Problem Sheet for Discrete Time Markov Chain

1. Suppose that whether or not it rains today depends on previous weather conditions through the last three days. Show how this system may be analysed using a Markov chain. How many states are needed?

   Suppose

   - that if it has rained for the past three days, then it will rain today with probability 0.8;
   - if it did not rain for any of the past three days then it will rain today with probability 0.2;
   - and in any other case the weather today woth be the same as yesterday with probability 0.6.

   Determine P for this Markov chain.

2. Suppose that we have two coins.

   - Coin 1 has probability 0.7 of coming up heads,
   - and coin 2 has probability 0.6 of coming up heads.
   - if the coin flipped today comes up heads, then we select coin 1 to flip tomorrow,
   - if it comes up tails, then we select coin 2 to flip tomorrow.
   - if the coin initially flipped is equally likely to be coin 1 or coin 2

   What is the probability that the coin flipped on the third day is coin 1?

3. Specify the classes of these Markov chains, and determine whether they are transient or recurrent:

$$P_1 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4. A transition matrix P is said to be doubly stochastic if the sum over each colum equals on; that is,

$$\sum_i P_{ij} = 1, \text{ for all } j.$$

If such a chain is irreducible and aperiodic and consists of $M+1$ states, 0, 1, ..., M, show that the limiting probabilities are given by

$$\pi_j = \frac{1}{M+1}, \; j = 0, 1, \ldots, M$$

5. A particle moves on a circle through points which have been marked 0,1,2,3,4 (in a clockwise order). At each step is has a probability p of moving to the right (clockwise) and $1-p$ to the left (counterclockwise). Let X_n denote the location on the circle after the nth step.

   (a) Find the transition matrix P.

   (b) Calculate the limiting probabilities.

6. A professor continually gives exams to her students.

   - She can give three possible types of exams,
   - and her class is graded as either having done well or badly.
   - Let p_i denote the probability that the class does well on a type i exam, and suppose that $p_1 = 0.3$, $p_2 = 0.6$, $p_3 = 0.9$.
   - If the class does well on an exam then the next exam is equally likely to any of the three.
   - If the class does badly then the next exam will always be of type 1.

   What proportion of exams are type i?

7. Each of two switches is either on or off during a day. On day n, each switch will independently be on with probability

$$(1 + \text{ number of on switches during day } n-1)/4.$$

   for instance, if both switches are on during day $n-1$, then each will independently be on during day n with probability $3/4$.

   What fraction of days are both switches on? What fraction are both off?