How to Find an Example

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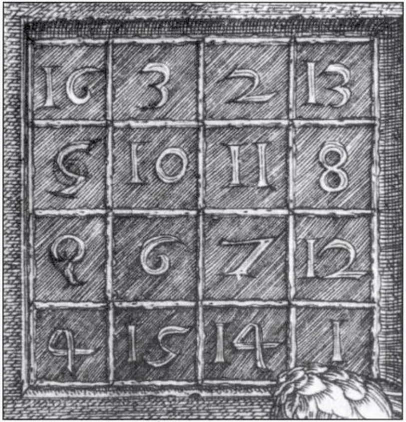
Outline

Be creative!

Albrecht Duerer,
Melancholia, 1514

Magic Square: Definition



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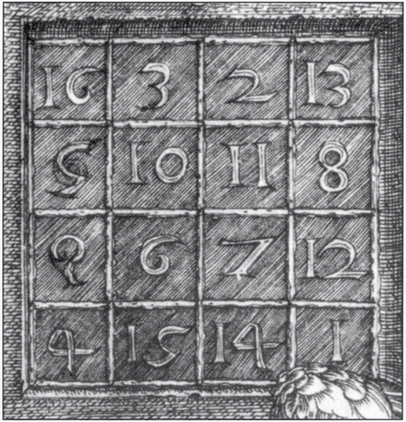
- $1, 2, 3, \ldots, 15, 16$



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- $1, 2, 3, \ldots, 15, 16$
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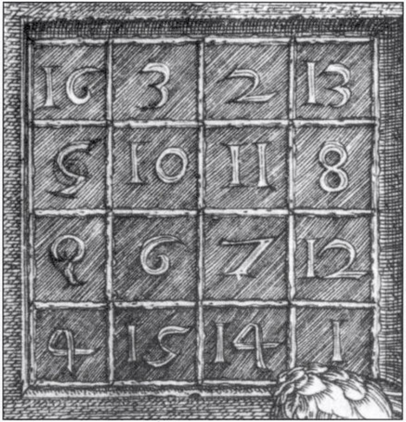

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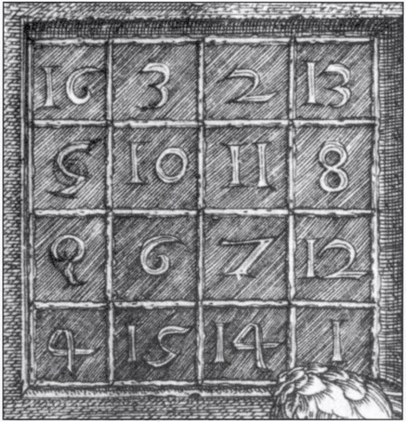
- $1, 2, 3, \ldots, 15, 16$
- $1, 2, 3, \ldots, n^2$ for $n \times n$
- the same sum in columns, rows, diagonals



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- what about size 3 – made of $1, 2, 3, \ldots, 9$?

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Magic Squares

Narrowing the Search

Multiplicative Magic Squares

More Puzzles

Integer linear combinations

Paths in a Graph

Looking for 3×3 Magic Square

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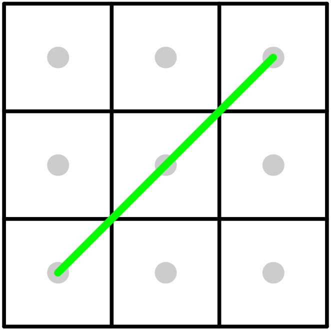
- $45/3 = 15$

Hint: the Center



summing up four lines…

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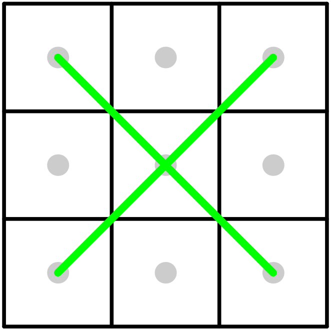

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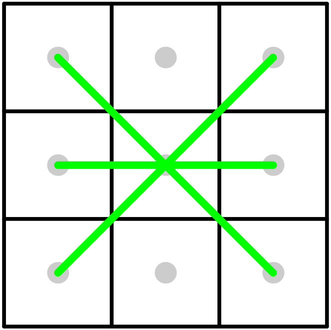

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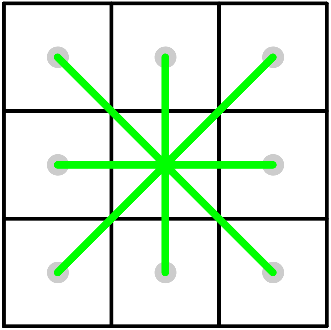

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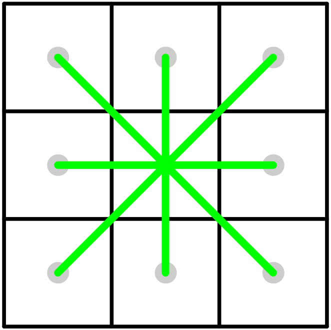

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$$4S = \text{total sum} + 3 \cdot \text{center}$$

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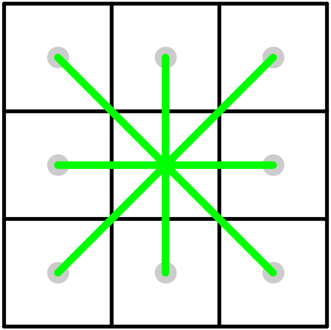

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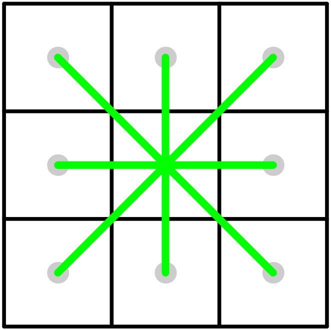

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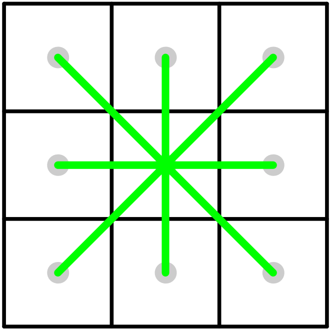

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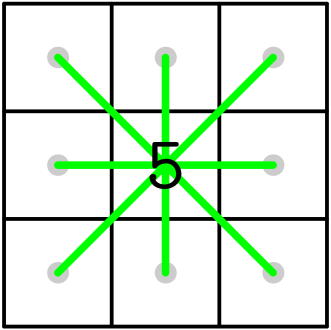

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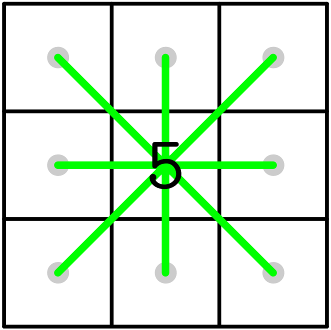

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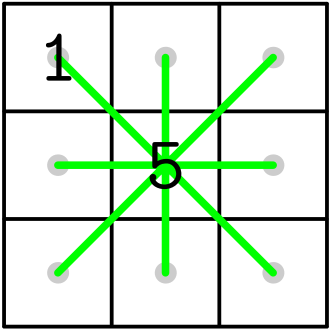

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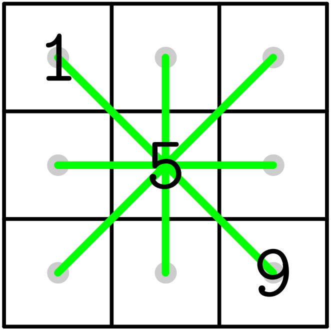

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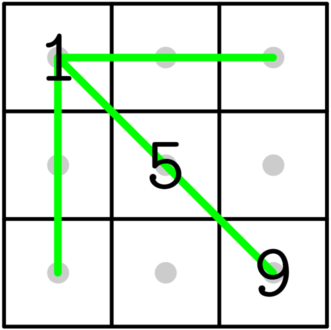

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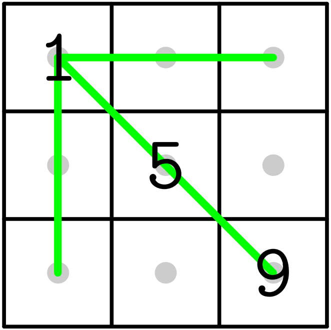

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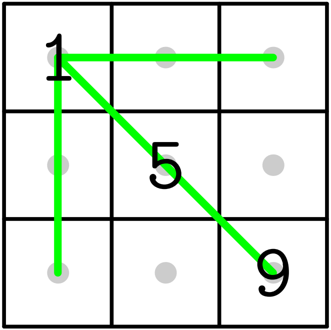

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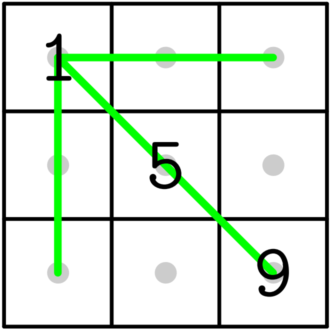

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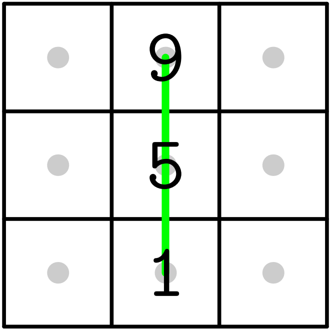

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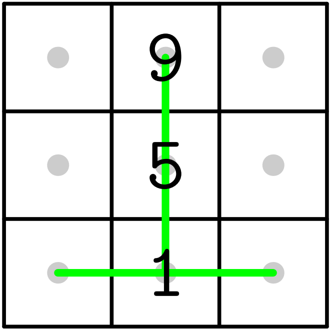

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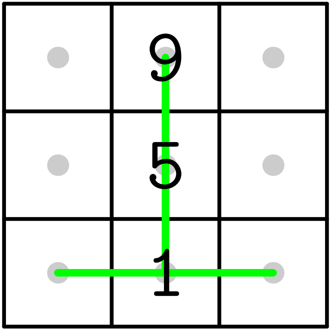

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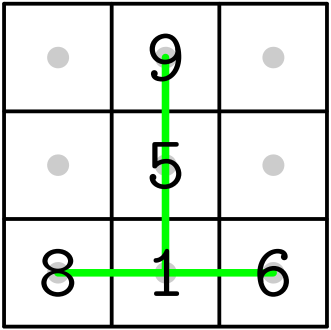

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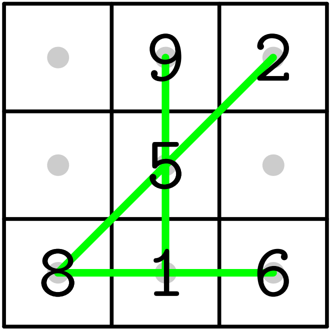

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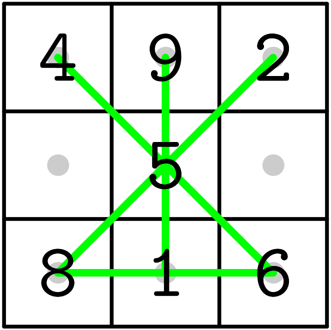

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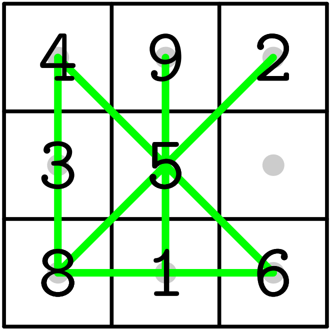

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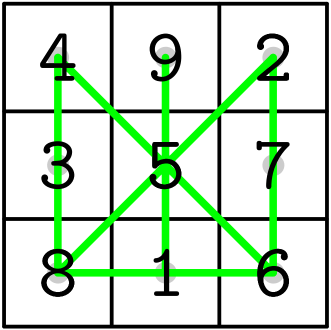

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$$\begin{array}{|c|c|c|}\hline 1 & 2 & 0 \\ \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|}\hline 2 & 4 & 1 \\ \hline 1 & 2 & 4 \\ \hline 4 & 1 & 2 \\ \hline \end{array}$$

Spoiler–2

1	2	0
0	1	2
2	0	1

$\rightarrow$

2	4	1
1	2	4
4	1	2

0	2	1
2	1	0
1	0	2

$\rightarrow$

1	9	3
9	3	1
3	1	9

Spoiler–2

1	2	0
0	1	2
2	0	1

$\rightarrow$

2	4	1
1	2	4
4	1	2

$\times$

0	2	1
2	1	0
1	0	2

$\rightarrow$

1	9	3
9	3	1
3	1	9

2	36	3
9	6	4
12	1	18

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- $2 \times 2 \times 3 \times 5 \times 7 = 420$

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- others?

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- 420×3 is too big

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- $177.242771^2 = 31414.9998718\ldots$

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- $\sqrt{31415} = 177.242771\ldots$
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- $177.243^2 = 31415.081049$

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- answer: 177 243 and 560 491.

Outline

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Two people, each has many coins of each type.
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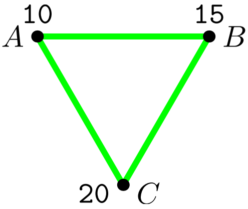
Outline

Hotels and Paths

- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10+15+20=45$ nights



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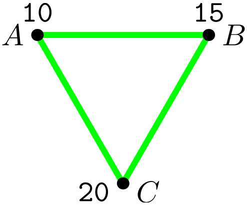

- repeat 5 times a path of length 9

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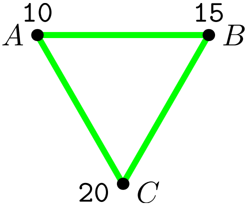

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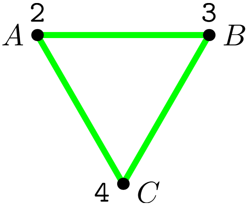

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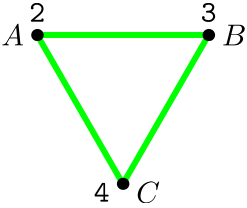

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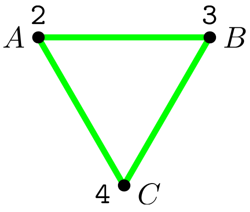

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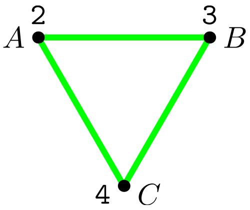

- repeat 5 times a path of length 9
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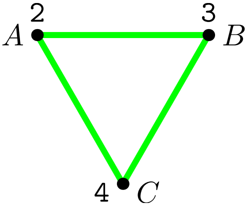

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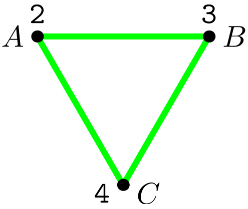

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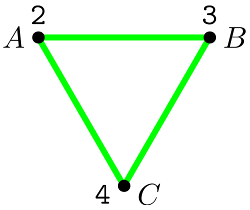

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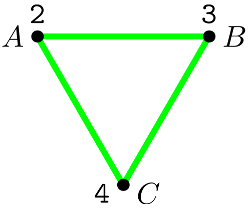

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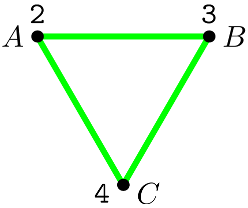

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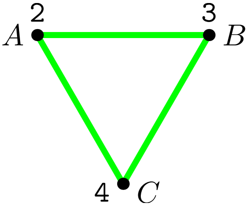

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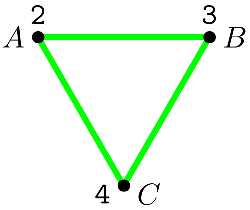

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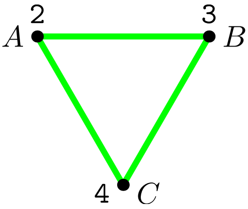

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- $ACACBCBCB$ (5 times)

When a Path Does Not Exist

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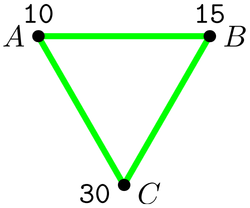

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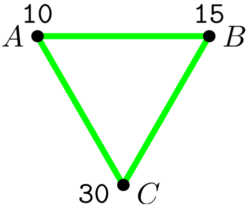

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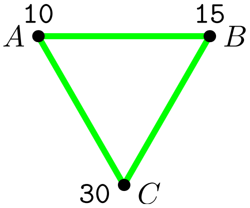

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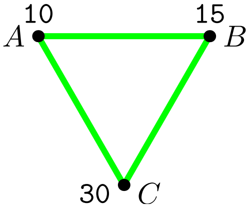

- possible or not?
- obstacle: too many vouchers for C
- to use them all we need at least 29 others