CHAPTER 1: CONVEX SETS

1. Convex sets

1.1. Convex sets.

Definition 1.1. Let $x_1, x_2, \cdots, x_m$ be m points of $\mathbb{R}^n$. A finite convex combination of the points $x_1, x_2, \cdots, x_m$ is the point

$$x = \sum_{i=1}^{m} \lambda_i x_i$$

such that for any $i = 1, \cdots, m$, $\lambda_i \in [0, 1]$ and $\sum_{i=1}^{m} \lambda_i = 1$.

Remark 1.2 (Special case $m = 2$). A convex combination of two points $x_1, x_2 of \mathbb{R}^n$ is $x = \lambda x_1 + (1 - \lambda)x_2$ and $\lambda \in [0, 1]$. If n = 1, then the points x for $\lambda \in [0, 1]$ describe the interval $[x_1, x_2]$ or $[x_2, x_1]$. If $n \geq 2$, then the points x for $\lambda \in [0, 1]$ describe the segment $[x_1, x_2]$.

Remark 1.3 (Special case $m = 3$). . A convex combination of three points x_1, x_2, x_3 of $\mathbb{R}^m$ is $x = \lambda x_1 + \mu x_2 + (1 - \lambda - \mu)x_3$ and $\lambda, \mu \in [0, 1]$. If $n = 1$, then the points x for $\lambda \in [0, 1]$ describe the interval $[min_i x_i, max_i x_i]$. If $n = 2$ and the points are not aligned, then the points x for $\lambda \in [0, 1]$ write the triangle with vertex x_1, x_2 and x_3.

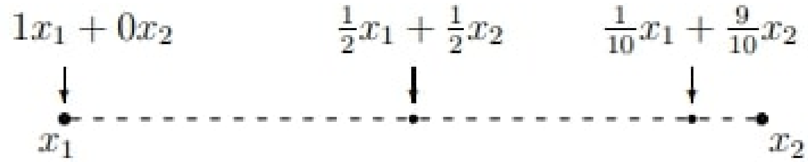


Figure 1. Convex combination of two elements

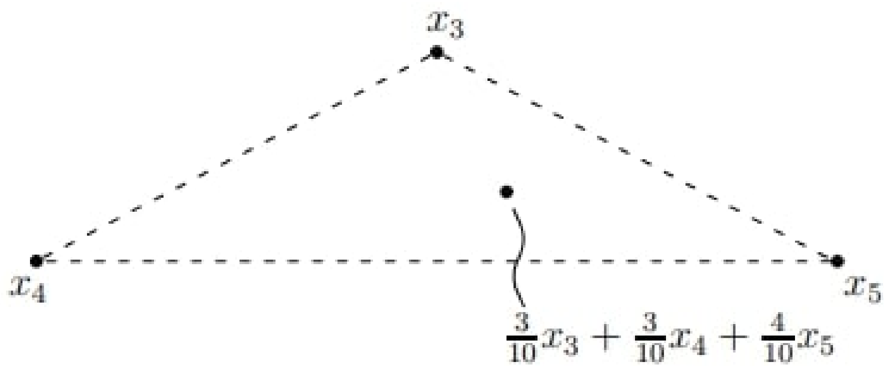


FIGURE 2. Convex combination of three elements

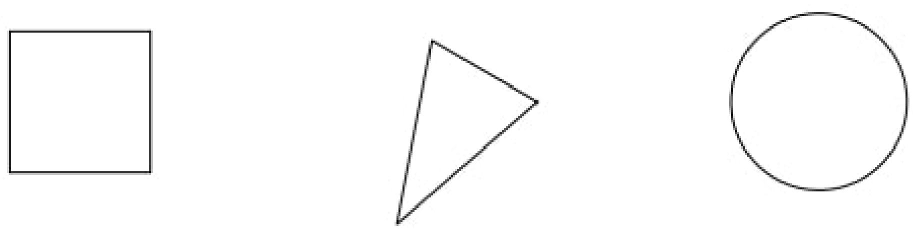


FIGURE 3. Examples of convex sets

Definition 1.4. A set $E \subset \mathbb{R}^n$ is said to be a convex set of $\mathbb{R}^n$ if any convex combination of two points in E is a point in E, i.e.,

$$\forall x, y \in E, \forall \lambda \in [0,1], \lambda x + (1-\lambda)y \in E.$$

Proposition 1.5. *E is a convex set if and only if* $\forall x_1, \cdots, x_m \in E, \forall \lambda \in [0,1]$,

$$\sum_{i=1}^{m} \lambda_i = 1 \Rightarrow \sum_{i=1}^{m} \lambda_i x_i \ E.$$

Proof. Exercise. □

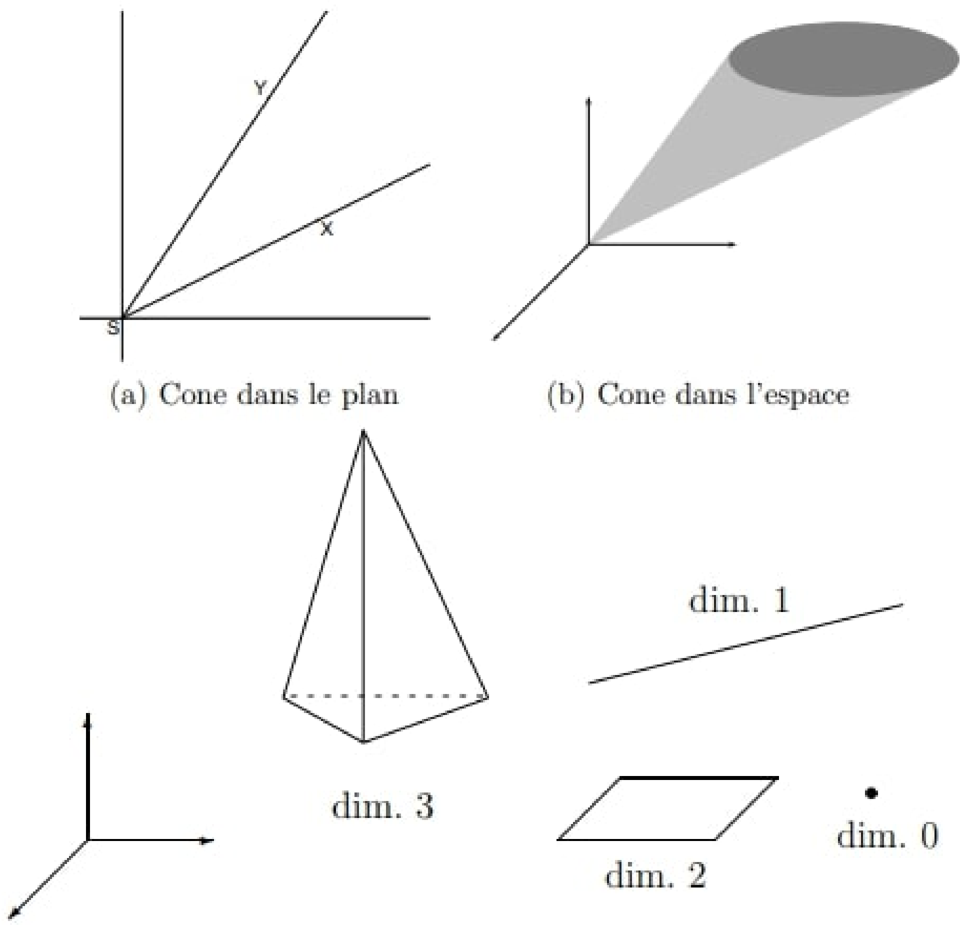
Proposition 1.6. *If E and F are two convex sets then $E+F$ and $E \cap F$ are convex sets.*

Proof. Exercise. □

1.2. **Cones, polyhedra, polytopes.**

Definition 1.7. A set $E \in \mathbb{R}^n$ is a **convex cone** if any positive combination of two elements of E is a point of E, i.e.,

$$\forall x, y \in E, \forall \lambda, \mu \geq 0, \lambda x + \mu y \in E.$$



(a) Cone dans le plan (b) Cone dans l'espace

(c) Polytopes

FIGURE 4. Cones and polytopes of plane and space

Definition 1.8. A **polyhedron** of $\mathbb{R}^n$ is the set $\{x \in \mathbb{R}^n, Ax \leq b\}$ for $A \in \mathbb{R}^{p\times n}$ and $b \in \mathbb{R}^p$. In other words, it is the set of points satisfying the linear inequalities of their components.

Example 1.9. If $n = 1$, the convex cones are the half-lines of the type $[0, \infty($ or $(\infty, 0]$. If $n = 2$, the convex cones are the cones determined by the vertex $E = (0, 0) \in R^2$ And two lines EX and EY for 2 points $X = (x_1, x_2)$ and $Y = (y_1, y_2)$.

Proposition 1.10. *The cones and polyhedra are convex sets.*

Proof. Exercise. □

Proposition 1.11. *E is a convex cone if and only if* $\forall x, y \in E, \forall \lambda_i \geq 0, \sum_{i=1}^m \lambda_i x_i \in E$.

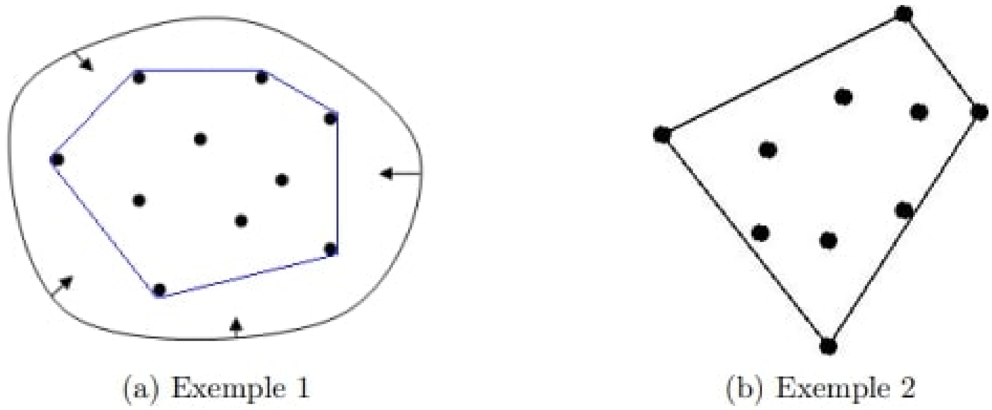


FIGURE 5. Convex envelopes

Definition 1.12. A polyhedral cone of $\mathbb{R}^n$ is the set $\{x \in \mathbb{R}^n, Ax \leq 0\}$. It is of the form

$$\left\{ \sum_{i=1}^{m} \lambda_i x_i, \forall i =, \cdots, m, \lambda_i \geq 0 \right\}.$$

1.3. **Convex envelopes.**

Definition 1.13. For a set $E \subset \mathbb{R}^n$, the convex envelope of E denoted by $co(E)$ is the set of all convex combination of elements of E, i.e,

$$co(E) = \left\{ e, \exists \lambda_1, \cdots, \lambda_m \in [0,1], \exists x_1, \cdots, x_m \in E, \sum_{i=1}^{m} \lambda_i = 1 \Rightarrow e = \sum_{i=1}^{m} \lambda_i x_i \right\}$$

Proposition 1.14. *The convex envelope of E is a convex set.*

Proof. Immediate since it is built by convex combinations. □

Proposition 1.15. *The convex envelope of E is the smallest convex set containing E.*

Proof. $co(E)$ is a convex set and $E \subset co(E)$ because $\forall e \in E$, $e = 1 \times e + 0 \times e$. So the intersection W of all convex sets containing E is such that $W \subset co(S)$. Let C be a convex set containing E. E contains any convex combination of its elements, in particular those of E since $E \subset C$. Therefore $co(E) \subset C$. Thus $\forall C$ convex, $co(E) \subset \cap C = W$. □

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Definition 1.16. A polytope of $\mathbb{R}^n$ is the convex envelope of m points $x_1, \cdots, x_m$ of $\mathbb{R}^n$, i.e. $co(x_1, \cdots, x_m)$.

Proposition 1.17. *For any set $E \subset F$, $co(E) \subset co(F)$.*

Proof. Let $x \in co(E)$. We know that there exists $\lambda_1, \cdots, \lambda_m \in [0,1]$ and $x_1, \cdots, x_m \in E$ such that $\sum_{i=1}^m \lambda_i = 1 \Rightarrow x = \sum_{i=1}^m \lambda_i x_i$. Since $E \subset F$, then for any $i = 1, \cdots, m$, $x_i \in F$ then $\sum_{i=1}^m \lambda_i x_i = x \in F$ □

Definition 1.18. For a set $E \in \mathbb{R}^n$, the canonical envelop of E denoted by $cone(E)$ is the set of any positive combination of elements of E, i.e.,

$$cone(E) = \left\{ s, \exists \lambda_1, \cdots, \lambda_m \geq 0, \exists x_1, ..., x_m \in E, e = \sum_{i=1}^m 1\lambda_i x_i \right\}.$$

Proposition 1.19. *The conical envelope of E is a cone.*

Proof. Immediate because it is built on positive combinations. □

Proposition 1.20. *The conical envelope of E is the smallest cone containing E.*

Proof. $cone(E)$ is a conical set and $E \subset cone(E)$ since $\forall e \in E$, $e = 1 \times e$. Therefore the intersection W of all conical sets containing E is such that $W \subset cone(E)$. Let C be a cone containing E. C contains any positive combination of its elements, in particular those of E since $E \subset C$. So $cone(E) \subset C$. Thus $\forall C$ cone, $cone(E) \subset \cap C = W$. □

Example 1.21. The (standard) simplex of $\mathbb{R}^n$ given by

$$S_n = \left\{ x \in \mathbb{R}^n, \forall i = 1, \cdots, n, x_i \in [0,1], \sum_{i=1}^n xi = 1 \right\}$$

is a polyhedron. In other words, $S_n = co(e-1, ..., e-n)$ where $(e_1, \cdots, e_n)$ are independent unit vectors.

Example 1.22. (Simplex (conic) of m points). For m points of $\mathbb{R}^n$ (linearly independent), the simplex of m points is the polytope of these m points. For m points of $\mathbb{R}^n$ (independent refinement), the conical simplex of m points is the conical envelope of these m points.

Proposition 1.23. *Let $x_1, \cdots, x_m \in \mathbb{R}^n$.*

- *when these points are linearly independent, we denote by $P = co(x_1, ..., X_m)$ the simplex corresponding. For each point of P, there exists a unique representation of x as convex combination of m points.*
- *when these points are finely independent, we denote by $P = cone(x_1, ..., x_m)$ the simplex corresponding conical. For each point of P, there exists a unique representation of x as a positive combination of m points.*

1.4. **Caratheodory theorem.**

Theorem 1.24 (Caratheodory for convex envelopes)**.** *Let $E \in \mathbb{R}^n$. Any point $x \in co(E)$ can be written as a convex combination of m independent refinement points in E such that $m \leq n+1$.*

Theorem 1.25 (Caratheodory pour les cones)**.** *Let $E \in \mathbb{R}^n$. Any point $x \in cone(E)$ can be written as a positive combination of m linearly independent points in E such that $m \leq n$.*