# Method of Partial Fractions

## ■ Partial Fraction Decomposition

Suppose we have a rational function, i.e. a function of the form $R(x) = P(x)/Q(x)$ where $P$ and $Q$ are polynomials. Under certain conditions we can decompose $R$ as a sum of simpler rational functions. This will usually make $R$ easier to integrate.

▶ **Example 1** We can add two rational functions by getting a common denominator:

$$\frac{2}{x+1} + \frac{5}{x-2} = \frac{2(x-2)+5(x+1)}{(x+1)(x-2)} = \frac{7x+1}{x^2-x-2}.$$

The method of partial fraction decomposition reverses this process, i.e. we start with the fraction

$$\frac{7x+1}{x^2-x-2}$$

and rewrite it as the sum

$$\frac{2}{x+1} + \frac{5}{x-2}.$$

**Theorem** Let $P(x)/Q(x)$ be a rational function where the degree of $P$ is less than the degree of $Q$. Then we can express this rational function as a sum

$$\frac{P(x)}{Q(x)} = F_1(x) + F_2(x) + \cdots + F_n(x)$$

where $F_1(x), F_2(x), \cdots, F_n(x)$ are rational functions of the form

$$\frac{A}{(ax+b)^k} \qquad \text{or} \qquad \frac{Ax+B}{(ax^2+bx+c)^k}$$

in which the denominators are factors of the denominator $Q(x)$.

The following rules help determine the form of the partial fraction decomposition:

**Linear Factor Rule**

If $(ax+b)^m$ is a factor of $Q(x)$, the partial fraction decomposition contains the sum

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}.$$

**Quadratic Factor Rule**

If $(ax^2+bx+c)^m$ is a factor of $Q(x)$, the partial fraction decomposition contains the sum

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}.$$

These rules tell us the what the partial fraction decomposition will look like, but not the values of constants in the numerator. These we must solve for separately.

▶ **Example 2** Find the form of the partial fraction decompositions for the following fractions:

(1) $\dfrac{3x+5}{(x+2)(x-1)}$

(2) $\dfrac{3}{(x^2+1)(x+2)}$

(3) $\dfrac{2x^2+x+1}{(x+1)^3}$

***Solution***

(1) In this case we have two linear factors, each only raised to the first power. Applying the linear factor rule to each one we get
$$\frac{3x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

(2) In this case we have a linear factor and a quadratic factor, each only raised to the first power. Applying both rules above we get
$$\frac{3}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2}$$

(3) In this case we have a linear factor raised to the power $m = 3$. Applying the linear factor rule we get
$$\frac{2x^2+x+1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

Once we have established the form of the partial fraction decomposition we can solve for the constants in the numerators by combining the fractions in the decomposition using $Q(x)$ as a common denominator. By equating the coefficients in the numerator of each side, we obtain equations that can be used to solve for the constants.

▶ **Example 3** Find the constants for the partial fraction decomposition of (1) and (2) in the above example.

***Solution***

(1) Combining the fractions in the decomposition from (1) above gives

$$\frac{3x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}.$$

Equating the numerators (and collecting powers of $x$) gives

$$3x+5 = (A+B)x + (2B-A).$$

Equating the coefficients of each power of $x$ gives the system

$$A+B=3$$
$$2B-A=5$$

Solving this system gives $A = 1/3$ and $B = 8/3$. This means the partial fraction decomposition is

$$\frac{3x+5}{(x+2)(x-1)} = \frac{1/3}{x+2} + \frac{8/3}{x-1} = \frac{1}{3(x+2)} + \frac{8}{3(x-1)}.$$

(2) Combing the fractions in the decomposition from (2) above gives

$$\frac{3}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2} = \frac{(Ax+B)(x+2)+C(x^2+1)}{(x^2+1)(x+2)}.$$

Equating the numerators (and collecting powers of $x$) gives

$$3 = (A+C)x^2 + (2A+B)x + (2B+C).$$

Equating the coefficients of each power of $x$ (most of coefficients on the left are 0) gives the system

$$A+C=0$$
$$2A+B=0$$
$$2B+C=3$$

Solving this system gives $A = -3/5$, $B = 6/5$, and $C = 3/5$. This means the partial fraction decomposition is

$$\frac{3}{(x^2+1)(x+2)} = \frac{(-3/5)x+(6/5)}{x^2+1} + \frac{3/5}{x+2} = \frac{-3x+6}{5(x^2+1)} + \frac{3}{5(x+2)}$$

▶ **Example 4** Use the method of partial fractions to evaluate the integral

$$\int \frac{3}{x^3 + 2x^2 + x + 2}\, dx$$

***Solution*** We can factor the denominator as

$$x^3 + 2x^2 + x + 2 = (x^2 + 1)(x + 2)$$

Then using the decomposition from (2) in the previous example this integral becomes

$$\begin{aligned}
\int \frac{3}{x^3 + 2x^2 + x + 2}\, dx &= \int \frac{-3x + 6}{5(x^2 + 1)} + \frac{3}{5(x + 2)}\, dx \\
&= -\frac{3}{5} \int \frac{x}{x^2 + 1}\, dx + \frac{6}{5} \int \frac{1}{x^2 + 1}\, dx + \frac{3}{5} \int \frac{1}{x + 2}\, dx \\
&= -\frac{3}{5} \left( \frac{1}{2} \ln(x^2 + 1) \right) + \frac{6}{5} \tan^{-1} x + \frac{3}{5} \ln(x + 2) + C
\end{aligned}$$