# CS 270 : Induction

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# Mathematical Induction

- Conjecture: The sum of the first $n$ odd natural numbers equals $n^2$
- How can be prove this?
- There are an infinite number of possible cases.

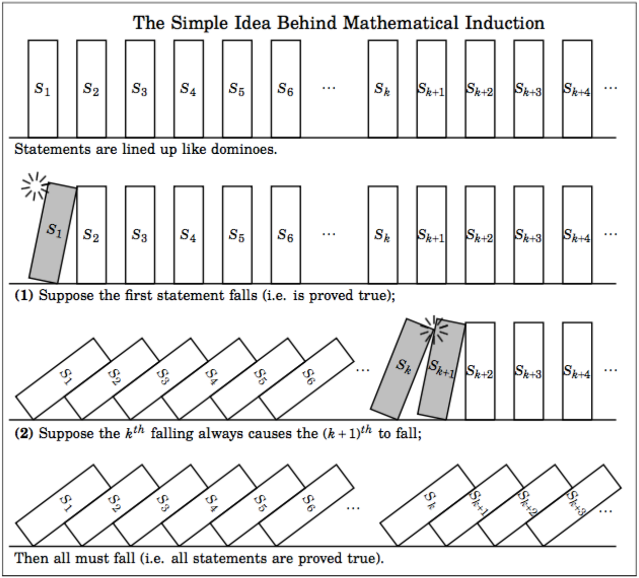
| $n$ | sum of the first $n$ odd natural numbers | $n^2$ |
|---|---|---|
| 1 | $1 =$ | 1 |
| 2 | $1 + 3 =$ | 4 |
| 3 | $1 + 3 + 5 =$ | 9 |
| 4 | $1 + 3 + 5 + 7 =$ | 16 |
| 5 | $1 + 3 + 5 + 7 + 9 =$ | 25 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | $1 + 3 + 5 + 7 + 9 + 11 + \cdots + (2n - 1) =$ | $n^2$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

# Mathematical Induction

- We can see that the first five lines were correct
- There is a pattern $\sum_{i=1}^{n}(2i-1)$
- How can we prove this conjecture is always true?
- Are all of the following statements true $\forall n \in \mathbb{N}$?

$$\begin{aligned}
S_1 &: 1 = 1^2 \\
S_2 &: 1 + 3 = 2^2 \\
S_3 &: 1 + 3 + 5 = 3^2 \\
&\vdots \\
S_n &: \sum_{i=1}^{n}(2n-1) = n^2 \\
&\vdots
\end{aligned}$$

# Mathematical Induction



# Mathematical Induction

**Outline for Proof by Induction.**

**Proposition** The statements $S_1, S_2, S_3, S_4, \cdots$ are all true.

*Proof.* (Induction)

(1) Prove that the first statement $S_1$ is true.

(2) Given any integer $k \geq 1$, prove that the statement $S_k \Rightarrow S_{k+1}$ is true.

It follows by mathematical induction that every $S_n$ is true. ■

# Inductive Steps

An Induction Proof has two steps

Step 1 The Basis Step

- Prove the starting point $S_1$ is true
- This is often trivial
- Many times we can just show by evaluation

Step 2 The Inductive Step

- Prove that $S_k \Rightarrow S_{k+1}$
- Give a direct proof that one answer implies the next
- Many times this will be an algebraic proof
- We assume $S_k$ is true during this proof
- This assumption is called the **inductive hypothesis**

# Example

## The sum of the first $n$ odd numbers is $n^2$.

**Proposition** If $n \in \mathbb{N}$ then $\sum_{i=1}^{n}(2i-1) = n^2$.

*Proof.* We will prove this with mathematical induction.

(1) Observer that if $n = 1$, this statement is $1 - 1^2$, which is obviously true.

(2) We must now prove $S_k \Rightarrow S_{k+1}$ for an $k \geq 1$.
**Inductive Hypothesis**
Assume that $S_k$ is true, $\sum_{i=1}^{k}(2i-1) = k^2$.
**Inductive Proof**
See Next Slide for proof that $S_k \Rightarrow S_{k+1}$.
If follows by induction that $\sum_{i=1}^{n}(2i-1) = n^2$ for every $n \in \mathbb{N}$.

∎

# Example

**Inductive Hypothesis**
Assume that $S_k$ is true, $\sum_{i=1}^{k}(2i-1) = k^2$.
**Inductive Proof**

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i-1) = &\left(\sum_{i=1}^{k}(2i-1)\right) + 2(k+1) - 1 \\
=&k^2 + 2(k+1) - 1 \text{(By inductive Hypothesis)} \\
=&k^2 + 2k + 2 - 1 \\
=&k^2 + 2k + 1 \\
=&(k+1)(k+1) \\
=&(k+1)^2
\end{aligned}$$

# Example 2

- Last week we saw the sum

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$$

- We justified this using some reasoning about the summation
- We can use induction to prove it is true.
- The smallest number $n$ can take $n = 1$, this will be the base case

# Example 2 (Base Case)

We want to prove

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$$

Base Case: $n = 1$

$$\sum_{i=1}^{1} i = 1$$

$$\frac{1}{2}(1)((1)+1) = \frac{1}{2}2 = 1$$

At $n = 1$ both the left and right side of the equation evaluate to 1. This proves the base case is correct.

We want to prove

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$$

We need to make an inductive hypothesis to prove $S_k \Rightarrow S_{k+1}$.

**Inductive Hypothesis**

Assume that

$$\sum_{i=1}^{k} i = \frac{1}{2}k(k+1)$$

# Example 2 (Inductive Case)

**Inductive Hypothesis**

$$\sum_{i=1}^{k} i = \frac{1}{2}k(k+1)$$

**Inductive Proof**

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \left(\sum_{i=1}^{k} i\right) + k + 1 = \frac{1}{2}k(k+1) + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k^2+3k+2}{2} = \frac{k^2+3k+2}{2} \\ &= \frac{(k+1)(k+2)}{2} = \frac{1}{2}(k+1)((k+1)+1)\end{aligned}$$

# Strong Induction

Sometimes, one base cases isn't enough to prove a statement.

**Outline for Proof by Strong Induction.**

**Proposition** The statement $S_1, S_2, S_3, S_4, \cdots$ are all true.

*Proof.* (Strong Induction)

(1) Prove that the first $x$ statements are true. $(S_1, S_2, \cdots S_x)$

(2) Given any integer $k \geq x$, prove that $(S_1 \wedge S_2 \wedge S_3 \wedge \cdots \wedge S_x) \Rightarrow S_{k+1}$

■

# Strong Induction Example

Prove that any postage amount greater than 11 cents can be created using only 4 and 5 cent stamps.
The first value greater than 11 cents is 12 cents.

$$4 + 4 + 4 = 12$$

What happens if we use this as our only base case?

**Inductive Hypothesis**
Assume that for some value $k$, we can come up with values $a, b$ such that

$$a * 4 + b * 5 = k$$

For example, $a = 3, b = 0$ got us $k = 12$.
**Inductive Proof**

$$\begin{aligned} k + 1 =& k + 1 - 4 \\ =& k - 3 \end{aligned}$$

If we have assumed $S_{k-3}$ was true, this proof would work.
We can only assume $S_k$ because we only proved 1 base case.

# Strong Induction

We need more base cases!

$$\begin{aligned}
S_{12} :& 3 * 4 = 12 \\
S_{13} :& 5 + 2 * 4 = 13 \\
S_{14} :& 2 * 5 + 4 = 14 \\
S_{15} :& 3 * 5 = 15
\end{aligned}$$

Now, we can assume $S_k, S_{k-1}, S_{k-2}, S_{k-3}$
This can make our previous statement true.

$$S_{k-3} \Rightarrow S_{k+1}$$

# Strong Induction

Prove that any postage amount greater than 11 cents can be created using only 4 and 5 cent stamps.
What Happened?
Our inductive Proof Required

$$S_{k-3} \Rightarrow S_{k+1}$$

We need this to be true for all numbers, we can't have a gap!
What we needed to prove was

$$(S_{k-3} \land S_{k-2} \land S_{k-1} \land S_k) \Rightarrow S_{k+1}$$

We can now cover all numbers $k > 11$

# Smallest Counterexample

## Outline of Proof by Smallest Counterexample.

**Proposition** The statements $S_1, S_2, S_3, S_4, \ldots$ are all true.

*Proof.* (Smallest counterexample)

(1) Check that the first statement $S_1$ is true.

(2) For the sale of contradiction, suppose not every $S_n$ is true.

(3) Let $k > 1$ be the smallest integer for which $S_k$ is **false**.

(4) Then $S_{k-1}$ is true and $S_k$ is false. Use this to get a contradiction.

■

# Counterexample

**Fundamental Theorem of Arithmetic.**

Any integer $n > 1$ has a unique prime factorization. That is, if $n = p_1 \cdot p_2 \cdot p_3 \cdots p_k$ and $n = a_1 \cdot a_2 \cdot a_3 \cdots a_l$ are two prime factorizations of $n$, then $k = l$, and the primes $p_i$ and $a_i$ are the same, except that they may be in a different order. ■

The base case works the same way. The smallest integer $n > 1$ is 2. I has a prime factorization of

$$2 = 2$$

The inductive hypothesis is similar.

**Inductive Hypothesis**

Assume that for all numbers $2 \leq k < n$ the theorem is true.

# Counterexample

The inductive proof is different. We now show that the claim $S_k$ is false cannot hold.

Let $n$ be the first time this theorem is false.

Then $n$ has two prime factorizations

$$n = a_1 \cdot a_2 \cdot a_3 \cdots a_k$$
$$n = p_1 \cdot p_2 \cdot p_3 \cdots p_l$$

Since these are primes, we have

$$\frac{n}{p_1} = \frac{a_1 \cdot a_2 \cdot a_3 \cdots a_k}{p_1} = p_2 \cdot p_3 \cdots p_l$$

Therefore, we have $p_1 = a_i$ for some $i$.

# Contradiction

We have divided out one number from both prime factorizations.

$$\frac{n}{p_1} = \frac{a_1 \cdot a_2 \cdot a_3 \cdots a_k}{p_1} = p_2 \cdot p_3 \cdots p_l$$

We now have a new number

$$\begin{aligned} \frac{n}{p_1} =& p_2 \cdot p_3 \cdots p_l \\ \frac{n}{p_1} =& a_1 \cdot a_2 \cdots a_{i-1} \cdot a_{i+1} \cdots a_k \end{aligned}$$

The number $\frac{n}{p_1}$ is clearly an integer and $1 < \frac{n}{p_1} < n$. This means it only has one prime factorization, but this is a contradiction to our assumption that $n$ had two unique factorizations.

# Fibonacci Series

The Fibonacci Series is a famous integer sequence.

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \cdots$$

It is defined recursively

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \end{aligned}$$

It also has a closed form

$$\begin{aligned} F_n &= \frac{\varphi^n - \psi^n}{\sqrt{5}} \\ \varphi &= \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2} \end{aligned}$$

This can be proven by induction. This proof has multiple cases cases.

**Base Case**

$$
\begin{aligned}
F_0 =& 0 \\
\frac{\varphi^0 - \psi^0}{\sqrt{5}} =& \frac{1-1}{\sqrt{5}} = 0 \\
F_1 =& 1 \\
\frac{\varphi^1 - \psi^1}{\sqrt{5}} =& \frac{\varphi - \psi}{\sqrt{5}} = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} \right) \\
=& \frac{1}{2\sqrt{5}} \left( 1 + \sqrt{5} - 1 + \sqrt{5} \right) = 1
\end{aligned}
$$

**Inductive Hypothesis**
$\forall k \in \{0 \leq k \leq n\}$ the closed form of the fibonacci series is correct.
**Inductive Case**

$$\begin{aligned} F_{n+1} =& F_n + F_{n-1} \\ =& \frac{\varphi^n - \psi^n}{\sqrt{5}} + \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} \\ =& \frac{\varphi^n + \varphi^{n-1} - \left(\psi^n + \psi^{n-1}\right)}{\sqrt{5}} \end{aligned}$$

This algebra requires some additional work.

# Additional Formulas

We show that $\varphi^2 = \varphi + 1$

$$\begin{aligned}
\varphi =& \frac{1+\sqrt{5}}{2} \\
\varphi^2 =& \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) \\
=& \frac{1+2\sqrt{5}+5}{4} = \frac{\left(3+\sqrt{5}\right)}{2} \\
=& \frac{2}{2} + \frac{1-\sqrt{5}}{2} \\
=& 1+\varphi
\end{aligned}$$

# Additional Formulas

It is also true that $\psi^2 = \psi + 1$

$$\begin{aligned}
\psi =& \frac{1-\sqrt{5}}{2} \\
\psi^2 =& \left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) \\
=& \frac{1-2\sqrt{(5)}+5}{4} = \frac{3-\sqrt{5}}{2} \\
=& \frac{2}{2} + \frac{1-\sqrt{5}}{2} \\
=& 1+\psi
\end{aligned}$$

# Inductive Case

We now complete the proof.

$$
\begin{aligned}
F_{n+1} =& F_n + F_{n-1} \\
=& \frac{\varphi^n + \varphi^{n-1} - \left(\psi^n + \psi^{n-1}\right)}{\sqrt{5}} \\
=& \frac{\varphi^{n-1}\left(\varphi + 1\right) - \psi^{n-1}\left(\psi + 1\right)}{\sqrt{5}} \\
=& \frac{\varphi^{n-1}\left(\varphi^2\right) - \psi^{n-1}\left(\psi^2\right)}{\sqrt{5}} \\
=& \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}} \\
=& F_{n+1}
\end{aligned}
$$