CS270 - Week 6 Lecture Notes

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1 Logical Deduction

Logical Deduction is a method of formalizing the process of drawing conclusions from a set of premises. (Some of these examples are from a wiki! `https://en.wikipedia.org/wiki/Deductive_reasoning`)

1.1 Example 1

1. All men are mortal
2. Socrates is a man
3. Socrates is mortal

1.2 Example 2

1. If Larry is sick, then he will be absent.
2. If Larry is absent, then he will miss his classwork.
3. If Larry is sick, then he will miss his classwork.

1.3 Summary

These two examples look a lot more like the proofs you will generally see. A lot of people won't even write out the line numbers. These writers assume the reader can follow the rules of logic used. For now, we will be very formal about our rule usage.

A deductive proof starts with any Premises. These are things we assume to be true within the proof. If they are true in practice is not part of the deduction. We just work under the assumption they are true. Our proof is only correct if under these assumptions.

The last line of the Proof is the Conclusion. This is what we can show is true if the Premises are true.

In between, we use the laws of deduction to get from the premises to the conclusion.

2 Introduction Rules

2.1 And (Conjunction) Introduction

1.	A	Premise
2.	B	Premise
3.	$A \wedge B$	$\wedge I1, 2$

This proof can also be read as

$$A \wedge B \rightarrow (A \wedge B) \tag{1}$$

If we know that A is true and B is true, it is trivial to say that means $A \wedge B$ must also be true.

The right side of the deduction specifies where the rule came from. The first two lines are premises and assumed to be true. The $\vee I$ says we are using the **Conjunction Introduction** rule. The two numbers tells us which lines of the proof the two inputs came from. This should always be written with the left input as the first number and the right input as the second number.

2.2 Or (Disjunction) Introduction

1.	A	Premise
2.	$A \vee B$	$\vee I1$

If we know that A is true, we can draw the conclusion that $A \vee B$ is true. Note that B has never appeared in the proof up till now. It is a completely new variable. We can add ANYTHING with **Disjunction Introduction**.

1.	A	Premise
2.	$(A \vee (X \rightarrow (C \wedge Q)))$	$\vee I1$

This is true because the proof translates to

$$A \rightarrow (A \vee B) \tag{2}$$

The truth table for this expression is as follows

A	B	$A \rightarrow (A \vee B)$
T	T	T
T	F	T
F	T	T
F	F	T

If we know A is true because it is a premise, the value of B doesn't matter.

The note for Disjunction Introduction only takes one line number, the half of the or that is know to be true.

2.3 Implies (Conditional) Introduction

1.	A	Premise
2.	M	Assumption
3.	$M \wedge A$	$\wedge I1, 2$
4.	A	$\wedge E3$
5.	$M \rightarrow A$	$\rightarrow I2 - 4$

With a conditional introduction, the failure case is $F \rightarrow T$. We assume the left input is true and prove the right side must be true in that case.

3 Elimination Rules

Everything that can be introduced can also be eliminated.

3.1 Conjunction Elimination

1.	$(A \wedge B)$	Premise
2.	A	$\wedge E1$
3.	B	$\wedge E1$
4.	$B \wedge A$	$\wedge I2, 3$

If we know that $A \wedge B$ is true, we can draw the conclusion that both A and B are independently true.

Note, this above proves that Conjunctions are commutative!

3.2 Conditional Elimination

1.	$A \rightarrow B$	Premise
2.	A	Premise
3.	B	$\rightarrow E1, 2$

If we know that an implies is true and we know the left side is true, we can draw the conclusion that the right side MUST be true. The note lists the implies first and the left side justification second.

3.3 Disjunction Elimination

Disjunction Elimination requires making assumptions. If we know a disjunction, $A \vee B$ is true, we have no way to know which of the two values made it true. We have to assume both sides are true and see what happens in each case.

The subproof based in the assumptions is drawn in a smaller area.

1.	$A \to X$	Premise
2.	$B \to X$	Premise
3.	$(A \vee B)$	Premise
4.	A	Assumption
5.	X	$\to E1, 4$
6.	B	Assumption
7.	X	$\to E2, 6$
8.	X	$\vee E3, 4-5, 6-7$

The note for Disjunction Elimination requires three values. The original or, the proof when the left side is assumed true, and the proof when the right side is assumed true.

4 Reverse Introduction

When starting at with a conclusion, we can use any introduction rule backwards. We assume whatever would have been need to make the introduction happen.

The conclusion must have come from a introduction. We assume A is true and true to use this prove $(B \to C) \to D$.

1.	$(B \to C) \to (D \vee E)$	Premise
2.	$(B \vee C)$	Premise
3.	$(C \vee A) \to D$	Premise
4.	A	Assumption
5.	$(C \vee A)$	$\vee I4$
6.	D	$\to E3, 5$
7.	$B \to C$	Assumption
8.	$(B \to C) \wedge D$	$\wedge I6, 7$
9.	D	$\wedge E8$
10.	$(B \to C) \to D$	$\to I7-9$
11.	$(A \to ((B \to C) \to D))$	$\to I4-10$

5 Summary

These rules can be used together to create proofs.

1.	$(\sim K \to M)$	Premise
2.	$(N \to M)$	Premise
3.	$(\sim K \vee N)$	Premise
4.	$\sim K$	Assumption
5.	M	$\to E1, 4$
6.	$(M \vee R)$	$\vee I5$
7.	N	Assumption
8.	M	$\to E2, 7$
9.	$(M \vee R)$	$\vee I8$
10.	$(M \vee R)$	$\vee E3, 4-6, 7-9$

Revising Example 2

1. If Larry is sick, then he will be absent.
2. If Larry is absent, then he will miss his classwork.
3. If Larry is sick, then he will miss his classwork.

Let

$$\begin{aligned} S &= \text{Larry is Sick} \\ A &= \text{Larry is Absent} \\ \sim C &= \text{Miss Classwork} \end{aligned}$$

Then we have the following proof.

1.	$S \to A$	Premise
2.	$A \to C$	Premise
3.	S	Assumption
4.	A	$\to E1, 3$
5.	C	$\to E2, 4$
6.	$S \to C$	$\to I5-5$

6 Terminology

- **Valid** - It is impossible for the premises to be true and conclusion to be false.
- **Invalid** - There exists at least one case where the premises are all true but conclusion is false.
- **Sound** - The argument is valid AND all the premises are actually true.

- **Soundness** - A proof method that cannot prove things that are not true in the truth table sense
- **Completeness** - A proof method that can prove anything that is true in the truth table sense

Your proof can be valid but not actually correct if your premises are bad. This would be an unsound proof. It relied on a flawed assumption.