I. USING S AS A PROBE

Let's use Eq. (2) along with the assumption the S leads is a probe. In this case, we have:

$$J_L = (\mathcal{L}_{LS,\mu} + \mathcal{L}_{LR,\mu})\delta\mu_L - \mathcal{L}_{LS,\mu}\delta\mu_S - \mathcal{L}_{LR,\mu}\delta\mu_R \tag{1}$$

$$J_S = -\mathcal{L}_{SL,\mu}\delta\mu_L + (\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu})\delta\mu_S - \mathcal{L}_{SR,\mu}\delta\mu_R \tag{2}$$

$$J_R = -\mathcal{L}_{RL,\mu}\delta\mu_L - \mathcal{L}_{RS,\mu}\delta\mu_S + (\mathcal{L}_{RL,\mu} + \mathcal{L}_{RS,\mu})\delta\mu_R \tag{3}$$

by using the condition $J_S = 0$, the superconductor lead as a probe, then we can eliminate $\delta\mu_S$ from the equations above. In fact, from the second equation we have:

$$J_S = 0 = -\mathcal{L}_{SL,\mu}\delta\mu_L + (\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu})\delta\mu_S - \mathcal{L}_{SR,\mu}\delta\mu_R$$

which allows us to write:

$$\delta\mu_S = \frac{\mathcal{L}_{SL,\mu}\delta\mu_L + \mathcal{L}_{SR,\mu}\delta\mu_R}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}} \tag{4}$$

and by plugging this into Eqs. for J_L and J_R, we have:

$$J_L = (\mathcal{L}_{LS,\mu} + \mathcal{L}_{LR,\mu})\delta\mu_L - \mathcal{L}_{LS,\mu}\left[\frac{\mathcal{L}_{SL,\mu}\delta\mu_L + \mathcal{L}_{SR,\mu}\delta\mu_R}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\right] - \mathcal{L}_{LR,\mu}\delta\mu_R$$

which leads to

$$J_L = \frac{(\mathcal{L}_{LS,\mu} + \mathcal{L}_{LR,\mu})(\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu})}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\delta\mu_L - \frac{\mathcal{L}_{LS,\mu}\mathcal{L}_{SL,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\delta\mu_L - \frac{\mathcal{L}_{LS,\mu}\mathcal{L}_{SR,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\delta\mu_R - \mathcal{L}_{LR,\mu}\delta\mu_R$$

which leads to

$$J_L = \frac{(\mathcal{L}_{LS,\mu} + \mathcal{L}_{LR,\mu})(\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}) - \mathcal{L}_{LS,\mu}\mathcal{L}_{SL,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\delta\mu_L - \left[\frac{\mathcal{L}_{LS,\mu}\mathcal{L}_{SR,\mu} + \mathcal{L}_{LR,\mu}(\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu})}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\delta\mu_R\right]$$

$$\begin{aligned} J_L = {} & \frac{\mathcal{L}_{LS,\mu}\mathcal{L}_{SL,\mu} + \mathcal{L}_{LS,\mu}\mathcal{L}_{SR,\mu} + \mathcal{L}_{LR,\mu}\mathcal{L}_{SL,\mu} + \mathcal{L}_{LR,\mu}\mathcal{L}_{SR,\mu} - \mathcal{L}_{LS,\mu}\mathcal{L}_{SL,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\delta\mu_L \\ & - \left[\frac{\mathcal{L}_{LS,\mu}\mathcal{L}_{SR,\mu} + \mathcal{L}_{LR,\mu}(\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu})}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\right]\delta\mu_R \end{aligned}$$

and performing some simplifications, we have:

$$J_L = \frac{\mathcal{L}_{LS,\mu}\mathcal{L}_{SR,\mu} + \mathcal{L}_{LR,\mu}\mathcal{L}_{SL,\mu} + \mathcal{L}_{LR,\mu}\mathcal{L}_{SR,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\delta\mu_L - \left[\frac{\mathcal{L}_{LS,\mu}\mathcal{L}_{SR,\mu} + \mathcal{L}_{LR,\mu}\mathcal{L}_{SL,\mu} + \mathcal{L}_{LR,\mu}\mathcal{L}_{SR,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\right]\delta\mu_R$$

and by defining,

$$D_L = \mathcal{L}_{LS,\mu}\mathcal{L}_{SR,\mu} + \mathcal{L}_{LR,\mu}\mathcal{L}_{SL,\mu} + \mathcal{L}_{LR,\mu}\mathcal{L}_{SR,\mu}$$

we can write the equation above as:

$$J_L = \frac{D_L}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}(\delta\mu_L - \delta\mu_R)$$

which can be put into the form:

$$J_L = \frac{D_L}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\Delta\mu_{LR}$$

which leads to:

$$\Delta\mu_{LR} = \frac{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}{D_L}J_{LR}. \tag{5}$$

where $J_{LR} = J_L = -J_R$ from the paper. We notice that Eq. (5) corresponds to Eq. (B1) from the paper (we must assume the time-reversal symmetry so the coefficient are symetric under exchanging of index).

The other equations in (B1) are determined by using Eq. (5)

We should obtain the same result if we substitute Eq. (4) into Eq. (3)[1]. Next, we calculate:

$$\begin{aligned}\Delta\mu_{LS} = \delta\mu_L - \delta\mu_S = \delta\mu_L - \frac{\mathcal{L}_{SL,\mu}\delta\mu_L + \mathcal{L}_{SR,\mu}\delta\mu_R}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}} \\ = \delta\mu_L\frac{(\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu})}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}} - \frac{\mathcal{L}_{SL,\mu}\delta\mu_L + \mathcal{L}_{SR,\mu}\delta\mu_R}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\end{aligned}$$

which leads to

$$\Delta\mu_{LS} = \frac{\mathcal{L}_{SR,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}(\delta\mu_L - \delta\mu_R)$$

then, we have:

$$\Delta\mu_{LS} = \frac{\mathcal{L}_{SR,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\Delta\mu_{LR}. \tag{6}$$

By substituting Eq. (5) into Eq. (7) we obtain:

$$\Delta\mu_{LS} = \frac{\mathcal{L}_{SR,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\frac{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}{D_L}J_{LR}, \tag{7}$$

which leads to

$$\Delta\mu_{LS} = \frac{\mathcal{L}_{SR,\mu}}{D_L}J_{LR}. \tag{8}$$

Next, we calculate $\Delta\mu_{RS}$ as follows:

$$\Delta\mu_{RS} = \delta\mu_R - \delta\mu_S = \delta\mu_R - \delta\mu_L + \delta\mu_L - \delta\mu_S = -\Delta\mu_{LR} + \Delta\mu_{LS}$$

and by using (5) and (8) we obtain:

$$\Delta\mu_{RS} = -\frac{(\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu})}{D_L} J_{LR} + \frac{\mathcal{L}_{SR,\mu}}{D_L} J_{LR}$$

which leads to

$$\Delta\mu_{RS} = -\frac{\mathcal{L}_{SL,\mu}}{D_L} J_{LR}. \tag{9}$$

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[1] Indee let's substitute Eq. (4) into Eq. (3):

$$J_R = -\mathcal{L}_{RL,\mu}\delta\mu_L - \mathcal{L}_{RS,\mu}\left[\frac{\mathcal{L}_{SL,\mu}\delta\mu_L + \mathcal{L}_{SR,\mu}\delta\mu_R}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\right] + (\mathcal{L}_{RL,\mu} + \mathcal{L}_{RS,\mu})\delta\mu_R$$

$$J_R = -\mathcal{L}_{RL,\mu}\delta\mu_L - \frac{\mathcal{L}_{RS,\mu}\mathcal{L}_{SL,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\delta\mu_L - \frac{\mathcal{L}_{RS,\mu}\mathcal{L}_{SR,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\delta\mu_R + (\mathcal{L}_{RL,\mu} + \mathcal{L}_{RS,\mu})\delta\mu_R$$

$$J_R = -\left[\frac{\mathcal{L}_{RL,\mu}(\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}) + \mathcal{L}_{RS,\mu}\mathcal{L}_{SL,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\right]\delta\mu_L + \left[\frac{-\mathcal{L}_{RS,\mu}\mathcal{L}_{SR,\mu} + (\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu})(\mathcal{L}_{RL,\mu} + \mathcal{L}_{RS,\mu})}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\right]\delta\mu_R$$

$$J_R = -\left[\frac{\mathcal{L}_{RL,\mu}\mathcal{L}_{SL,\mu} + \mathcal{L}_{RL,\mu}\mathcal{L}_{SR,\mu} + \mathcal{L}_{RS,\mu}\mathcal{L}_{SL,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\right]\delta\mu_L + \left[\frac{-\mathcal{L}_{RS,\mu}\mathcal{L}_{SR,\mu} + \mathcal{L}_{SL,\mu}\mathcal{L}_{RL,\mu} + \mathcal{L}_{SL,\mu}\mathcal{L}_{RS,\mu} + \mathcal{L}_{SR,\mu}\mathcal{L}_{RL,\mu} + \mathcal{L}_{SR,\mu}\mathcal{L}_{RS,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\right]\delta\mu_R$$

$$J_R = -\left[\frac{\mathcal{L}_{RL,\mu}\mathcal{L}_{SL,\mu} + \mathcal{L}_{RL,\mu}\mathcal{L}_{SR,\mu} + \mathcal{L}_{RS,\mu}\mathcal{L}_{SL,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\right]\delta\mu_L + \left[\frac{\mathcal{L}_{SL,\mu}\mathcal{L}_{RL,\mu} + \mathcal{L}_{SL,\mu}\mathcal{L}_{RS,\mu} + \mathcal{L}_{SR,\mu}\mathcal{L}_{RL,\mu}}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\right]\delta\mu_R$$

and by using the definition of D_L we have:

$$J_R = \frac{D_L}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}(-\delta\mu_L + \delta\mu_R) = -\frac{D_L}{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}\Delta\mu_{LR}$$

which leads to:

$$\Delta\mu_{LR} = \frac{\mathcal{L}_{SL,\mu} + \mathcal{L}_{SR,\mu}}{D_L} J_{LR}$$

which is indeed Eq. (5).