Exercises for ICTP School 2025

Exercise 1: Gravitational wave properties

The propagation of gravitational waves in a curved background is described by

$$\Box\bar{h}_{\mu\nu} + 2\bar{R}_{\alpha\mu\beta\nu}\bar{h}^{\alpha\beta} = 0\,. \tag{0.0.1}$$

Using the plane wave ansatz

$$h_{\mu\nu}(x) = \mathrm{Re}\left[\left(A_{\mu\nu}(x) + \varepsilon A^{(1)}_{\mu\nu}(x) + \varepsilon^2 A^{(2)}_{\mu\nu}(x) + \cdots\right) e^{i\theta(x)/\varepsilon}\right]\,, \tag{0.0.2}$$

explicitly derive that:

1. GWs propagate in null geodesics,
2. the number of gravitons is preserved,
3. the polarization tensor is orthogonal to the propagation direction and parallel transported.

Exercise 2: Gravitational wave inspiral in frequency domain

The strain can be written as

$$h(t) = \frac{4}{d_{\rm eff}}\left(\frac{G\mathcal{M}_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t)}{c}\right)^{2/3}\cos\left[\Phi(t) + \chi\right] \tag{0.0.3}$$

where $d_{\rm eff}$ is an effective distance that include the dependence in inclination, $\mathcal{M}_c$ is the chirp mass,$f_{\rm gw}$ is the GW frequency, $\Phi(t)$ is the phase evolution, and the there is a phase χ. The phase evolution is determined by the chirping of the signal

$$\Phi(t) = -2\left(\frac{5G\mathcal{M}_c}{c^3}\right)^{-5/8}(t_c - t)^{5/8} + \Phi_0\,. \tag{0.0.4}$$

The goal of this exercise is to compute the frequency domain waveform using the Fourier transform

$$\tilde{h}(f) = \int dt\, h(t)e^{i2\pi ft}\,. \tag{0.0.5}$$

The inverse Fourier transform is defined as

$$h(t) = \int_{-\infty}^{\infty} df\tilde{h}(f)e^{-i2\pi ft}\,. \tag{0.0.6}$$

1. Show that for a real time-domain strain $h(t)$, the frequency domain signal must satisfy

$$\tilde{h}(-f) = \tilde{h}^*(f)\,. \tag{0.0.7}$$

2. Compute the Fourier transform of the strain using the stationary phase approximation. Follow the steps described at the end of Sec. 5.2.

3. Show that $\tilde{h}(f)$ only depends on four parameters: an effective distance d, the chirp mass $\mathcal{M}_c$, a reference time t_0 and a reference phase ϕ_0. Collectively, they form a 4D parameter space $\vec{\theta} = \{d, \mathcal{M}_z, t_0, \phi_0\}$.

4. What is the scaling of the amplitude and complex phase $\tilde{h}(f) = \tilde{A}(f, \vec{\theta})e^{i\Psi(f,\vec{\theta})}$ with frequency?

5. What is the range of frequencies in which this expression is valid?

Exercise 3: Gravitational wave damping by the Hubble friction

If one introduces the spatial wave vector k, and looks for wave solutions $h_A \propto e^{i\vec{k}\vec{x}}$, the cosmological wave propagation can also be written as

$$h''_A + 2\mathcal{H}h'_A + c^2k^2a^2h_A = 0\,, \tag{0.0.8}$$

where $\mathcal{H} \equiv a'/a$ is the Hubble parameter in terms of conformal time. We can also rewrite this equation in terms of comoving time t:

$$\ddot{h}_A + 3H\dot{h}_A + c^2k^2h_A = 0\,, \tag{0.0.9}$$

where $\dot{h} = c^{-1}\partial h/\partial t$. From this point of view, the expansion of the Universe $H > 0$ introduces a damping of the GWs, which is sometimes referred as the *Hubble friction*.

1. Solve this equation using the WKB approximation.

2. What is the leading order solution?

3. How is it corrected?

Exercise 4: Radiative degrees of freedom

On a global vacuum spacetime we have seen that the equations of motion only contain two physical propagating degrees of freedom. These are *radiative* modes. On a global spacetime with matter sources, however, this is not generically true. There could be physical, *non-radiative* degrees of freedom. These modes are characterized by non-wave equations. Schematically, instead of $g_{\alpha\beta}\nabla^\alpha\nabla^\beta\phi_A = \cdots$ one has $\gamma_{ij}\nabla^i\nabla^j\phi_A = \cdots$, where γ_{ij} is the spatial metric. Derive such equations!

1. What are the equations of motion followed by each degree of freedom?

2. Are they coupled?

If you are looking for inspiration, this is nicely discussed in Flanagan and Hughes review [1] as well as Carroll's book [2].

Exercise 5: Detecting a compact binary coalescence

During the course we have studied the gravitational waves produced during the inspiral phase of a compact binary. We have then learned how these waves propagate in the Universe and what imprint they leave on a ground-base detector. It is now the time to explore what are the capabilities of such detectors to measure the properties of the signal. This is essential if we want to understand how such detections could be used to learn about astrophysics, cosmology and fundamental physics.

A compact binary at cosmological distances emits a GW during the inspiral phase whose polarizations are given (at leading order) by the following expressions in the detector's frame:

$$h_+(t) = \frac{4}{d_L}\left(\frac{G\mathcal{M}_z}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t)}{c}\right)^{2/3}\left(\frac{1+\cos^2\iota}{2}\right)\cos(\Phi(t))\,, \tag{0.0.10}$$

$$h_\times(t) = \frac{4}{d_L}\left(\frac{G\mathcal{M}_z}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t)}{c}\right)^{2/3}\cos\iota\sin(\Phi(t))\,, \tag{0.0.11}$$

where d_L is the luminosity distance, ι is the inclination angle and the phase is given by

$$\Phi(t) = -2\left(\frac{5G\mathcal{M}_c}{c^3}\right)^{-5/8}(t_c - t)^{5/8} + \Phi_0\,. \tag{0.0.12}$$

Recall that $\mathcal{M}_z = (1+z)\mathcal{M}_c$ is the redshifted chirp mass and $f_{\rm gw}(t)$ the frequency evolution measured at the detector.

At the detector one measures a strain which is a projection into the antenna pattern functions,

$$h(t) = F_+ h_+(t) + F_\times h_\times(t)\,, \tag{0.0.13}$$

where $F_{+,\times}$ are a function of the sky position and polarization angle $\{\theta,\varphi,\psi\}$.

1. Show that the strain can be written as

$$h(t) = \frac{4}{d_{\rm eff}}\left(\frac{G\mathcal{M}_z}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t)}{c}\right)^{2/3}\cos\left[\Phi(t)+\chi\right] \tag{0.0.14}$$

   by finding the values of the effective distance $d_{\rm eff}$ and the phase χ. With a single detector can we disentangle the GW distance from the inclination and sky position? For generic sky positions, will we think that the source is closer or further away?

2. Compute the Fourier transform of the strain (using the stationary phase approximation),

$$\tilde{h}(f) = \int dt\, h(t) e^{i2\pi f t}\,, \tag{0.0.15}$$

   and show that it only depends on four parameters: an effective distance d, the redshifted chirp mass $\mathcal{M}_z$, a reference time t_0 and a reference phase ϕ_0. Collectively, they form a 4D parameter space $\vec{\theta} = \{d, \mathcal{M}_z, t_0, \phi_0\}$. What is the scaling of the amplitude and complex phase $\tilde{h}(f) = \tilde{A}(f,\vec{\theta})e^{i\Psi(f,\vec{\theta})}$ with frequency? What is the range of frequencies in which this expression is valid?

3. Using the frequency domain strain that you have just derived, compute the Fisher matrix

$$\Gamma_{ij} = (\partial_i h|\partial_j h)\,, \tag{0.0.16}$$

where $\partial_i = \partial/\partial\theta^i$ indicates a derivative in the parameter space $\vec{\theta} = \{\ln d^{-1}, \ln \mathcal{M}_z, t_0, \phi_0\}$ and $(a|b)$ is the noise-weighted inner product of $a(t)$ and $b(t)$:

$$(a|b) = 4\mathrm{Re}\int_0^\infty \frac{\tilde{a}^*(f)\tilde{b}(f)}{S_n(f)}df\,. \tag{0.0.17}$$

Note that the logarithms are useful to get fractional errors. Express each component of the matrix in terms of the signal-to-noise ratio $\rho^2 = (h|h)$ and introduce the notation for the frequency moments

$$\bar{f}^\alpha \equiv \frac{(f^\alpha h|h)}{(h|h)} \tag{0.0.18}$$

in order to conveniently encapsulate the integral terms with powers of the frequency. Since in the inspiral phase the frequency domain waveform scales as a power-law with the frequency, then $\bar{f}^\alpha$ only depends on the frequency and $S_n(f)$.

Bibliography

[1] E. E. Flanagan and S. A. Hughes, *The Basics of gravitational wave theory*, *New J. Phys.* **7** (2005) 204, [`gr-qc/0501041`].

[2] S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*. Cambridge University Press, 7, 2019. Based on gr-qc/9712019.