ECE/CS 532
Homework 2: Vectors and Matrices

1. Let $\boldsymbol{X} = [\boldsymbol{x}_1\boldsymbol{x}_2 \cdots \boldsymbol{x}_n] \in \mathbb{R}^{p\times n}$, where $\boldsymbol{x}_i \in \mathbb{R}^p$ is the ith column of $\boldsymbol{X}$. Consider the matrix

$$\boldsymbol{C} = \frac{\boldsymbol{X}\boldsymbol{X}^T}{n} .$$

a. Express $\boldsymbol{C}$ as a sum of rank-1 matrices (i.e., columns of $\boldsymbol{X}$ times rows of $\boldsymbol{X}^T$).

b. Assuming $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are linearly independent, what is the rank of $\boldsymbol{C}$?

2. Define the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}$ as follows. Let $G_1, G_2, \ldots, G_m$, with $m \leq n$, be a partition of the set $\{1, 2, \ldots, n\}$. In other words, split the set of integers into m disjoint groups. For every $\boldsymbol{x} \in \mathbb{R}^n$ define

$$\Phi(\boldsymbol{x}) = \sum_{j=1}^{m} \left(\sum_{i\in G_j} x_i^2 \right)^{1/2} ,$$

where x_i is the i-th entry of $\boldsymbol{x}$.

a. Is $\Phi(\boldsymbol{x})$ a norm?

b. What is $\Phi(\boldsymbol{x})$ equal to when $m = 1$ and when $m = n$?

3. Inequalities are very useful for bounding errors and making approximations. Prove that for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$|\boldsymbol{x}^T\boldsymbol{y}| \leq \|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2 .$$

Hint: Recall the definition of $\cos(\boldsymbol{x}, \boldsymbol{y})$.

4. Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a non-singular matrix. Suppose that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$.

a. Given $\boldsymbol{A}$ and $\boldsymbol{y}$, write an expression for $\boldsymbol{x}$.

b. Bound the 2-norm of $\boldsymbol{x}$ in terms of $\|\boldsymbol{y}\|_2$ and a function of the matrix $\boldsymbol{A}$.

5. Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} .$$

a. What is the rank of $\boldsymbol{A}$?

b. Suppose that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$. Derive an explicit formula for $\boldsymbol{x}$ in terms of $\boldsymbol{y}$.

6. Let

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} .$$

a. What is the rank of $\boldsymbol{X}$?

b. What is the rank of $\frac{\boldsymbol{X}\boldsymbol{X}^T}{n}$?

c. Find a set of linearly independent columns in $\boldsymbol{X}$.

7. Design an algorithm to remove noise from images.

Most denoising algorithms are based on some sort of average. The simplest methods simply replace each noisy pixel with an average of itself and its neighbors. The Matlab code below shows how this is done. The basic idea is this: Think of the spatial distance between pixels as a measure of similarity between them. A pixel is denoised by averaging it with other pixels that are "similar" in this sense. The weights in the averaging are proportional to the similarity (e.g., large for nearest neighbors and zero for far away pixels).

Spatial proximity is just one way in which we might quantify the similarity between pixels (and not necessarily the best way).

a. As we discussed in class, think about other ways in which we might measure the similarity between pixels in an image.

b. Implement a denoising algorithm in Matlab based on similarity measure other than simple spatial proximity, and compare your results to the standard smoothing method in the code below.

```
clear
close all
n = 64;

% noise free image
x = double(phantom(n))*256;

% add noise
y = x+randn(size(x))*15;

% denoise by distance-based averaging
w = [1/16 1/8 1/16;1/8 1/4 1/8;1/16 1/8 1/16];

for i=1:n
    for j=1:n
        if (i==1)|(i==n)|(j==1)|(j==n)
            xavg(i,j) = y(i,j);  % don't process pixels at edge
        else
            b = [y(i-1,j-1) y(i,j-1) y(i+1,j-1);
                 y(i-1,j) y(i,j) y(i+1,j);
                 y(i-1,j+1) y(i,j+1) y(i+1,j+1)];

            xavg(i,j) = sum(sum(b.*w));
        end
    end
end

figure(1);clf;
subplot(131);imagesc(x,[0,256]);axis image;colormap gray
subplot(132);imagesc(y,[0,256]);axis image;colormap gray
subplot(133);imagesc(xavg,[0,256]);axis image;colormap gray
linkaxes
```