Firm-to-Firm Trade and Growth with Long-Term Relationships*

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Abstract

In the face of weak formal contract enforcement, firms that use relationship-specific inputs may engage their suppliers in relational contracts, rewarding suppliers for good performance by remaining in the relationship for longer. This has the static benefit of improving the supplier performance, but the dynamic cost that when a new supplier that can offer its variety at a low cost comes along, the buyer may not switch because of its promise to remain in the contract. This paper explores the implications of such a response to weak formal enforcement on firm dynamics. We build a model of firm dynamics where the dynamics arise from the arrival of new potential firm-to-firm relationships. We show that when relational contracts are used to mitigate weak formal enforcement, firm dynamics are slower, in the sense that firm sales are less volatile, there is less mean reversion, exit rates are lower, and the right tail of the firm size distribution is thinner. We corroborate these predictions with production data on Indian manufacturing plants, using variation across states in court congestion as a proxy for weak formal enforcement and variation across industries in whether the output requires customization. Finally, we show that the dynamic cost is significant, with the increased court congestion between the state with the fastest courts and the state with the slowest courts reducing aggregate productivity by roughly 15%.

KEYWORDS:
JEL:

*We thank...

1 Introduction

In many developing countries, industries are characterized by a lack of dynamism. Small firms, even when productive, remain small while large incumbents tend to stay large, and firms rarely exit the market (Hsieh and Klenow, 2014, ...). In this paper we explore one mechanism that may contribute to this phenomenon: firms choose to engage in long-term relational contracts with suppliers to overcome hold-up problems as a substitute for contract enforcement by formal judicial institutions. While these relational contracts have the virtue of improving performance within relationships, they may inhibit firms from switching to cheaper suppliers (Johnson, McMillan and Woodruff, 2002, Hémous and Olsen, 2018). If firms are reluctant to switch suppliers, young, productive firms may not grow as fast as they otherwise would because potential customers tend to stay in their current relationships. At the same time, older large firms with many customers will keep those customers for longer. Exit rates and the volatility of firms growth rates will be lower as well; long term relationships reduce dynamism.

In this paper we attempt to understand and quantify the role of relational sourcing for firm dynamics and for allocative efficiency in the aggregate economy. We exploit data from India's manufacturing sector to show that when firms are subject to hold-up problems due to the relationship-specific nature of their products and their institutional environment, the dynamism of firm growth and relationship dynamics is depressed: not only are average growth rates of small, young firms lower, but the *variance* of growth rates is also lower, and the size distribution of firms is less skewed. We supplement these results with data from Pakistan's firm-to-firm transactions data to show that in such situations relationships between firms are on average longer, indicating that firms are more likely to source relationally. The use of relational contracts, while providing static benefits, depresses dynamism and leads to lower rates of reallocation (growth, contraction, and exit).

In Section 2 we construct a model of firm dynamics that features relationships between buyers and suppliers, which we argue are key for understanding the patterns of firm dynamics that are present in the data. The model features a continuum of firms that each draw suppliers for their inputs and need to decide when to switch to new suppliers. In general this decision would require taking not only the current supplier's and the new supplier's productivity into account, but also those of the entire chain of suppliers, since cost reductions are passed on to suppliers. Under a set of assumption about the matching and productivity process the cost of production becomes a random walk, which makes the supplier decision tractable because only the current cost of the current and alternative suppliers need to be compared. New firms are born continually, so that the distribution of cost is ergodic. Demand for firms' output arises endogenously from firms being chosen as suppliers. The model generates predictions on the probability of exit, which is what happens when a firm loses its last buyer.

The model is parsimonious in the sense that there is just a single shock that drives all of the dynamics. Despite this simplicity, the model is consistent with a range of empirical regularities that are not present in canonical firm dynamics models. This happens because the shock plays a number of different roles, depending on a firm's relationship to the location of the shock. For example: cost shock for buyer, demand shock to supplier. intensive margin for supplier's supplier. important to a firm with one buyer, not so important for another buyer.

In Section 3, we show the model is consistent with a number of empirical regularities about firm dynamics. Firm volatility declines with size, but more slowly than one would expect if the firm were simply the sum of independent components of similar size. The distribution of firm growth rates has fat right and left tails. Large firms occasionally exit, but the exit rate declines smoothly with size and approaches zero as firm size goes to infinity.

Section 4 introduces contracting frictions and relational contract and describes the model's predictions for firm dynamics. We show that more severe contracting frictions reduce firm volatility, reduce the rate of mean reversion, reduce the exit rate, and reduce the thickness of the right tail of the firm size distribution.

Section 5 uses production data on Indian manufacturing plants and transactions data on Pakistani firms to test these predictions. We use court congestion as a proxy for weak contract enforcement. We show that more congested courts have a larger impact on firm dynamics in industries that produce goods that are relationship-specific ((Rauch, 1999). In line with the models predictions, we find that more severe contracting frictions reduce firm volatility, reduce the rate of mean reversion, reduce the exit rate, and reduce the thickness of the right tail of the firm size distribution.

Finally, Section 6 studies the impact of slow firm dynamics induced by weak contract enforcement on aggregate output. The model predicts that contracting frictions weaken the relationship between firm cost and firm size, as firms become less likely to switch to good suppliers. We find that in the state with the most congested court, output is 15% lower than it would be if its courts congestion were reduced to the level of the state with the least congested courts.

1.1 Related Literature

A number of paper study the impact of court congestion on contract enforcement, in India ((Boehm and Oberfield, 2020), (Amirapu, 2017), (Chemin, 2012)) and elsewhere.

In our model, firms grow by accumulating customers, as in Luttmer (2011). This is consistent with Foster, Haltiwanger and Syverson (2016), who find that firms grow by accumulating demand over time. Here, demand is simply customers that choose to buy a firm's good.

Hsieh and Klenow (2014) have documented slow firm dynamics in India. Akcigit, Alp and Peters (2021) posit that this is caused by delegation frictions that make it costly for some firms to grow. In our model, the relational contracts slows the reallocation of customers across firms, similar to how firing costs slow the reallocation of workers across firms in Hopenhayn and Rogerson (1993).

Our paper builds on the literature on relational contracting, such as Kranton (1996), Macchiavello and Morjaria (2015), Macchiavello and Morjaria (2021).

Most of the literature on firm-to-firm trade that has focused on the patterns of firm heterogeneity have done so in a static environment Oberfield (2018), Bernard, Moxnes and Ulltveit-Moe (2018), Eaton, Kortum and Kramarz (2022), Bernard et al. (2022). Here dynamics are critical because can't talk about long term relationships without it, and get a lot of extra information from dynamics like variance of growth rates and exits.

Chaney (2014) has firm to firm trade, dynamics, size distribution. In his model, firms have a continuum of customers, so size evolves deterministically.

Most of the firm-to-firm with dynamics has been focused on understanding the pass through of shocks Lim (2018) and Huneeus (2018), and do not focus on firm dynamics. Miyauchi (2018) has some dynamics, mostly focused on recovery from shocks. Martin, Mejean and Parenti (2023) and Fontaine, Martin and Mejean (2023) incorporate frictions to adjustment of firm-to-firm relationships to study the the impact of trade shocks.

A number of papers have documented that switching of suppliers is relatively frequent ((Gopinath and Neiman, 2014), (Lu, Mariscal and Mejía, 2024), (Damijan, Konings and Polanec, 2014)), that it is slower for relation-specific inputs ((Monarch, 2022)), and that it is an important channel for changes in a firm's cost ((Baqaee et al., 2023)).

In our model, the thickness right tail of the firm size distribution is endogenous. Kwon, Ma and Zimmermann (2023) and Chen (2023) have documented that this varies over time and space.

2 Simple Model

The economy consists of a representative household, firms that produce, and retailers. Firms produce goods using labor and intermediate inputs, and sell their output to other firms and to retailers. Retailers purchase goods from firms and compete monopolistically in selling goods to the household. The household supplies labor to firms and purchases goods from retailers.

We first describe an environment in which the economy consists of a single industry of firms. In Section 2.6 we extend the model to allow for many industries.

2.1 Firms

A firm is a variety. To produce, a firm uses a technique. A technique is a triple: a buyer, b, a supplier, s, and a match-specific productivity, z. A technique is a production function for the buyer

$$y_b = A(z_{bs}x_s)^{\alpha}l^{1-\alpha}, \qquad A \equiv \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$$

where y_b is output of the buyer's good, x_s is the units of supplier's good used as an intermediate input, and l is labor.

At a point in time, a firm has access to a single technique, but is constantly searching for new, better techniques. When it discovers a new technique, it can either switch to the new one or remain with its old one.

There is a representative household with a labor endowment that grows at rate γ, so that the measure of labor available at t is $L_t = L_0 e^{\gamma t}$. Labor can be used for production or to create new firms. If a fraction ρ of labor is used to create firms, $\rho L_t \chi$ new firms are created.

Each new firm is endowed with a single technique with supplier randomly drawn and match-specific productivity drawn from a distribution with distribution function Z_0.

There is no overhead cost, so firms have no reason to exit. But a firm may have no customers, in which case it will have no employment or revenue. However, if such a firm later acquires a customer, its revenue will be positive again.

2.2 Static Equilibrium

We begin by characterizing a static equilibrium, and then proceed to characterize a balanced growth path.

We focus on a static equilibrium concept in which there is monopolistic competition among retailers in selling to the household and stable contracting arrangements among firms and retailers.

A contracting arrangement consists of a quantity and a payment for each buyer-supplier pair. A contracting arrangement is stable with respect to a coalition of firms if the firms in the coalition do not wish to change the terms of the contracts among them and do not wish to drop contracts with others that are not in the coalition. A contracting arrangement is countably stable if it is stable with respect to any countable coalition of firms.

Proposition 1 shows that in any countably stable contracting arrangement, contracts are such that there is no double marginalization.

Proposition 1 *In any stable contracting arrangement, for any buyer supplier pair, the buyers marginal cost is*

$$c_b = \left(\frac{c_s}{z_{bs}}\right)^{\alpha} w^{1-\alpha}$$

There are many stable equilibria that have the same marginal costs and same allocation of labor and intermediate goods, but differ in how profit is split across buyers and suppliers. We assume here that the supplier's bargaining power is α. That is the supplier gets a fraction α of the profit of the buyer.

Suppose a buyer b uses supplier s_1, and that supplier s_1 uses supplier s_2, etc. These buyer-supplier pairs form a supply chain. We define the supply chain's productivity to be

$$q = z_{b,s_1} z_{s_1,s_2}^{\alpha} z_{s_2,s_3}^{\alpha^2} \dots$$

In any stable equilibrium, a firm that uses a supply chain with productivity q will have marginal cost wq^{α}.

If a firm spends wl_j on labor, its total revenue is $\frac{\varepsilon}{\varepsilon-1}\frac{1}{1-\alpha}wl_j$ and its payment to its supplier is $\frac{\varepsilon}{\varepsilon-1}\frac{\alpha}{1-\alpha}wl_j$, so its static profit is $\frac{1}{\varepsilon-1}wl_j$.

2.3 Dynamics

For each firm, new techniques arrive randomly according to a Poisson process. The identity of the supplier is randomly drawn from all existing firms. The match-specific productivity is random.

We assume that when a firm gets a new potential supplier, there is a spillover from its existing match. In particular, we follow Buera and Oberfield (2020) in assuming that the new match specific productivity is the product of an original component, b, and a component that is inspired by the firm's existing supply chain. We assume, that the arrival rate of new techniques with original component great than z is $\kappa b^{-\beta}$.

As a result, the arrival rate of a new supplier that offers an reduction in effective cost larger than x is

$$\phi x^{-\beta}$$

where $\phi \equiv \kappa \int \left(\frac{c}{w}\right)^{-\beta} dF(c)$, where $F(c)$ is the fraction of firms in the cross-section with cost weakly less than c.

There is no recall. Once a buyer switches to a new supplier, there is no option to switch back to the supplier it left.

2.3.1 Discussion of Model Assumption

The mechanics of the model are governed by the random arrival of techniques and one key economic decision: when a new potential supplier arrives, should the firm switch or remain with its current supplier? In principle this is a complicated decision because a firm's state is extraordinarily large.

The main impediment to a simple decision rule is mean reversion in cost. To understand why mean reversion can cause problems, consider the following example. A firm b is choosing between two potential suppliers, its new supplier s_1 and a new potential supplier s_2, that respectively offer the buyer effective costs c_1/z_{bs_1} and c_2/z_{bs_2}. With mean reversion in cost, it might be the case that the supplier who offers a lower effective cost now is likely to offer a higher effective cost in the future. A buyer thinking about which of those suppliers to choose will have to weigh its current demand relative to future demand. If the buyer has a low effective cost but few customers, so that its future demand is expected to be higher than current demand, it may value a low future effective cost more than a low current effective cost. Or if it has a high current cost so that it does not expect to gain many new customers, but many current customers, it may place a high value on current effective cost cost. And the decision gets even more complicated when one considers whether a firm expects its customers' customers to grow, etc.

The critical features that keeps the decision simple and tractable are modeling assumptions that ensure that each firm's marginal cost follows a random walk. In that case, a firm would adopt a new supplier if it offers a lower effective cost. In that case, a firm should simply choose the supplier that offers a lower effective cost, as the distribution of that supplier's effective cost at any point in the future would first-order-stochastically dominate that of the potential supplier currently offers a higher effective cost. Thus there is never any trade-off between current and future effective cost. When comparing two potential suppliers, current efficiency delivered is a sufficient statistic for stochastic dominance of that supplier in the future.

Many of the modeling choices have been made with an eye toward ensuring that firm's cost follow random walk. The spillovers from current matches to the match productivity of a new match ensure that the arrival rate of new suppliers that deliver a given cost reduction is independent of current cost. And if each supplier's cost follows a random walk, then the distribution of cost reductions further upstream are independent of current costs. The assumption that there is no recall is also important. If a supplier could recall one of its suppliers, than a buyer would want to know whether a supplier had a close alternative to *its* suppliers when choosing whether to switch. We also assumed that a firms never die, when they have no customers they are only dormant. We say a firm exits when it has no customers. In this way, exit does not introduce any mean reversion in cost.

2.4 Changes in Cost

A firm's production cost falls if it finds a new better supplier, if its supplier finds a new better supplier, or if any supplier in the firm's supply chain finds a new better supplier. Since supply chains are infinite, there are an infinite number of events that can reduce the firm's cost.

A firm's cost thus follows a stochastic process called a jump process with infinite activity. In any strictly positive interval of time, almost surely an infinite number of supplier's in the firm's supply chain will find new suppliers, reducing the firm's cost. Nevertheless, most of these events are so far upstream that they have only a small impact on the firm's cost, as the new suppliers account for such a small fraction of the supply chain's value added. That being said, when the firm itself finds a new supplier, or if a supplier not too far upstream finds a new supplier, this can have a big effect on the firm's cost. Altogether, even though there are an infinite number of potential events, the cumulative impact of these events is a well behaved distribution of changes in cost.

We now characterize those changes in cost. For $x \geq 1$, let $M(x,t)$ be the probability that a firm's cost declines by a factor weakly less than x in an interval of length t. Because changes in cost accumulate geometrically, it will be useful to work with the Mellin transforms of these distributions. In particular, define $\varphi^M(s,t)$ to be the Mellin transform of $M(x,t)$, i.e., $\varphi^M(s,t) \equiv \int_1^\infty x^{-s} M_x(x,t)dx$.

Lemma 1 $\varphi^M(s,t) = e^{-\phi t \sum_{k=0}^{\infty} \frac{s}{\beta\alpha^{-k}+s}}$

The kth term in the summation reflects the possibility of there being a new supplier k steps upstream.

At short horizons, the distribution of changes is lumpy. At longer horizons, the suitably normalized distribution of changes converges to a standard normal, in accordance with the central limit theorem.

Claim 1 *Let $X_j(t)$ be the proportional decline of firm j's cost. As t grows large,*

$$\frac{1}{\sqrt{t}}\left(\frac{\log X_j(t)-\frac{\phi t}{\beta(1-\alpha)}}{\frac{2\phi}{\beta^2(1-\alpha^2)}}\right)$$

converges in distribution to a standard normal random variable.

Recall that, along a BGP, the measure of entrants grows at rate γ Let $F(c)$ be the fraction of firms with cost weakly less than c. Suppose that the distribution of new firms that arrive has CDF $F_0(c)$. Let φ^F and φ_0^F be their respective Mellin transforms.

Proposition 2 *Mellin transform of F is*

$$\varphi^F(s)=\frac{\gamma}{\gamma+\phi\sum_{k=0}^{\infty}\frac{s}{s-\beta\alpha^{-k}}}\varphi_0^F(s)$$

Let ν be the unique solution of $\frac{\gamma}{\phi}=\sum_{k=0}^{\infty}\frac{\alpha^k}{\beta/\nu-\alpha^k}$. Then if the distribution of cost among newborn firms has a sufficiently thin left tail, i.e., $\lim_{c\to 0}F_0(c)/c^{\nu}=0$, then the left tail of the distribution of cost in the cross-section has a power law left tail

$$\lim_{c\to 0}\frac{\log F(c)}{\log c}=\nu.$$

2.5 Retailers

Retailers purchase goods from a single firm and sell their goods to the household. The household has Dixit-Stiglitz preferences across retailers, with elasticity of substitution $\varepsilon>1$. The retailers compete monopolistically in sales to the household. At any point, each retailer has a single supplier (a producing firm). Over time, new potential suppliers arrive in exactly the same manner as they do for firms. Retailers have the same production function as firms except that the their labor share is zero, so that the retailer's cost is the supplier's production cost divided by the match-specific productivity.

2.6 Many Industries

We now introduce multiple industries. In each industry $\omega\in\Omega$, output is produced using labor and a fixed set of intermediate inputs according to the Cobb-Douglas production function

$$y_{\omega b}=A_\omega l^{\alpha_{\omega l}}\prod_{\omega'\in\Omega}\left(z_{\omega b,\omega' s'}x_{\omega' s'}\right)^{\alpha_{\omega\omega'}}$$

with $\alpha_{\omega l}+\sum_{\omega'}\alpha_{\omega\omega'}=1$ and $A_\omega\equiv\alpha_{\omega l}^{-\alpha_{\omega l}}\prod_{\omega'}\alpha_{\omega\omega'}^{-\alpha_{\omega\omega'}}$. That is, for each industry ω', the firm has a supplier s' and a match-specific productivity $z_{\omega b,\omega' s'}$ specific to using that supplier.

Retailers now purchase one input from each industry and combines them according to a Cobb-Douglas production function.

At any point in time, each firm has a single supplier in each supplying industry. For each supplying industry, new potential suppliers arrive randomly and independent of the arrival in other supplying industries. Each new potential supplier comes with a match-specific productivity with an original component and a component inspired by the firm's current supply chain for that input. With multiple industries, the efficiency of the current supply chain for input ω can be expressed recursively according to

$$q_{\omega' s'} = \prod_{\omega''} \left(z_{\omega' s', \omega'' s''} q_{\omega'' s''} \right)^{\alpha_{\omega', \omega''}}$$

This formulation remains tractable because within each industry, the log cost of every firm follows the same random walk (although the stochastic process for changes in cost differs across industries). To see why, note that a weighted average of random walks is still a random walk. If the weights are the same for all firms in an industry, then the log cost of every firm in an industry will evolve according to the same random walk. This happens provided that all firms in each industry produce using the same Cobb-Douglas production function.

3 Firm Dynamics with Firm-to-Firm Trade

In this section we describe several features of the model. In particular we show that the model is consistent with a number of well-documented empirical regularities

The model is simple in the sense that there is just one type of shock in the model. Size depends on how many customers a firm has and how big those customers are, which itself depends on how many customers those customers have, how big those customers' customers are, etc. The number of customers a firm depends on how many potential customers have arrived, which is random with an arrival rate that is uniform across firms, how many of those potential customers chose to switch to the firm, which depends on the firm's cost (which evolves over time) and the draw of match specific productivity, which is random, and how long that customer chooses to stay with the firm. Changes in size come from gaining or losing customers or customers growing and shrinking, which comes from them gaining or losing customers, etc.

Again, all this is driven by a single shock, the arrival rate of a new potential match. This is a supply shock to firms downstream from the potential match and a demand shock for firms upstream from the potential match.

The patterns of firm size, growth, and survival are quite rich. In this section we show that the model matches a number of well-documented stylized facts about firm dynamics that canonical models of firm dynamics do not speak to.

3.1 Calibration

This section describes our calibration, as summarized by Table I.

Parameter	Value	Target	Target value	Data source
Population growth (γ)	0.04	Employment share by age		Hsieh & Klenow (2014)
New technique shape (β)	3.52	Cost reduction from new suppliers	-0.284	Baqaee et al. (2023)
New supplier arrival rate (ϕ)	0.58	Mean relationship length	1.72 years	Pakistan data
Observation threshold	varies	$\frac{\text{Median sales above threshold}}{\text{Threshold}}$	6.36	Pakistan data
Number of retailer firms ratio	60	Annual exit probability	0.05	
Household EoS (ε)	4			

Table I Parameterization

ϕ is the arrival rate of new suppliers which reduce effective cost. So if firms swap suppliers at this rate (i.e., they all suppliers who improve cost), then the mean relationship length is $1/\kappa$. We set $1/\kappa$ to match a mean relationship length of 1.72, taken from the Pakistan data.

We turn next to β. Recall that when a firm gets a new supplier, the reduction in effective cost follows a Pareto distribution with shape parameter β. Denote the effective cost of an input as λ. The expected change in log cost is $\mathbb{E}[\log(\frac{\lambda}{\lambda'})] = \frac{1}{\beta}$. The overall effect on marginal cost is scaled by the firm's expenditure share of that input. Baqaee et. al (2023) estimate an object using Belgian data that is a close match to this. The estimate that when a firm gets a new supplier that represent $x\%$ of the firm's unit cost (for reason's unrelated to changes in the firm's productivity), the firm's log unit cost declines by -0.284. This corresponds to $\beta = 3.52$.[1]

The rate of population growth, γ, which, along a BGP is equal to the rate of growth of the number of firms, determines the shares of industry employment accounted for by old and young firms. We set $\gamma = 0.04$ to match these shares in India as reported by Hsieh and Klenow (2014). See Appendix A.1.1 for details.

Firms in the retail industry are supplied by intermediate industries. We assume that sales the retailer's expenditure shares are the same as each industry's share of final sales. Households have Dixit-Stiglitz preferences across retailers. We set the elasticity of substitution across retailers to 4.

The number of retailers relative to those in other industries is important because it determines how granular each retailer is. If there are many retailers relative to producing firms, then most producers would sell to a large number of retailers. This would mean that most producers would be active and would seldom exit. Further, firms would be diversified enough that gains and losses of retailer customers, limiting the volatility of firm sales. The parameter value ρ_R is the ratio of the number of retail firms to the average number of firms in each non-retail industry. There are 3509 non-retail industries, so there are $\frac{\rho_R}{3509}$ retail firms for every non-retail firm in the rest of the economy. We set ρ_R to match an exit rate of 0.05.

In our data consists of firms that are sufficiently large to pass a statutory threshold for reporting. The threshold varies with time but is the same across all industries. We impose a similar threshold so that the moments we calculate are more easily compared with those in the data. In particular, we set a threshold for firms to be "observed" in the model so that the ratio of median output of all firms above the threshold to the threshold itself is 6.36.

[1]This parameter plays virtually no role in the patterns of firm dynamics, but is important for the magnitude of the losses from misallocation.

We calibrate production functions by constructing input-output tables from Indian ASI data from 2003-2008, focusing on manufacturing industries. We make some minor modifications to the input-output tables that increase the speed with which we can solve the model by making the input output tables more sparse. See Appendix A.2 for details. make the that make the

We describe the simulation procedure in Appendix A.3.

3.2 Volatility and Firm Size

The strong version of Gibrat's law (1931)—that the distribution of firm growth rates is independent of firm size—has been repeatedly rejected. In particular, the large firms tend to be less volatile than small firms. This fact, pointed out as early as Meyer and Kuh (1957), Hymer and Pashigian (1962), and Mansfield (1962), has been corroborated across many different contexts.[2]

Hymer and Pashigian (1962) suggested that a possible explanation for the negative size-volatility relationship is that firms are composed of subunits and larger firms are composed of more subunits. If each subunit has the same volatility, then larger firms would be less volatile because thy are more diversified. This type of mechanism is captured by Klette and Kortum (2004), in which a firm is a collection of products over which the firm has patents, and each product evolves independently.

One challenge for this class of explanations are findings from Koren and Tenreyro (2013) and Yeh (2023). They respectively find that the size-volatility relationship is basically unchanged when comparing firms with the same number of segments (Compustat data) or 6-digit products (US Census data), or among firms with a single segment/product. It is of course possible that the relevant subunits are not well captured by segments/products. Large firms also have more establishments than small firms, which may lead to some diversification across establishment-level shocks, but Yeh (2023) finds that the size-volatility relationship holds even after controlling for the number of establishments.[3]

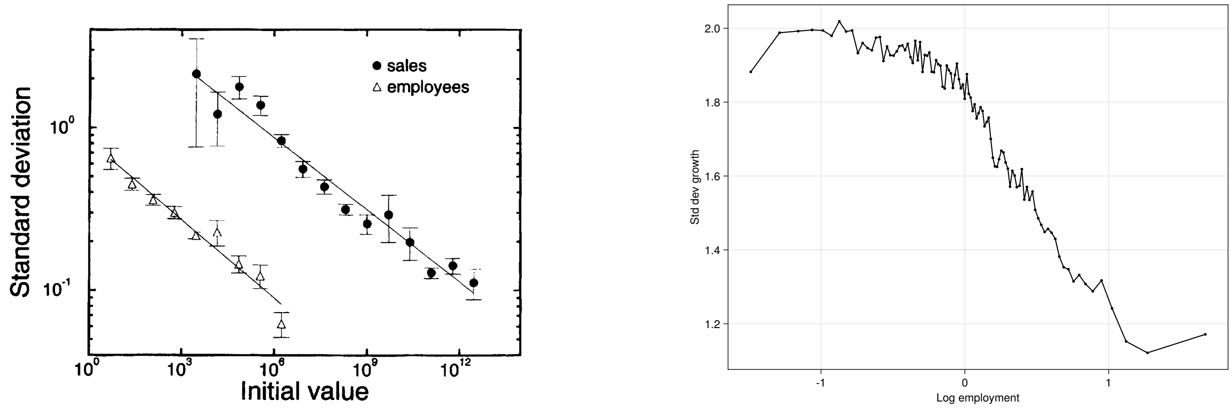
In the model here, a firm's size depends on its sales to its customers. Whether a firm gains or loses one customer is independent of its size, and whether a one customer gains or loses customers is largely independent across customers. As emphasized by Kramarz, Martin and Mejean (2020), since larger firms tend to have more customers they are also more diversified.

A second important feature is that the size variance relationship is less sharp then what would would be predicted by a model where a firm were composed of IID subunits. As noted by Hymer and Pashigian (1962), in that case, volatility would decline at rate of $\sigma(S) \propto S^{-1/2}$. But the rate of decline is typically slower.

Stanley et al. (1996) proposed a log linear relationship, $\sigma(S) \propto S^{-\alpha}$, using firms in Compustat,

[2]See Yeh (2023) and Coad (2007) for good reviews of the literature.

[3]A second explanation is that volatility depends directly on age as it would in a Jovanovic (1982), and the size-volatility relationship simply reflects the correlation of age and size. While volatility does indeed decline with age, Yeh (2023) shows that the size-variance is nearly unchanged when controlling for age, either parametrically or using cohort fixed effects. Yeh (2023) advances a third explanation that the pass through of productivity shocks into price declines with firm size. In this view, IID productivity growth rate shocks translate into less than one-for-one changes in sales/employment because of imperfect pass through of cost into price. There is a demand system with the right match between pass-through and size that delivers the size-variance relationship.



(a) Data

(b) Model

Figure 1 Firm Size and Volatility

Note: Panel (a) shows the relationship between the standard deviation of firm growth rates and firm size among firms in Compustat, taken from Stanley et al. (1996). Panel (b) shows the relationship between firm size and volatility in the calibrated model.

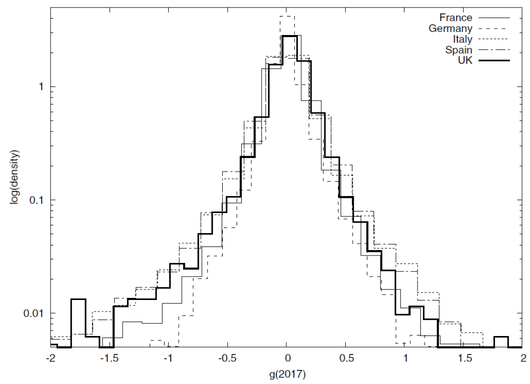
which incorporates only publicly traded firms. When Yeh (2023) used LBD which includes all firms, he found a relationship close to log-linear, although the relationship was weaker among small firms. Part of this may be survivor bias, which is more important for small firms.

We are aware of two types of explanations. First, as suggested by Hymer and Pashigian (1962) and adapted by Stanley et al. (1996), there is some correlation of shocks across units. Sutton (2002) examines segments data of Compustat firms and argues that the correlation of growth rates across segments is too weak to explain the flatness of the relationship.

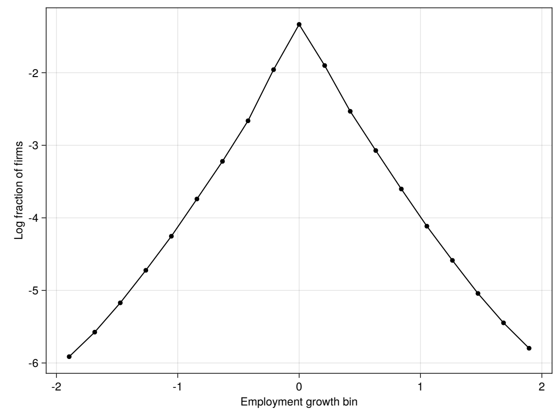
A second explanation, advanced by Amaral et al. (1998) and Sutton (2002), is that individual segments vary in size, with some segments much larger than others. As Gabaix (2011) later explained, when individual components are granular, so that the size distribution of components has a fat tail, the law of large numbers kicks in more slowly, so that the standard deviation of growth rates declines more slowly than $S^{-1/2}$. As an extreme example, if almost all of a firm's sales are from a single subunit, then its volatility will be almost the same as a small firm with a single subunit, no matter how big the large firm is.

In the model, volatility declines with log size at a rate smaller than $1/2$ mostly for this second reason. Customers vary in size. Some larger firms have few customers, and a firm with many customers may have most of its sales accounted for by few customers. While large customers are likely to be less volatile, gaining or losing a large customer is a large event. Those shocks to large customers are not diversified with shocks to smaller customers. In line with Kramarz, Martin and Mejean (2020), a natural prediction of this model is that once one controls for the HHI of customers, the size variance-relationship weakens.[4]

[4] In addition, a second reason for the weaker size variance relationship is that there is some correlation in growth rate across a firm's customers that comes from a firm's own changes in cost. In our calibration, this is less important quantitatively.



(a) Data



(b) Model

Figure 2 The distribution of firm growth rates

Note: Panel (a) shows the distribution of firm growth rates among firms in various countries using data from ORBIS, taken from Bottazzi, Kang and Tamagni (2023). Panel (b) shows the distribution of growth rates in the calibrated model.

3.3 Fat Tails of the Growth Rate Distribution

Another striking feature of firm growth rates is that the distribution features fat tails. This is true for the unconditional distribution, as can be seen in panel (a) of Figure 3.3 as well as conditional on initial size.

Some studies have argued that the distribution of changes in log size follows a Laplace distribution (back to back exponentials), e.g., Stanley et al. (1996), Bottazzi and Secchi (2003, 2006), and Bottazzi, Kang and Tamagni (2023). This would mean that on a log-log plot such as in Figure 3.3, the distribution is piecewise linear. Some studies find even fatter tails than Laplace.

The model here has the property that the distribution of growth rates features fat tails. Extreme increases or decreases in size can come from gaining or losing a very large customer. Thus even conditional on size, there can be extreme growth events.

3.4 Firm Exit

Another well-known fact about firm dynamics is that larger firms are more likely to survive than smaller firms, Mansfield (1962), Hall (1987), Evans (1987*b*,*a*). Some models, such as Hopenhayn (1992) or Luttmer (2007), feature a simple cutoff rule, so that a firm exits when its size drops below the cutoff. This pattern of exit predicted by these models is, of course, much more stark than data. In the data, smaller firms are, in fact, more likely to exit. But there are many very small firms that survive, and some very large firms that exit.

A number of papers that intend to capture this in a simple way by incorporating both exogenous exit with a fixed opportunity cost of operations, and an exogenous death shock that is IID across firms and over time, which causes a firm to exit. Again, this does not capture the gradual decline

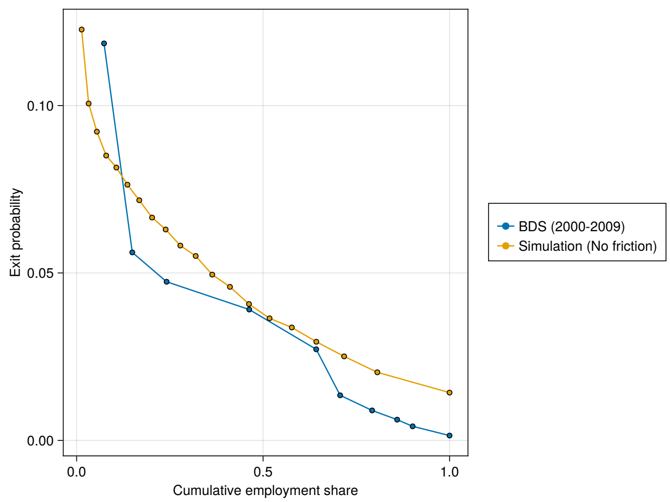


Figure 3 Firm Size and Probability of Exit

Note: This figure shows exit rates for US firms using BDS data and exit rates as predicted by the calibrated model.

of exit rates with size.

In this model, a firm exits once its last customer switches to an alternative supplier. Some large firms may exit if they are large because they have one large customer, and the customers switches away to a new supplier. But most large firms will have many customers, so the probability of exit is small. The probability of exit is smoothly declining in firm size because the probabilities of having a single customer and of not gaining any additional customers are both smoothly declining in firm size.

This logic is also used by Garcia-Macia, Hsieh and Klenow (2019) who use a model in the spirit of Klette and Kortum (2004) but with dispersion in market size across products. They observe only survival and changes in employment, but they discipline the dispersion in market size across products by looking at how the exit rate varies with firm size.

3.5 Type Dependence

Many models of the firm size distribution or of the wealth distribution have trouble matching the quickness with which individuals end up in the right tail (Luttmer, 2011, Gabaix et al., 2016).[5] Luttmer (2011) proposed introducing a persistent (but not permanent) state that he labeled quality that determined the firm's growth rate. Gabaix et al. (2016) call this heterogeneity in expected growth rates "type dependence." This made it possible for some young firms to move into the right tail of the firm size distribution quickly. And random transitions from high quality to low quality made it possible for the right tail of firm size distribution to follow a power law.

[5] According to Luttmer (2011), the median age of firms above 10,000 employees is 75 years. He argues that a model in which the volatility of firm growth is independent of size can get close to this when the volatility is calibrated to that of small firms, but models that match the volatility of large firms predict that the largest firms would be much older than in the data.

The model here naturally features this kind of "type dependence." Firms differ in expect growth rates because they differ in production cost; a firm with a low production cost expects to acquire customers more quickly. Further, as in Luttmer (2011), a firm's type—its cost—is persistent but not permanent.

3.6 Persistence and Firm Age

Many have estimated the persistence of firm growth rates, and it is safe to say that there are is no consensus on whether firm growth rates are positively or negatively autocorrelated. See Coad (2007). Recent work has extracted more subtle patterns. For example, Coad, Daunfeldt and Halvarsson (2018) find that autocorrelation is positive for young firms but negative for older firms.

This naturally arises in the model if initial differences in costs across newborn firms are large. In general, firms expand by adding customers, which depends on the firm's production cost, and contract when customers leave, which depends on the number fo customers. When the rate of inflow of customers is equal to the outflow, the autocorrelation of growth rates is likely to be negative. The rate of outflow depends on the number of customers. Given a firm's unit cost, there is a number of customers that will make it so that the rate of inflow of customers will equal the outflow rate. If a firm has many fewer customers than that, it is likely to grow. New firms that happen to have a low unit cost in particular are likely to be in a situation where the number of customers is far below the level where inflows equals outflows, so growth is likely, and will continue to be likely in the next few periods, until it gains sufficient number of customers for he rates to equalize. This leads to more positive autocorrelation for young firms.

4 Contracting Frictions and Long Term Relationships

In this section we introduce the contracting frictions and relational contracts.

The result will be that if courts are less efficient, firms will switch suppliers of relationship-specific goods less frequently. In particular, for users of relationship-specific inputs, the arrival rate of new suppliers of those inputs declines uniformly.

At this point, the decline in the arrival rate of new suppliers is a behavioral assumption. In the next section we discuss a potential microfoundation for this behavior as resulting for the use of relational contracting as a substitute for courts. A relational contract is a repeated game, and there are many equilibria. At this point, we can show for a special case of the model that a uniform decline in κ for relationship-specific inputs is indeed an equilibrium. We conjecture that this result extends to the full model, but we do not yet have a proof.

4.1 Microfoundation

A contract between a buyer-supplier pair specifies, in addition to a quantity and a transfer, a level of defectiveness $\delta \in [0, 1]$. With commitment, surplus between the buyer and supplier is maximized at $\delta = 0$. The supplier can produce a defective input. Doing so reduces the supplier's cost, but increases the probability that the output will be defective.

If the output is defective, the buyer has a claim on the supplier. This claim can be enforced in court. But delay in court reduces value of payment, and the cost of the delay is proportional to value of transaction.

In a one shot game, both the buyer and supplier would anticipate that the supplier would not customize the good. While the enforcement in court helps a bit. The reduced surplus from the relationship is built into the price of the input.

As usual, there is an equilibrium in which the static Nash outcome is played at each instant. But there is another, equilibrium that pareto dominates. In this relational contract, the supplier chooses $\delta = 0$, and the buyer chooses a lower arrival rate of new suppliers (which is observable to supplier, but not the court). Doing this makes the relationship likely to last longer, raising the supplier's surplus from the relationship in a way that is backloaded.

The relational contract is enforced with trigger strategies. If supplier does not customize, the buyer does not reduce arrival of new suppliers. If buyer does not reduce arrival rate, supplier stops customizing. The supplier's punishment for defective inputs is that the relationship ends faster *and* enforcement in court. As a result, better formal enforcement reduces the need for the buyer to make the relationship last longer.

4.2 Slow Firm Dynamics

To explore the implications of weak contract enforcement, we turn to a numerical simulation of the model. There is only one change in the model, a reduction in the arrival rate of new suppliers of relationship-specific inputs. But this manifests itself in a number of ways.[6]

Each figure below shows patterns of firm dynamics for two scenarios: In the first, labeled "no friction," court congestion is such that the average case age is one year, corresponding roughly to the fastest courts in India. In the second, labeled "friction," courts are more congested and the average case age is four years, corresponding roughly to the slowest courts in India of roughly four years. Our empirical findings below indicate that for each additional year of court delay, buyer-supplier pairs with relationship-specific inputs, relationships last about 0.25 years longer. In all figures we show statistics related to firms in industries that produce relationship-specific goods.

First, we show how contracting frictions change the overall distribution of firm growth rates. Figure 4 shows the density of changes in log size among all firms in industries that produce relationship-specific goods.With more sever contracting frictions, the distribution of firm growth rates is more concentrated. there are more firms that have smaller growth rates, and fewer firms with extremely large or extremely small growth rates. This is a natural consequence of slower gain and loss of customers. Figure 5 shows the impact of relational contracting on firm-level volatility. It shows

[6]For now, we hold entry fixed in counterfactuals. While the adjustment of entry will affect aggregate output, it will not affect patterns of firm dynamics that we document in this section. We are currently working on incorporating changes in entry.

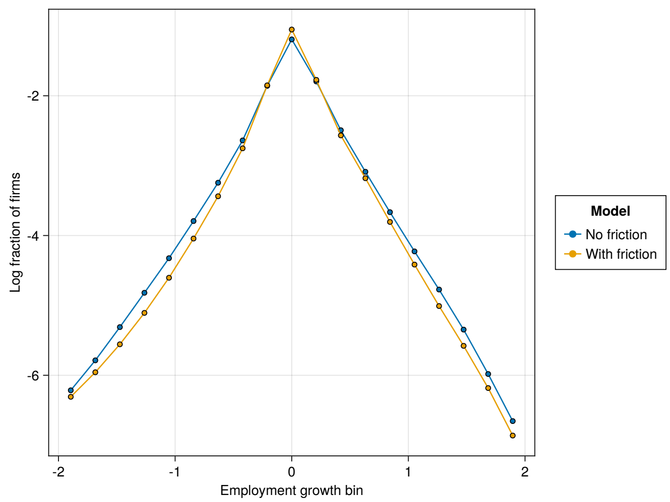
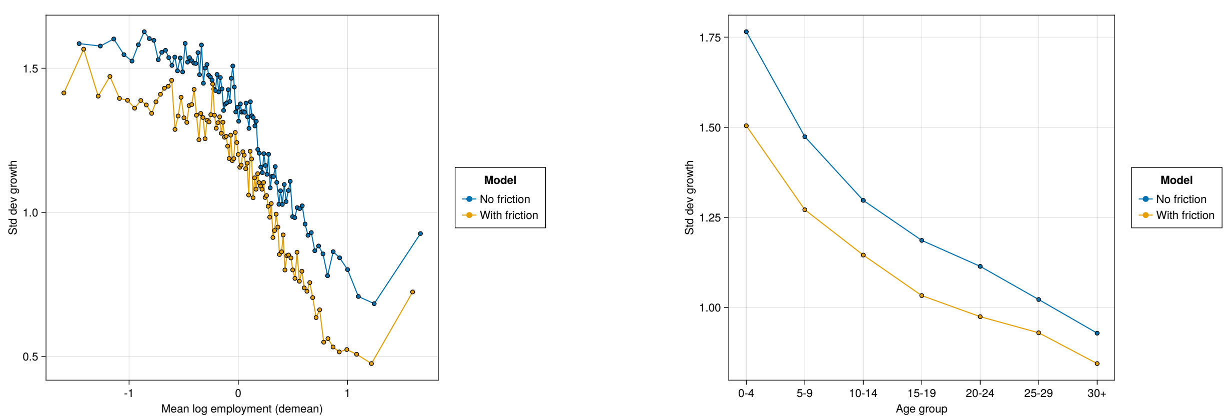


Figure 4 Distribution of Changes in Log Size

Note: This figure shows the density of changes in log size among firms in industries that produce relationship-specific industries.

that with more severe contracting frictions, the standard deviation of changes in log sales lower, as firms gain and lose customers more slowly. Panel (a) shows that lower volatility conditional on size, and panel (b) shows lower volatility conditional on age.



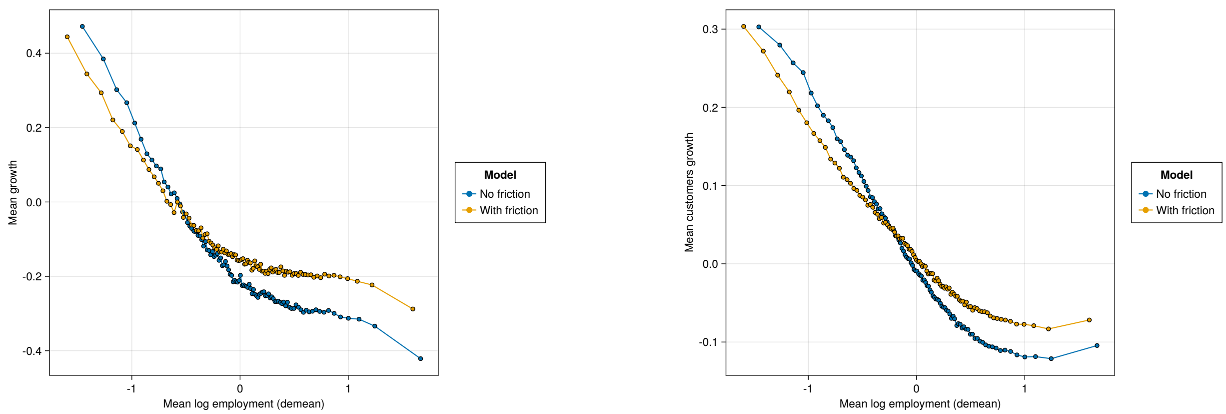
(a) Volatility by Age

(b) Volatility by Age

Figure 5 Volatility

Note: For each size or age bin, this figure shows the standard deviation of the log change in size among firms in industries that produce relationship-specific industries.

Figure 6 shows that relational contracting leads to slower mean reversion. The two panels show the change in log size conditional on initial size, where size is measured either by sales in panel (a) or by number of customers in panel (b). In either case, small firms grow more slowly, as customers are less likely to switch to them. And large firms shrink more slowly, as they are more likely to keep each customer for longer.



(a) Sales

(b) Number of Customers

Figure 6 Mean Reversion

Note: This figure shows changes in log sales and changes in the log of number of customers for firms of various initial sizes.

Figure 7 shows the right tail of the within-industry size distribution for industries that produce relationship-specific inputs on a log-log plot. Here, a steeper curve means a thicker right tail. The figure shows that relational contracting leads to a firm size distribution with a thinner right tail. Without the contracting friction, the firms that are in the right tail of the firm size distribution are ones that have a low cost and have been lucky enough to have had many potential customers arrive. With long-term relationships, potential customers do not arrive as quickly, so that firms with a very low cost cannot attract new customers as quickly, and therefore do not grow as large.

Figure 8 shows exit rates among those that produce relationship-specific goods, by size and by age. With imperfect enforcement, exit rates are lower. A firm exits when it loses its last customer. With relational contracting, firms are less likely to lose their customers in equilibrium, so that a firm that is down to its last customer is less likely to lose that customer and hence less likely to exit.

To summarize, in industries that produce relationship specific inputs, when enforcement is worse:

- Lower variance of firm growth
- Less mean reversion
- Less skewed size distribution
- Lower exit rate

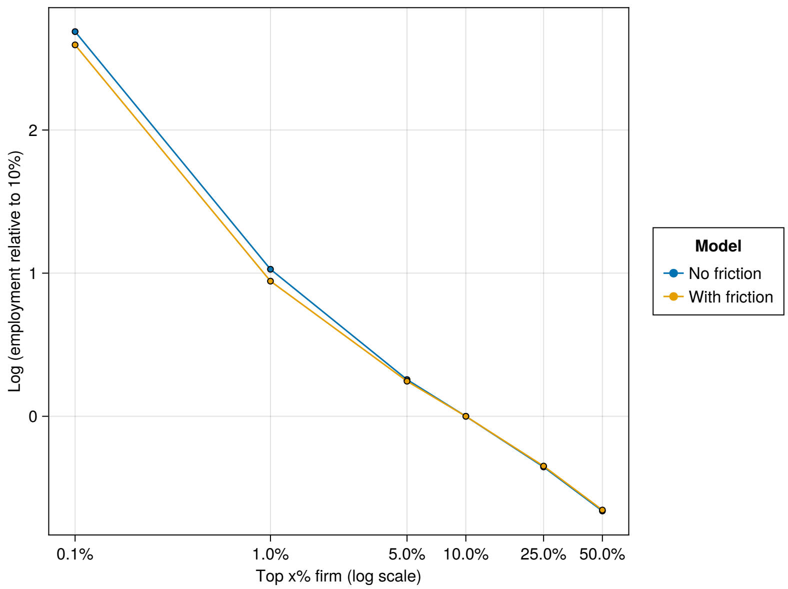
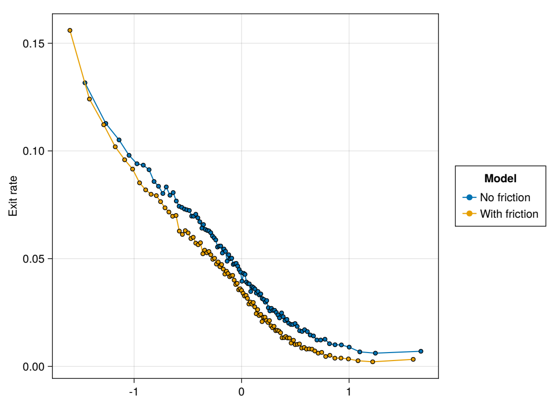
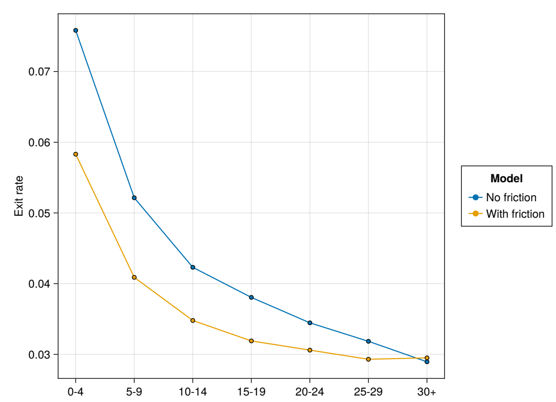


Figure 7 Right Tail of the Firm Size Distribution

Note: This figure shows changes in log sales and changes in the log of number of customers for firms of various initial sizes.



(a) Exit Rates By Size



(b) Exit Rates by Age

Figure 8 Exit Rates by Size and Age

Note: This figure shows exit rates for industries that produce relationship-specific inputs with perfect contract enforcement and with imperfect enforcement. Panel (a) shows exit rates by size. Panel (b) shows exit rates by age.

5 Testing the Model's Predictions

In this section we use data on India and Pakistan to assess the predictions of the model. We first describe the data and the setting, discuss our approach, and then proceed to contrast the model's predictions with empirical patterns in the data.

5.1 Data and Approach

Production and Transactions Data—We use two types of data. First, we use the Annual Survey of Industry (ASI), the official annual survey of India's formal manufacturing sector. This is a panel that covers all establishments with more than 100 employees, and, every year, a fifth of all establishments with more than 20 employees (or more than 10 if they use power).

We also use firm-to-firm transactions data from firms in Pakistan.

Court Congestion—For each Indian State, we construct a measure of court congestion from microdata on pending civil cases in High Courts, which the Indian NGO Daksh collects from causelists and other court records (Narasappa and Vidyasagar (2016)). These records show the status and age of pending and recently disposed cases, along with characteristics of the case, such as the act under which the claim was filed or a case type categorization. Our measure of high court congestion is the average age of pending civil cases in each court, at the end of the calendar year 2016.[7] The average age of pending civil cases varies substantially across high courts – from less than one year in Goa and Sikkim, to about four and a half years in Uttar Pradesh and West Bengal. The cross-state average is two and a half years.

The main explanation for why court speed varies so much across states lies in the history of India's political subdivisions. The first high courts (Madras, Bombay, and Calcutta) were set up by the British in the 1861 Indian High Courts Act, and served as the precursor for India's post-independence high courts. Upon independence, India was divided into a number of federated states, with the Constitution of India (1947) mandating a high court for each state. Throughout the twentieth century and beyond, India has frequently subdivided its states, often because of ethno-nationalist movements. These subdivisions were often accompanied with new high courts being set up, which then start without any existing backlog of cases. The age of the high court is hence a strong determinant of its speed of enforcement; the F-statistic in that regression is 11.1). We will later use the age of high courts as an instrument for its level of congestion.

Our measure of court congestion is, similarly to India, the average age of pending civil cases, which we construct from Committee (n.d.). In contrast to India, we construct this from cases in district courts due to the lower number of provinces (the most closely corresponding administrative unit to Indian states in Pakistan). We also have no indication that parties are able to bypass district courts in Pakistan. We have data, on the aggregate provincial level, for Sindh, Khyber-Pakhtunkhwa, Balochistan, and Islamabad (ICT), and at the district level for the 37 districts of the

[7] Whenever a high court has jurisdiction over two states and a separate bench in each of them (such as the Bombay High Court, which has jurisdiction over Maharashtra and Goa), we construct the statistic by state.

Punjab. We exclude Balochistan because the data do not appear internally consistent (numbers not adding up to reported totals, strange patterns).

The raw data contains the number of civil cases pending at the end of 2020 that were instituted in each year, between 2010 and 2020, and before 2010. From that we calculate the average age of pending cases, assuming that (i) institution of cases is uniformly distributed within the year; (ii) cases instituted up to the end of 2010 are, on average, 11.5 years old.

Approach—The regressions below study the differential impact of court congestion on industries that produce relationship specific goods relative to industries that produce standardized goods.

All regressions control for district fixed effects, to control for the impact of such as the level of development, population density, the state of technology, to the extent that they have a similar impact on standardized and relationship specific industries.

Each data source offers advantages and disadvantages. For the ASI data from India we have precise information on the 5-digit products sold by the plant, whereas in the Pakistan data we have the two-digit industry of the firm, and we cannot see the products in each transaction. There is also more variation in court congestion across states within India, and we have an instrument for court congestion. The advantage of the VAT data from Pakistan is that we can see the firm-to-firm transactions, and the data comes at the firm level rather than the plant level.

5.2 Court Congestion and Relationship Length

We first provide some preliminary evidence on how court congestion affects the length of relationships using VAT data from Pakistan. The dependent variable in Table **??** is the time elapsed between the date of the first transaction and the date of the last transaction of the firm pair, in years.

When a firm-to-firm transaction is between firms that is across different districts, the firms have the freedom to sign contracts that will be enforced in any district they choose. In Table II, we assess the impact of court congestion in the buyer's district, the supplier's district, and the minimum of the two. In each case, we find that a more congested court is associated with longer duration of when the supplier produces a relationship-specific good compared to when the supplier produces a standardized good.

The first three columns use a measure of relationship specificity from Rauch (1999). The last two columns used a measure of enforcement intensity that is specific to the supplier industry-buyer industry pair from Boehm (2022), as it is based on the frequency of litigation between buyers and suppliers in those industries.

5.3 Firm Volatility

We next turn to manufacturing data from the ASI to assess the predictions described in Section II. Table III shows the impact of court congestion on firm volatility The dependent variable is the standard deviation of annualized, residualized sales growth within each state-industry-year. These sales growth rates are residualized with respect to age, year, state and industry.

Table II Relationship Duration and Court × Enforcement Intensity interaction

	Dependent variable: Age of Relationship (in Years)				
	(1)	(2)	(3)	(4)	(5)
Age of pending cases (S) × RelSpec_S	0.238^{**}				
	(0.048)				
Age of pending cases (B) × RelSpec_S	0.0589				
	(0.045)				
Age of pending cases (Min(B,S)) × RelSpec_S		0.208^{**}	0.141^{*}		
		(0.056)	(0.065)		
Age of pending cases (Min(B,S)) × $\text{EnforcementIntensity}_{b,s}$				312.2^{*}	287.3^{+}
				(142.0)	(160.8)
B × S Industry FE	Yes	Yes	Yes	Yes	Yes
B District FE	Yes	Yes	Yes	Yes	Yes
S District FE	Yes	Yes		Yes	
S District × S Industry FE			Yes		Yes
R^2	0.132	0.133	0.153	0.130	0.151
Observations	2007266	2037575	2037252	1951625	1951307

Standard errors in parentheses, clustered at the origin district × destination district level.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

RelSpec_S is a industry-level average of Rauch's conservative measure of relationship-specificity (where a good is considered relationship-specific if it's not traded on an organized exchange, and there is no reference price). $\text{EnforcementIntensity}_{b,s}$ is $z^{(1)}$ from Boehm (2022).

The first two columns use OLS, while the third and fourth columns use the age of the court as an instrument for court congestion. The second and fourth columns control for average growth of those firms. In all cases, more sever contracting frictions reduce the volatility of firm size.

Table III Lower variance of sales growth when frictions are large

	Dependent variable: $\sigma(\Delta \log \text{Sales})_{d\omega}$			
	(1)	(2)	(3)	(4)
Avg age of civil cases × Rel. spec.	-0.0253^{*}	-0.0257^{*}	-0.0446^{*}	-0.0439^{*}
	(0.012)	(0.012)	(0.021)	(0.020)
$\overline{(\Delta \log \text{Sales})_{d\omega}}$		-0.290^{**}		-0.290^{**}
		(0.022)		(0.022)
State FE	Yes	Yes	Yes	Yes
5-digit Industry FE	Yes	Yes	Yes	Yes
Estimator	OLS	OLS	IV	IV
R^2	0.284	0.306	0.000326	0.0303
Observations	7028	7028	7028	7028

Regression at the state × industry level. Only state-industry cells with more than 3 observations used.

5.4 Mean Reversion

Table IV shows the impact of contracting frictions on the rate of mean reversion. The first shows the baseline rate of mean reversion for plants in industries that produce standardized goods. The coefficient is negative, meaning that larger firms grow slowly. The second row shows the differential impact firms that produce relationship-specific goods. across specifications, the point estimate is positive, meaning that the rate of mean reversion is smaller.[8]

Table IV Mean Reversion

	Dependent variable: Change in log Sales					
	(1)	(2)	(3)	(4)	(5)	(6)
$\log \text{Sales}_{t-1}$	-0.403**	-0.427**	-0.555**	-0.403**	-0.436**	-0.583**
	(0.011)	(0.025)	(0.037)	(0.012)	(0.028)	(0.038)
$\log \text{Sales}_{t-1} \times$ Age civ. cases $\times$ relspec	0.00709+	0.0206*	0.0249+	0.00687	0.0256*	0.0405*
	(0.0037)	(0.0096)	(0.015)	(0.0044)	(0.012)	(0.019)
Plant $\times$ 5-digit Industry FE	Yes	Yes	Yes	Yes	Yes	Yes
State FE	Yes			Yes		
Year $\times$ Previous Year FE	Yes			Yes		
Age FE		Yes	Yes		Yes	Yes
Industry $\times$ District $\times$ Year FE		Yes			Yes	
Industry $\times$ District $\times (t, t-1)$ FE			Yes			Yes
Method	OLS	OLS	OLS	IV	IV	IV
R^2	0.457	0.636	0.671	0.256	0.250	0.278
Observations	204518	78053	51401	204518	78053	51401

Standard errors in parentheses, clustered at the state $\times$ industry level.

5.5 Size Distribution

Table V shows the impact of contracting frictions on the skewness of the plant size distribution. To measure skewness in the right tail of the plant size distribution, we use a statistic that measures the slope of the log-rank, log-size plot, following Chen (2023). In particular, for any two quantiles of the overall size distribution, S_0 and S_1, we compute skewness of the size distribution in a state-industry-year as

$$\frac{\log(\text{Share of plants above } S_1) - \log(\text{Share of plants above } S_0)}{\log S_1 - \log S_0}$$

The various columns of the table use different combinations of quantiles of the overall plant size distribution (25th, 50th, and 75th, 90th).

Across specifications, we find that more severe contracting frictions are associated with less skewed plant-size distributions.

5.6 Exit Rates

Table VI shows the impact of contracting frictions on exit rates. The dependent variable is the exit rate for each size quartile of the industry-district-year. Columns (1), (2), and (3) are included to show the baseline exit rates. Column (4) shows the differential impact on the exit rates for

[8] That is, the magnitude of the sum of the first and second rows is smaller than the magnitude of the first row.

	Dependent variable: Skewness of log Sales					
	(1)	(2)	(3)	(4)	(5)	(6)
Relspec x Court Congestion	-0.0720*	-0.0696*	-0.142**	-0.125+	-0.136*	-0.161
	(0.0335)	(0.0298)	(0.0523)	(0.0698)	(0.0620)	(0.103)
R^2	0.540	0.435	0.554	0.001	0.000	0.007
State FE	Yes	Yes	Yes	Yes	Yes	Yes
5-digit Industry FE	Yes	Yes	Yes	Yes	Yes	Yes
Estimator	OLS	OLS	OLS	IV	IV	IV
Statistic	2575	5075	5090	2575	5075	5090
Observations	3008	3008	1448	3008	3008	1448

Table V Firm Size Distribution

industries that produce relationship-specific goods. We find that more severe contracting frictions are associated with lower exit rates.

	Dependent variable: P(exit)			
	(1)	(2)	(3)	(4)
Q1 Dummy	0.0738***	0.0717***		
	(0.0023)	(0.0057)		
Q2 Dummy	0.0255***	0.0208***	-0.0460***	-0.0469***
	(0.0018)	(0.0033)	(0.0013)	(0.0042)
Q3 Dummy	0.0131***	0.00979***	-0.0576***	-0.0567***
	(0.00099)	(0.0016)	(0.0016)	(0.0043)
Q4 Dummy	0.00800***	0.00677***	-0.0611***	-0.0586***
	(0.00071)	(0.0011)	(0.0018)	(0.0044)
Q1 × Relspec × AvgAgeCourts		0.00129		-0.00539*
		(0.0026)		(0.0025)
Q2 × Relspec × AvgAgeCourts		0.00299*		-0.00501**
		(0.0014)		(0.0019)
Q3 × Relspec × AvgAgeCourts		0.00221*		-0.00627***
		(0.00099)		(0.0016)
Q4 × Relspec × AvgAgeCourts		0.000871		-0.00755***
		(0.00087)		(0.0016)
Industry × Year FE			Yes	Yes
R^2	0.0525	0.0526	0.0460	0.0462
Observations	417711	411541	417698	411528

Standard errors in parentheses, clustered at the industry-region level.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

Table VI Exit Rates by Size

6 Aggregate Productivity

In this section we return to calibrated model to assess the impact of weak contract enforcement on aggregate productivity. In some sense, there is a misallocation of suppliers across buyers.

In rough terms, contracting frictions weaken the relationship between aggregate productivity will be lower when there if supplier that are more efficient account for more of the buyer-supplier relationships.

We can see this misallocation in two complementary ways: (i) the correlation between log cost and log size, and (ii) dispersion in size among those with the same cost.[9]

Table VII shows the correlation of cost and size among firms that produce relationship-specific inputs. The correlation is measured across all such firms in two ways, first by subtracting the industry mean from each observation, and second by subtracting the industry mean and dividing by the industry standard deviation. In either case, the correlation is weaker in the economy with more severe contracting frictions, indicating more severe misallocation.[10]

Model	**Correlation (demeaned)**	**Correlation (normalized)**
No friction	-0.281	-0.370
With friction	-0.260	-0.340

Table VII Correlation of Log Cost and Log Size

Note: This table shows the correlation of log cost and log size among firms that produce relationship-specific inputs. The correlation is measured across all such firms. In the column labeled "demeaned," we subtract from each firm's log cost and log size the respective industry mean. In the column labeled "normalized," we subtract the industry mean and divide by the industry standard deviation.

Figure 9 shows the a second measure of misallocation, the dispersion in size among firms of in the same cost decile. For almost every cost decile, there is more dispersion in size when contracting frictions are more severe.[11]

Finally, Table VIII shows the implications for aggregate productivity. As discussed in Section 2, the growth rate of aggregate productivity is invariant to long term relationships, as this depends only on the population growth rate (the model is one of semi-endogenous growth). Nevertheless, the contracting frictions reduce the *level* of aggregate productivity. In particular, by reducing the rate of reallocation of buyers across suppliers, court congestion in India's state with the most congested courts reduces aggregate productivity in the state by about 15% (16.2 log points) relative to the court congestion in the state with the least congested courts.[12]

[9] There is a parallel between these two complementary measures of misallocation with the slope coefficient and R-squared of a regression as complementary measures of explanatory power.

[10] Even without contracting frictions , there will be an imperfect correlation between log cost and size, because of the random arrival of customers and the random sizes of those customers.

[11] The exception is the lowest cost decile, in which dispersion in size in the economy with less severe contracting frictions is driven by the thicker right tail of the firm size distribution, as discussed in Section 4.

[12] Two notes: the magnitude of the loss in aggregate productivity scales one-fore-one with $1/\beta$. Because we only have evidence a single study, Baqaee et al. (2023), there is some uncertainty about the parameter value. Second, in the counterfactuals, we do not adjust the measure of firms that are created.

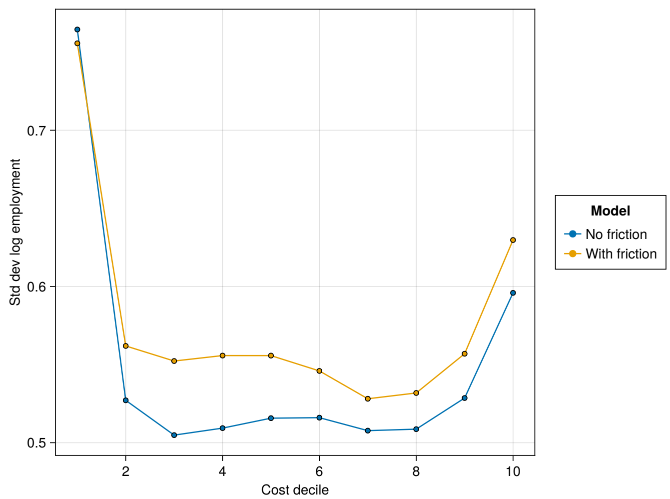


Figure 9 Dispersion in Size for each Cost Bin

Note: This figure shows the standard deviation of log employment among firms with the same cost, among firms in industries that produce relationship-specific industries.

	No friction	**With friction**
Mean income growth	0.015	0.015
Log real income difference	0.000	-0.162

Table VIII Aggregate Productivity

7 Conclusion

TBD...

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A Numerical simulations

A.1 Calibration

A.1.1 Population growth

The employment share of young firms is greater with a high population growth rate. Hsieh and Klenow (2014) give statistics on the employment share of plants by age in 2010-11 from the ASI and NSS in Figure III. But they also show that cohort sizes are larger post 1997. They also include a calculation for steady-state employment shares based on measured employment growth and exit rates assuming that all cohorts are equally sized in Figure VI.

A.2 Input Output Tables

To facilitate the numerical simulation, we make some modifications to the input output tables so that it is model consistent and more sparse. We first remove some links and industries.

We use the following notation: Expenditure between a buying and supplying industries, $E(b, s)$, is the total expenditure by single product plants in b on s. Production, $P(\omega)$ is the total ex-factory value of produced goods. Final consumption, $C(\omega)$, is production less total ASI expenditure on the product.

1. Only manufacturing products are included.
2. **Important links:** links that are unimportant for both supply and demand are removed to increase sparsity.

    - (Supply) For each purchasing industry b, let M be the set of supply industries. The supply-important pairs (b, s) are those with s in the smallest set $S \subseteq M$ such that

$$\sum_{s \in S} E(b, s) \geq 0.99 \times \sum_{s \in M} E(b, s)$$

Table IX Hsieh & Klenow employment shares

Age group	Employment share (2010-11)	Employment share (SS)
< 5	0.26	0.18
$5 - 9$	0.22	0.15
$10 - 14$	0.18	0.12
$15 - 19$	0.10	0.10
$20 - 24$	0.08	0.08
$25 - 29$	0.05	0.06
$30 - 34$	0.04	0.04
$35 - 39$	0.02	0.04
≥ 40	0.05	0.23

- (Demand) For each supplying industry s, let M be the set of purchasing industries. The demand-important pairs (b, s) are those with b in the smallest set $B \subseteq M$ such that

$$\sum_{b \in B} E(b, s) \geq 0.99 \times \sum_{b \in M} E(b, s)$$

- Expenditure on pairs which are neither supply-important nor demand-important is set to zero.

3. **Slow industries:** For some industries, calculating the firms' costs is very slow. These are industries with either very high expenditure share on manufacturing inputs and more generally those associated with large eigenvalues of the IO matrix.

    - We remove industries with total expenditure on manufacturing outputs greater than 95% of the ex-factory value of production.
    - We remove the industries associated with eigenvalues of the IO matrix above 0.5.

4. **Industries with no suppliers:** We remove industries which do not purchase inputs from any included industries.
5. **No path to final consumption:** We remove any industries with zero direct final consumption and zero (potentially indirect) customers with strictly positive final consumption.

Then given the remaining industries and buyer-supplier links, we reconstruct consistent IO tables as follows:

- We assume any purchases from industries or through links that are not in our final sample are spent on primary inputs.
- We assume any sales to industries or through links that are not in our sample are sales to retailers.
- $\alpha_{b,s} = \frac{E(b,s)}{P(b)}$
- $\alpha^H_\omega = \max\{0, C(\omega)\}$ (normalized to sum to 1 across industries)
- Given the IO matrix A, the fraction of firms in each industry ρ_ω is set to equalize the average value of output per firm across industries.

A.3 Simulation Procedure

We simulate the model using a discrete approximation using a finite number of firms.

In time step Δ, Given an arrival rate ϕ of techniques that dominate one's current supplier,

At each point in time, we have recorded for each firm the identity of each supplier and the match specific productivity. We compute each firm's cost as a fixed point.

If the time step is small enough, it is likely there is at most one new technique that dominates the existing technique. Among all techniques that arrive, the identity of the supplier is random. However, among techniques that are larger than any threshold, suppliers that have a low cost are over-represented. In particular, among techniques that beat the current supplier, the probability that it comes from supplier j is

$$\frac{c_j^{-\beta}}{\sum_{\tilde{j}\in J_{\hat{\omega}}} c_{\tilde{j}}^{-\beta}} \tag{1}$$

Thus we assume that in each time step, a new, better supplier arrives with probability $\Delta\phi$, and if it does arrive, the reduction in cost x is drawn from a Pareto with shape parameter β and the identity of the supplier is randomly drawn according to (1).

In practice, we use a time step Δ of one year. Our simulations are insensitive to varying the length of the time step.

We simulate the model for 150 years. We discard the first 100 years, as by this time the model has converged sufficiently to its balanced growth path, and the statistics we compute have stabilized. We then use the remaining 50 years to collect outcomes.

B Proofs

B.1 Static Equilibrium

B.2 Simple Model

First, we define the infinitesimal generator of the process. Let $m(x) \equiv M_t(x,0)$, where M_t is the partial derivative of M with respect to its second argument.

B.2.1 Proof of Proposition ??

Lemma 2

$$m(x) = -\phi \sum_{k=0}^{\infty} x^{-\beta\alpha^{-k}}$$

Proof. (Heuristic)

For a short enough time period t, there are two ways events that can happen: either the firm finds a new supplier that delivers a jump in efficiency larger than x, or the existing supplier's efficiency improves by more than $x^{\frac{1}{\alpha}}$ (the full proof shows that the probability of some mixture of the two events is negligible).

$$\begin{aligned} M(x,t) &= \left\{M\left(x^{\frac{1}{\alpha}},t\right)\right\}\left\{e^{-t\phi\int_x^\infty \beta b^{-\beta-1}db}\right\} \\ &= M\left(x^{\frac{1}{\alpha}},t\right)e^{-t\phi x^{-\beta}} \end{aligned}$$

Differentiating and evaluating at $t=0$

$$\begin{aligned}
M_t(x,0) &= \frac{d}{dt}\left\{M\left(x^{\frac{1}{\alpha}},t\right)e^{-t\phi x^{-\beta}}\right\}\Big|_{t=0} \\
&= M_t\left(x^{\frac{1}{\alpha}},t\right)e^{-t\phi x^{-\beta}} - \phi x^{-\beta}M\left(x^{\frac{1}{\alpha}},t\right)e^{-t\phi x^{-\beta}}\Big|_{t=0} \\
&= M_t\left(x^{\frac{1}{\alpha}},0\right) - \phi x^{-\beta}
\end{aligned}$$

where the last line used $M\left(x^{\frac{1}{\alpha}},0\right)=1$.

Next, using the fact that $\lim_{x\to\infty} M_t(x,0)=0$, we can compute M recursively

$$\begin{aligned}
m(x) &= -\phi x^{-\beta} + m\left(x^{\frac{1}{\alpha}}\right) \\
&= -\phi x^{-\beta} - \phi x^{-\frac{\beta}{\alpha}} + m\left(x^{\frac{1}{\alpha^2}}\right) \\
&= -\sum_{k=0}^{K-1}\phi x^{-\beta\alpha^{-k}} + m\left(x^{\alpha^{-K}}\right) \\
&\to -\phi\sum_{k=0}^{\infty}x^{-\beta\alpha^{-k}}
\end{aligned}$$

■

Proof. (Full)

What is the probability that a firm's efficiency increase by weakly less than the proportion x in an interval of length t? This could happen either the firm does not find a new supplier and the existing supplier improves by no more than $x^{1/\alpha}$, an even which occurs with probability $e^{-\phi t}M\left(x^{\frac{1}{\alpha}},t\right)$, or if there is a jump with increment z at some time $\tau\in[0,t]$, which occurs with density $\beta z^{-\beta-1}\phi e^{-\phi\tau}$, that the new supplier improved by y between τ and t, and that the initial existing supplier improved by less than $\left(\frac{x}{yz}\right)^{1/\alpha}$ between 0 and τ. Thus

$$M(x,t) = e^{-\phi t}M\left(x^{\frac{1}{\alpha}},t\right) + \int_0^t \phi e^{-\phi\tau}\int_1^x\int_1^{\frac{x}{y}} M\left(\left(\frac{x}{yz}\right)^{\frac{1}{\alpha}},\tau\right)\beta z^{-\beta-1}M_x(y,t-\tau)\,dzdyd\tau$$

Taking the derivative with respect to t gives

$$\begin{aligned}
M_t(x,t) &= -\phi e^{-\phi t}M\left(x^{\frac{1}{\alpha}},t\right) + e^{-\phi t}M_t\left(x^{\frac{1}{\alpha}},t\right) \\
&\quad +\phi e^{-\phi t}\int_1^x\int_1^{\frac{x}{y}} M\left(\left(\frac{x}{yz}\right)^{\frac{1}{\alpha}},t\right)\beta z^{-\beta-1}M_x(y,0)\,dzdy \\
&\quad \int_0^t \phi e^{-\phi\tau}\int_1^x\int_1^{\frac{x}{y}} M\left(\left(\frac{x}{yz}\right)^{\frac{1}{\alpha}},\tau\right)\beta z^{-\beta-1}M_{xt}(y,t-\tau)\,dzdyd\tau
\end{aligned}$$

Evaluating this at $t = 0$ and gives

$$M_t(x,0) = -\phi M\left(x^{\frac{1}{\alpha}},0\right) + M_t\left(x^{\frac{1}{\alpha}},0\right) + \phi \int_1^x \int_1^{\frac{x}{y}} M\left(\left(\frac{x}{yz}\right)^{\frac{1}{\alpha}},0\right) \beta z^{-\beta-1} M_x(y,0)\,dz dy$$

Using the fact that $M(x,0) = 1$, $\forall x \geq 1$ and that $M_x(y,0)$ is the Dirac delta function at $y = 1$, this is

$$\begin{aligned}
M_t(x,0) &= -\phi + M_t\left(x^{\frac{1}{\alpha}},0\right) + \phi \int_1^x \left[1 - \left(\frac{x}{y}\right)^{-\beta}\right] M_x(y,0)\,dy \\
&= -\phi + M_t\left(x^{\frac{1}{\alpha}},0\right) + \phi\left(1 - x^{-\beta}\right) \\
&= M_t\left(x^{\frac{1}{\alpha}},0\right) - \phi x^{-\beta}
\end{aligned}$$

Next, using the fact that $\lim_{x\to\infty} M_t(x,0) = 0$, we can compute M recursively

$$\begin{aligned}
m(x) &= -\phi x^{-\beta} + m\left(x^{\frac{1}{\alpha}}\right) \\
&= -\phi x^{-\beta} - \phi x^{-\frac{\beta}{\alpha}} + m\left(x^{\frac{1}{\alpha^2}}\right) \\
&= -\sum_{k=0}^{K-1} \phi x^{-\beta\alpha^{-k}} + m\left(x^{\alpha^{-K}}\right) \\
&\to -\phi \sum_{k=0}^{\infty} x^{-\beta\alpha^{-k}}
\end{aligned}$$

■

To get at the distribution of efficiency growth we define $\varphi(s,t)$ to be the Mellin transform of $M(x,t)$, i.e., $\varphi(s,t) \equiv \int_1^\infty x^{-s} M_x(x,t)\,dx$.

$$\varphi_s(s,t) \equiv \int_1^\infty (-\log x)\, x^{-s} M_x(x,t)\,dx.$$

Claim 2 $\varphi(s,t) = e^{-\phi t \sum_{k=0}^{\infty} \frac{s}{\beta\alpha^{-k}+s}}$

Proof. We first derive an expression for the time derivative of the Mellin transform at $t = 0$. To do this, we first integrate by parts

$$\begin{aligned}
\varphi(s,t) &= \int_1^\infty x^{-s} M_x(x,t)\,dx = M(x,t)\, x^{-s}\Big|_1^\infty + \int_1^\infty s x^{-s-1} M(x,t)\,dx \\
&= \int_1^\infty s x^{-s-1} M(x,t)\,dx
\end{aligned}$$

we then differentiate with respect to time and eveluate at $t=0$

$$\begin{aligned}
\varphi_t(s,0) &= \int_1^\infty s x^{-s-1} M_t(x,0)\,dx \\
&= \int_1^\infty s x^{-s-1}\left[-\phi\sum_{k=0}^\infty x^{-\beta\alpha^{-k}}\right]dx \\
&= -\phi\sum_{k=0}^\infty \frac{s}{\beta\alpha^{-k}+s}
\end{aligned}$$

Finally, we use the fact the Mellin transform of independent random variables is the product of their transforms. Therefore

$$\begin{aligned}
\log\varphi(s,t) &= \lim_{n\to\infty} n\log\varphi\left(s,\frac{t}{n}\right) \\
&= t\lim_{n\to\infty}\frac{n}{t}\log\varphi\left(s,\frac{t}{n}\right) \\
&= t\lim_{\Delta\to 0}\frac{\log\varphi(s,\Delta)}{\Delta} \\
&= t\varphi_t(s,0) \\
&= t\left[-\phi\sum_{k=0}^\infty\frac{s}{\beta\alpha^{-k}+s}\right]
\end{aligned}$$

Exponentiating both sides gives the result. ■

Claim 3 *Let $X_j(t)$ be the (gross) growth rate of j's efficiency. As t grows large,* $\frac{1}{\sqrt{t}}\left(\log X_j(t)-\frac{\phi t}{\beta(1-\alpha)}\right)\sim N\left(0,\frac{2\phi}{\beta^2(1-\alpha^2)}\right)$

Proof. This is just the central limit theorem. For the proof in this case, let $y_j(t)=\log X_j(t)-\frac{\phi t}{\beta(1-\alpha)}$. Using the Mellin transform of $x_j(t)$, the laplace transform of y is

$$\begin{aligned}
\tilde{\varphi}(s,t) &= e^{-\phi t\sum_{k=0}^\infty\frac{s}{\beta\alpha^{-k}+s}+\frac{\phi t}{\beta(1-\alpha)}s} = e^{-\phi t\sum_{k=0}^\infty\left(\frac{s}{\beta\alpha^{-k}+s}-\frac{s}{\beta\alpha^{-k}}\right)} \\
&= e^{-\phi t\sum_{k=0}^\infty\frac{s^2}{(\beta\alpha^{-k}+s)\beta\alpha^{-k}}}
\end{aligned}$$

Note that

$$\tilde{\varphi}\left(\frac{s}{\sqrt{t}},t\right)=e^{-\phi t\sum_{k=0}^\infty\frac{\frac{s^2}{t}}{\left(\beta\alpha^{-k}+\frac{s}{\sqrt{t}}\right)\beta\alpha^{-k}}}=e^{-\phi\sum_{k=0}^\infty\left(\frac{s}{\beta\alpha^{-k}}\right)^2}=e^{-\phi\frac{s^2}{\beta^2(1-\alpha^2)}}$$

which is the laplace transform of a normal with mean zero and variance $\frac{2\phi}{\beta^2(1-\alpha^2)}$. ■

B.3 Full Model