CS 270 : Introduction to Logic

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Logic

- Logic is a systematic way of thinking that allows us to deduce new information from old information and to parse the meaning of sentences.
- You use logic constantly in your everyday life.
- Formal Logical gives a consistent framework to communicate ideas.

Logic

- We use facts to deduce new information
- Example:
  1. Circle X has radius equal to 3
  2. If any circle has radius r, then its area is πr^2
  3. Therefore: Circle X has area 9π square units
- Logic is a process of deducing information correctly
- We have deduced that if 1 and 2 are true then 3 is true

Information

- Both our Logic and Information must be true
- Example: if the circle's radius is different than the area is also wrong
- Information
  - This is believed to be true at the start of deduction
  - Can be trivially true (5 is a number)
  - or a proven fact ($\sum_{i=0}^{\infty} \frac{1}{2^i} = 2$)
- Logic uses information to draw conclusions
- We assume our information is known to be true or false
- There is no middle ground, everything must be true or false

Statements

- A statement is a sentence or expression that is either definitely true or false
- Statements may be dependent on variables
- We assign letters or names to statements when doing a proof
- A statement is Boolean, it can be true or false

$$S \text{ :The solution of } 2x = 84 \text{ is } 42.$$

$$P(x) \text{ : If an integer x is a multiple of 6, then x is even}$$

- a statement that is not always true or false is an called a contingent statement
    - $Q(x)$: x is even

Types of Statements

- **Contingent**: True or False based on values of variables.
- **Tautology**: Always true regardless of variables.
- **Contradictory**: Always false regardless of variables.
- **Satisfiable**: Can be made true.
- **Falsifiable**: Can be made false.

Boolean Operators

- Boolean Logic has a set of operators.
- We combine statements using Boolean operators.
- These are all based on their normal usage in language.
- **And**: Both inputs are true
- **Or**: At least one of the two is true
- **Not**: Get the opposite

And

- The number X is even **and** the number Y is odd.
- And: is true when both statements are true
- A true table show all possible outcomes based on statements.

$$P(X) : \text{The number X is even}$$
$$Q(Y) : \text{The number Y is odd}$$

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

- If X=2 and Y=4, look at the second row of the table

Or

- The number X is even **or** the number Y is odd.
- Or: Either one of both of the statements are true

$$P(X) : \text{The number X is even}$$
$$Q(Y) : \text{the number Y is odd}$$

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Not

- It is not true that the number X is even
- Not: returns the opposite

$$P(X) : \text{The number X is even}$$

P	$\sim P$
T	F
F	T

Other Common Symbols

- Difference fields use different symbols for these
- Personal Preference: $\wedge, \vee, \sim$.

Meaning	Operators
A and B	$A \wedge B$, $A \cap B$, $A\dot{B}$, AB, $A\&B$, $A\&\&B$
A or B	$A \vee B$, $A \cup B$, $A + B$, $A\vert B$, $A\vert\vert B$
not A	$\sim A$, $-A$, $\bar{A}$, $!A$

Conditional Statements

- Conditional Statements are used with the implies operator
- You can think of $P \Rightarrow Q$ as being a promise that whenever P is true, Q will be true also.
- if the integer a is a multiple of 6, then a is divisible by 2.

$$P : \text{The integer } a \text{ is a multiple of 6.}$$
$$Q : \text{The integer } a \text{ is a multiple of 2.}$$
$$R : \text{If P, the Q}$$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Implies

- An implies is only false when $T \Rightarrow F$
- **If** you pass the final exam, **then** you will pass the course
  - If you pass the final and pass the class, the statement was true.

$$(T \Rightarrow T) = T$$

  - If you fail the final, you could still pass the class.

$$(F \Rightarrow T) = T$$

  - If you fail the final, you could still fail the class.

$$(F \Rightarrow F) = T$$

  - If you pass the final and do not pass the course, then the statement was a false.

$$(T \Rightarrow F) = F$$

Implies

- An implies statement is either **valid** or **invalid**
- **Valid**: The condition $T \Rightarrow F$ is impossible
- **Invalid**: The condition $T \Rightarrow F$ can happen
- If a statement is **valid** we can use it to draw conclusions.
- Alternatives:
  - "P only if Q" is the same as "if P then Q"
  - "P if Q" is the same as "If Q then P"

Bi-conditional Statements

- $P \Rightarrow Q$ only handles one direction
- What if we want to say "If P then Q and If Q then P"
- P if and only if Q
- $P \iff Q$
- P iff Q is used as a shorthand.
- This is a Boolean equals!

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

- We can write expressions using the Boolean operators
- Proofs can be done by truth table
- Proofs can be done algebraically

Example: P or Q is true, and it is not the case that both P and Q are true.

$$(P \vee Q) \wedge \sim (P \wedge Q)$$

P	Q	$(P \vee Q)$	$(P \wedge Q)$	$\sim (P \wedge Q)$	$(P \vee Q) \wedge \sim (P \wedge Q)$
T	T	T	T	F	**F**
T	F	T	F	T	**T**
F	T	T	F	T	**T**
F	F	F	F	T	**F**

Truth Tables

- A truth table is a proof
- It shows all possible outcomes
- A row of the truth table shows if an expression is true of false based on initial values
- If two expressions have the same truth table, they are the same

Example: Proof that $P \Rightarrow Q$ and $\sim P \vee Q$ are the same.

P	Q	$P \Rightarrow Q$	$\sim P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Logical Equivalence

- If two statements have the same truth table then they are logically equivalent
- We can do algebra on statements
- Logical Equivalence is show using the $=$ symbol.
- This makes it clear we are doing math $=$ instead of stating equivalence $\iff$

$$
\begin{aligned}
\mathbf{T} \Rightarrow P &= \sim \mathbf{T} \vee P \\
&= \mathbf{F} \vee P \\
&= P
\end{aligned}
$$

Logical Equivalences

- The following are commonly used logical equivalences involving constants

$$
\begin{aligned}
T \wedge A &= A \\
F \wedge A &= F \\
T \vee A &= T \\
F \vee A &= A
\end{aligned}
$$

Logical Equivalences

- Implies and "If and only if" can be written in terms of simpler operators

$$\begin{aligned} A \Rightarrow B &= \sim A \vee B \\ A \iff B &= (A \Rightarrow B) \wedge (B \Rightarrow A) \\ &= (A \wedge B) \vee (\sim A \wedge \sim B) \end{aligned}$$

- We can prove these with truth tables.

If and Only If

- If the following statement is a tautology then the two expressions are equivalent.

$$(A \iff B) \iff ((A \wedge B) \vee (\sim A \wedge \sim B))$$

A	B	$A \iff B$	$(A \wedge B) \vee (\sim A \wedge \sim B)$	$\cdots \iff \cdots$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	T	T

DeMorgan's Law

- DeMorgan's Law is one of the most important logical equivalences
- Allows for distribution of a not into expressions

$$\sim (P \wedge Q) = (\sim P) \vee (\sim Q)$$
$$\sim (P \vee Q) = (\sim P) \wedge (\sim Q)$$

P	Q	$\sim (P \wedge Q)$	$(\sim P) \vee (\sim Q)$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

DeMorgan's Law

$$\sim (P \vee Q) = (\sim P) \wedge (\sim Q)$$

P	Q	$\sim (P \vee Q)$	$(\sim P) \wedge (\sim Q)$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Other Important Laws

- The following laws are also commonly used
- Notice **and** works like **multiplication** and **or** works like **plus**

$$
\begin{aligned}
\sim\sim P =& P \\
P \Rightarrow Q =& (\sim Q) \Rightarrow (\sim P) \\
P \wedge Q =& Q \wedge P \\
P \vee Q =& Q \vee P \\
P \wedge (Q \vee R) =& (P \wedge Q) \vee (P \wedge R) \\
P \vee (Q \wedge R) =& (P \vee Q) \wedge (P \vee R) \\
P \wedge (Q \wedge R) =& (P \wedge Q) \wedge R \\
P \vee (Q \vee R) =& (P \vee Q) \vee R
\end{aligned}
$$