# Useful Probability Cheatsheet Part 1

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## 1 Properties of the Integral and Expected Values

**Definition 1.1** *A real valued function $X$ defined on $\Omega$ is said to be a* ***random variable*** *if for every Borel set $B \subset \mathbb{R}$ we have $X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F}$. When we need to emphasize the $\sigma$-field, we will say that $X$ is* $\mathcal{F}$***-measurable*** *or write $X \in \mathcal{F}$*

**Theorem 1.1 (Jensen's inequality.)** *Suppose $\phi$ is convex, that is,*

$$\lambda\phi(x) + (1-\lambda)\phi(y) \geq \phi(\lambda x + (1-\lambda)y)$$

*for all $\lambda \in (0,1)$ and $x, y \in \mathbb{R}$. If $\mu$ is a probability measure, and $f$ and $\phi(f)$ are integrable then*

$$\phi\Big(\int f d\mu\Big) \leq \int \phi(f) d\mu$$

**Theorem 1.2 (Jensen's inequality.)** *Suppose $\phi$ is convex, then*

$$E(\phi(X)) \geq \phi(EX)$$

*provided both expectations exist, i.e., $E|X|$ and $E|\phi(X)| < \infty$*

**Theorem 1.3 (Bounded convergence theorem.)** *Let $E$ be a set with $\mu(E) < \infty$. Suppose $f_n$ vanishes on $E^c$, $|f_n(x)| \leq M$, and $f_n \to f$ in measure. Then*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

**Theorem 1.4 (Fatou's lemma.)** *If $f_n \geq 0$ then*

$$\liminf_{n\to\infty} \int f_n d\mu \geq \int \Big(\liminf_{n\to\infty} f_n\Big) d\mu$$

**Theorem 1.5 (Fatou's lemma.)** *If $X_n \geq 0$ then*

$$\liminf_{n\to\infty} EX_n \geq E(\liminf_{n\to\infty} X_n)$$

**Theorem 1.6 (Monotone convergence theorem.)** *If $f_n \geq 0$ and $f_n \uparrow f$ then*

$$\int f_n d\mu \uparrow \int f d\mu$$

**Theorem 1.7 (Monotone convergence theorem.)** *If $0 \leq X_n \uparrow X$ then $EX_n \uparrow EX$*

**Theorem 1.8 (Dominated convergence theorem.)** *If $f_n \to f$ a.e., $|f_n| \leq g$ for all $n$, and $g$ is integrable, then*

$$\int f_n d\mu \to \int f d\mu$$

**Theorem 1.9 (Dominated convergence theorem.)** *If $X_n \to X$ a.s., $|X_n| \leq Y$ for all $n$, and $EY < \infty$, then $EX_n \to EX$*

**Theorem 1.10 (Chebyshev's inequality.)** *Suppose $\phi : \mathbb{R} \to \mathbb{R}$ has $\phi \geq 0$, let $A \in \mathcal{R}$ and let $i_A = inf\{\phi(y) : y \in A\}$.*

$$i_A P(X \in A) \leq E(\phi(X); X \in A) \leq E\phi(X)$$

**Theorem 1.11 (Fubini's theorem.)** *If $f \geq 0$ or $\int |f| d\mu < \infty$ then*

$$\int_X \int_Y f(x,y)\mu_2(dy)\mu_1(dx) = \int_{X \times Y} f d\mu = \int_Y \int_X f(x,y)\mu_1(dx)\mu_2(dy)$$