Problem Sheet for Conditional Expectation and Random Walk

1. Suppose X and Y are independent and exponentially distributed with parameter $\lambda > 0$. Their common probability density function is

$$f(t) = \lambda e^{-\lambda t} \text{ for } t \geq 0.$$

   (a) Calculate $P(X > 5|X > 3)$.

   (b) Calculate $P(X + Y \leq 1)$.

2. There are two TA's for a certain course. For a particular assignment handed in, if it were marked by the first TA, the mark would be random with mean 75% and variance $(0.1)^2$; while if it were marked by the second TA, the mark would be random with mean 70% and variance $(0.05)^2$. The first TA has 40% chance to mark any single assignment. Let X be the mark of the particular assignment. Calculate the mean and variance of X.

3. an urn contains three white, 6 red and 5 black balls. six of these balls are randomely selected from this urn. Let X and Y denote, respectively, the number of white and black balls selected. Find the probability mass function of X given that $Y = 3$. Also Find $\mathrm{E}[X|Y = 1]$.

4. Let $Y = X_1 X_2 \cdots X_N$ where the random variables X_k are independent of one another with the same mean μ, and N is a positive integer-valued random variable independent of the X_ks. Prove (justify your steps) that $\mathrm{E}(Y) = G_N(\mu)$ where G_N is the probability generating function of N. Then find a simple sufficient condition on μ in order for $\mathrm{E}(Y)$ to always exist and be finite.

5. Let $X_1, X_2, X_3, \ldots$ be independently distributed random variables such that X_n has probability mass function

$$f_n(k) = P(X_n = k) = \binom{n}{k} p^k (1-p)^{n-k} \text{ for } k = 0, 1, \ldots, n.$$

   (a) Find the probability generating function of X_n.

   (b) Find the probability generating function of $X_1 + X_2 + X_3$.

   (c) Let N be a positive integer valued random variable with probability generating function $G(s)$ and assume it is independent of $X_1, X_2, \ldots$. Find the probability generating function of X_N.

   (d) Continuation of (c), find the probability generating function of $X_N + X_{N+1}$.

6. Derive the probability generating function of a Poisson random variable with mean μ. Indicate the interval of convergence.

7. Let Y be a random variable with probability mass function $P(Y = 0) = p$ and $P(Y = k) = p^{k-1}(1-p)^2$ for $k = 1, 2, 3, \ldots$, where $0 < p < 1$. Verify that this is a probability mass function and then obtain the probability generating function $G_Y(s)$. Evaluate $\mathrm{E}(Y)$ using $G_Y(s)$.

8. Consider a random walk on the integers, starting at the origin 0, with probability $p = 0.65$ of a jump to the right. Evaluate the following:

   (a) What is the probability that the walk ever passes through state 15?

   (b) What is the expected number of steps required to first visit state 15?

   (c) What is the probability that the walk ever passes through state -15?

   (d) What is the probability of a return to the origin?

   (e) What is the expected number of returns to 0?

   (f) Find the values of p for which the expected number of returns to 0 is 200.

   (g) Suppose the walk is balanced, i.e., $p = 0.5$, and the walk is currently in state 10. What is the probability that it will hit state 15 before it hits state 0?

9. Consider the random walk $X_0 = 0$, $X_n = X_{n-1} + Z_n$ where $P(Z_n = +1) = p$, $P(Z_n = -1) = q$, $n = 1, 2, \ldots$, independently ($p + q = 1$). Find the probability that the event $X_n = r$ will ever occur where r is a fixed positive integer. If $p > q$, find the expected time until its first occurrence.

10. Verify that the generating finction of $\{u_n\}$ (return to zero at trial n) is given by:

$$U(s) = (1 - 4pqs^2)^{-1/2}.$$

11. Mary is gambling. On each game she either wins or loses \$100. She starts out with \$1500. (Assume that each game is an independent trial.) Identify 1 unit = \$100, and let X0 = 15 and let Xn be her total worth (divided by 100) after n games. In this way she goes bankrupt when she hits 0. (Alternatively, you can define Xn to be her net worth so that X0 = 0. In this case bankruptcy occurs at -15.)

   (a) If Mary′s chance of winning on any game is 0.4 and she continues to play as long as she has money, find the average number of games required for her to go bankrupt.

   (b) Find the probability that Mary′s total worth reaches \$3000 at some point. (We assume here that Mary has to quit playing if she goes bankrupt so this is a gamblers ruin problem.)

   (c) Now suppose Mary has a rich aunt who constantly resupplies her with \$1500 whenever she happens to go bankrupt. Now what is the probability that her total worth reaches \$3000 at some point?

   (d) Suppose there is no rich aunt but we enable Mary to play indefinitely by letting her go arbitrarily deep into debt. (Could this really happen? I think they break your kneecaps after a certain point.) Now what is the probability that her total worth manages to reach \$3000? Why is this answer different from your answer in (c)?

12. Jane and John are playing a card game. Jane is an expert and, on each game, has probability p = 0.7 of beating John.

   (a) Suppose that Jane and John each begin with \$10 and agree that, on each game, the loser pays the winner \$1. They agree to play until one of them is bankrupt. What is the probability that it will be John who goes bankrupt?

   (b) Repeat (a) above only assuming that Jane starts with \$10 and John starts with \$30.