ECE 532: Homework 1

Due on Tuesday, Sept. 9

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September 16, 2014

Problem 1

1a

Given $\boldsymbol{X} = [\boldsymbol{x}_1 \boldsymbol{x}_2 \dots \boldsymbol{x}_n] \in \mathbb{R}^p$, we can express the matrix $\boldsymbol{C}$ where

$$\boldsymbol{C} = XX^T \tag{1}$$

as the following sum of rank-1 matrices

$$\boldsymbol{C} = \sum_{i=1}^{\infty} \frac{\boldsymbol{x}_i \boldsymbol{x}_i^T}{n} \tag{2}$$

1b

The rank of $\boldsymbol{C}$ will be n.

Problem 2

2a

To determine if $\Phi(\boldsymbol{x})$ is a norm where

$$\Phi(\boldsymbol{x}) = \sum_{j=1}^{m} \left(\sum_{i \in G_j} x_i^2 \right)^{1/2} \tag{3}$$

we first recognize that $\Phi(\boldsymbol{x})$ is simply a sum over an instance of the p-norm where $p = 2$ because $i \in G_j$ will include all elements in the sent $\{1, 2, \dots, n\}$. The sum over the p-norm is also a 1-norm. The norm of a norm, is in fact a norm, thus $\Phi(\boldsymbol{x})$ is a norm.

2b

When $m = 1$, $\Phi(\boldsymbol{x})$ is the Euclidean norm. When $m = n$, $\Phi(\boldsymbol{x})$ is the 1-norm.

Problem 3

Given

$$\cos(\boldsymbol{x}, \boldsymbol{y}) = \frac{\boldsymbol{x}^T \boldsymbol{y}}{\|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2} \tag{4}$$

and that $|\cos(\boldsymbol{x}, \boldsymbol{y})| \leq 1$, the absolute value of the numerator cannot be larger than the denominator, thus $|\boldsymbol{x}^T \boldsymbol{y}| \leq \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2$.

Problem 4

4a

Given $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ we can write $\boldsymbol{x}$ as

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{y} \tag{5}$$

4b

To bound the 2-norm of $\boldsymbol{x}$ with a function of $\boldsymbol{A}$ and $\boldsymbol{y}$ we first take $\|\boldsymbol{x}\| = \|\boldsymbol{A}^{-1}\boldsymbol{y}\|$ which can be expressed as

$$\frac{\|\boldsymbol{A}^{-1}\boldsymbol{y}\|}{\|\boldsymbol{y}\|} \|\boldsymbol{y}\| \tag{6}$$

where

$$\frac{\|\boldsymbol{A}^{-1}\boldsymbol{y}\|}{\|\boldsymbol{y}\|} \tag{7}$$

is the matrix norm. Eq. 8 will always be less than $\|\boldsymbol{A}^{-1}\|$, thus

$$\|\boldsymbol{x}\| \leq \|\boldsymbol{A}^{-1}\| \|\boldsymbol{y}\| \tag{8}$$

Problem 5

5a

The rank of $\boldsymbol{A}$ is 3.

5b

$\boldsymbol{x}$ can be expressed as

$$\boldsymbol{x} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \boldsymbol{y} \tag{9}$$

Problem 6

6a

The rank of $\boldsymbol{X}$ is 3.

6b

The rank of $\frac{\boldsymbol{X}\boldsymbol{X}^{\boldsymbol{T}}}{n}$ is 3.

6c

A set of linearly independent columns of $\boldsymbol{X}$ are

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$