Astro 735: Cosmology
Lecture 4: Virial Theorem; Galaxy Clusters; The Cosmological Distance Ladder

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1 The virial theorem

One of the most important and widely used relationships in astrophysics is the **virial theorem**: for gravitationally bound systems in equilibrium, the total energy is always one-half of the time averaged potential energy:

$$\langle E \rangle = \frac{1}{2}\langle U \rangle. \tag{1}$$

A derivation of this can be found in most mechanics textbooks and many astrophysics textbooks. Because the total energy is the sum of the kinetic and potential energies, and because the potential energy is negative for gravitationally bound objects, this can also be expressed as the fact that the kinetic energy is twice the negative of the potential energy,

$$-2\langle K \rangle = \langle U \rangle. \tag{2}$$

This applies to all gravitationally bound objects: stars, galaxies, dark matter halos, and galaxy clusters. It is particularly important in the discussion of dark matter halos. The halo radius at which objects are bound is called the **virial radius**, and we also commonly refer to the **virial temperature**, which is a measurement of the binding energy of the halo such that the halo can only trap gas with $T < T_{\rm vir}$.

2 Galaxy clusters and large scale structure

The distribution of galaxies is not uniform; observations and cosmological simulations show **large scale structure**, in which galaxies live in groups and clusters with voids between them. The Milky Way is a member of the "Local Group," which contains ~ 30 small galaxies and three large spirals (the Milky Way, Andromeda, and M101). Larger collections of galaxies—galaxy clusters—are the largest gravitationally bound structures in the universe. The largest galaxy clusters (called "rich" clusters) contain thousands of galaxies and have masses of up to $10^{15} h_{70}^{-1}$ $\rm M_\odot$ in volumes of a few Mpc^3.

Galaxy clusters provided the first indirect evidence of dark matter, as realized by the astronomer Fritz Zwicky in 1933, well ahead of his time. Zwicky measured the radial velocities of galaxies in the Coma cluster, a nearby rich cluster, calculated the velocity dispersion of the galaxies in the cluster, and used the virial theorem to estimate the mass of the cluster:

For a cluster of N identical galaxies of mass m and with no coherent motions (or for a galaxy of N identical stars of mass m), $\langle v \rangle = 0$, the kinetic energy is

$$\langle K \rangle = \sum_{i=1}^{N} \frac{1}{2} m v_i^2 \tag{3}$$

$$= \frac{N}{2} m\sigma^2 \tag{4}$$

$$= \frac{M_{\rm tot}\sigma^2}{2}. \tag{5}$$

To find the potential energy, we consider the potential energy of a spherical shell at radius r, surrounding a mass $M(<r)$. The mass of the shell is $dm = 4\pi r^2 dr \times \rho(r)$. The potential energy U is

$$\begin{aligned} dU(r) &= -\frac{GM(<r)}{r} dm && (6) \\ &= -\frac{GM(<r)}{r}\, 4\pi r^2 dr \times \rho(r) && (7) \end{aligned}$$

We integrate over all shells out to some radius R:

$$U(R) = -4\pi G \int_0^R M(<r)\rho(r) r\, dr \tag{8}$$

For a uniform sphere of radius R ($\rho =$ constant),

$$M(<r) = \rho \frac{4\pi}{3} r^3 = M_{\rm tot} \frac{r^3}{R^3} \tag{9}$$

and the potential energy is

$$\langle U \rangle = -\frac{3}{5}\left(\frac{GM_{\rm tot}^2}{R}\right) \tag{10}$$

From the virial theorem:

$$-2\,\langle K \rangle = \langle U \rangle. \tag{11}$$

So

$$-2\left[\frac{M_{\rm tot}\sigma^2}{2}\right] = \frac{3}{5}\left(\frac{GM_{\rm tot}^2}{R}\right) \tag{12}$$

for a spherical, constant density cluster (we can work out more accurate versions for any $\rho(r)$).

Rearranging:

$$M_{\rm tot} = \frac{5R}{G}\left[\frac{\sigma^2}{3}\right] \tag{13}$$

Now consider the total velocity dispersion. For a spherical system, the velocities have three components:

$$\sigma^2 = \sigma_r^2 + \sigma_\theta^2 + \sigma_\phi^2 \tag{14}$$

σ_r^2 is the radial velocity dispersion and is the only one we can measure (the other two are perpendicular to our line of sight). In general,

$$\sigma_r^2 \neq \sigma_\theta^2 \neq \sigma_\phi^2. \tag{15}$$

However, if $\sigma_r^2 = \sigma_\theta^2 = \sigma_\phi^2$ (isotropic velocity dispersion),

$$\sigma^2 = 3\sigma_r^2 \tag{16}$$

and

$$M_{\rm tot} = \frac{5R\sigma_r^2}{G}. \tag{17}$$

The mass obtained this way is called the **virial mass**.

For the Coma cluster, the velocity dispersion is $\sigma = 977$ km s^{-1} , and the cluster's radius is 3 Mpc, so

$$M \approx \frac{5\sigma_r^2 R}{G} = 3.3 \times 10^{15}\ \mathrm{M}_\odot. \tag{18}$$

Zwicky observed that this is considerably larger than the sum of the masses of the galaxies in the cluster, and that therefore some other source of mass is required.

Most of the baryonic mass of the cluster is in the form of hot, X-ray emitting gas called the **intracluster medium**; clusters are usually more well-defined when observed in the X-rays than in the optical. The X-rays are produced by **thermal bremsstrahlung emission** (braking radiation, also called free-free emission). This is emission which occurs when a free electron passes near an ion, emits a photon, and slows down (the ion is necessary for conservation of energy and momentum). Bremsstrahlung radiation has a characteristic, easily identifiable spectrum, and the luminosity density (luminosity per unit volume) depends on the electron density and the temperature:

$$\mathcal{L}_{\rm vol} = 1.42 \times 10^{-40}\, n_e^2 T^{1/2} \text{ W m}^{-3}. \tag{19}$$

(Important parts: $\mathcal{L}_{\rm vol} \propto n_e^2 T^{1/2}$) The temperature can be measured from the X-ray spectrum of the gas; for the Coma cluster it is 8.8×10^7 K. The total X-ray luminosity of the cluster is

$$L_x = \frac{4}{3}\pi R^3 \mathcal{L}_{\rm vol}. \tag{20}$$

We can measure R, L_x and T, so these two equations can be combined to solve for n_e. For the Coma cluster, $n_e = 300$ m^{-3}. Very diffuse!

We can then compute the total mass of the gas:

$$M_{\rm gas} = \frac{4}{3}\pi R^3 n_e m_H \tag{21}$$

since there is one proton for every electron. For the Coma cluster, $M_{\rm gas} = 1.05 \times 10^{14}$ M$_\odot$. This is a lot of gas, but much less than the total cluster mass 3.3×10^{15} M$_\odot$ we computed earlier. Most of the baryonic mass is in the hot gas rather than in galaxies, but most of the total mass is dark matter.

Observations of a galaxy cluster also provide some of the best evidence that dark matter is non-baryonic and collisionless, as predicted by CDM models. The "Bullet cluster" is actually two colliding galaxy clusters. The mass distributions of the clusters can be determined by gravitational lensing (the massive clusters act as a lens, bending the light of background galaxies in a way that can be used to determine the cluster mass), and the gas in the clusters can be observed in x-rays. This shows that the gas in the two clusters is concentrated between the two and shocked by the collision, while most of the mass of the clusters is offset from the gas. This means that most of the mass in the clusters does not interact; the dark matter (and galaxies) in the cluster have passed through each other, with only the gas interacting.

3 The extragalactic distance scale

In order to study the structure of the universe, we need to be able to measure how far away things are. We also need to measure distances in order to measure the Hubble constant: $v = H_0 d$, so we need to measure both v (easy, for things that are far enough away so that peculiar velocities don't matter) and d (hard).

There are many methods of measuring distances, making up what's called the **extragalactic distance scale** or **cosmological distance ladder**. For objects within a few hundred pc we can measure **trigonometric parallaxes** (a geometric method involving changes in the apparent position of a star due to the rotation of the Earth around the Sun; highly accurate but only works in the general solar neighborhood. We need other methods. There are many of these, and we'll only discuss a few of the more important here.

Most astronomical distance indicators are what are called "standard candles." A standard candle is an object with a fixed luminosity, or with some other property we can measure that will tell us what its luminosity is. If we know the luminosity, we can determine the distance by measuring the flux we receive from the object.

Also used but less common are "standard rulers," objects with a fixed physical size; if we can measure the angular size and know the physical size, we can determine the distance.

Of particular importance is the distance to the Large Magellanic Cloud, a small satellite galaxy of the Milky Way. We measure the distance to the LMC with as many different methods as possible, in order to calibrate the distance indicators, and so many distances to more distant objects depend on the LMC distance.

3.1 Red clump

Earlier we discussed the HR diagram, in which stars are plotted by temperature vs. luminosity. In observational terms this is called the color-magnitude diagram (color = temperature and magnitude = luminosity). The stars in certain prominent features of this diagram, in particular the clump of very bright He-burning giant stars that have moved off the main sequence, appear to have roughly the same luminosity in different galaxies. If a galaxy is close enough that we can resolve individual stars and plot them on a color-magnitude diagram, we can then use the roughly constant luminosity of these stars to determine the distance to the galaxy. An example of this is given in the text for the distance to the LMC, which is found to be ~ 50 kpc using this method, correct to within 10%.

3.2 Cepheid variables

Cepheids are a type of post-main sequence pulsating variable star whose period is directly related to their luminosity (a relationship discovered by Henrietta Swan Leavitt in the early 20th century). These stars are very bright, about 100 times brighter than the red clump giant stars, and so can be seen to greater distances. Using the few Cepheids with distances determined from parallax measurements, we find the following period-luminosity relation:

$$M_V = -1.43 - 2.81 \log P, \tag{22}$$

where P is the period in days. Because there are many more Cepheids in the LMC, we usually use a period-luminosity relation based on the LMC instead. Of course this requires an independent distance measurement to the LMC, from clump giants or some other method.

Cepheids can be seen to large distances, as far as the Coma cluster (~ 90 Mpc away). A Hubble Space Telescope project to measure the distance to Coma with Cepheids was begun but unfortunately cut short due to the failure of the camera (which has since been fixed, but the project won't be continued unless additional telescope time is approved).

Very important for this and all standard candles: we need to know and account for extinction, the absorption of light by dust. This is usually done by measuring changes in color, since dust absorption reddens the spectrum.

One of the major science goals of the Hubble Space Telescope was to make an accurate measurement of the Hubble constant by measuring large numbers of Cepheid variables. This was called the Hubble Key Project and was recently completed, finding $H_0 = 72 \pm 8$ km s^{-1} Mpc^{-1}.

3.3 Galaxy scaling relations

Spiral and elliptical galaxies exhibit well-defined empirical relationships between their luminosities and kinematics (rotational velocity in the case of spirals, and velocity dispersion in the case of ellipticals, which are dominated by random stellar motions rather than rotation). These are useful for measuring the distances to galaxies, but also tell us important things about galaxy evolution.

The correlation between the luminosity and maximum rotational velocity of spiral galaxies is known as the **Tully-Fisher relation** (1977). In luminosity terms, this is approximately $L \propto v_{\max}^4$, though the details depend somewhat on the type of galaxy and the wavelength at which we observe it. As usual, we plot this logarithmically using the absolute magnitude M and $\log v$. One observed form of the Tully-Fisher relation is

$$M_B = -10.2 \log v_{\max} + 2.71. \tag{23}$$

According to the definition of absolute magnitude, $M = M_{\text{Sun}} - 2.5 \log(L/L_{\odot})$, so $L \propto v^4$ is equivalent to $M = -10 \log v +$ const.

The correlation is tighter at IR wavelengths, since they are less affected by dust (which preferentially absorbs and scatters blue light), and the light comes from stars that are a better tracer of the overall luminous mass distribution (blue light is mostly from young stars in regions of recent star formation).

This effectively works as a distance indicator: If we can measure the maximum rotational velocity of a spiral galaxy and have reason to believe it should fall on the Tully-Fisher relation, we can estimate its absolute magnitude (its luminosity) and thus its distance.

Can we understand where this relationship comes from?

For flat rotation curves,

$$v_c^2 = \frac{GM}{r}. \tag{24}$$

We write $M = (M/L) \times L$, and $L = \Sigma \pi r^2$, where Σ is the average surface brightness (luminosity per unit area of the galaxy), so $r^2 = L/(\Sigma\pi)$. Then square both sides and substitute:

$$v_c^4 = \frac{G^2 M^2}{r^2} = G^2 (M/L)^2 L^2 \frac{\Sigma\pi}{L} = \left[G^2 (M/L)^2 \Sigma\pi \right] L, \tag{25}$$

or $L \propto v_c^4$. Notice that we've assumed that all galaxies have the same M/L and the same average surface brightness, neither of which is true—so the fact that this is actually observed is somewhat surprising. Another way to think about this: the radius in our original expression for velocity is the radius enclosing all the mass, while the radius we've used for the luminosity (surface brightness) is the radius enclosing all the light—these aren't the same. So this correlation indicates a direct relationship between the stellar mass and the total gravitational mass including dark matter, an important conclusion for galaxy formation.

Elliptical galaxies obey a similar relationship called the **Faber-Jackson relation**, with the substitution of the velocity dispersion σ for the rotational velocity. This has more scatter than the Tully-Fisher relation, probably because elliptical galaxies vary more widely than spiral galaxies, and a tighter relationship is obtained by also including the size of the galaxy; this is called the **Fundamental Plane**.

3.4 Type Ia supernovae lightcurves

Type Ia supernovae are the most important extragalactic distance indicator for cosmological purposes, because they are the brightest and can be seen to the largest distances.

Supernova review: There are two different types of supernove, core collapse and Type Ia (core collapse supernovae are Type II and Types Ib and Ic; the classification was originally based on the presence or absence of hydrogen lines in the spectra, before the causes were understood).

A core collapse supernova is the explosion at the end of the life of a massive star ($M \gtrsim 8\ \mathrm{M}_\odot$). These stars are hot enough to continue to fuse heavier elements when their fuel becomes exhausted: after running out of hydrogen, they fuse helium into carbon, carbon into neon, neon into oxygen, oxygen into silicon, and silicon into iron at the core. Because iron has the highest binding energy per nucleon, the fusion chain can't go any further without losing energy instead of gaining it. The star suddenly loses its internal energy supply and undergoes a catastrophic gravitational collapse: a core collapse or Type II supernova. Most of the energy is released in the form of neutrinos.

The standard model of a Type Ia supernova is that a carbon-oxygen white dwarf star (the endpoint of stellar evolution for stars not massive enough to undergo a Type II supernova) accretes material from a binary companion which pushes it over the Chandrasekhar limit and causes it to explode (white dwarfs are supported by electron degeneracy pressure rather than thermal pressure, and the Chandrasekhar limit is the limit at which electron degeneracy pressure is not strong enough to prevent gravitational collapse. $M_{\mathrm{Chandra}} \simeq 1.4\ \mathrm{M}_\odot$). The true picture may be more complicated than this; recent work suggests that there are some problems with this scenario. Type Ia SNe almost certainly have to do with the explosion of a CO white dwarf, but they could involve a merger of two WDs or some other scenario.

Whatever the cause, the useful fact for measuring distances is that the brightnesses and light curves (a plot of luminosity vs. time) of Type Ia SNe are very similar. They have average absolute magnitudes at maximum light of $\langle M_B \rangle \simeq \langle M_V \rangle \simeq -19.3 \pm 0.03$. Note: small scatter, and very bright; this is as bright as an entire galaxy.

What we actually measure is the brightness of the supernova at various points in time (the "light curve") and at different wavelengths, and there is a well-defined inverse correlation between the maximum luminosity, the color and the rate of decline of the light curve: brighter supernova take longer to decline. We can use this to determine the intrinsic peak luminosity. An empirical relation is

$$M_B \sim -19.2 - 1.52(s-1) + 1.57c \tag{26}$$

where c is related to the color of the supernova: $c = (B - V) + 0.57$, where B and V are the magnitudes in the blue and visual (green, roughly) bands. The "stretch" s is a measurement of the duration of the supernova relative to the mean time duration of SN Ia. This means that by measuring the color and the duration we can determine the absolute magnitude, and therefore the distance. Calibration is done with reference to Type Ia SNe that have occurred in galaxies with Cepheid distances.

Type Ia supernovae can be used to measure distances to > 1000 Mpc ($z \sim 0.25$) with an uncertainty of $\sim 5\%$. SN Ia measurements were the first sign that the universe appears to be accelerating due to a non-zero cosmological constant.