CS 270 Lab 8 (Recursion and Induction)

Week 8 - Nov 13 – Nov 17, 2017.

Name 1:_____________Solution Key____

Drexel Username 1:____

Instructions: For this exercise you are encouraged to work in groups of two or three so that you can discuss the problems, help each other when you get stuck and check your partners work. There are four problems that ask you to form conjectures and prove your conjectures using induction. The last problem requires a recursive construction and inductive proofs based on the construction.

Note arguments about equalities and inequalities use the following properties of <, = and ≤

1. A = B implies B = A [symmetry of =]
2. A = B and B = C implies that A=C [transitivity of =]
3. A < B and B < C implies A < C [transitivity of <]
4. A < B implies A+C < B+C [relation of < and +]
5. A < B and C > 0 then A·C < B·C [relation of < and ·]
6. A ≤ B and B ≤ C implies A ≤ C [A ≤ B ≡ A < B ∨ A = B and its properties follow from < and =]
7. A ≤ B implies A + C ≤ B + C
8. A ≤ B and C ≥ 0 implies A·C ≤ B·C

1. Create a table of values for n^2 and 2^n and verify that eventually $n^2 < 2^n$. Use induction to prove $n^2 < 2^n$, $\forall n \geq 5$.

n	n^2	2^n
1	1	2
2	4	4
3	9	8
4	16	16
5	25	32
6	36	64
7	49	128

Base case. (n=5). See table.

Inductive Hypothesis (IH). Assume $n^2 < 2^n$

Show that IH implies $(n+1)^2 < 2^{n+1}$

$(n+1)^2 = n^2 + 2n + 1$

$< n^2 + 3n$ since $1 < n$ [assumption that $n \geq 5$] and property 4 of <

$< 2n^2$ since $3 < n$, $3n < n^2$ [property 5 of <] and property 4 of <

$< 2 \cdot 2^n$ by IH

$= 2^{n+1}$

From the chain of inequalities and equalities, we conclude, by transitivity, $(n+1)^2 < 2^{n+1}$. A similar argument will show that eventually $n^k < 2^n$ for all nonnegative integers k.

2. Create a table of values for n! and 2^n and verify that eventually $2^n < n!$ Use induction to prove $2^n < n!$, $\forall\, n \geq 4$.

Note that n! is defined recursively by $n! = n\cdot(n-1)!$ For $n>0$ and $0! = 1$

n	n!	2^n
1	1	2
2	2	4
3	6	8
4	24	16
5	120	32
6	720	64
7	5040	128

Base case. (n=4). See table.

Inductive Hypothesis (IH). Assume $n! > 2^n$

Show that IH implies $(n+1)! > 2^{n+1}$

$(n+1)! = (n+1)\cdot n!$ by definition of factorial

$> (n+1)\cdot 2^n$ by IH

$> 2\cdot 2^n = \cdot\, 2^{n+1}$ since $(n+1) > 2$ [assumption that $n \geq 4$] and property 5 of <

3. Use induction to prove that any postage amount greater than 11 can be created using only 4 and 5 cent stamps. First determine which values upto 15 can be done with 4 and 5 cent stamps. How many base cases do you need? What if instead you use 4 and 6 cent stamps?

You can think of this as a recursive procedure to find a and b such that $n = 4a+5b$.

The following table shows how to create postage for amounts ranging from 1 to 15 cents.

Amount	Is it possible to create postage of n with 4 and 5 cent stamps
1	No
2	No
3	No
4	4
5	5
6	No
7	No
8	4+4
9	4+5
10	5+5
11	No
12	4+4+4
13	4+4+5
14	4+5+5
15	5+5+5

We now argue by induction that it is always possible for $n \geq 12$.

Base case. (n=12, 13, 14, and 15 from the table. Four base cases are necessary because the inductive argument uses the four sizes smaller than the current size.

Assume that it is possible for all amounts $m < n$. That is $\exists a,b \geq 0$ such that $m = 4a+5b$. [IH]

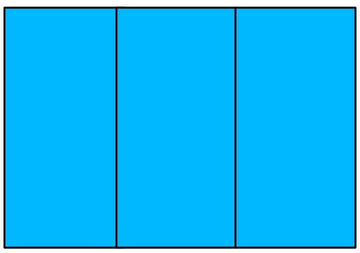
Show that this implies that it is possible to find integers a' and b' such that $n = 4a' + 5b'$.

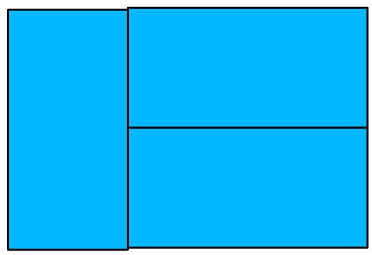
By the IH we can find a and b such that $(n-4) = 4a + 5b$. Therefore $n = 4(a+1) + 5b$.

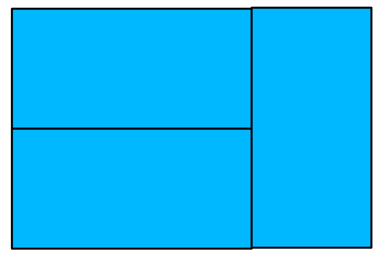
If we did not have four consecutive base cases then we would not always be able to find a and b for n-4 [The recursion might not hit the base case]. More formally, let $S(n)$ be the hypothesis that $\exists a,b \geq 0$ such that $n = 4a+5b$, then $\forall n \geq 12$ $S(n)$ follows from $S(12)$, $S(13)$, $S(14)$, $S(15)$ and $S(n) \rightarrow S(n+4)$.

For 4 and 6 cent stamps the result is not true (there is no way to produce an odd number $= 4a+6b$). The inductive argument will fail because we will not be able to find four consecutive base cases. More generally it is only possible for stamps of values s_1 and s_2 when $\gcd(s_1,s_2)=1$.

4. Given a set of 1 × 2 dominoes, how many different ways are there to tile and 2 × n rectangle placing a single domino vertically (2 × 1) or stacking two dominoes horizontally? For example, for n=3, there are 3 ways.







Think recursively: What are the base cases? How do you solve the 2 × n using solutions to smaller problems?

- Derive a recurrence relation for D(n) the number of tilings of a 2 × n rectangle.
- Produce a table of values of D(n) and compare to 2^n and $(3/2)^n$
- Formulate a conjecture and use induction to prove your result.

Note that any tiling must start with either a vertical tile or two stacked horizontal tiles. The all tilings of size n are either a vertical tile followed by a tiling of size n-1 or two horizontal tiles followed by a tiling of size n-2. For n=1 there is only one tiling and for n=2 there are two tilings (two vertical tiles or two horizontal tiles).

Therefore the number of tilings of size n, D(n) satisfies the recurrence relation D(n) = D(n-1)+D(n-2) for n > 2 and D(1)=1, D(2)=2. These are the Fibonacci numbers shifted by one position, i.e. D(n) = the (n+1)-st Fibonacci number F_n. The following table suggests D(n) < 2^n for n $\geq$ 1 and D(n) > $(3/2)^n$ for n $\geq$ 5.

n	D(n)	2^n	$(3/2)^n$
1	1	2	3/2 = 1.5
2	2	4	9/4 = 2.25
3	3	8	27/8 = 3.375
4	5	16	81/16 = 5.0625
5	8	32	243/32 $\approx$ 7.59
6	13	64	729/64 $\approx$ 11.391
7	21	128	2187/128 $\approx$ 17.086
8	34	256	6561/256 $\approx$ 25.629

This can be proven by induction.

Base case (n=1 and n=2) from the table. Why are there two base cases?

Assume the induction hypothesis $D(n-1) < 2^{n-1}$ and $D(n-2) < 2^{n-2}$ and show $D(n) < 2^n$.

$D(n) = D(n-1) + D(n-2)$ By the definition of D(n).

$< 2^{n-1} + 2^{n-2}$ By IH and property 4 of < [applied twice with transitivity, i.e.

$D(n-1) < 2^{n-1} \rightarrow D(n-1)+D(n-2) < 2^{n-1} + D(n-2)$ and $D(n-2) < 2^{n-2} \rightarrow$

$2^{n-1} + D(n-2) < 2^{n-1} + < 2^{n-2}$

$< 2^{n-1} + 2^{n-1} = 2^n$ By property 4 of < and property of powers.

Similarly, we prove $D(n) > (3/2)^n$ for $n \geq 5$ by induction on n. The base cases n=5 and n=6 follow from the calculation in the table. Assume the induction hypothesis $D(n-1) > (3/2)^{n-1}$ and $D(n-2) > (3/2)^{n-2}$ and show $D(n) > (3/2)^n$.

$D(n) = D(n-1) + D(n-2)$ By the definition of D(n).

$> (3/2)^{n-1} + (3/2)^{n-2}$ By IH and property 4 of < [

$= (3/2)^{n-2} (3/2 + 1)$ By property of powers and distributive property.

$> (3/2)^{n-2}(3/2)^2 = (3/2)^n$ Since $(3/2+1) = 2.5 > (3/2)^2 = 2.25$.

Note that these properties of D(n) can be sharpened to get tighter bounds. The following property holds for sufficiently large n. $(F_{2k+2}/F_{2k+1})^n < D(n) < (F_{2k+1}/F_{2k})^n$, for k=1,2,… and F_j is the j-th Fibonacci number. This follows from Cassini's identity which states that $F_{n+1} F_{n-1} – F_2 = -1^n$ which can be proved by induction. Further note that the ratio of consecutive Fibonacci numbers approaches the golden ratio, $\phi \approx 1.618$ which is the positive root of x^2-x-1. Therefore the number of domino configurations grows like ϕ^n.