NONCOOPERATIVE GAME THEORY — ECE270: HOMEWORK #3

Exercise 6. In a *quadratic zero-sum game* the player P_1 selects a vector $u \in \mathbb{R}^{n_u}$, the player P_2 selects a vector $d \in \mathbb{R}^{n_d}$, and the corresponding outcome of the game is the quadratic form

$$J(u,d) := \begin{bmatrix} u' & d' & x' \end{bmatrix} \begin{bmatrix} P_{uu} & P_{ud} & P_{ux} \\ P'_{ud} & P_{dd} & P_{dx} \\ P'_{ux} & P'_{dx} & P_{xx} \end{bmatrix} \begin{bmatrix} u \\ d \\ x \end{bmatrix} \tag{2}$$

that P_1 wants to minimize and P_2 wants to maximize, where $x \in \mathbb{R}^n$ is a vector, $P_{uu} \in \mathbb{R}^{n_u \times n_u}$ a symmetric positive definite matrix, $P_{dd} \in \mathbb{R}^{n_d \times n_d}$ a symmetric negative definite matrix, and $P_{xx} \in \mathbb{R}^{n \times n}$ a symmetric matrix.

1. Show that

$$\begin{bmatrix} u^* \\ d^* \end{bmatrix} := - \begin{bmatrix} P_{uu} & P_{ud} \\ P'_{ud} & P_{dd} \end{bmatrix}^{-1} \begin{bmatrix} P_{ux} \\ P_{dx} \end{bmatrix} x \tag{3}$$

   is the unique solution to the system of equations

$$\frac{\partial J(u^*, d^*)}{\partial u} = 0, \qquad \frac{\partial J(u^*, d^*)}{\partial d} = 0.$$

   Hint: If the matrices A and $D - CA^{-1}B$ are invertible, then $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is also invertible.

2. Show that the pair of policies $(u^*, d^*) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_d}$ is a saddle-point equilibrium for the quadratic zero-sum game and its value is given by

$$J(u^*, d^*) = x' \left(P_{xx} - \begin{bmatrix} P'_{ux} & P'_{dx} \end{bmatrix} \begin{bmatrix} P_{uu} & P_{ud} \\ P'_{ud} & P_{dd} \end{bmatrix}^{-1} \begin{bmatrix} P_{ux} \\ P_{dx} \end{bmatrix} \right) x. \qquad \square$$

Exercise 7 (Mixed security levels/policies – Graphical method)**.** For the following zero-sum matrix game, compute the average security levels and all mixed security policies for both players.

$$A = \underbrace{\begin{bmatrix} 2 & 6 & -2 & 10 \\ -6 & -4 & -3 & -6 \\ 0 & 4 & -3 & -8 \end{bmatrix}}_{\mathsf{P}_2 \text{ choices}} \left.\vphantom{\begin{bmatrix} 2 \\ -6 \\ 0 \end{bmatrix}}\right\} \mathsf{P}_1 \text{ choices},$$

Use policy domination and the graphical method. $\square$

Exercise 8 (Mixed security levels/policies – LP method)**.** For each of the following two zero-sum matrix games compute the average security levels and a mixed security policy

$$A = \underbrace{\begin{bmatrix} 2 & 6 & -2 & 10 \\ -6 & -4 & -3 & -6 \\ 0 & 4 & -3 & -8 \end{bmatrix}}_{\mathsf{P}_2 \text{ choices}} \left.\vphantom{\begin{bmatrix} 2 \\ -6 \\ 0 \end{bmatrix}}\right\} \mathsf{P}_1 \text{ choices}, \qquad B = \underbrace{\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}}_{\mathsf{P}_2 \text{ choices}} \left.\vphantom{\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}}\right\} \mathsf{P}_1 \text{ choices},$$

Solve this problem numerically using MATLAB®. $\square$

Exercise 9 (Resistor Design)**.** Consider a design problem where a circuit designer must choose a nominal value for a resistor from the finite set of 11 values

$$R_{\text{nom}} \in \{.70, .75, .80, .85, \cdots, 1.10, 1.15, 1.20\} \quad [\Omega] \qquad (.05\Omega \text{ equally spaced values})$$

and an opponent selects a percentage deviation from the nominal value from the following set of 21 values

$$\delta \in \{-10\%, -9\%, -8\%, \dots, 9\%, 10\%\}. \qquad \text{(1\% equally spaced values)}$$

The outcome J of this "game" is the absolute value of the difference between a desired current of $1A$ and the actual current that flows through the resistance when a $1V$ voltage is applied to its terminals, i.e.,

$$J = \left|1 - \frac{1}{R}\right|, \quad R \coloneqq R_{\text{nom}}(1+\delta).$$

Use MATLAB® to compute:

1. The 11×21 matrix that represents this matrix game.
2. The pure security values for this game.
3. The mixed security value and corresponding saddle-point equilibrium for this game. □