Class project

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The project

- Maximize

$$E_t \sum_{s=0}^{\infty} \beta^t \frac{\left(c_{t+s}^{\mu} \left(1 - l_{t+s}\right)^{1-\mu}\right)^{1-\sigma}}{1-\sigma}, \qquad \beta < 1,$$

subject to

$$\begin{aligned} c_t + k_{t+1} - (1-\delta) k_t &\leq z_t f\left(k_t, l_t\right) \; ; \\ \ln z_t &= \rho \ln z_{t-1} + \varepsilon_t ; \varepsilon \sim N\left(0, \sigma^2\right) \end{aligned}$$

- The corresponding dynamic program

$$v(z,k) = \max_{k'} \left\{ \begin{array}{c} u\left(zf(k) - k' + (1-\delta) k, 1 - l\right) \\ +\beta E \left\{ v\left(z', k'\right) \right\} \end{array} \right\}.$$

On the margin

- Euler

$$u_c(t) = \beta E_t \left\{ u_c(t+1) \left[\theta z_{t+1} \left(\frac{k_{t+1}}{l_{t+1}} \right)^{\theta-1} + 1 - \delta \right] \right\}$$

- Labor-leisure

$$\frac{u_l(t)}{u_c(t)} = f_l(t) \Rightarrow \frac{1-\mu}{\mu} \frac{c_t}{1-l_t} = (1-\theta) z_t k_t^{\theta} l_t^{-\theta};$$

- Budget constraint

$$c_t = z_t f(k_t, l_t) - k_{t+1} + (1-\delta) k_t$$

- Labor-leisure + BC

$$\frac{1-\mu}{\mu} \frac{k_{t+1} - (1-\delta) k_t}{z_t k_t^{\theta}} = l_t^{1-\theta} \left[\frac{1-\mu}{\mu} + 1 - \theta - \frac{1-\theta}{l_t} \right]$$

Deterministic steady state

- $z_t = 1$ for all t
- Labor-leisure + BC

$$\frac{1-\mu}{\mu}\frac{\delta}{\left(\frac{\bar{k}}{\bar{l}}\right)^{\theta-1}} = \left[\frac{1-\mu}{\mu} + 1 - \theta - \frac{1-\theta}{\bar{l}}\right]$$

- Euler

$$\left(\frac{\bar{k}}{\bar{l}}\right)^{\theta-1} = \frac{\frac{1}{\beta} - (1-\delta)}{\theta}$$

- Combine

$$\bar{l} = \frac{1-\theta}{\frac{1-\mu}{\mu} + 1 - \theta - \frac{1-\mu}{\mu}\frac{\delta\theta}{\frac{1}{\beta}-(1-\delta)}}; \bar{k} = \bar{l}\left(\frac{\frac{1}{\beta} - (1-\delta)}{\theta}\right)^{\frac{1}{\theta-1}}$$

Discretize

- Use the code to generate a Markov $nz * nz$ transition matrix for nz discrete z possibility
- Divide the state space of k into a finite nk number of grid points
- Suppose $nz = 3$ and $nk = 3$

$v(z, k) =$

	k_1	k_2	k_3
z_1	$v(z_1, k_1)$	$v(z_1, k_2)$	$v(z_1, k_3)$
z_2	$v(z_2, k_1)$	$v(z_2, k_2)$	$v(z_2, k_3)$
z_3	$v(z_3, k_1)$	$v(z_3, k_2)$	$v(z_3, k_3)$

The return matrix

- Is an array with $nz * nk * nk$ dimensions
- The first page with $k' = k_1$

$$R(z,k,k') = \begin{array}{|c|c|c|}\hline R(z_1,k_1,k_1) & R(z_1,k_2,k_1) & R(z_1,k_3,,k_1) \\ \hline R(z_2,k_1,,k_1) & R(z_2,k_2,,k_1) & R(z_2,k_3,,k_1) \\ \hline R(z_3,k_1,,k_1) & R(z_3,k_2,,k_1) & R(z_3,k_3,,k_1) \\ \hline \end{array}$$

- The second page with $k' = k_2$

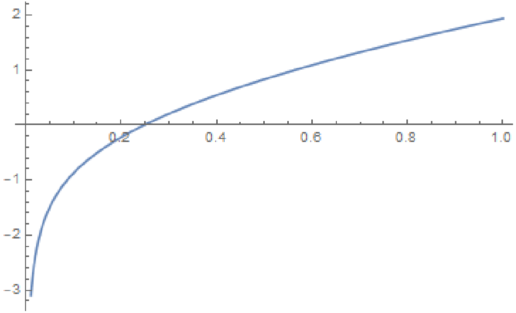
$$R(z,k,k') = \begin{array}{|c|c|c|}\hline R(z_1,k_1,k_2) & R(z_1,k_2,k_2) & R(z_1,k_3,,k_2) \\ \hline R(z_2,k_1,,k_2) & R(z_2,k_2,,k_2) & R(z_2,k_3,,k_2) \\ \hline R(z_3,k_1,,k_2) & R(z_3,k_2,,k_2) & R(z_3,k_3,,k_2) \\ \hline \end{array}$$

Nuts and bolts of the utility values

- The question is how do we compute the elements of this matrix
- For each value of z, k, k', l must solve

$$\frac{1-\mu}{\mu}\frac{k'-(1-\delta)k}{zk^{\theta}} = l^{1-\theta}\left[\frac{1-\mu}{\mu}+1-\theta-\frac{1-\theta}{l}\right]$$

- The RHS is well behaved



- This needs a separate function to solve for l for each z, k, k'

Finding maximum

- The RHS of Bellman equation is

$$R + \beta E \left\{ v \left(z', k' \right) \right\}$$

- We begin with a guess of course with

$$v_0 \left(z, k \right) = \begin{array}{|c|c|c|} \hline v_0 \left(z_1, k_1 \right) & v_0 \left(z_1, k_2 \right) & v_0 \left(z_1, k_3 \right) \\ \hline v_0 \left(z_2, k_1 \right) & v_0 \left(z_2, k_2 \right) & v_0 \left(z_2, k_3 \right) \\ \hline v_0 \left(z_3, k_1 \right) & v_0 \left(z_3, k_2 \right) & v_0 \left(z_3, k_3 \right) \\ \hline \end{array}$$

Finding maximum

- Suppose you are choosing $k' = k_1$. Current state is $z = z_1$ and $k = k_1$. Then, a particular row in the transition matrix gives you $\pi\left(z_1|z_1\right)\ \pi\left(z_2|z_2\right)\ \pi\left(z_3|z_3\right)$
- This row gets multiplied with the first column of v_0 above

$$\left[\pi\left(z_1|z_1\right)\ \pi\left(z_2|z_1\right)\ \pi\left(z_3|z_1\right)\right]\begin{bmatrix}\boxed{v_0\left(z_1, k_1\right)} \\ \boxed{v_0\left(z_2, k_1\right)} \\ \boxed{v_0\left(z_3, k_1\right)}\end{bmatrix}$$

- Add this to the first row, first column element in the first page, i.e., $R\left(z_1, k_1, k_1\right)$. Repeat this for all z, k, k'
- Find maximum for each z, k by looking at which page has got the maximum
- This value replaces the current $v_0\left(z, k\right)$
- Repeat this process

Markov chain plus policy function

- With three points for z – the transition matrix for z is $nz * nz$:

$P =$

	z_1	z_2	z_3
z_1	π_{11}	π_{12}	π_{13}
z_2	π_{21}	π_{22}	π_{23}
z_3	π_{31}	π_{32}	π_{33}

- Let the policy function for k' (similarly for c, l, i) be:

$g(z, k) =$

	k_1	k_2	k_3
z_1	$g(z_1, k_1)$	$g(z_1, k_2)$	$g(z_1, k_3)$
z_2	$g(z_2, k_1)$	$g(z_2, k_2)$	$g(z_2, k_3)$
z_3	$g(z_3, k_1)$	$g(z_3, k_2)$	$g(z_3, k_3)$

State transition matrix

- With $\{z, k\}$ jointly there are $nz * nk$ total possible states are:
- State transition matrix

$G =$

	$z_1 k_1$	$z_1 k_2$	$z_1 k_3$		$z_3 k_1$	$z_3 k_2$	$z_3 k_3$
$z_1 k_1$	$\pi_{11,11}$	$\pi_{11,12}$					
$z_1 k_2$							
$z_1 k_3$						$\pi_{13,32}$	$\pi_{13,33}$
$z_2 \ldots$	..	..	..				
$z_3 k_1$							
$z_3 k_2$						$\pi_{32,32}$	$\pi_{32,33}$
$z_3 k_3$						$\pi_{33,32}$	$\pi_{33,33}$

- We get these elements by combining g with P

State transition matrix

- Let current state be z_3 and k_1. First look at $g(z_3, k_1)$, which will be one of $\{k_1, k_2, k_3\}$
- Suppose $g(z_3, k_1) = k_1$
- Then, only columns that have k_1 are possible

	$z_1 k_1$	$z_1 k_2$	$z_1 k_3$	$z_2 k_1$	$z_2 k_2$	$z_2 k_3$	$z_3 k_1$	$z_3 k_2$	$z_3 k_3$
$z_1 k_1$									
$z_1 k_2$									
$z_1 k_3$									
$z_2 \ldots$	..	..	..						
$z_3 k_1$	π_{31}	0	0	π_{32}	0	0	π_{33}	0	0
$z_3 k_2$									
$z_3 k_3$									

- Do the same for all current z, k combinations

Stationary distribution

- Recall

$$p'_{t+1} = p_t G$$

- Stationary distribution $p_{t+1} = p_t$
- With computer, once we have G, we can iterate to convergence
- Guess initial p (vector of $nz * nk$)

$$p_0 = \left\{ \frac{1}{nz * nk}, \frac{1}{nz * nk} ..., \frac{1}{nz * nk} \right\}'$$

- Then iterate until p converges.

$$\max \{ p_{n+1} - p_n G \} \leq \text{ tol}$$

Distribution for k

- Now that we have $p = \{p_{11}, p_{12}, p_{13}, p_{21}, p_{22}, p_{23}, p_{31}, p_{32}, p_{33}\}$.
- These are joint probabilities. One can integrate over z to get unconditional probabilities for k
- Unconditional for k

$$p_k = \{p_{11} + p_{21} + p_{31}, p_{12} + p_{22} + p_{32}, p_{13} + p_{23} + p_{33}\}$$

- Plot the distibution
- Use it to get mean and standard deviation of k

Second moments for c,l,i,y

- We know the stationary distriburtion of all $\{z, k\}$
- We know c, l, i, y as functions of $\{z, k\}$
- Then organize the policy functions

$g(z,k) =$

	k_1	k_2	k_3
z_1	$g(z_1, k_1)$	$g(z_1, k_2)$	$g(z_1, k_3)$
z_2	$g(z_2, k_1)$	$g(z_2, k_2)$	$g(z_2, k_3)$
z_3	$g(z_3, k_1)$	$g(z_3, k_2)$	$g(z_3, k_3)$

- As vectors corresponding to the states:

$$\left\{ \begin{array}{c} g(z_1, k_1), g(z_1, k_2), g(z_1, k_3), g(z_2, k_1), g(z_2, k_2), \\ g(z_2, k_3), g(z_3, k_1), g(z_3, k_2), g(z_3, k_3) \end{array} \right\}'$$

Moments of output

- The vector of output and vector p

$$\begin{array}{|c||c|}
p & y \\
p(z_1, k_1) & y(z_1, k_1) \\
p(z_1, k_2) & y(z_1, k_2) \\
p(z_1, k_3) & y(z_1, k_3) \\
p(z_2, k_1) & y(z_2, k_1) \\
p(z_2, k_2) & y(z_2, k_2) \\
p(z_2, k_3) & y(z_2, k_3) \\
p(z_3, k_1) & y(z_3, k_1) \\
p(z_3, k_2) & y(z_3, k_2) \\
p(z_3, k_3) & y(z_3, k_3)
\end{array}$$

- Sufficient to get mean $\bar{y}$ and variance σ_y^2 and correlation with all other variables.

Autocorrelation of output

- We need a period lag (or lead):

	$z_1 k_1$	$z_1 k_2$	$z_1 k_3$		$z_3 k_1$	$z_3 k_2$	$z_3 k_3$
$z_1 k_1$	$\pi_{11,11}$	$\pi_{11,12}$					
$z_1 k_2$							
$z_1 k_3$						$\pi_{13,32}$	$\pi_{13,33}$
$z_2 \ldots$	..	..	..				
$z_3 k_1$							
$z_3 k_2$						$\pi_{32,32}$	$\pi_{32,33}$
$z_3 k_3$						$\pi_{33,32}$	$\pi_{33,33}$

$$\begin{vmatrix} y\left(z_1, k_1\right) \\ y\left(z_1, k_2\right) \\ y\left(z_1, k_3\right) \\ y\left(z_2, k_1\right) \\ y\left(z_2, k_2\right) \\ y\left(z_2, k_3\right) \\ y\left(z_3, k_1\right) \\ y\left(z_3, k_2\right) \\ y\left(z_3, k_3\right) \end{vmatrix}$$

- Now combining this with the two vectors on the previous slide is what will get you the autocorrelation:

$$\left(p * \left(y - \bar{y}\right)\right)' \bullet \left(G \bullet \left(y - \bar{y}\right)\right)$$

- The first bracket product is an element by element product. Dot $(\bullet)$ product is the standard matrix multiplication.