Sequential modelling of von-Mises Fisher distributions

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1 Motivation

Articulated joints, such as those seen in verterbrates, lie on the surface of a sphere centered at the position of their 'parent' joint. Each joint is constrained by the rigid length of a bone fixed at the parent joint. This probability distribution of the position of this joint, given its parent's position, can be succinctly modeled with a von-Mises-Fisher (vMF) distribution. The vMF distribution is characterized by some mean direction vector $\mu \in \mathbb{R}^D$ and concentration parameter $\tau \in \mathbb{R}$. The variance is isomorphic about the mean angle (direction vector).

When this joint moves over time, we can attempt to model its observed trajectory as a state-space model operating on the vMF distribution. The appeal of this is that, modelling the dynamics of the joint position coordinates can be highly non-linear and require many parameters to constrain, particularly as the number of articulated joints grow. We could alternatively model the position as the coordinate pointed to by the direction vector from a joint's parent to itself. Then, given the parent joint's current position, the joint's current direction vector would be linear to the direction vector from the previous time-step. Its position trajectory on the $\mathbb{S}^2$ sphere can thus be modelled as a linear state-space model with a dynamics matrix encoding rotation.

2 Model

The model will consist of T observed points, $\{x_1, \ldots, x_T\}$ in a 2D space. These observed points actually all fall on the perimeter/surface of a circle (S^1 sphere, and so can be modelled with a latent direction vectors $\{u_1, \ldots, u_T\}$.

Assuming a fixed angular velocity, the distribution of the next direction u_{t+1} is also vMF but with a mean centered on $R_{z_k} u_t$. $R_{z_t=k}$ is the dynamics matrix for this system, and specifically it is a rotations matrix. A sudden switch in direction can be characterized by a switch in discrete state $z_t = k$. In $\mathbb{S}^1 \subset \mathbb{R}^2$, there are only $2^{D-1} = 2$ discrete rotational states for a given angular velocity. In $\mathbb{S}^2 \subset \mathbb{R}^3$, there would be $2^{D-1} = 4$ discrete rotational directions.

3 Datasets

I will produce synthetic data is illustrate by the generative model in Figure 1. After randomly initializing variables, z_{t+1} is sampled from its Bernoulli distribution, the corresponding R matrix is selected and the next direction vector and subsequent observation are evolved and generated.

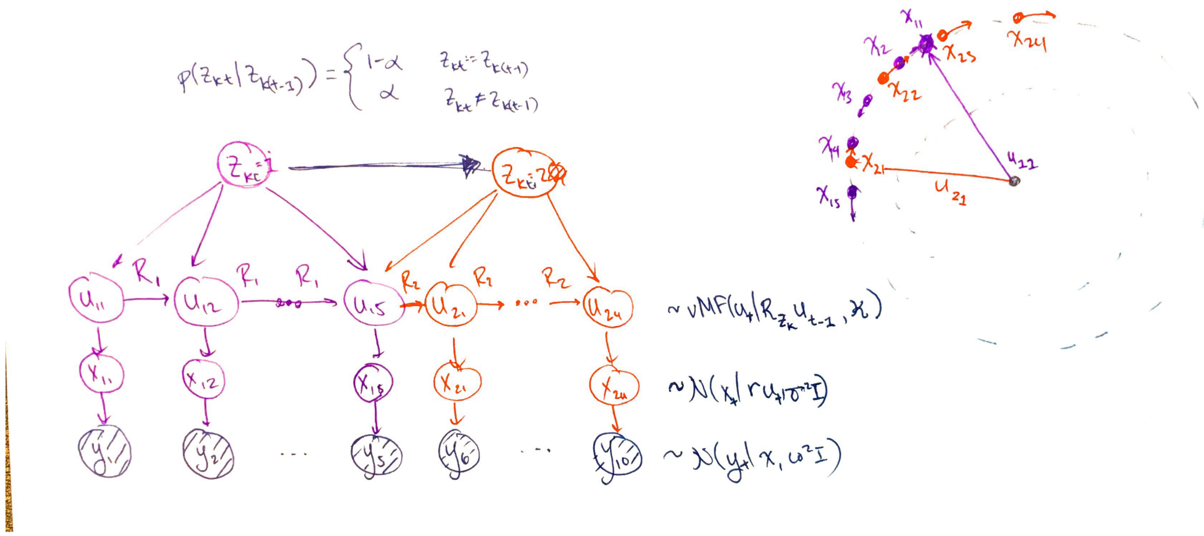

Figure 1: Proposed HMM and continue state-space model for vMF sequencing. *(left)*. Corresponding cartoon of true points X evolving over time, and the underlying direction vectors (only 2 sketched out).

I will make observations $y_1, \ldots, y_T$ with random observation noise, and attempt to infer the latent dynamis (rotation matrix) and hence latent discrete state.

4 Extensions

This model can be extended in one of two directions (with the remaining time in the quarter in mind). One direction is extending it into $\mathbb{R}^3$ and continuing to work with enumerated discrete states. As previously mentioned, only 4 discrete states would need to be enumerated. The second direction which I think is more interesting is incorporating some sort of 'time-warping' variable into the model as well. This addition would (ideally) allow us to a) model dynamics with varying angular velocity, and b) ultimately replace the "discrete state" $z_t \in \triangle_K$ (since a change in direction is just a de-acceleration in the old direction followed by an acceleration in the new direction. This would be useful in 3-dimensions because most motion is a combination of the 4 elemental discrete rotations. This would also allow us to have a fully continuous model (of a continuous process), which seems like a reasonable objective to build towards.

References