Problem Sheet for Discrete Time Markov Chain

1. Let D denote "Dry" and R denote "Rain". The transition matrix P is as follows:

$$P = \begin{array}{c} (RRR) \\ (RRD) \\ (RDR) \\ (RDD) \\ (DRR) \\ (DRD) \\ (DDR) \\ (DDD) \end{array} \begin{pmatrix} 0.8 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

2. Let the state on any day be the number of the coin that is flipped on that day:

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix}$$

and so,

$$P^2 = \begin{pmatrix} 0.67 & 0.36 \\ 0.66 & 0.34 \end{pmatrix}$$

and,

$$P^3 = \begin{pmatrix} 0.667 & 0.333 \\ 0.666 & 0.334 \end{pmatrix}$$

. Hence the probability is

$$\frac{1}{2}(P^3_{11} + P^3_{21}) = 0.6665$$

3. (a) $\{0, 1, 2\}$ recurrent
   (b) $\{0, 1, 2, 3\}$ recurrent
   (c) $\{0, 2\}$ recurrent, $\{1\}$ transient, $\{3, 4\}$ recurrent
   (d) $\{0, 1\}$ recurrent, $\{2\}$ recurrent, $\{3\}$ transient, $\{4\}$ transient

4. If $\sum_{i=1}^{m} P_{ij} = 1$ for all i, then $\pi_j = 1/(M+1)$ satisfies $\pi_j = \sum_{i=0}^{M} \pi_i P_{ij}$, $\sum_{j=0}^{M} \pi_j = 1$. Hence, by uniqueness these are the limiting probabilities.

5. (a)

$$P = \begin{pmatrix} 0 & p & 0 & 0 & 1-p \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & p & 0 & 1-p \\ p & 0 & 0 & 1-p & 0 \end{pmatrix}.$$

   (b)

6. Let $X_n = i$ if the nth exam is type i for $i = 1,2,3$. Then X_n is a Markov chain. For $i = 1,2,3$

$$
\begin{aligned}
p_{i1} &= P(X_1 = 1|X_0 = i) \\
&= P(X_1 = 1, \textit{done well}|X_0 = i) + P(X_1 = 1, \textit{done badly}|X_0 = i) \\
&= P(X_1 = 1|\textit{done well}, X_0 = i)P(\textit{done well}|X_0 = i) + P(X_1 = 1|\textit{done badly}, X_0 = i)P(\textit{done badly}|X_0 = i) \\
&= \frac{1}{3}p_i + (1 - p_i) = 1 - \frac{2}{3}p_i.
\end{aligned}
$$

Similiarly, we get $p_{i2} = p_{i3} = \frac{1}{2}p_i$, for $i = 1,2,3$. So the transition matrix is

$$
P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}
$$

Let $\pi = (\pi_1, \pi_2, \pi_3)$ be the stationary distribution. Then $\pi P = \pi$ and $\pi_1 + \pi_2 + \pi_3 = 1$, so we get: $\pi_1 = \frac{5}{7}$, $\pi_2 = \pi_3 = \frac{1}{7}$. So the proportions of type 1, 2 and 3 are : 71.4, 14.3 and 14.3 % respectively.

7. Let X_n be the number of on switches on day n. X_n can take values $0,1,2$. We then get the transition matrix

$$
P = \begin{pmatrix} 9/16 & 3/8 & 1/16 \\ 1/4 & 1/2 & 1/4 \\ 1/16 & 3/8 & 9/16 \end{pmatrix}
$$

Thus,

$$
\begin{aligned}
\pi_0 &= \frac{9}{1}6\pi_0 + \frac{1}{4}\pi_1 + \frac{1}{1}6\pi_2 \\
\pi_1 &= \frac{3}{8}\pi_0 + \frac{1}{2}\pi_1 + \frac{3}{8}\pi_2 \\
\pi_0 &+ \pi_1 + \pi_2 = 1
\end{aligned}
$$

The solution of the system is

$$
\pi_0 = \frac{2}{7}, \pi_1 = \frac{3}{7}, \pi_2 = \frac{2}{7}
$$