**Astro 735: Cosmology**
**Lecture 6: Newtonian Cosmology II**

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## 1 The expansion rate of the universe and the flatness problem

The rate at which the universe is expanding is a function of time—the value of the Hubble "constant" isn't constant, $H(z) \neq H_0$ and

$$H(z) = \frac{\dot{a}(t)}{a(t)} \neq \frac{\dot{a}(t_0)}{a(t_0)} \tag{1}$$

We return to our equation for the evolution of the scale factor

$$\left[\left(\frac{\dot{a}(t)}{a(t)}\right)^2 - \frac{8\pi}{3}G\rho\right]a^2(t) = -kc^2. \tag{2}$$

$kc^2$ is constant and $H(z) = \dot{a}(t)/a(t)$, so

$$\left[H^2(z) - \frac{8\pi}{3}G\rho\right]a^2(t) = \left[H_0^2 - \frac{8\pi}{3}G\rho_0\right]a^2(t_0). \tag{3}$$

Since $a(t_0) = 1$ and $a(t) = 1/(1+z)$,

$$H^2(z) - \frac{8\pi}{3}G\rho = \left[H_0^2 - \frac{8\pi}{3}G\rho_0\right](1+z)^2. \tag{4}$$

From the definition of the critical density

$$\Omega_0 = \frac{\rho_0}{\rho_c} = \frac{8\pi G\rho_0}{3H_0^2}, \tag{5}$$

$$\frac{8\pi G\rho_0}{3} = H_0^2\Omega_0, \tag{6}$$

and because mass is conserved,

$$\rho(z)a^3(t) = \rho_0 a^3(t_0) \Rightarrow \rho(z) = \rho_0(1+z)^3. \tag{7}$$

The universe was denser at higher redshifts. So

$$\frac{8\pi G\rho(z)}{3} = H_0^2\Omega_0(1+z)^3 \tag{8}$$

and we have

$$H(z)^2 - H_0^2\Omega_0(1+z)^3 = H_0^2[1-\Omega_0](1+z)^2 \tag{9}$$

and

$$H(z)^2 = H_0^2[\Omega_0(1+z) + 1 - \Omega_0](1+z)^2 \tag{10}$$

which simplifies to

$$\boxed{H(z)^2 = H_0^2(1+z)^2(1+\Omega_0 z)}. \tag{11}$$

This tells us about the evolution of the expansion rate of the universe.

It is also useful to calculate the evolution of the density of the universe. As noted above, the universe used to be smaller and denser, so $\Omega(z) \neq \Omega_0$. We can calculate $\Omega(z)$ starting from the same point at which we calculated $H(z)$. The algebra is left as an exercise for the reader, but the result is

$$\boxed{1 - \Omega(z) = \frac{1 - \Omega_0}{1 + \Omega_0 z}} \tag{12}$$

The general behavior of Equation 12 is shown in Figure 1, for an open universe with $\Omega_0 = 0.3$ and a closed universe with $\Omega_0 = 1.5$. In both cases, $\Omega(z) \to 1$ at high redshift. Flat universes are always flat, but open and closed universes also used to be very close to flat. This is a problem which we'll talk more about later. $\Omega = 1$ is unstable: if the universe was even slightly more dense than $\rho_c$, then at late times it's much denser than $\rho_c$, and if it were slightly less dense than $\rho_c$, it should now be much less dense than $\rho_c$. So the current matter density of the universe ($\Omega_0 \sim 0.3$) is very unlikely! In order to have this density today, the universe would have to have formed with a density within a very tiny fraction of the critical density—one part in $10^{62}$ or less. This is called the **flatness problem**. This is one of the main motivations for inflation, and one of the reasons why theorists favored a flat universe even when observations favored an open universe.

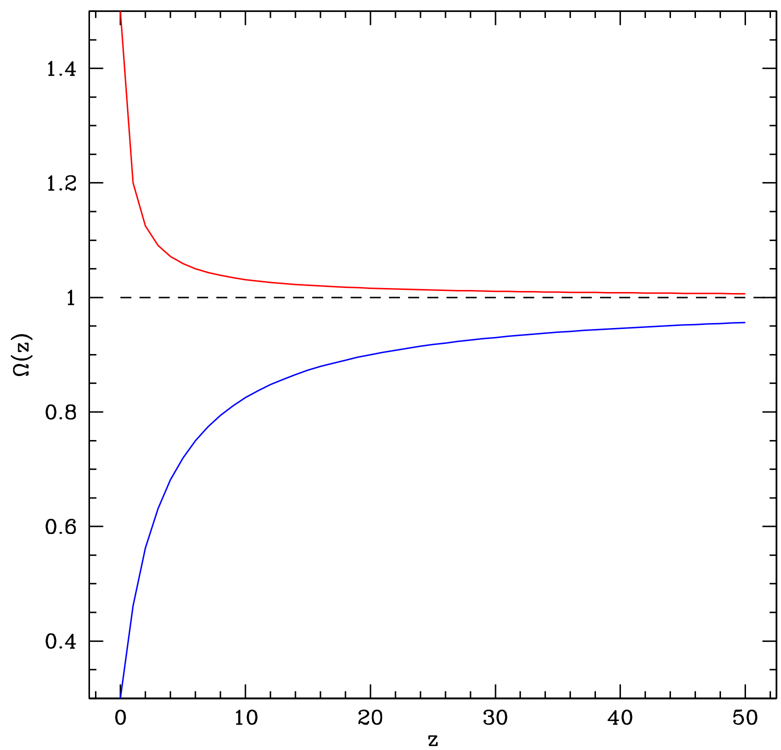


Figure 1: Evolution of the density parameter with redshift.

## 2 Adding pressure to the model of the universe

So far we have only considered the gravitational effect of matter in our model of the universe. We will now broaden this a bit. We start with our now familiar differential equation for the evolution of the scale factor

(the Friedmann equation)

$$\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\rho}{3}\right]a^2 = -kc^2. \tag{13}$$

We will expand the definition of $\rho$ to include relativistic particles like photons and neutrinos as well as normal matter. For normal matter, $\rho$ is just the usual mass density. For relativistic particles, we make use of the equivalence of mass and energy: $\rho$ is the energy density divided by $c^2$.

Remember that this equation is essentially a statement of the conservation of energy; it says that the sum of the gravitational potential energy and the kinetic energy of expansion of the universe is constant. We will use another expression of conservation of energy to calculate the effects of components of the universe that produce pressure. Our universe is now filled with a fluid of density $\rho$ (this is now the equivalent mass density), temperature $T$, and pressure $P$. We will apply the first law of thermodynamics to this fluid:

$$dU = dQ - PdV \tag{14}$$

(This says that the change in the internal energy of a system is equal to the amount of heat supplied to the system, minus the amount of work performed by the system on its surroundings. The work is equal to the pressure times the change in volume.) The universe has the same temperature everywhere, so there is no heat flow: $dQ = 0$. This says that the expansion of the universe is adiabatic. We write this as a time derivative,

$$\frac{dU}{dt} = -P\frac{dV}{dt} \tag{15}$$

and substitute $V = \frac{4}{3}\pi r^3$ to obtain

$$\frac{dU}{dt} = -\frac{4}{3}\pi P\frac{d(r^3)}{dt}. \tag{16}$$

We then define the internal energy per unit volume:

$$u = \frac{U}{\frac{4}{3}\pi r^3} \tag{17}$$

so that

$$\frac{dU}{dt} = \frac{4\pi}{3}\frac{d(r^3 u)}{dt}. \tag{18}$$

Setting this equal to equation 14 above, we find

$$\frac{d(r^3 u)}{dt} = -P\frac{d(r^3)}{dt}. \tag{19}$$

We then write $u$ in terms of the equivalent mass density $\rho$:

$$\rho = \frac{u}{c^2} \tag{20}$$

to get

$$\frac{d(r^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(r^3)}{dt}. \tag{21}$$

We then substitute $r = a\chi$ and cancel $\chi$ because it's constant:

$$\boxed{\frac{d(a^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(a^3)}{dt}} \tag{22}$$

This is called the **fluid equation**, and is the second key equation describing the expansion of the universe. Note that for the pressureless model we considered earlier, $P = 0$ and $a^3\rho =$ constant, as we expect.

The fluid equation and the Friedmann equation are both statements of energy conservation. We can combine them to derive an equation that describes how the expansion of the universe behaves with time. We multiply the Friedmann equation by $a$:

$$a\left(\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G\rho a^3 = -kc^2 a \tag{23}$$

and take a time derivative:

$$\left(\frac{da}{dt}\right)^3 + 2a\left(\frac{da}{dt}\right)\left(\frac{d^2a}{dt^2}\right) - \frac{8\pi}{3}G\frac{d(\rho a^3)}{dt} = -kc^2\frac{da}{dt}. \tag{24}$$

We replace $kc^2$ using the Friedmann equation again, and use the fluid equation to replace the $d(\rho a^3)/dt$ term, which gives us, after a bunch of algebra,

$$\boxed{\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)a} \tag{25}$$

This is the **acceleration equation**. Note that the effect of pressure is to slow down the acceleration; for positive matter density and pressure, the acceleration is negative, so the universe is slowing down.

We'll write down the Friedmann equation again, so that we have all three important equations together:

$$\boxed{\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G\rho\right]a^2 = -kc^2} \tag{26}$$

These equations have three unknowns, $a$, $\rho$ and $P$, but they are not independent: we can use any two to derive the third, as we just did for the acceleration equation. So to solve for $a$, $\rho$ and $P$ we need another equation: an **equation of state** that links the variables. We write the equation of state as

$$P = w\rho c^2, \tag{27}$$

where $w$ is a constant, so the pressure is proportional to the energy density of the fluid. For matter with no pressure, as in our first model of the universe, $w = 0$. A fluid of photons or other massless particles is relativistic, and has the equation of state

$$P = \frac{1}{3}\rho c^2, \tag{28}$$

so $w = 1/3$. We will return to this question of pressure and the equation of state when we discuss the cosmological constant.

## 3 Evolution of the temperature

A key point of the Big Bang theory is that the early universe was very dense and hot. We expect that this hot, dense universe would have been in thermodynamic equilibrium , and that therefore the radiation field had a blackbody spectrum. We can compute the cooling of this radiation as the universe expands.

The energy density of blackbody radiation is

$$u = a_{\rm rad}T^4, \tag{29}$$

where $a_{\rm rad} = 4\sigma_{\rm SB}/c$ is the radiation constant (not to be confused with the scale factor $a$!). By replacing $P$ in the fluid equation with the equation of state $P = w\rho c^2$ we find

$$a^{3(1+w)}u = a^4 u = u_0, \tag{30}$$

since $w = 1/3$ for photons and $a_0 = 1$. This tells us that the energy density of the universe today is smaller by a factor of $a^4$ than it was at some earlier time with scale factor $a$. A factor of $a^3$ comes from the change in volume of the universe, and an additional factor of $a$ comes from the lower energy of the longer wavelength photons we see today, because of the cosmological redshift. Therefore

$$a_{\rm rad} a^4 T^4 = a_{\rm rad} T_0^4 \tag{31}$$

and the current temperature of the blackbody radiation is related to the temperature at an earlier time by

$$aT = T_0. \tag{32}$$

When the universe was half as large it was twice as hot. Recalling that

$$a = \frac{1}{1+z} \tag{33}$$

we can also write

$$T = (1+z)T_0 \tag{34}$$

for the dependence of the temperature of the radiation on redshift.

We can make an order of magnitude estimate of the current temperature of the blackbody radiation by considering the conditions needed to produce helium in the early universe. The early universe was hot and dense enough for nuclear reactions to take place, and the heaviest element that was formed in these reactions was He (and a very small amount of Li). This fusion requires approximately $T \simeq 10^9$ K and $\rho_b \simeq 10^{-2}$ kg m$^{-3}$, where the $b$ subscript refers to the baryon density. (If the temperature were higher the deuterium nuclei needed for the reaction would photodisassociate, and if the temperature were lower it would be too difficult to overcome the Coulomb barrier. The density is needed to produce the observed amount of He.) We can therefore estimate the value of the scale factor at the time of helium formation:

$$a \simeq \left(\frac{\rho_{b,0}}{\rho_b}\right)^{1/3} = 3.5 \times 10^{-9} \tag{35}$$

We can then combine this with the temperature required, $T(a) = 10^9$ K, to determine the current temperature of the radiation:

$$T_0 = aT(a) \simeq 3.5 \text{ K} \tag{36}$$

This simple calculation gives a very good estimate of the actual temperature of the radiation, $T_{\rm CMB} = 2.725$ K. Also note that this was predicted in 1948, well before it was discovered.

## 4 A two-component model of the universe

A complete model of the universe needs to include the effects of radiation and relativistic matter as well as the non-relativistic matter we have considered up to this point. Radiation—the cosmic microwave background—has a negligible effect on the dynamics of the universe now, but at very early times it was dominant.

We already saw that

$$a^{3(1+w)}\rho = \rho_0, \tag{37}$$

which means that particles with different equations of state are diluted differently by the expansion of the universe. We will neglect the very early universe when even massive particles were relativistic, and consider photons and neutrinos to be the only relativistic particles.

The energy density of radiation (photons) is

$$u_{\rm rad} = a_{\rm rad} T^4, \tag{38}$$

which we will rewrite as

$$u_{\rm rad} = \frac{1}{2} g_{\rm rad} a_{\rm rad} T^4, \tag{39}$$

where $g_{\rm rad}$ is the number of degrees of freedom of a photon. In general,

$$g = (\text{\# of types}) \times n_{\rm anti} \times n_{\rm spin}, \tag{40}$$

where $n_{\rm anti}$ indicates the possible existence of an antiparticle, $n_{\rm spin}$ is the number of spin states, and # of types indicates the number of different types of a particle. For photons, $g_{\rm rad} = 2$, since $n_{\rm anti} = 1$ (a photon is its own antiparticle) and $n_{\rm spin} = 2$, corresponding to the two possible polarizations with spin parallel or antiparallel to the direction of motion.

Next we will consider neutrinos. We'll neglect their small masses and treat them as massless particles. The early universe was sufficiently dense that neutrinos were in thermal equilibrium, with a spectrum similar to that of blackbody radiation (though not identical because photons are bosons and neutrinos are fermions). We are confident that there is a cosmic neutrino background, even though we haven't detected it (not surprising, given the difficulty of detecting even neutrinos from the Sun).

There are three flavors of neutrinos (electron, muon and tau), each with a corresponding antineutrino. The total energy density of all three flavors is

$$u_\nu = 3 \times \frac{7}{8} \times a_{\rm rad} T_\nu^4, \tag{41}$$

where the factor of 7/8 arises from the fact that neutrinos are described by Fermi-Dirac statistics rather than the Bose-Einstein statistics of photons. We write the total energy density as

$$u_\nu = \frac{1}{2}\left(\frac{7}{8}\right) g_\nu a_{\rm rad} T_\nu^4, \tag{42}$$

where $g_\nu = 6$ for neutrinos. Neutrinos have 3 types, each with an antineutrino, so $n_{\rm anti} = 2$, and they have one spin state (all neutrinos are left-handed), so $n_{\rm spin} = 1$. Therefore $g_\nu = 3 \times 2 \times 1 = 6$.

$T$ in cosmology generally refers to the temperature of the blackbody photons, which is not necessarily the neutrino temperature. In the early universe, for $T > 3.5 \times 10^{10}$ K, $T = T_\nu$, but as the universe expanded the number density of neutrinos was diluted until they stopped interacting with other particles. Since this time of **neutrino decoupling** the neutrinos have expanded and cooled at their own rate, independently of the photons.

Now we'll calculate the total energy density of relativistic particles. The neutrino temperature is somewhat lower than the CMB temperature because electron-positron annihilation supplied energy to the photons but not to the neutrinos. We will see later, when we discuss the thermal history of the universe in more detail, that the neutrino temperature $T_\nu$ is related to the CMB temperature $T$ by

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T. \tag{43}$$

Therefore the total neutrino energy density is

$$u_\nu = \frac{1}{2}\left(\frac{7}{8}\right) g_\nu \left(\frac{4}{11}\right)^{4/3} a_{\rm rad} T^4 = 0.681 a_{\rm rad} T^4. \tag{44}$$

The total energy density for relativistic particles, both photons and neutrinos, is

$$u_{\rm rel} = \frac{1}{2} g_* a_{\rm rad} T^4, \tag{45}$$

where

$$g_* = g_{\rm rad} + \left(\frac{7}{8}\right) g_\nu \left(\frac{4}{11}\right)^{4/3} = 3.363 \tag{46}$$

is the *effective* number of degrees of freedom of the relativistic particles. We also define the equivalent mass density of relativistic particles

$$\rho_{\rm rel} = \frac{u_{\rm rel}}{c^2} = \frac{g_* a_{\rm rad} T^4}{2c^2}. \tag{47}$$

This value of $g_*$ becomes valid at about $t = 1.3$ s after the Big Bang. In the very early universe the number of relativistic particles is larger and $g_*$ needs to be modified accordingly.

Using the definition of the critical density, we can calculate the current density parameter for relativistic particles:

$$\Omega_{\rm rel} = \frac{\rho_{\rm rel}}{\rho_c} = \frac{8\pi G \rho_{\rm rel}}{3H^2} = \frac{4\pi G g_* a_{\rm rad} T^4}{3H^2 c^2}. \tag{48}$$

For $T_0 = 2.725$ K, we find

$$\Omega_{\rm rel,0} = 8.24 \times 10^{-5}, \tag{49}$$

which is much less than $\Omega_{m,0} = 0.27$. So the contribution of relativistic particles to the current density of the universe is negligible.