Astro 735: Cosmology
Lecture 5: Newtonian Cosmology

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1 The evolution of the universe

Now we will mathematically describe the evolution of a universe containing only non-relativistic matter. We will do this using only Newtonian physics; this is not really correct, but will give us the right answer anyway.

We'll consider the evolution of the universe to be essentially a battle between expansion and gravity. The universe we consider is homogeneous and isotropic, with constant density ρ. Consider a spherical shell of radius r and mass m. The shell is expanding due to the expansion of the universe, with velocity $v = dr/dt = \dot{r}$.

As the universe expands, the mass inside the shell $M(< r)$ is constant—the expansion is uniform, so mass that starts within the shell will always be within the shell. So $\rho r^3 =$ constant.

Because the mass is uniformly distributed, mass outside the shell doesn't matter.

The total energy of the shell as it expands is the sum of its kinetic and potential energies:

$$\frac{1}{2}mv^2 - \frac{GM(< r)m}{r} = E \tag{1}$$

Now let's parameterize the total energy of the shell:

$$E = -k\chi^2 \times \frac{1}{2}mc^2, \tag{2}$$

where $k\chi^2$ are parameters we'll discuss later. So

$$\frac{1}{2}mv^2 - \frac{GM(< r)m}{r} = -k\chi^2\,\frac{1}{2}mc^2. \tag{3}$$

Next we substitute $M(< r) = (4\pi/3)r^3\rho$, cancel the shell mass m, and multiply by 2:

$$v^2 - \frac{8\pi}{3}G\rho r^2 = -k\chi^2c^2. \tag{4}$$

Now we write the radius r in terms of a global **scale factor** $a(t)$, which describes the overall expansion of the universe:

$$r(t) = a(t)\chi \tag{5}$$

$a(t)$ is called the "scale factor" of the universe, and χ is called the **comoving coordinate**. The comoving coordinate describes the distance between objects, but it stays constant as the universe expands; this is a useful way to talk about distances when the whole universe is expanding.

For instance, at time t_1, two galachies are separated by a distance $r(t_1) = a(t_1)\chi$, and at t_2 they are separated by a distance $r(t_2) = a(t_2)\chi$. Their comoving separation stays the same, but their physical separation r increases due to the increasing scale factor $a(t)$.

Now we rewrite the expansion velocity

$$v = \frac{dr}{dt} = \chi \frac{da}{dt} = \chi \dot{a} \tag{6}$$

and substitute this and $r = a\chi$ into Equation 4:

$$\chi^2 \dot{a}^2 - \frac{8\pi}{3} G \rho a^2 \chi^2 = -kc^2 \chi^2. \tag{7}$$

We cancel χ:

$$\boxed{\dot{a}^2 - \frac{8\pi}{3} G \rho a^2 = -kc^2} \tag{8}$$

This is an expression for the evolution of only the scale factor a — no more references to specific shells or coordinates. This equation is in fact the solution to Einstein's field equations, and in that context is called the Friedmann equation.

Now we'll do a few more things. Divide out a^2 from the left side:

$$\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi}{3} G\rho\right] a^2 = -kc^2 \tag{9}$$

We can choose the normalization of $a(t)$ in any way we like. For convenience, let's set the scale factor today equal to 1: $a(t_0) = 1$ (t_0 = time today—0 subscript refers to the value of something at the present time, e.g. H_0 is the value of the Hubble constant (which isn't constant) today, and ρ_0 is the density of the universe today). So

$$\left[\frac{\dot{a}(t_0)}{a(t_0)}\right]^2 - \frac{8\pi}{3} G\rho_0 = -kc^2 \tag{10}$$

at the present time.

Now let's look at the first term. According to the Hubble law,

$$v(t) = H(t) r(t) = H(t) a(t) \chi. \tag{11}$$

Also,

$$v(t) = \frac{dr(t)}{dt} = \chi \frac{da(t)}{dt}. \tag{12}$$

Equating these two expressions for v, we find an expression for the Hubble constant at any time t:

$$H(t) = \frac{da(t)/dt}{a}. \tag{13}$$

So in general,

$$H(t) = \frac{\dot{a}(t)}{a(t)}, \tag{14}$$

and at the present day

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)}. \tag{15}$$

We recognize this as the left term in Equation 10 and substitute it in:

$$\boxed{H_0^2 - \frac{8\pi}{3} G\rho_0 = -kc^2} \tag{16}$$

In order from left to right, the three terms in this equation describe the kinetic, potential and total energies.

Now we'll discuss the meaning of the constant k, which as we see above describes the total energy of the universe. It tells us whether the universe is gravitationally bound or unbound:

- Bound: $k > 0$. Total energy $E < 0$, so universe can recollapse. We call this **closed**.
- Unbound: $k < 0$. Total energy $E > 0$, so kinetic energy wins and universe keeps expanding. We call this **open**.
- $k = 0$: **Critical** or **flat**. Perfect balance between expansion and gravitational resistance.

We'll consider the "critical" universe with $k = 0$, where expansion and gravity are perfectly balanced. We can calculate the density we need to make this happen. This is the **critical density** ρ_c. If $k = 0$,

$$H_0^2 - \frac{8\pi}{3} G\rho_c = 0 \tag{17}$$

and

$$\boxed{\rho_c = \frac{3H_0^2}{8\pi G}} \tag{18}$$

This is about 9.5×10^{-27} kg m^{-3}. For comparison, our best estimate of the density of baryonic matter is 4.17×10^{-28} kg m^{-3}; this is about 4% of the critical density.

If $\rho_0 < \rho_c$, there isn't enough mass to reverse the expansion and the universe is open. If $\rho_0 > \rho_c$, gravity wins and the universe is closed.

As discussed earlier, we parameterize the density of the universe in terms of a dimensionless number Ω:

$$\Omega_0 \equiv \frac{\rho_0}{\rho_c} \tag{19}$$

- $\Omega_0 = 1$: flat universe with $k = 0$
- $\Omega_0 < 1$: open universe with $k < 0$
- $\Omega_0 > 1$: closed universe with $k > 0$

We can then also rewrite Equation 18 as

$$H_0^2(\Omega_0 - 1) = kc^2 \tag{20}$$

at the present day.

2 Evolution of the scale factor

We return to our equation for the evolution of the scale factor $a(t)$,

$$\left[\frac{da(t)}{dt}\right]^2 - \frac{8\pi}{3} G\rho a(t)^2 = -kc^2. \tag{21}$$

Remember that the mass within the original shell was constant,

$$\rho(t)r^3(t) = \text{ constant.} \tag{22}$$

Writing this in terms of the scale factor,

$$\rho(t)a^3(t)\chi^3 = \text{ constant,} \tag{23}$$

where the comoving coordinate χ is also constant. So

$$\rho(t)a^3(t) = \rho(t_0)a^3(t_0) = \rho_0 \tag{24}$$

since $a(t_0) = 1$.

Therefore

$$\left[\frac{da(t)}{dt}\right]^2 - \frac{8\pi G}{3}\frac{\rho(t)a(t)^3}{a(t)} = -kc^2 \tag{25}$$

which can be written

$$\boxed{\left[\frac{da(t)}{dt}\right]^2 - \frac{8\pi G\rho_0}{3a(t)} = -kc^2.} \tag{26}$$

This is a differential equation for $a(t)$, and we can solve it to get an expression for the evolution of the size of the universe.

2.1 Flat universe

We'll start with the case in which the universe is flat, with $k = 0$ and density equal to the critical density,

$$\rho_0 = \rho_{c,0} = \frac{3H_0^2}{8\pi G}. \tag{27}$$

Then our differential equation is

$$\left[\frac{da(t)}{dt}\right]^2 - \frac{H_0^2}{a(t)} = 0. \tag{28}$$

We solve:

$$\begin{aligned} \frac{da}{dt} &= \frac{H_0}{\sqrt{a(t)}} && (29)\\ \sqrt{a(t)}\; da &= H_0\; dt && (30)\\ \int_0^a \sqrt{a}\; da &= H_0 \int_0^t dt && (31)\\ \frac{2}{3}a^{3/2}(t) &= H_0 t && (32) \end{aligned}$$

and

$$\boxed{a(t) = \left(\frac{3}{2}\right)^{2/3}\left(\frac{t}{t_H}\right)^{2/3}} \tag{33}$$

where $t_H \equiv 1/H_0$ is the Hubble time.

So, for a critical, flat universe ($\Omega_0 = 1$), $a(t) \propto t^{2/3}$. Note that this expands forever. Also note that, since $a(t_0) = 1$,

$$t_0 = \frac{2}{3H_0}. \tag{34}$$

This is the age of the universe.

2.2 Closed universe

Things get more complicated if the universe isn't flat. For a closed universe with $\Omega_0 = \rho_0/\rho_c > 1$ we can express the solution to our differential equation in parametric form:

$$\begin{aligned} a &= \frac{4\pi G\rho_0}{3kc^2}(1-\cos\theta) &(35)\\ &= \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1-\cos\theta) &(36)\end{aligned}$$

and

$$\begin{aligned} t &= \frac{4\pi G\rho_0}{3k^{3/2}c^3}(\theta-\sin\theta) &(37)\\ &= \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(\theta-\sin\theta) &(38)\end{aligned}$$

We increase the value of θ, and see what happens to a and t:

$\theta = 0$:

$$\begin{aligned} a &= 0 &(39)\\ t &= 0 &(40)\end{aligned}$$

$\theta = \pi$:

$$\begin{aligned} a &= \frac{8\pi G\rho_0}{3kc^2} = \frac{\Omega_0}{\Omega_o - 1} &(41)\\ t &= \frac{4\pi G\rho_0}{3k^{3/2}c^3} = \frac{\pi}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} &(42)\end{aligned}$$

$\theta = 2\pi$:

$$\begin{aligned} a &= 0 &(43)\\ t &= 2\times t(\theta=\pi) &(44)\end{aligned}$$

So the universe recollapses! (These are equations for a cycloid; see Figure 1.)

What's the current age of the universe in this model? Messy! $a_0 = 1$, so

$$\frac{2(\Omega_0 - 1)}{\Omega_0} - 1 = -\cos\theta_0 \quad (45)$$

and

$$\begin{aligned} \theta_0 &= \cos^{-1}\left[1 - \frac{2(\Omega_0-1)}{\Omega_0}\right] = \cos^{-1}\left[\frac{2}{\Omega_0} - 1\right] &(46)\\ \sin\theta_0 &= \sqrt{1 - \left[1 - \frac{2(\Omega_0-1)}{\Omega_0}\right]^2} = \sqrt{1 - \left[\frac{2}{\Omega_0} - 1\right]^2} &(47)\end{aligned}$$

So

$$t_0 = \frac{\pi}{2H_0}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}\left[\cos^{-1}\left[\frac{2}{\Omega_0} - 1\right] - \sqrt{1 - \left[\frac{2}{\Omega_0} - 1\right]^2}\right] \quad (48)$$

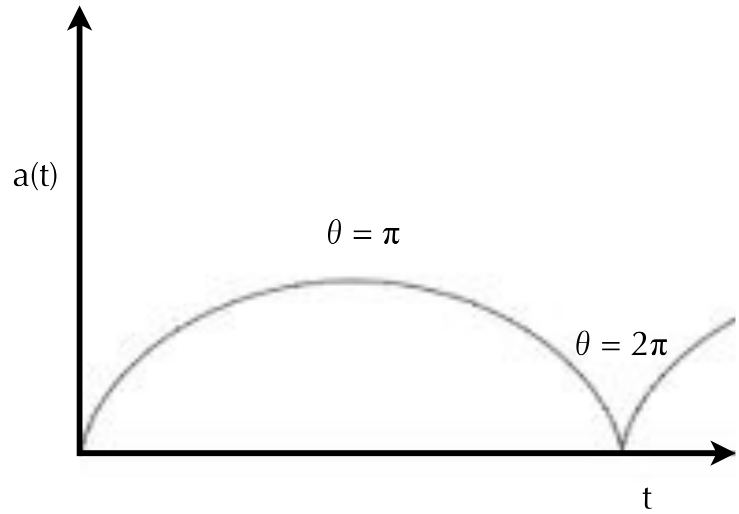

Figure 1: Evolution of the scale factor for a closed universe.

2.3 Open universe

For an open universe with $k < 0$ and $\Omega_0 < 1$, we again have parametric solutions for the evolution of the scale factor:

$$a = \frac{4\pi G\rho_0}{3|k|c^2}(\cosh\theta - 1) \tag{49}$$

$$= \frac{1}{2}\frac{\Omega_0}{1-\Omega_0}(\cosh\theta - 1) \tag{50}$$

and

$$t = \frac{4\pi G\rho_0}{3|k|^{3/2}c^3}(\sinh\theta - \theta) \tag{51}$$

$$= \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\sinh\theta - \theta) \tag{52}$$

where $\sinh\theta$ and $\cosh\theta$ are the hyperbolic sine and cosine functions:

$$\cosh\theta \equiv \frac{e^\theta + e^{-\theta}}{2} \geq 1 \tag{53}$$

$$\sinh\theta \equiv \frac{e^\theta - e^{-\theta}}{2} \geq \theta \tag{54}$$

In this case a is monotonically increasing, so the universe expands forever.

For all of these expressions, you can see that the age of the universe depends on Ω_0. More dense universes are younger.

3 Cosmological redshift

The evolution of the scale factor affects more than galaxies. The expansion of the universe also changes the wavelength of light; light stretches as the universe expands.

Suppose light is emitted with wavelength $\lambda_1 = \lambda_{\rm em}$ at some time $t_1 \ll t_0$, when the universe was smaller than its current size—say $a_1 = 1/3$. At some later time t_2 the universe has doubled in size to $a_2 = 2/3$, changing the wavelength of light along with it, so the wavelength of the light is also twice as big, $\lambda_2 = 2\lambda_{\rm em}$. We detect the light at time t_0, when $a_0 = 1 = 3a_1$, and the wavelength of the light is now $\lambda_3 = 3\lambda_{\rm em}$. So, recalling the definition of redshift,

$$\frac{\lambda_{\rm obs}}{\lambda_{\rm em}} \equiv (1+z) = \frac{a(t_0)}{a(t_1)}. \tag{55}$$

For this example, the redshift $z = 2$.

So the redshift and scale factor are directly related, and when we talk about the history of the universe we usually just parameterize it in terms of the redshift we would measure today for photons which were emitted at some earlier time $t < t_0$ when the universe was more compact.

$$(1+z) = \frac{a(t_0)}{a(t_1)}, \tag{56}$$

so

$$a(t) = \frac{1}{(1+z)}, \tag{57}$$

independent of any of the cosmological parameters like H_0 or Ω_0. So, for example, at a redshift $z = 3$, the universe was 1/4 of its current size.

For a flat universe we derived

$$a(t) = \left(\frac{t}{\frac{2}{3}t_H}\right)^{2/3} \tag{58}$$

which can be rearranged to

$$t(z) = \left(\frac{2}{3}t_H\right)\left(\frac{1}{(1+z)^{3/2}}\right). \tag{59}$$

So as we found before, today at $z = 0$ $t_0 = (2/3)t_H$, and at $z = 3$, $t = t_0/8$. If $\Omega_0 = 1$ and we observe a galaxy at redshift $z = 3$, we are observing a galaxy at 1/8 the current age of the universe.

4 Lookback time

It's also useful to define the "lookback time" to redshift z, the amount of time which has passed between when a redshifted photon was emitted and when we detect it today.

$$t_{\rm lookback} \equiv t_0 - t(z) \tag{60}$$

In other words, this is just the difference in age between the universe today and the universe at time $t(z)$. The lookback time depends on Ω_0 and H_0.