Exercises for ICTP School 2025

Exercise 1: Gravitational wave properties

The propagation of gravitational waves in a curved background is described by

$$\Box \bar{h}_{\mu\nu} + 2\bar{R}_{\alpha\mu\beta\nu}\bar{h}^{\alpha\beta} = 0. \tag{0.0.1}$$

Using the plane wave ansatz

$$h_{\mu\nu}(x) = \operatorname{Re}\left[\left(A_{\mu\nu}(x) + \varepsilon A_{\mu\nu}^{(1)}(x) + \varepsilon^2 A_{\mu\nu}^{(2)}(x) + \cdots\right) e^{i\theta(x)/\varepsilon}\right], \qquad (0.0.2)$$

explicitly derive that:

- 1. GWs propagate in null geodesics,
- 2. the number of gravitons is preserved,
- 3. the polarization tensor is orthogonal to the propagation direction and parallel transported.

Exercise 2: Gravitational wave inspiral in frequency domain

The strain can be written as

$$h(t) = \frac{4}{d_{\text{eff}}} \left(\frac{G\mathcal{M}_c}{c^2}\right)^{5/3} \left(\frac{\pi f_{\text{gw}}(t)}{c}\right)^{2/3} \cos\left[\Phi(t) + \chi\right]$$
(0.0.3)

where d_{eff} is an effective distance that include the dependence in inclination, \mathcal{M}_c is the chirp mass, f_{gw} is the GW frequency, $\Phi(t)$ is the phase evolution, and the there is a phase χ . The phase evolution is determined by the chirping of the signal

$$\Phi(t) = -2\left(\frac{5G\mathcal{M}_c}{c^3}\right)^{-5/8} (t_c - t)^{5/8} + \Phi_0.$$
 (0.0.4)

The goal of this exercise is to compute the frequency domain waveform using the Fourier transform

$$\tilde{h}(f) = \int dt \, h(t)e^{i2\pi ft} \,. \tag{0.0.5}$$

The inverse Fourier transform is defined as

$$h(t) = \int_{-\infty}^{\infty} df \tilde{h}(f) e^{-i2\pi f t}. \qquad (0.0.6)$$

1. Show that for a real time-domain strain h(t), the frequency domain signal must satisfy

$$\tilde{h}(-f) = \tilde{h}^*(f). \tag{0.0.7}$$

- 2. Compute the Fourier transform of the strain using the stationary phase approximation. Follow the steps described at the end of Sec. 5.2.
- 3. Show that $\tilde{h}(f)$ only depends on four parameters: an effective distance d, the chirp mass \mathcal{M}_c , a reference time t_0 and a reference phase ϕ_0 . Collectively, they form a 4D parameter space $\vec{\theta} = \{d, \mathcal{M}_z, t_0, \phi_0\}$.
- 4. What is the scaling of the amplitude and complex phase $\tilde{h}(f) = \tilde{A}(f, \vec{\theta})e^{i\Psi(f, \vec{\theta})}$ with frequency?
- 5. What is the range of frequencies in which this expression is valid?

Exercise 3: Gravitational wave damping by the Hubble friction

If one introduces the spatial wave vector k, and looks for wave solutions $h_A \propto e^{i\vec{k}\vec{x}}$, the cosmological wave propagation can also be written as

$$h_A'' + 2\mathcal{H}h_A' + c^2k^2a^2h_A = 0, \qquad (0.0.8)$$

where $\mathcal{H} \equiv a'/a$ is the Hubble parameter in terms of conformal time. We can also rewrite this equation in terms of comoving time t:

$$\ddot{h}_A + 3H\dot{h}_A + c^2k^2h_A = 0\,, (0.0.9)$$

where $\dot{h} = c^{-1}\partial h/\partial t$. From this point of view, the expansion of the Universe H > 0 introduces a damping of the GWs, which is sometimes referred as the *Hubble friction*.

- 1. Solve this equation using the WKB approximation.
- 2. What is the leading order solution?
- 3. How is it corrected?

Bibliography