1 Heat Conduction Theoretical Background

The general form of the heat conduction equation in a three-dimensional domain, considering a transient state and anisotropic thermal conductivity (k_x, k_y, k_z), is:

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right) - q = \rho c \frac{\partial T}{\partial t}, \tag{1}$$

where:

- T is the temperature field,
- k_x, k_y, k_z are the thermal conductivities in the x, y, z directions, respectively,
- q is the volumetric heat generation rate ($\mathrm{W/m^3}$),
- ρ is the material density,
- c is the specific heat capacity,
- $\frac{\partial T}{\partial t}$ is the transient term.

Special Cases

1. Transient State, Isotropic Thermal Conductivity

If the thermal conductivity is isotropic ($k_x = k_y = k_z = k$), the equation reduces to:

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) - q = \rho c \frac{\partial T}{\partial t}. \tag{2}$$

2. Steady-State with Heat Generation

In the steady-state case with heat generation, the equation becomes the **Poisson equation**:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} - \frac{q}{k} = 0. \tag{3}$$

3. Transient State, No Heat Generation

In the transient state with no heat generation, the equation becomes the **diffusion equation**:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \tag{4}$$

where $\alpha = \frac{k}{\rho c}$ is the thermal diffusivity.

4. Steady-State, No Heat Generation

In the steady-state case with no heat generation, the equation becomes the **Laplace equation**:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0. \tag{5}$$

2 Heat Conduction in 2D with heat generation – MECE 563 Thermal project

The steady-state heat conduction equation with internal heat generation is given by:

$$k\frac{\partial^2 T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2} - q = 0, \tag{6}$$

where:

- $T(x, y)$ is the temperature field,
- k is the thermal conductivity,
- q in the case of 2D heat conduction is the uniform heat generation per unit area ($\mathrm{W/m^2}$).

The temperature $T(x, y)$ is approximated as:

$$T(x, y) = \sum_{j=1}^{4} \phi_j(x, y) T_j, \tag{7}$$

where:

- $\phi_j(x, y)$ are the shape functions,
- T_j are the nodal temperatures at the four nodes of the rectangular element.

Derivation of Shape Functions from a Polynomial Approximation

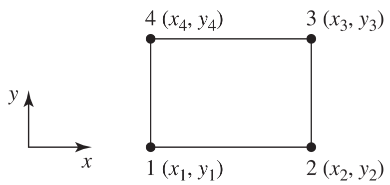
The temperature field $T(x, y)$ is approximated using a bilinear polynomial:

$$T(x, y) = c_o + c_1 \cdot x + c_2 \cdot y + c_3 \cdot x \cdot y, \tag{8}$$

where c_0, c_1, c_2, c_3 are coefficients to be determined.

The nodal conditions are:

$$T(x_1, y_1) = T_1, \quad \text{at Node 1,} \tag{9}$$
$$T(x_2, y_2) = T_2, \quad \text{at Node 2,} \tag{10}$$
$$T(x_3, y_3) = T_3, \quad \text{at Node 3,} \tag{11}$$
$$T(x_4, y_4) = T_4, \quad \text{at Node 4.} \tag{12}$$



The exact behaviors of the shape functions will depend on how you set up your labels. For example, if you start labeling the element nodes starting at the bottom left and proceed in the clockwise direction, the following substitutions can be made in the above equations:

$$\begin{aligned} x_1 = x_4, \quad x_2 = x_3 \\ y_1 = y_2, \quad y_3 = y_4 \end{aligned} \tag{13}$$

$$
\begin{aligned}
T(x_1, y_1) &= T_1, \quad \text{at Node 1}, &(14)\\
T(x_2, y_1) &= T_2, \quad \text{at Node 2}, &(15)\\
T(x_2, y_3) &= T_3, \quad \text{at Node 3}, &(16)\\
T(x_1, y_3) &= T_4, \quad \text{at Node 4}. &(17)
\end{aligned}
$$

Substituting these into $T(x, y)$, we get the following system of equations:

$$
\begin{aligned}
T_1 &= c_0 + c_1 \cdot x_1 + c_2 \cdot y_1 + c_3 \cdot x_1 \cdot y_1, &(18)\\
T_2 &= c_0 + c_1 \cdot x_2 + c_2 \cdot y_1 + c_3 \cdot x_2 \cdot y_1, &(19)\\
T_3 &= c_0 + c_1 \cdot x_2 + c_2 \cdot y_3 + c_3 \cdot x_2 \cdot y_3, &(20)\\
T_4 &= c_0 + c_1 \cdot x_1 + c_3 \cdot y_3 + c_3 \cdot x_1 \cdot y_3. &(21)
\end{aligned}
$$

In matrix form:

$$
\begin{bmatrix} 1 & x_1 & y_1 & x_1 \cdot y_1 \\ 1 & x_2 & y_1 & x_2 \cdot y_1 \\ 1 & x_2 & y_3 & x_2 \cdot y_3 \\ 1 & x_1 & y_3 & x_1 \cdot y_3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}. \tag{22}
$$

Let:

$$
A = \begin{bmatrix} 1 & x_1 & y_1 & x_1 \cdot y_1 \\ 1 & x_2 & y_1 & x_2 \cdot y_1 \\ 1 & x_2 & y_3 & x_2 \cdot y_3 \\ 1 & x_1 & y_3 & x_1 \cdot y_3 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{23}
$$

the above equation can be written in shorthand form as:

$$
A \cdot \mathbf{C} = \mathbf{T}. \tag{24}
$$

To solve for $\mathbf{C}$, we rearrange the matrix equation:

$$
\mathbf{C} = A^{-1} \cdot \mathbf{T}. \tag{25}
$$

which yields the following results for the constants:

$$
\begin{aligned}
c_0 &= \frac{T_3\, x_1\, y_1 + T_1\, x_2\, y_3 - T_2\, x_1\, y_3 - T_4\, x_2\, y_1}{(x_1 - x_2)\,(y_1 - y_3)}\\
c_1 &= -\frac{T_1\, y_3 + T_3\, y_1 - T_2\, y_3 - T_4\, y_1}{(x_1 - x_2)\,(y_1 - y_3)}\\
c_2 &= -\frac{T_1\, x_2 - T_2\, x_1 + T_3\, x_1 - T_4\, x_2}{(x_1 - x_2)\,(y_1 - y_3)}\\
c_3 &= \frac{T_1 - T_2 + T_3 - T_4}{(x_1 - x_2)\,(y_1 - y_3)}
\end{aligned} \tag{26}
$$

Substituting the expressions of $c_0, c_1, c_2,$ and c_3 back into $T(x, y)$ in Equation 8 and rearranging the result as a liner combination of $T_1, T_2, T_3,$ and T_4 yields in the familiar form:

$$
T(x, y) = \phi_1(x, y)T_1 + \phi_2(x, y)T_2 + \phi_3(x, y)T_3 + \phi_4(x, y)T_4, \tag{27}
$$

where:

$$
\begin{aligned}
\phi_1(x, y) &= \frac{(x_2 - x)(y_3 - y)}{(x_2 - x_1)(y_3 - y_1)}, &(28)\\
\phi_2(x, y) &= \frac{(x - x_1)(y_3 - y)}{(x_2 - x_1)(y_3 - y_1)}, &(29)\\
\phi_3(x, y) &= \frac{(x - x_1)(y - y_1)}{(x_2 - x_1)(y_3 - y_1)}, &(30)\\
\phi_4(x, y) &= \frac{(x_2 - x)(y - y_1)}{(x_2 - x_1)(y_3 - y_1)}. &(31)
\end{aligned}
$$

Weak Form of the Governing Equation

To derive the finite element formulation, multiply the governing equation by a test function $\phi_i(x, y)$ and integrate over the domain Ω:

$$\int_\Omega \phi_i \left(k\frac{\partial^2 T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2} - q \right) dx\, dy = 0. \tag{32}$$

Using integration by parts to transfer derivatives from $T(x, y)$ to $\phi_i(x, y)$, the weak form becomes:

$$\int_\Omega k\frac{\partial \phi_i}{\partial x}\frac{\partial T}{\partial x}\, dx\, dy + \int_\Omega k\frac{\partial \phi_i}{\partial y}\frac{\partial T}{\partial y}\, dx\, dy - \int_\Omega q\phi_i\, dx\, dy = 0. \tag{33}$$

Substituting $T(x, y) = \sum_{j=1}^{4} \phi_j(x, y)T_j$:

$$\int_\Omega k\frac{\partial \phi_i}{\partial x}\sum_{j=1}^{4}\frac{\partial \phi_j}{\partial x}T_j\, dx\, dy + \int_\Omega k\frac{\partial \phi_i}{\partial y}\sum_{j=1}^{4}\frac{\partial \phi_j}{\partial y}T_j\, dx\, dy - \int_\Omega q\phi_i\, dx\, dy = 0. \tag{34}$$

Rearranging, the equation becomes:

$$\sum_{j=1}^{4} T_j \int_\Omega k\left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x}\, dx\, dy + \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y}\, dx\, dy\right) - \int_\Omega q\phi_i\, dx\, dy = 0. \tag{35}$$

The first integral term is the i^{th} row and j^{th} column component of the stiffness matrix and is designated by the shorthand K_{ij} for readability. The second term is the nodal heat generation F_i, i.e.,

$$\sum_{j=1}^{4} T_j K_{ij} - F_i = 0. \tag{36}$$

where: K_{ij} is the stiffness matrix:

$$K_{ij} = \int_\Omega k\left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y}\right) dx\, dy. \tag{37}$$

and F_i is the thermal load at node i:

$$F_i = \int_\Omega q\phi_i\, dx\, dy. \tag{38}$$

For a rectangular element with domain (x_1, y_1) to (x_2, y_3), the stiffness matrix and load vector entries are computed as:

$$K_{ij} = \int_{x_1}^{x_2}\int_{y_1}^{y_3} k\left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y}\right) dy\, dx, \tag{39}$$

$$F_i = \int_{x_1}^{x_2}\int_{y_1}^{y_3} q\phi_i\, dy\, dx. \tag{40}$$

For example, for $i = 1$ and $j = 3$, K_{ij} and F_i are given by:

$$\begin{aligned}
K_{13} &= \int_{x_1}^{x_2}\int_{y_1}^{y_3} k\left(\frac{\partial \phi_1}{\partial x}\frac{\partial \phi_3}{\partial x} + \frac{\partial \phi_1}{\partial y}\frac{\partial \phi_3}{\partial y}\right) dy\, dx, \\
&= \int_{x_1}^{x_2}\int_{y_1}^{y_3} k\left(\frac{\partial}{\partial x}\left[\frac{(x_2 - x)(y_3 - y)}{(x_2 - x_1)(y_3 - y_1)}\right]\frac{\partial}{\partial x}\left[\frac{(x - x_1)(y - y_1)}{(x_2 - x_1)(y_3 - y_1)}\right]\right. \\
&\qquad \left. + \frac{\partial}{\partial y}\left[\frac{(x_2 - x)(y_3 - y)}{(x_2 - x_1)(y_3 - y_1)}\right]\frac{\partial}{\partial y}\left[\frac{(x - x_1)(y - y_1)}{(x_2 - x_1)(y_3 - y_1)}\right]\right) dy\, dx. \\
&= -\frac{k\left(x_1{}^2 - x_2{}^2\right) + 2(y_1{}^2 - y_3{}^2)}{6\left(x_1 - x_2\right)\left(y_1 - y_3\right)}
\end{aligned} \tag{41}$$

$$
\begin{aligned}
F_1 &= \int_{x_1}^{x_2} \int_{y_1}^{y_3} q \, dy \, dx, \\
&= \int_{x_1}^{x_2} \int_{y_1}^{y_3} \frac{(x_2 - x)(y_3 - y)}{(x_2 - x_1)(y_3 - y_1)} \, dy \, dx. \\
&= \frac{q \, (x_1 - x_2) \, (y_1 - y_3)}{4}
\end{aligned} \tag{42}
$$

Remember that the product $(x - x_1)(y - y_1)$ represents the area of the element A_e. The expressions for K_{13} and F_3 become:

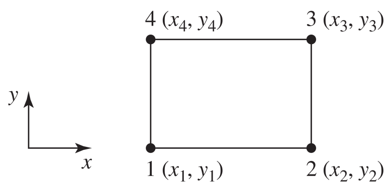
$$
K_{13} = - \frac{k \left({x_1}^2 - {x_2}^2 \right) + 2({y_1}^2 - {y_3}^2)}{6 \, A_e} \tag{43}
$$

$$
F_1 = \frac{q \, A_e}{4} \tag{44}
$$

If we designate the width and height of the element as:

$$
a = x_2 - x_1, \tag{45}
$$

$$
b = y_3 - y_1, \tag{46}
$$



the element stiffness matrix could be written in terms of a and b as:

$$
\mathbf{K_{(e)}} = \begin{bmatrix}
\frac{k\left(a^2+b^2\right)}{3ab} & \frac{k\left(a^2-2b^2\right)}{6ab} & -\frac{k\left(a^2+b^2\right)}{6ab} & -\frac{k\left(2a^2-b^2\right)}{6ab} \\
\frac{k\left(a^2-2b^2\right)}{6ab} & \frac{k\left(a^2+b^2\right)}{3ab} & -\frac{k\left(2a^2-b^2\right)}{6ab} & -\frac{k\left(a^2+b^2\right)}{6ab} \\
-\frac{k\left(a^2+b^2\right)}{6ab} & -\frac{k\left(2a^2-b^2\right)}{6ab} & \frac{k\left(a^2+b^2\right)}{3ab} & \frac{k\left(a^2-2b^2\right)}{6ab} \\
-\frac{k\left(2a^2-b^2\right)}{6ab} & -\frac{k\left(a^2+b^2\right)}{6ab} & \frac{k\left(a^2-2b^2\right)}{6ab} & \frac{k\left(a^2+b^2\right)}{3ab}
\end{bmatrix} \tag{47}
$$

Notice that $ab = A_e$ is simply the area of the rectangular element, which further simplifies the stiffness matrix and force vector as:

$$
\mathbf{K_{(e)}} = \frac{k}{6A_e} \begin{bmatrix}
2\left(a^2+b^2\right) & \left(a^2-2b^2\right) & -\left(a^2+b^2\right) & -\left(2a^2-b^2\right) \\
\left(a^2-2b^2\right) & 2\left(a^2+b^2\right) & -\left(2a^2-b^2\right) & -\left(a^2+b^2\right) \\
-\left(a^2+b^2\right) & -\left(2a^2-b^2\right) & 2\left(a^2+b^2\right) & \left(a^2-2b^2\right) \\
-\left(2a^2-b^2\right) & -\left(a^2+b^2\right) & \left(a^2-2b^2\right) & 2\left(a^2+b^2\right)
\end{bmatrix} \tag{48}
$$

$$
\mathbf{F_{(e)}} = \frac{qA}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{49}
$$

The resulting system of equations for the element is:

$$
\mathbf{K_{(e)}} \cdot \mathbf{T_{(e)}} = \mathbf{F_{(e)}}, \tag{50}
$$

where:

- $\mathbf{K_{(e)}}$ is the stiffness matrix of the element,
- $\mathbf{T_{(e)}} = [T_1, T_2, T_3, T_4]^T$ is the vector of nodal temperatures for the element,
- $\mathbf{F_{(e)}}$ is the load vector.

Matlab Code

The following code generates the components of the stiffness matrix and the force vector.

```
clear all
clc

% Define your symbols:
syms c_0 c_1 c_2 c_3 real
syms x_1 x_2 x_3 x_4 y_1 y_2 y_3 y_4 real
syms T_1 T_2 T_3 T_4 real
syms x y real
syms q k real
syms A_e real
syms w h real
% Write down the equations:

% Define the coefficient matrix A and the vector B
A = [1, x_1, y_1, x_1*y_1;
     1, x_2, y_2, x_2*y_2;
     1, x_3, y_3, x_3*y_3;
     1, x_4, y_4, x_4*y_4]

A = subs(A, [x_3, x_4, y_2, y_4], [x_2, x_1, y_1, y_3])

T_vector = [T_1; T_2; T_3; T_4]

abcd_solution = simplify(A \ T_vector);

% Assign the solutions for clarity
c0_sol = abcd_solution(1)
c1_sol = abcd_solution(2)
c2_sol = abcd_solution(3)
c3_sol = abcd_solution(4)

T_bar = c0_sol + c1_sol*x + c2_sol*y + c3_sol*x*y;

Phi1 = simplify(diff(T_bar, T_1))
Phi2 = simplify(diff(T_bar, T_2))
Phi3 = simplify(diff(T_bar, T_3))
Phi4 = simplify(diff(T_bar, T_4))

phi_vector = sym(zeros(4,1));

for i = 1:4
    phi_vector(i) = diff(T_bar, T_vector(i));
end

K = sym(zeros(4,4));
for i = 1:4
    for j = 1:4
        int_x = simplify(int(k*(diff(phi_vector(i),x) * diff(
            phi_vector(j), x) + diff(phi_vector(i),y) * diff(
            phi_vector(j), y)), x, x_1, x_2));
        int_y = int(int_x, y, y_1, y_3);
```

```
        K(i,j) = int_y;
    end
end
%K = subs(K, [(x_1-x_2), (y_1-y_3)], [w,h])
K = simplify(K, 20)

F = sym(zeros(4,1));
for i = 1:4
    int_x = simplify(int(phi_vector(i)*q, x, x_1, x_2));
    int_y = int(int_x, y, y_1, y_3);
    F(i) = int_y;
end
F = simplify(F)
```