Stat 860: HW 2

Due on Tuesday, Sept. 18

Grace Wahba 4:00 PM

Elijah Bernstein-Cooper

September 18, 2014

Problem 1

1a

A positive definite matrix requires that the matrix is Hermitian and its eigenvalues are positive. Our problem is then reduced to, given positive definite matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, show that the Kronecker product of the matrices is Hermitian and has positive eigenvalues.

A property of the Kronecker product goes as follows: given $\boldsymbol{A} \in \mathbb{R}^{m,n}$ and $\boldsymbol{B} \in \mathbb{R}^{p,q}$, then $(\boldsymbol{A} \otimes \boldsymbol{B})^* = \boldsymbol{A}^* \otimes \boldsymbol{B}^*$, which shows that a Kronecker product will be Hermitian if the two matrices in the product are Hermitian.

Next given, $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{v} \in \mathbb{R}^p$ and

$$\boldsymbol{A}\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i \qquad i = 1, \ldots, m \qquad \boldsymbol{B}\boldsymbol{v}_j = \mu_j \boldsymbol{v}_j \qquad j = 1, \ldots, p$$

where λ and μ are the eigenvalues for $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. For $i = 1, \ldots, m, j = 1, \ldots, p$

$$(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{u}_i \otimes \boldsymbol{v}_j) = \lambda_i \mu_j (\boldsymbol{u}_i \otimes \boldsymbol{v}_j)$$

thus if the eigenvalues for $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive, the eigenvalue of $\boldsymbol{A} \otimes \boldsymbol{B}$ will be positive.

We are now able to assert that the Kronecker product of two positive definite matrices will be positive definite.

1b

We again want to show that $\boldsymbol{A} \circ \boldsymbol{B}$ (where $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same size) is positive definite by showing that the product is Hermitian and has positive eigenvalues. The Shur product of $\boldsymbol{A}$ and $\boldsymbol{B}$ can be represented as a principle submatrix of the Kronecker product of $\boldsymbol{A}$ and $\boldsymbol{B}$ where

$$\boldsymbol{A} \circ \boldsymbol{B} = \boldsymbol{A}(\otimes \boldsymbol{B})(\alpha, \beta)$$

$$\alpha : 1, m+2, \ldots, m^2 \qquad \beta : 1, n+2, \ldots, n^2 \qquad \alpha = \beta, m = n$$

A principle submatrix of a Hermitian matrix will be Hermitian itself. Finally we know that the eigenvalues of the principle submatrix of the Kronecker product will be greater than the eigenvalues of the Kronecker product due to the Eigenvalue Interlacing for Principal Submatrices theorem. The Shur product of positive definite matrices is therefore positive definite.