Exercises for ICTP School 2025

Exercise 1: Gravitational wave properties

The propagation of gravitational waves in a curved background is described by

$$\Box \bar{h}_{\mu\nu} + 2\bar{R}_{\alpha\mu\beta\nu}\bar{h}^{\alpha\beta} = 0\,. \tag{0.0.1}$$

Using the plane wave ansatz

$$h_{\mu\nu}(x) = \mathrm{Re}\left[\left(A_{\mu\nu}(x) + \varepsilon A^{(1)}_{\mu\nu}(x) + \varepsilon^2 A^{(2)}_{\mu\nu}(x) + \cdots\right) e^{i\theta(x)/\varepsilon}\right]\,, \tag{0.0.2}$$

explicitly derive that:

1. GWs propagate in null geodesics,
2. the number of gravitons is preserved,
3. the polarization tensor is orthogonal to the propagation direction and parallel transported.

Exercise 2: Gravitational wave inspiral in frequency domain

The strain can be written as

$$h(t) = \frac{4}{d_{\mathrm{eff}}}\left(\frac{G\mathcal{M}_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\mathrm{gw}}(t)}{c}\right)^{2/3}\cos\left[\Phi(t) + \chi\right] \tag{0.0.3}$$

where d_{eff} is an effective distance that include the dependence in inclination, $\mathcal{M}_c$ is the chirp mass,f_{gw} is the GW frequency, $\Phi(t)$ is the phase evolution, and the there is a phase χ. The phase evolution is determined by the chirping of the signal

$$\Phi(t) = -2\left(\frac{5G\mathcal{M}_c}{c^3}\right)^{-5/8}(t_c - t)^{5/8} + \Phi_0\,. \tag{0.0.4}$$

The goal of this exercise is to compute the frequency domain waveform using the Fourier transform

$$\tilde{h}(f) = \int dt\, h(t) e^{i2\pi f t}\,. \tag{0.0.5}$$

The inverse Fourier transform is defined as

$$h(t) = \int_{-\infty}^{\infty} df \tilde{h}(f) e^{-i2\pi f t}\,. \tag{0.0.6}$$

1. Show that for a real time-domain strain $h(t)$, the frequency domain signal must satisfy

$$\tilde{h}(-f) = \tilde{h}^*(f)\,. \tag{0.0.7}$$

2. Compute the Fourier transform of the strain using the stationary phase approximation. Follow the steps described at the end of Sec. 5.2.

3. Show that $\tilde{h}(f)$ only depends on four parameters: an effective distance d, the chirp mass $\mathcal{M}_c$, a reference time t_0 and a reference phase ϕ_0. Collectively, they form a 4D parameter space $\vec{\theta} = \{d, \mathcal{M}_z, t_0, \phi_0\}$.

4. What is the scaling of the amplitude and complex phase $\tilde{h}(f) = \tilde{A}(f, \vec{\theta})e^{i\Psi(f,\vec{\theta})}$ with frequency?

5. What is the range of frequencies in which this expression is valid?

Bibliography