DU 10

1. a) $\int \frac{2}{e^{4x}+e^{2x}-2} = \left| \begin{array}{l} e^{2x} = t \\ 2e^{2x} = dt \\ 2 = \frac{dt}{t} \end{array} \right| = \int \frac{dt}{t(t^2+t-2)}$

$\frac{1}{t(t-1)(t+2)} = \frac{A}{t} + \frac{B}{t-1} + \frac{C}{t+2} =$

$= \left| \begin{array}{l} B: \frac{1}{1(1+2)} = \frac{1}{3} \\ C: \frac{1}{-2(-2-1)} = \frac{1}{6} \end{array} \right| = \frac{A}{t} + \frac{1}{3(t-1)} + \frac{C}{6(t+2)} =$

$= \left| \begin{array}{l} 1 = A(t-1)(t+2) + \frac{t(t+2)}{3} + \frac{t(t-1)}{6} \\ \Rightarrow 6At^2 + 2t^2 + t^2 = 0 \\ A = -\frac{1}{2} \end{array} \right| =$

$= -\frac{1}{2t} + \frac{1}{3(t-1)} + \frac{1}{6(t+2)}$

$\int \frac{2}{e^{4x}+e^{2x}-2} = -\frac{1}{2}\int\frac{1}{t} + \frac{1}{3}\int\frac{1}{t-1} + \frac{1}{6}\int\frac{1}{t+2} =$

$= -\frac{1}{2}\ln e^{2x} + \frac{1}{3}\ln|e^{2x}-1| + \frac{1}{6}\ln(e^{2x}+2) =$

$= -x + \frac{1}{3}\ln|e^{2x}-1| + \frac{1}{6}\ln(e^{2x}+2) + c, \quad x \in (-\infty; 0) \cup (0; +\infty)$

b) $\int \frac{1}{e^x+e^{-x}} = \left| \begin{array}{l} e^x = t \\ e^x = dt \end{array} \right| = \int \frac{1}{t(t+t^{-1})} dt =$

$= \int \frac{1}{t^2+1} dt = \operatorname{arctg}(t) = \operatorname{arctg}(e^x) + c, \quad x \in \mathbb{R}$

c) $\int \frac{1}{x \ln 3x} = \left| \begin{array}{l} \ln 3x = t \\ \frac{3}{3x} = dt \\ \frac{1}{x} = dt \end{array} \right| = \int \frac{1}{t} dt = \ln|\ln 3x| + c$

$x \in (-\infty; \frac{1}{3}) \cup (\frac{1}{3}; +\infty)$

d) $\int \frac{\cos x}{2+\sin x}\,dx = \left| \begin{matrix} \sin x = t \\ \cos x = dt \end{matrix} \right| = \int \frac{1}{2+t}\,dt = \ln|\sin x| + c,$

$x \in \mathbb{R} \setminus \{\pi n, n \in \mathbb{Z}\}$

e) $\int \frac{1}{(2+\cos x)\sin x}\,dx = \left| \begin{matrix} \cos x = t \\ -\sin x = dt \\ 1 = \frac{dt}{-\sin x} \end{matrix} \right| = \int \frac{1}{(2+t)\sin^2 x}\,dt =$

$= -\int \frac{1}{(2+t)(1-t^2)}\,dt = -\int \frac{1}{(2+t)(1-t)(1+t)}$

$\frac{1}{(2+t)(1-t)(1+t)} = \frac{A}{2+t} + \frac{B}{1-t} + \frac{C}{1+t} = -\frac{1}{3(2+t)} + \frac{1}{6(1-t)} + \frac{1}{2(1+t)}$

$A: \frac{1}{(1-(-2))(1-2)} = \frac{1}{-3} = -\frac{1}{3}$

$B: \frac{1}{3 \cdot 2} = \frac{1}{6}$

$C: \frac{1}{1 \cdot 2} = \frac{1}{2}$

$\int \frac{1}{(2+t)(1-t)(1+t)} = -\frac{1}{3}\int \frac{1}{2+t} + \frac{1}{6}\int \frac{1}{1-t} + \frac{1}{2}\int \frac{1}{1+t} =$

$= -\frac{1}{3}\ln|2+\cos x| + \frac{1}{6}\ln|1-\cos x| + \frac{1}{2}\ln|1+\cos x| + c$

$x \in \mathbb{R} - \{\pi n, n \in \mathbb{Z}\}$

f) $\int \frac{\sin x}{\cos x + \sin^2 x + 1} = \left| \begin{matrix} \cos x = t \\ -\sin x = dt \end{matrix} \right| = -\int \frac{1}{t + (1-t^2) + 1} =$

$= -\int \frac{1}{-(t^2-t-2)} = \int \frac{1}{t^2-t-2} = \int \frac{1}{(t-2)(t+1)}$

$\frac{1}{(t-2)(t+1)} = \frac{A}{t-2} + \frac{B}{t+1} =$

$= \left| A: \frac{1}{2+1} = \frac{1}{3} \quad B: \frac{1}{-1-2} = -\frac{1}{3} \right| = \frac{1}{3(t-2)} - \frac{1}{3(t+1)}$

$$\int \frac{t}{(t-2)(t+1)} = \frac{1}{3}\int \frac{1}{t-2} - \frac{1}{3}\int \frac{1}{t+1} =$$

$$= \frac{1}{3}\left(\ln|\cos x - 2| - \ln|\cos x + 1|\right) + C, \quad x \in \mathbb{R} \setminus \{\pi + 2\pi$$

8) $$\int \frac{\sin^2 x + 2}{\cos x + \cos x \sin x} = \int \frac{\sin^2 x + 2}{\cos x\,(1+\sin x)} = \left| \begin{matrix} \sin x = t \\ \cos x = dt \end{matrix} \right| =$$

$$= \int \frac{t^2+2}{(1-t^2)(1+t)}$$

$$\frac{t^2+2}{(1-t^2)(1+t)} = \cancel{\frac{Ax+B}{1-t^2}} \; \frac{t^2+2}{(1-t)(1+t)^2} =$$

$$= \frac{A}{(1-t)} + \frac{B}{(1+t)^2} + \frac{C}{(1+t)}$$

$$A: \frac{1+2}{(2)^2} = \frac{3}{4}$$

$$B: \frac{3}{2}$$

$$\frac{t^2+2}{(1-t)(1+t)^2} = \frac{3}{4(1-t)} + \frac{3}{2(1+t)^2} + \frac{C}{1+t}$$

$$t^2+2 = \frac{3(1+t)^2}{4} + \frac{3(1-t)}{2} + C(1+t)(1-t)$$

$$\cancel{4t^2+2 = 3(1+t^2) + 6(1-t) + C(1-t^2)}$$

$$\cancel{4t^2+8 = 3t^2+3+6t+3+C-Ct^2}$$

$$4t^2 = 3t^2 - Ct^2 \qquad C = -1$$

$$\cancel{4t^2+8 = 3t^2+3+6-6t+4C(1-t^2)}$$

$$\cancel{4t^2+8 = 3t^2+9-6t+4C-4Ct^2}$$

$$4t^2+8 = 3(t^2+2t+1) + 6(1-t) + 4C(1-t^2)$$

$$4t^2+8 = 3t^2 \cancel{+6t} + 3 + 6 \cancel{-6t} + 4C - 4C\cdot t^2$$

$$4t^2 = 3t^2 - 4Ct^2$$

$$t^2 = -4Ct^2 \qquad C = -\frac{1}{4}$$

$$\frac{t^2+2}{(1-t)(1+t^2)} = \frac{3}{4(1-t)} + \frac{3}{2(1+t)^2} - \frac{1}{4(1+t)}$$

$$\int \frac{t^2+2}{(1-t)(1+t)^2} = -\frac{3}{4}\int \frac{-1}{1-t} + \frac{3}{2}\int \frac{1}{t^2+2t+1} - \frac{1}{4}\int \frac{1}{1+t} =$$

$$= -\frac{3}{4}\ln|1-\sin x| - \frac{1}{4}\ln|1+\sin x| + \frac{3}{2}\int \frac{1}{(1+t)^2}$$

$\cancel{\frac{1}{(1+t)^2} = \frac{A}{t}}$

$$\int \frac{1}{(1+t)^2} = \left| \begin{matrix} z = 1+t \\ dz = 1 \end{matrix} \right| = \int \frac{1}{t^2} dz = \int t^{-2} dz = t^{-1} + c$$

$$= -\frac{3}{4}\ln|1-\sin x| - \frac{1}{4}\ln|1+\sin x| + \frac{3}{2\cdot\sin x} + c, \; x \in \mathbb{R} \setminus \{0\} \setminus \{\tfrac{\pi}{2} + \pi n, n \in \mathbb{Z}\}$$

2.

~~a) $\int_1^3 x\,dx = [x]_1^3 = 3-1 = 2$~~

~~b) $\int_0^1 x^2\,dx = [x^2]_0^1 = 1-0 = 1$~~

~~c) $\int_0^1 \sqrt{x}\,dx = [\sqrt{x}]_0^1 = 1-0 = 1$~~

e) $\int_0^{\pi} x \cos x \, dx = [x \cos x \, dx]_0^{\pi} =$

a) $\int_1^3 x\,dx = \left[\frac{x^2}{2}\right]_1^3 = \frac{9}{2} - \frac{1}{2} = \frac{8}{2} = 4$

$\int x = \frac{x^2}{2}$

b) $\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

$\int x^2 = \frac{x^3}{3}$

c) $\int_0^1 \sqrt{x}\,dx = \left[\frac{2}{3}x\sqrt{x}\right]_0^1 = \frac{2\cdot 1}{3} - 0 = \frac{2}{3}$

$\int x^{\frac{1}{2}} = \frac{2}{3}x^{\frac{3}{2}}$

d) $\int_0^1 \arcsin x \, dx$

$$\int \arcsin x \, dx = \left| \begin{matrix} u = \arcsin x & v' = 1 \\ u' = \frac{1}{\sqrt{1-x^2}} & v = x \end{matrix} \right| = x \arcsin x -$$

$$- \int u'v = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx$$

$$\int \frac{x}{\sqrt{1-x^2}} = \left| \begin{matrix} 1 - x^2 = t \\ -2x \, dx = dt \\ x \, dx = \frac{dt}{-2} \end{matrix} \right| = -2 \int \frac{1}{\sqrt{t}} \, dt =$$

$$= -2 \int \cancel{4} \; t^{-\frac{1}{2}} \, dt = \cancel{-2\int t} \; -2 \cdot \left(\frac{1}{2} t^{\frac{1}{2}} \right) = -\sqrt{t} =$$

$$= -\sqrt{1-x^2}$$

$$\int \arcsin x \, dx = x \arcsin x + \sqrt{1-x^2} + C, \quad x \in [-1; 1]$$

$$\int_0^1 \arcsin x \, dx = \left[x \arcsin x + \sqrt{1-x^2} \right]_0^1 = \left(\frac{\pi}{2} + 0 \right) - 1 =$$

$$= \frac{\pi}{2} - 1$$

e) $\int_0^\pi x \cos x \, dx$

$$\int x \cos x \, dx = \left| \begin{matrix} u = x & v' = \cos x \\ u' = 1 & v = \sin x \end{matrix} \right| = x \sin x -$$

$$- \int \sin x = x \sin x + \cos x + C$$

$$\int_0^\pi x \cos x \, dx = \left[x \sin x + \cos x \right]_0^\pi = (\pi + 0) - 1 =$$

$$= -1 - 1 = -2$$

f) $\int_0^{\frac{1}{2}\ln 3} \frac{e^x}{e^{2x}+1}$

$$\int \frac{e^x}{e^{2x}+1} = \left| \begin{matrix} e^x = t \\ e^x dx = dt \end{matrix} \right| = \int \frac{1}{t^2+1} = \operatorname{arctg}(e^x) + C$$

$$\int_0^{\frac{1}{2}\ln 3} \frac{e^x}{e^{2x}+1} = \left[\operatorname{arctg} e^x\right]_0^{\frac{1}{2}\ln 3} = \operatorname{arctg} e^{\frac{1}{2}\ln 3} - \operatorname{arctg}(0)$$

$$= \operatorname{arctg} e^{\frac{1}{2}\ln 3} =$$

$$= \operatorname{arctg} 3^{\frac{1}{2}} = \operatorname{arctg} \sqrt{3} = \frac{\pi}{3}$$