Astro 735: Cosmology
Lecture 8: Cosmological distances

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1 The Hubble parameter and the age of the universe

We have described the contents of the universe using the three parameters Ω_m, $\Omega_{\rm rel}$ and Ω_Λ. Cosmologists sometimes use an additional density parameter Ω_k, which measures the curvature of space and is defined such that

$$\Omega_m + \Omega_{\rm rel} + \Omega_\Lambda + \Omega_k = 1. \tag{1}$$

An additional note on terminology: a cosmological model with $\Omega_m = \Omega = 1$ (flat and containing only non-relativistic matter) is known as **Einstein-de Sitter**, and our currently preferred Λ-dominated model is sometimes called WMAP or ΛCDM.

We saw that the Friedmann equation for a cosmological model including both relativistic and nonrelativstic matter and the cosmological constant can be written

$$H^2[1 - (\Omega_m + \Omega_{\rm rel} + \Omega_\Lambda)]a^2 = -kc^2 \tag{2}$$

or

$$H^2(1 - \Omega)a^2 = -kc^2. \tag{3}$$

By setting $H^2(1-\Omega)a^2 = H_0^2(1-\Omega_0)$ and including the redshift dependences of the different components of the universe, we can derive an expression for the evolution of the Hubble parameter with redshift:

$$H(z) = H_0(1+z)\left[\Omega_{m,0}(1+z) + \Omega_{{\rm rel},0}(1+z)^2 + \frac{\Omega_{\Lambda,0}}{(1+z)^2} + 1 - \Omega_0\right]^{1/2}. \tag{4}$$

We can derive the time dependence of the scale factor a for a flat universe by setting $k = 0$ in the Friedmann equation

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G(\rho_m + \rho_{\rm rel} + \rho_\Lambda)\right]a^2 = -kc^2. \tag{5}$$

We find

$$t = \sqrt{\frac{3}{8\pi G}}\int_0^a \frac{a\,da}{(\rho_{m,0}a + \rho_{{\rm rel},0} + \rho_{\Lambda,0}a^4)^{1/2}}. \tag{6}$$

An analytic solution requires neglecting the relativistic particle density $\rho_{\rm rel}$, and the eventual result is

$$t(a) = \frac{2}{3}\frac{1}{H_0\sqrt{\Omega_{\Lambda,0}}}\ln\left[\sqrt{\left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)a^3} + \sqrt{1 + \left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)a^3}\,\right]. \tag{7}$$

Since the scale factor $a = 1/(1+z)$, this is also an expression for the age of the universe as a function of redshift:

$$t(z) = \frac{2}{3}\frac{1}{H_0\sqrt{\Omega_{\Lambda,0}}}\ln\left[\sqrt{\left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)(1+z)^{-3}} + \sqrt{1 + \left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)(1+z)^{-3}}\,\right]. \tag{8}$$

We can find the current age of the universe by setting $z = 0$ or $a = 1$:

$$t_0 = 1.37 \times 10^{10}\ {\rm yr}, \tag{9}$$

in good agreement with the best-fit age of the universe from WMAP, 13.7 ± 0.2 Gyr. We can also use this expression to find the time at which the universe began to accelerate: $t_{\rm accel} = 7.08$ Gyr (corresponding to the redshift $z_{\rm accel} = 0.76$), so the universe has been accelerating for about the second half of its existence.

2 Cosmological distances

How do we measure distances in an expanding and potentially curved universe? This isn't an easy question, and the answer is different depending on exactly what we're trying to measure.

2.1 Hubble distance

An order of magnitude estimate for the size of the observable universe is easy: the speed of light multiplied by the age of the universe. The approximate age of the universe is the Hubble time t_H, and this distance is the Hubble distance,

$$d_H = c\,t_H = \frac{c}{H_0} \simeq 4.2 \text{ Gpc}. \tag{10}$$

2.2 Proper distance

Now we'll consider other distances. The **proper distance** is the distance between us and some object *now*—not the distance between us and the object when its light was emitted. We can derive this using the Robertson-Walker metric

$$(ds)^2 = (c\,dt)^2 - a^2(t)\left[\left(\frac{d\chi}{\sqrt{1-k\chi^2}}\right)^2 + (\chi\,d\theta)^2 + (\chi\sin\theta\,d\phi)^2\right]. \tag{11}$$

Because this is the distance to the object now, $dt = 0$, and $d\theta = d\phi = 0$ along a radial line from us to the object, which is at comoving coordinate χ. We set $a(t_0) = 1$ since we're interested in the distance right now. We can then find the proper distance to the object by integrating:

$$d_p = \int_0^\chi \frac{d\chi'}{\sqrt{1-k\chi'^2}} \tag{12}$$

The solution to this depends on the value of k.

For a flat universe with $k = 0$, $d_p = \chi$, and the proper distance to the object is its comoving coordinate. The distance given by χ is also called the **coordinate distance**. We will see how to calculate the comoving distance shortly.

For a closed universe with $k > 0$, the solution is

$$d_p = \frac{1}{\sqrt{k}}\sin^{-1}(\chi\sqrt{k}) \tag{13}$$

and for an open universe with $k < 0$, the proper distance is

$$d_p = \frac{1}{\sqrt{|k|}}\sinh^{-1}(\chi\sqrt{|k|}). \tag{14}$$

In a closed universe the proper distance to an object is greater than its coordinate distance, while in an open universe it is less. (This is an effect of the curvature, like the fact that for a closed universe the circumference of a circle is less than 2π times its radius.)

2.3 Horizon distance

A real calculation of the size of the observable universe must account for the fact that as the universe ages, photons from increasingly distant objects have more time to reach us. In other words, more of the universe may come into causal contact with us over time. The farthest observable point is called the **particle horizon**, and the proper distance to that point is the **horizon distance** d_h. This is the diameter of the largest causally connected region. The horizon distance at time t is

$$d_h(t) = a(t) \int_0^t \frac{c\ dt'}{a(t')} \tag{15}$$

As a photon moves toward us it travels a small distance $c\ dt$ in each interval of time dt. We can't just add these distances together because the universe expands as the photon travels, so we divide the small distance $c\ dt$ by the scale factor at the time t to account for the expansion.

During the radiation era, the scale factor was $a(t) = Ct^{1/2}$, where C is a constant. Substituting this into Equation 6, we find the time dependence of the horizon distance,

$$d_h(t) = 2ct. \tag{16}$$

During the matter era (assuming a flat universe with $k = 0$), $a(t) = Ct^{2/3}$, which gives

$$d_h(t) = 3ct. \tag{17}$$

This can be rewritten in terms of redshift as

$$d_h(z) = \frac{2c}{H_0\sqrt{\Omega_{m,0}}} \frac{1}{(1+z)^{3/2}} \tag{18}$$

We can then estimate the present horizon distance by setting $z = 0$:

$$d_{h,0} \approx \frac{2c}{H_0\sqrt{\Omega_{m,0}}} = 16.3 \text{ Gpc}. \tag{19}$$

In the current, Λ-dominated era, the expression for the scale factor as a function of time is

$$a(t) = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}H_0 t\sqrt{\Omega_{\Lambda,0}}\right) \tag{20}$$

(this is the inversion of the expression for $t(a)$ we saw earlier). For the horizon distance, this gives

$$d_h(t) = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}H_0 t\sqrt{\Omega_{\Lambda,0}}\right) \int_0^t \frac{c\ dt'}{\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}H_0 t'\sqrt{\Omega_{\Lambda,0}}\right)} \tag{21}$$

This has no simple analytic solution and must be integrated numerically. The result is that at the present time, the distance to the particle horizon in a flat universe is

$$d_{h,0} = 14.6 \text{ Gpc}. \tag{22}$$

Note that in the radiation and matter eras the horizon distance is proportional to t, while the scale factor is proportional to $t^{1/2}$ and $t^{2/3}$ respectively. This means that during those eras the size of the observable universe increased more rapidly that the universe expanded, and therefore the universe became increasingly causally connected as it aged.

In the Λ era, the integral portion of Equation 12, without the term in front, is the present distance to the point that will be at the particle horizon at time t. As $t \to \infty$, this integral converges to 19.3 Gpc. This means that the proper distance today to the farthest object that will ever be observable in the future is 19.3 Gpc. Everything within a sphere of radius 19.3 Gpc will eventually become visible, and everything beyond is hidden forever.

2.4 Angular diameter distance

Astronomers are often interested in the physical size of a galaxy (i.e., its size in kpc), but what we measure is the angular size (in arcsec or radians). Converting from angular to physical size requires the distance,

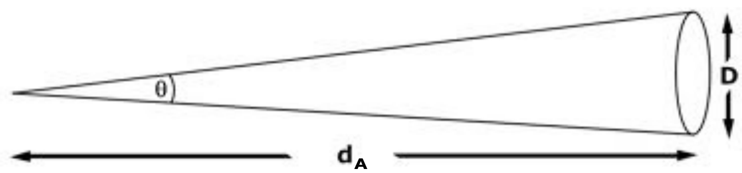

Figure 1: Angular diameter. $D = d_A\theta$, when θ is in radans.

and this is not so simple when the universe is curved and expanding. We need to use the Robertson-Walker metric to find the distance, as we did for the proper distance. Now $d\chi = 0$ (the object is at comoving coordinate χ), and we integrate over $d\theta$ since we are integrating over the angle subtended by the object.

$$ds = a(t)\chi d\theta \tag{23}$$

so

$$\begin{aligned} D &= \int ds \int_0^\theta a(t)\chi d\theta & (24)\\ &= a(t)\chi\theta & (25)\end{aligned}$$

In a flat and non-expanding universe, $D = d_A\theta$, where d_A is the distance to the object. In order to preserve this relationship between physical and angular size, we define by analogy a distance that works the same way, that we can use to translate between physical and angular size. This is called the **angular diameter distance**,

$$d_A = a(t)\chi, \tag{26}$$

or, in terms of redshift,

$$d_A = \frac{\chi(z)}{1+z}. \tag{27}$$

In order to evaluate this, we need to know the comoving coordinate as a function of redshift (the coordinate distance). We define

$$I(z) \equiv \int_0^z \frac{dz'}{\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\rm rel,0}(1+z')^4 + \Omega_{\Lambda,0} + (1-\Omega_0)(1+z')^2}}. \tag{28}$$

It is also common to define the function

$$E(z) \equiv \sqrt{\Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}, \tag{29}$$

which is the denominator of the integrand above with $\Omega_{\rm rel} = 0$ and with $1 - \Omega$ replaced with Ω_k. (Note also that from the expression for the Hubble parameter $H(z)$ above, $H(z) = H_0 E(z)$.)

Since $\Omega_k = 1 - \Omega$,

$$I(z) = \int_0^z \frac{dz'}{E(z')}. \tag{30}$$

Then the present proper distance is

$$d_{p,0}(z) = \frac{c}{H_0} I(z) \tag{31}$$

and the comoving coordinate $\chi(z)$ is

$$\chi(z) = \frac{c}{H_0} I(z) \qquad (\Omega_0 = 1) \tag{32}$$

$$\chi(z) = \frac{c}{H_0\sqrt{\Omega_0 - 1}} \sin\left[I(z)\sqrt{\Omega_0 - 1}\right] \qquad (\Omega_0 > 1) \tag{33}$$

$$\chi(z) = \frac{c}{H_0\sqrt{1 - \Omega_0}} \sinh\left[I(z)\sqrt{1 - \Omega_0}\right] \qquad (\Omega_0 < 1). \tag{34}$$

This **comoving distance** is perhaps the fundamental cosmological distance, since most other distances of interest are derived from it. This is the appropriate distance to use when discussing large scale structure or the distances between objects in the Hubble flow. Also important is **comoving volume**, which is just the volume defined by the comoving coordinate. In order to study the evolution of some type of object (quasars, elliptical galaxies, etc) as a function of redshift, we need to know how or if its number density changes with redshift. This is the number per unit comoving volume, which accounts for the change in density due solely to the expansion of the universe. The normalization of the luminosity function ϕ_* represents the number of galaxies per unit comoving volume.

Having determined the comoving distance, we substitute it into Equation 27,

$$d_A = \frac{\chi(z)}{1+z}, \tag{35}$$

to calculate d_A. This is done numerically, in practice; various online cosmological calculators are available to calculate the various distances as a function of redshift (and there's an iPhone app too).

The angular diameter distance is weird! In a static and Euclidean universe the angular diameter distance is equal to the proper distance, but in most other cosmological models the angular size of an object decreases with distance as we would expect until about $z \sim 1.5$, but beyond this distance, the angular size gets *bigger* again.

The universe acts like a gravitational lens, making a galaxy bigger than it would appear in a static, Euclidean universe.

3 Luminosity distance

We also want to be able to relate the flux we receive from an object to its intrinsic luminosity. In static, Euclidean space,

$$F = \frac{L}{4\pi d_L^2}, \tag{36}$$

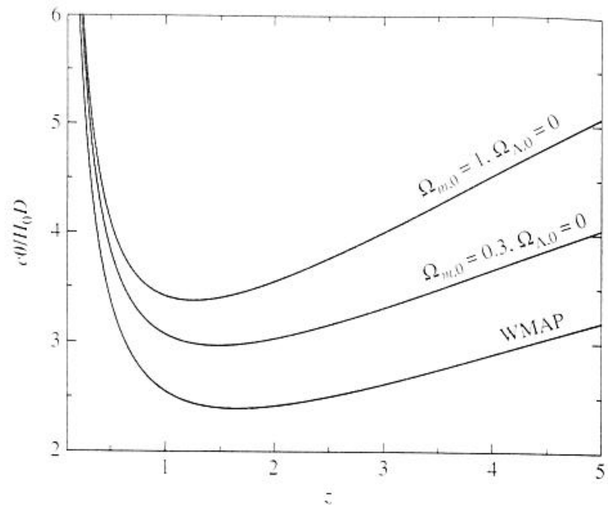

FIGURE 29.30 The angular diameter θ of a galaxy in units of H_0D/c for several values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$.

Figure 2: Angular diameter distance as a function of redshift for different cosmological models

where d_L is the distance to the object. But in our possibly curved and certainly expanding universe things are again more complicated.

- The energy of each photon from the object is less, because of the cosmological redshift, $\lambda_{\rm obs} = \lambda_{\rm em}(1+z)$
- The arrival rate of photons is lower because of **cosmological time dilation**,

$$\frac{\Delta t_{\rm obs}}{\Delta t_{\rm em}} = 1 + z \tag{37}$$

- We need to use the Robertson-Walker metric to calculate the distance to the object.

The result is

$$F = \frac{[L/(1+z)^2]}{4\pi\chi^2} \tag{38}$$

$$= \frac{L}{4\pi\chi^2(1+z)^2} \tag{39}$$

where the two factors of $(1+z)$ account for the lower energy and reduced arrival rate of the photons. To retain the usual relationship between flux and luminosity, we define the **luminosity distance** d_L, such that

$$F = \frac{L}{4\pi d_L^2}. \tag{40}$$

So

$$d_L(z) = \chi(z)(1+z) \tag{41}$$

where we use our previously calculated $\chi(z)$.

We can use this to construct the distance modulus as usual:

$$m - M = 5\log\left(\frac{d_L(z)}{10\ \text{pc}}\right) \tag{42}$$

This is called the **redshift-magnitude relation**.

So we have

$$d_L(z) = \chi(z)(1+z) \tag{43}$$

and

$$d_A = \frac{\chi(z)}{1+z}. \tag{44}$$

so

$$d_A(z) = \frac{d_L(z)}{(1+z)^2}. \tag{45}$$

This means that at $z \sim 1$, the angular diameter distance is a factor of ~ 4 smaller than the luminosity distance (d_L doesn't turn over like d_A does).

3.1 Surface brightness

This has strong implications for the surface brightness (luminosity per unit area) of objects as a function of redshift. The apparent surface brightness is

$$\Sigma = \frac{\text{flux}}{\text{angular area}} = \frac{L/4\pi d_L^2}{(D/d_A)^2} = \left[\frac{L}{4\pi D^2}\right]\frac{d_A^2}{d_L^2} \tag{46}$$

where the term in square brackets is constant because it contains only intrinsic properties of the galaxy.

So

$$\Sigma \propto \left(\frac{d_A}{d_L}\right)^2 \tag{47}$$

$$\propto \frac{1}{(1+z)^4} \tag{48}$$

This is a very strong dependence; at $z \sim 1$, a galaxy's surface brightness has dropped by a factor of $\sim 2^{-4} = 1/16$. This is another reason why distant galaxies are difficult to observe.

4 Type Ia supernovae and the cosmological constant

Having now discussed the extragalactic distance scale, the luminosity distance, and cosmological models including the cosmological constant, we will now look at the measurement of the accelerating universe from Type Ia SNe in more detail.

As already discussed, Type Ia supernovae are useful because they are standard candles and they're extremely bright. Their intrinsic luminosity can be estimated from their light curves, which allows us to determine their distance. Because they're so bright, we can see them to extremely large distances, $z > 1$. To see how this is useful in constraining cosmological parameters, we can look at how the luminosity distance changes for different cosmologies. As redshift increases, the luminosity distance becomes increasingly different for

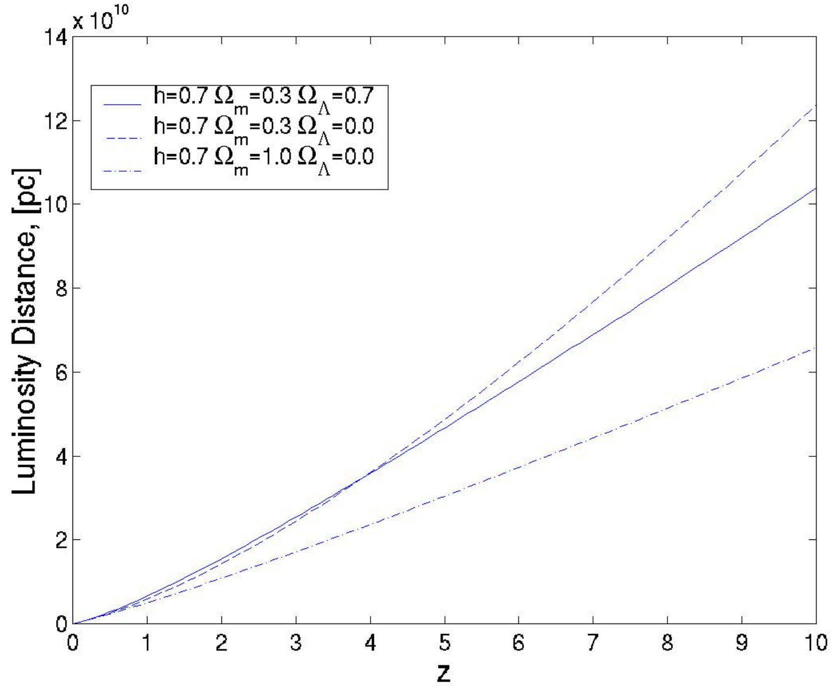

Figure 3: Luminosity distance for different cosmologies

different values of the cosmological parameters. Recall that the purpose of the luminosity distance is to preserve the usual relationship between flux and luminosity,

$$F = \frac{L}{4\pi d_L^2}. \tag{49}$$

We measure F and determine L by studying the light curve of the supernova, which means that we can measure the luminosity distance. If we have enough supernovae at high enough redshift, we can see which values of the cosmological parameters best matches the luminosity distances we measure.

Figure 3 shows the luminosity distance as a function of redshift for our currently favored cosmology with $\Omega_\Lambda = 0.7$ and $\Omega_m = 0.3$, and two matter only models with $\Omega_m = 1$ and $\Omega_m = 0.3$. Note that for $z \lesssim 3$, where our current measurements are made, the luminosity distance is largest for the flat, Λ-dominated universe, and smaller for both of the matter-only models. This larger luminosity distance means that the supernovae will appear fainter than they would in matter-only cosmologies, and that is indeed what is observed. Note also that the differences are small, particularly between the flat, Λ model and the open, matter-only model. Very precise photometric measurements are required!

In practice, astronomers usually plot the redshift-magnitude relation, i.e. the (extinction-corrected) distance modulus vs. redshift. Recall that the distance modulus is given by the difference between apparent and absolute magnitudes

$$m - M = 5\log(d_L/10\text{ pc}). \tag{50}$$

Writing the distance in units more convenient for cosmological measurements, we have

$$\mu = m - M = 5\log d_L + 25, \tag{51}$$

where d_L is measured in Mpc.

A recent plot of the distance modulus vs. redshift for Type Ia supernovae is shown in Figure 4.

- The dashed line on the main plot shows the best fit cosmological model, which has $\Omega_m = 0.29$ and $\Omega_\Lambda = 0.71$. The inset shows the differences between the binned data and various models (after an empty universe model with $\Omega = 0$ has been subtracted).

- $\boldsymbol{w}$. The acceleration of the universe depends on the density and pressure of dark energy, which in turn depend on the equation of state $P = w\rho c^2$. We can therefore constrain w by measurements of SNIa, as shown by some of the models on the inset figure. The data are consistent with $w = -1$.

- **Gray dust.** An important initial concern with the supernova data, when the SNe were found to be fainter than expected, was that there might be some sort of "gray" dust in the way, absorbing the light equally at all wavelengths and making the SNe fainter. Normal dust affects blue light more than red light, so we can see that it changes the color of objects; gray dust would be very difficult to detect because it would make objects fainter without changing their color. We don't actually know of any gray dust, but it didn't seem to be any less plausible than the cosmological constant when the fainter-than-expected supernovae were initially discovered. However, if gray dust were making the SNe fainter, they would continue to be fainter than expected as they got farther and farther away, and that's not what happened as more distant SNe were discovered. As can be seen from the solid black (best-fit) line in the inset panel, the supernovae do indeed get fainter than expected with redshift, but then they turn over and start getting brighter again, at about the redshift when the universe changed from being matter-dominated to being Λ-dominated. This trend rules out the gray dust. In other words, *the supernovae strongly favor a model with recent acceleration and previous deceleration.*

- **Evolution.** Another early counter-argument to the accelerating universe was that the Type Ia supernovae might not actually be standard candles: there might be something different about high redshift supernovae which makes them intrinsically fainter—lower metallicity, for example. However, like the gray dust case, if the intrinsic brightness of the SNe were evolving with redshift we would expect them to continually get fainter with distance, and they don't. The turnover in observed brightness also makes such evolutionary effects very unlikely. This is the "evolution $\sim z$" model on the plot.

- There were a lot of doubts about the SNe results when they first started coming out over 10 years ago, but subsequent data has made the accelerating universe model much more robust.

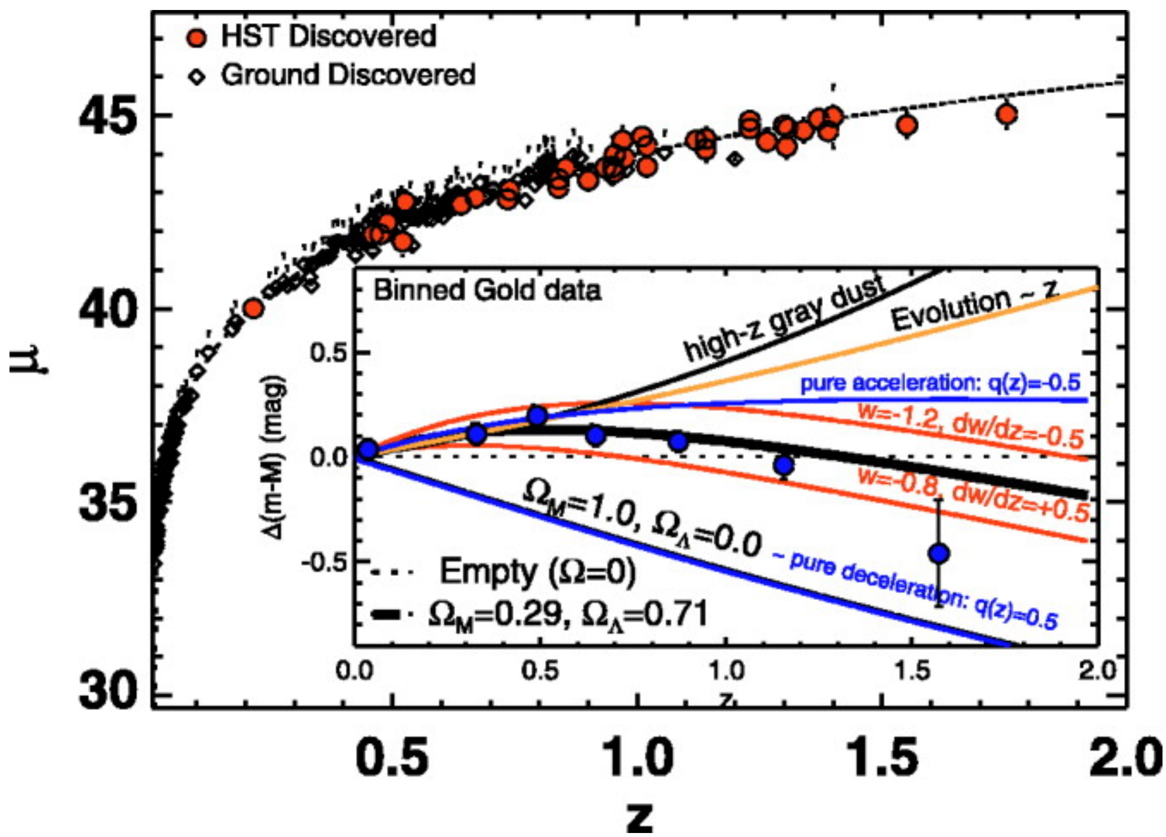

Figure 4: Distance modulus vs. redshift for Type Ia supernovae (Riess et al. 2007, *ApJ*, 659, 98)