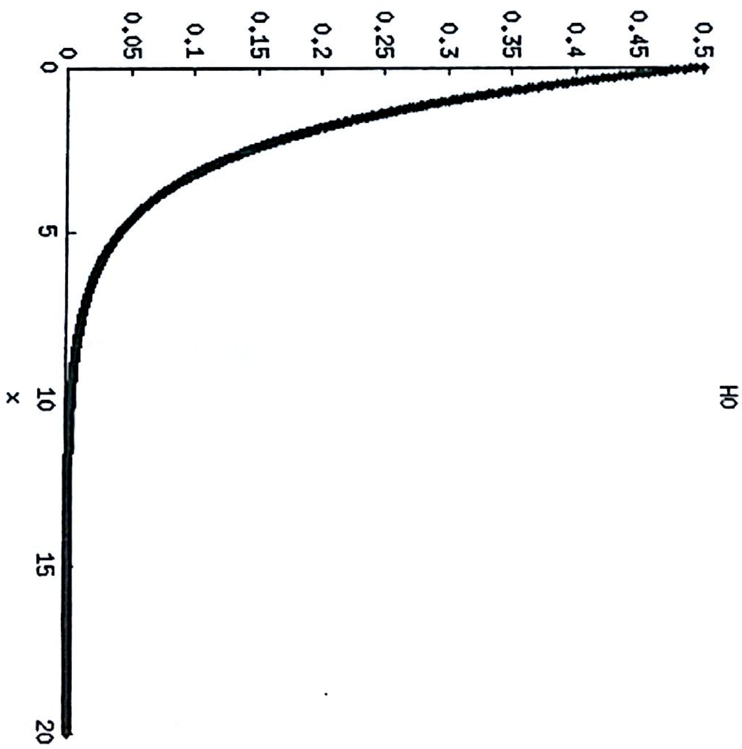
$$\alpha = Q\left(\frac{\gamma}{\sqrt{n}}\right)$$

8a)

$\langle p_0 \rangle = 0.5$



$\langle p_1 \rangle = 0.5$



8b) $P_{FP} = \int_{\gamma}^{\infty} e^{-x}\, dx$

$= -e^{-x} \Big|_{\gamma}^{\infty}$

$= e^{-\gamma}$

$\cancel{P_F}$ $P_{FP} \leq 0.05$

$e^{-\gamma} \leq 0.05$

$\gamma \geq -\log 0.05$

$P_D = \int_{\gamma}^{\infty} 0.5\, e^{-x/2}\, dx$

$= -e^{-x/2} \Big|_{\gamma}^{\infty}$

$= e^{-\gamma/2}$

SPRT :

$\gamma_1 \leq \frac{e^{-\gamma/2}}{e^{-\gamma}}$

$\gamma_1 \leq \frac{1}{e^{-\gamma/2}}$

$\gamma_1 \leq e^{\gamma/2}$

$\gamma_0 \geq \frac{1 - e^{-\gamma/2}}{1 - e^{-\gamma}}$

8c)

$$D(p_0 \| p_1) = \sum_{i=1}^{n} p_{0i} \log \frac{p_{0i}}{p_1}$$

$$E_0[K^*] = \frac{(1-P_{FA}) \log\left(\frac{1-P_{FA}}{1-P_D}\right) - P_{FA} \log\left(\frac{P_D}{P_{FA}}\right)}{D(p_0 \| p_1)}$$

$$E_1[K^*] = \frac{P_D \log\left(\frac{P_D}{P_{FA}}\right) + -(1-P_D) \log\left(\frac{1-P_{FA}}{1-P_D}\right)}{D(P_1 \| p_0)}$$

$\gamma = -\log(0.05) = 2.99$

$\gamma_1 = e^{\gamma/2} = 4.47$

$\gamma_0 = \frac{1-e^{-\gamma/2}}{1-e^{-\gamma}} = 0.81$

$P_{FP} = 0.05$ $\qquad$ $P_D = e^{-\gamma/2} = 0.22$

$E_0 = 0.6 \rightarrow E_0 = 1$

$E_1 = 1$