CVI 13 - Poslední

16.1 a) $f(x) = a^T x + b$

$$f((1-\alpha)x + \alpha y) = (1-\alpha)a^T x + \alpha a^T y + b$$

$$(1-\alpha) f(x) + \alpha f(y) = (1-\alpha)a^T x + (1-\alpha)b + \alpha a^T y + \alpha b =$$

$$= (1-\alpha) a^T x + \alpha a^T y + b$$

$=$ $\Rightarrow$ konvexní

$$-f((1-\alpha)x + \alpha y) = -(1-\alpha)a^T x - \alpha a^T y - b$$

$$(1-\alpha)(-f(x)) + \alpha(-f(y)) = -(1-\alpha)a^T x - (1-\alpha)b - \alpha a^T y - \alpha b$$

$$= -(1-\alpha)a^T x - \alpha a^T y - b$$

$=$ $\Rightarrow$ konkavní

Odpověď: kovexní a konkavní

b) $f(x) = x^T x$

$$\cancel{f((1-\alpha)x + \alpha y) = ((1-\alpha)x + \alpha y)^T((1-\alpha)x + \alpha y)}$$

$$(1-\alpha) f(x) + \alpha f(y) = (1-\alpha)x^T x + \alpha y^T y$$

$$= x^T x - \alpha x^T x + \alpha y^T y$$

$$f((1-\alpha)x + \alpha y) = ((1-\alpha)x + \alpha y)^T((1-\alpha)x + \alpha y) =$$

$$= (1-\alpha)^2 x^T x + 2(1-\alpha)\alpha x^T y + \alpha^2 y^T y$$

$$= x^T x - 2\alpha x^T x + \alpha^2 x^T x + 2\alpha x^T y - 2\alpha^2 x^T y + \alpha^2 y^T y$$

$$\cancel{x^T x - \alpha x^T x}$$

$$\cancel{x^T x} - 2\alpha x^T x + \alpha^2 x^T x + 2\alpha x^T y - 2\alpha^2 x^T y + \alpha^2 y^T y \overset{?}{\le} \cancel{x^T x} - \alpha x^T x + \alpha y^T y$$

$$-\cancel{2}x^T x + \alpha x^T x + 2x^T y - 2\alpha x^T y + \alpha y^T y \overset{?}{\le} \cancel{-x^T x} + y^T y$$

$$\underbrace{-x^T x + \alpha x^T x}_{\le 0} + 2x^T y - 2\alpha x^T y + \alpha y^T y \overset{?}{\le} y^T y$$

???

d) $f(x) = \text{median}\, x$

① $f((1-\alpha)x + \alpha y) = \text{median}((1-\alpha)x + \alpha y)$

② $(1-\alpha) f(x) + \alpha f(y) = (1-\alpha)\,\text{median}(x) + \alpha\,\text{median}(y)$

Pro $n = 1$: konvexní a konkavní

Pro $n = 2$:

$$f((1-\alpha)x + \alpha y) = \text{median}\left((1-\alpha)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \alpha \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) =$$

$$= \text{median}\left(\begin{bmatrix} (1-\alpha)x_1 + \alpha y_1 \\ (1-\alpha)x_2 + \alpha y_2 \end{bmatrix}\right) =$$

$$= \frac{(1-\alpha)x_1 + \alpha y_1 + (1-\alpha)x_2 + \alpha y_2}{2}$$

$$(1-\alpha) f(x) + \alpha f(y) = (1-\alpha) \frac{x_1 + x_2}{2} + \alpha \frac{y_1 + y_2}{2}$$

není děleno dvěma oproti levé části $\Rightarrow$ ① $\leq$ ② $\to$ konvex.

S $-f(x)$ je vš. stejné, tak $f(x)$ je konkavní taky

$n \geq 2$: není konvexní ani konkavní

Odpověď: $n \leq 2$: konvexní a konkavní

$n > 2$: není konvexní ani konkavní

e) $f(x) = \min_{i=1}^{n} |x_i|$

$n \geq 2$

Není konvexní: p.p: $x = \begin{bmatrix} -200 \\ 0 \end{bmatrix}$ $y = \begin{bmatrix} -100 \\ -200 \end{bmatrix}$ $\alpha = 0{,}5$

$$f((1-\alpha)x + \alpha y) = f\left(\begin{bmatrix} -100 \\ 0 \end{bmatrix} + \begin{bmatrix} -50 \\ -100 \end{bmatrix}\right) = 100$$

$$(1-\alpha) f(x) + \alpha f(y) = 0{,}5 \cdot 0 + 0{,}5 \cdot 100 = 50$$

$$100 \not\leq 50$$

$n = 1$: konvexní

$$|(1-\alpha)x + \alpha y| \leq (1-\alpha)|x| + \alpha |y|$$

konkavnost $n = 1$

$$-|(1-\alpha)x + \alpha y| \nleq (1-\alpha)\cdot|x| + \alpha|y|$$

p.p. $x = -5 \quad y = 5 \quad \alpha = 0{,}5$

$$-|-0{,}5\cdot 5 + 0{,}5\cdot 5| \nleq -0{,}5\cdot 5 - 0{,}5\cdot 5$$

$$= 0 \qquad\qquad = -5$$

$n \geq 2$ $\quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad y = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$$f\left((1-\alpha)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = 0 \qquad\qquad 0 \nleq -2\alpha$$

$$-(1-\alpha) f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \alpha \cdot f\begin{bmatrix} -1 \\ -1 \end{bmatrix} = -2\alpha$$

<u>16.3</u> a) $f(x) = e^{x^2}$

$f''(x) = H = \left[4e^{x^2}x^2 + 2e^{(x^2)}\right]$ – je pos-def $\Rightarrow$ f je <u>konvexní</u>

> 0 $\quad -f(x)$: $H = \left[-4e^{(x^2)}x^2 - 2e^{(x^2)}\right]$ – není konkavní

b) $f(x) = e^{-x^2}$

$$f''(x) = H = \left[\frac{4x^2-2}{e^{x^2}}\right] \qquad\qquad -f(x): H = \left[\frac{2-4x^2}{e^{x^2}}\right]$$

$x \in [-\sqrt{\tfrac{1}{2}}; \sqrt{\tfrac{1}{2}}]$ – není konvexní, ale konkavní

$x \in [-\infty; -\sqrt{\tfrac{1}{2}}) \cup (\sqrt{\tfrac{1}{2}}; +\infty)$ – je konvexní, není konkavní

c) $f(x,y) = |x-y|$

$x - y > 0$ $\qquad\qquad$ $x - y < 0$

$f(x,y) = x - y$ $\qquad\qquad$ $f(x,y) = -x + y$

$f'(x,y) = (1, -1)$ $\qquad\qquad$ $f'(x,y) = (-1, 1)$

$$H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad\qquad H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

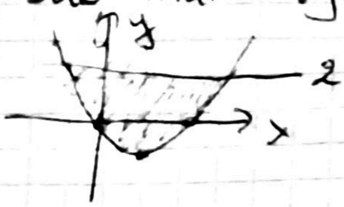
Hl. minory nejsou záporné

$\Rightarrow$ <u>konvexní</u> a konkavní

e) $f(x) = \|Ax - b\|_2^2$

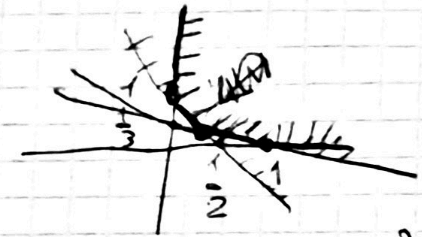
16.8 $f(x) = x^2 - x$

Subkontura výšky 2 je množina $\{x \in \mathbb{R} \mid x^2 - x \leq 2\}$



17.1 c) $\min\{\min\{x,y\} \mid x,y \geq 0,\ 2x+y \geq 1,\ x+3y \geq 1\}$

Transformace na LP ?



~~$x \geq \frac{2}{5}$ $y \geq \frac{1}{5}$~~

$$\begin{aligned} &\min z \\ &x - z \geq 0 \\ &y - z \geq 0 \\ &2x+y \geq 1 \\ &x+3y \geq 1 \\ &x,y \geq 0 \end{aligned}$$

$z = 0$?

$(x,y) = (1,0)$

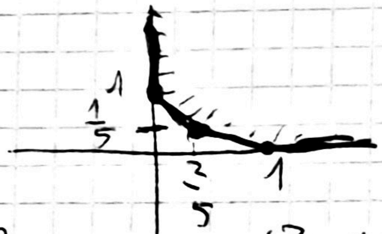
$(x,y) = (0,1)$

d) $f(x,y) = \max\{x,y\}$

$\min\{\max\{x,y\} \mid x,y \geq 0,\ 2x+y \geq 1,\ x+3y \geq 1\}$

Převod na LP

$$\begin{aligned} &\min z \\ &2x+y \geq 1 \\ &x+3y \geq 1 \\ &x - z \leq 0 \\ &y - z \leq 0 \\ &x,y \geq 0 \end{aligned}$$



$z = \frac{2}{5}$ $(x,y) = \left(\frac{2}{5}, \frac{1}{5}\right)$

17.5

$$\max\Big\{ \|(x_1,y_1)-(x_2,y_2)\| + \|(x_1,y_1)-(x_n,y_n)\| + \ldots$$

$$\vdots$$

$$+ \|(x_{n-1},y_{n-1})-(x_n,y_n)\| \ \Big|\ \begin{matrix} x,y \geq 0 \\ y \leq 4 \\ x \leq 5 \end{matrix} \Big\}$$