Statistique d'ordre

Exo 4

1- Y_1 est un loi exp de paramètre B/n

$$f(x) = \frac{1}{B} e^{-1/B\, x} \Rightarrow F(x) = 1 - e^{-1/B\, x}$$

$$g(y_1) = n[1 - F(x)]^{n-1} f(x)$$

$$= n[1 - (1 - e^{-1/B\, x})]^{n-1} (1/B\, e^{-1/B\, x})$$

$$= n(e^{1/B\, x(n-1)})(1/B\, e^{-1/B\, x})$$

$$g(y_1) = \frac{n}{B} e^{-\frac{n}{B} x} \Rightarrow Y_1 \sim \mathcal{E}\left(\frac{B}{n}\right)$$

$$G(y_1) = 1 - (1 - F(y_1))^n$$

$$= 1 - (1 - (1 - e^{-\frac{1}{B} y_1}))^n = 1 - e^{-\frac{n}{B} y_1}$$

$$\Rightarrow Y_1 \sim \mathcal{E}\left(-\frac{n}{B}\right)$$

- $f(x) = g(y_n) = n(F(x))^{n-1} f(x)$

$$= n(1 - e^{-1/B\, x})^{n-1} (1/B\, e^{-1/B\, x})$$

$$= \frac{n}{B}(1 - e^{-1/B\, x})^{n-1} e^{-1/B\, x}$$

2/3

- $g(y_n) = n(1 - F(x)^{n-1}) f(x)$

$$f(x) = \begin{cases} 1 & \text{si } x \in [0,1] \\ 0 & \text{sinon} \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ x & x \in [0,1] \\ 1 & x > 1 \end{cases}$$

REDMI NOTE 8 PRO

$g(y_1) = n(1-x)^{n-1}$

- $F(y_1) = 1-(1-F(y_1))^n = 1-(1-x)^n$
- $F(y_n) = [F(y)]^n = x^n$
- $f_{\frac{n}{2},n}(x) = \dfrac{n!}{(\frac{n}{2}-1)!\,(n-\frac{n}{2})!}\, x^{n/2-1}(1-x)^{n-\frac{n}{2}}$

$$= \frac{n!}{(\frac{n}{2}-1)!\,\frac{n}{2}!}\, x^{n/2-1}(1-x)^{n/2}$$

si n est paire : $n = 2k$

$$f_{\frac{n}{2},n}(x) = \frac{(2k)!}{(k-1)!\,k!}\, x^{k-1}(1-x)^{k}$$

si n est impaire $n = 2k+1$

$$f_{\frac{n+1}{2},n}(x) = \frac{(2k+1)!}{\left(\frac{2k+1}{2}\right)!\left(\frac{2k+1}{2}\right)!}\, x^{\frac{2k+1}{2}}(1-x)^{\frac{2k+1}{2}}$$

REDMI NOTE 8 PRO

l'espérances

$$E(Y_1) = \int_0^1 y\, f(y)\, dy$$

$$= \int_0^1 y\, n\, (1-y)^{n-1}\, dy$$

$$= n\, B(2, n) = n \frac{(2-1)!\,(n-1)!}{(2+n-1)!}$$

$$= n \frac{1!\,(n-1)!}{(n+1)!} = \frac{1}{(n+1)}$$

Variances

$$V(Y_1) = E(Y_1^2) - E^2(Y_1)$$

$$E(Y_1^2) = \int_0^1 n\, y_1^2\, [1-y_1]^{n-1}\, dy_1 = n\, Beta(3, n)$$

$$= \frac{n\, 2!\,(n-1)!}{(n+2)!} = \frac{2}{(n+1)(n+2)}$$

$$V(Y_1) = \frac{2}{(n+1)(n+2)} - \frac{1}{(n+1)^2} = \frac{n}{(n+1)^2(n+2)}$$

REDMI NOTE 8 PRO

Exo3

1/ La fct de répartition $\quad X_i \perp\!\!\!\perp \ \forall i = 1, n$

$$F_{n,n}(x) = P(X_{n,n} \leq x) = P\left(\bigcap_{i=1}^{n} X_i \leq x\right)$$

$$= \prod_{i=1}^{n} P(X_i \leq x) \text{ car iid}$$

$$= \prod_{i=1}^{n} F(x) = (F(x))^n$$

La densité

$$f_{n,n}(x) = (F_{n,n}(x))' = n f(x) (F(x))^{n-1}$$

2/ La fct de répartition :

$$F_{1,n}(x) = P(X_{1,n} \leq x) = 1 - P(X_{1,n} > x)$$

$$= 1 - P\left(\bigcap_{i=1}^{n} X_i > x\right) = 1 - \prod_{i=1}^{n} P(X_i > x)$$

$$= 1 - \prod_{i=1}^{n} [1 - P(X_i \leq x)] = 1 - [1 - F(x)]^n$$

La densité

$$f_{1,n}(x) = [F_{1,n}(x)]' = n [1 - F(x)]^{n-1} f(x)$$

3/ la loi de couple $(X_{(1)}, X_{(n)})$

$$P(X_{(1,n)} \leq x, X_{(n,n)} \leq x) = P(X_1 \leq x, X_n \leq x)$$

$$= P(X_n \leq x) - P(X_1 \geq x, X_n \leq x)$$

$$= P(X_n \leq x) - P\left(\bigcap_{i=1}^{n} X_i \in [x_1, x_n]\right)$$

$$= [F(x)]^n - \prod_{i=1}^{n} P(x_1 \geq x_i \geq x_n)$$

$$= [F(x)]^n - [F(x_n) - F(x_1)]^n$$

REDMI NOTE 8 PRO

Exo de cours

$Y_1 < Y_2 < Y_3$ $\qquad Z = Y_3 - Y_1$

$Y_i \sim U(0,1)$

$f_{y_i} = \begin{cases} 1 & \text{si } y_i \in [0,1] \\ 0 & \text{sinon} \end{cases}$, $F_{y_i} = \begin{cases} 0 & y_i < 0 \\ 1 & y_i > 1 \\ y_i & y_i \in]0,1[\end{cases}$

on prend

$h(Y_1, Y_3) = \begin{cases} U = Y_1 \\ Z = Y_3 - Y_1 \end{cases}$

$h^{-1}(U, Z) = \begin{cases} Y_1 = U \\ Y_3 = Z + U \end{cases}$

$$J = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$f_{Y_1 Y_n} = n(n-1)[F(y_n) - F(y_1)]^{n-2} f_{Y_n} f_{Y_1}$ $\quad 3(3-1)[Z + U - U]^{3-2} \cdot 1 \cdot 1 = 6Z$

$f_{U,Z} = 6Z$

$U \in [0,1]$, $Z \in [0,1]$

$F_Z = \int_0^1 6Z \, \mathbb{1}_{Z \in [0,1]}(z) \, du$

La suite pour exo 3

3/3

$$\frac{\partial F_{x_1 x_n}(x,x)}{\partial x_1} = n\left(F(x_n) - F(x_1)\right)^{n-1} f(x_1)$$

$$\frac{\partial^2 F_{x_1 x_n}}{\partial x_1 x_n} = n(n-1)\left(F(x_n) - F(x_1)\right)^{n-2} f(x_1) f(x_n)$$

- L'étendue

$$R = X_n - X_1$$

on prend

$$h(X_1, X_n) = \begin{cases} Q = X_1 \\ R = X_n - X_1 \end{cases}$$

inversement

$$h^{-1}(Q, R) = \begin{cases} X_1 = Q \\ X_n = R + Q \end{cases}$$

$$J = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$f_{X_1 X_n} = n(n-1)\left[F(X_n) - F(X_1)\right]^{n-2} f(X_n) f(X_1)$$

$$f_{QR} = n(n-1)\left[F(Q+R) - F(Q)\right]^{n-2} f(Q) f(R)$$

$$f_R = \int f_{Q,R}\, dQ$$

$$= \int n(n-1)\left[F(Q+R) - F(Q)\right]^{n-2} f(Q) \underbrace{f(R)}_{Q+R} dQ$$

$$F_R = \int f_R\, dR$$

REDMI NOTE 8 PRO

4|3

$$N_y = \sum_{i=1}^{n} 1\!\!1_{x_i \leq y}$$

$$1\!\!1_{x_i \leq y} = \begin{cases} 0 & x_i > y \\ 1 & x_i \leq y \end{cases} \sim Ber\left(p = P(1\!\!1_{x_i \leq y}) = F(x_i)\right)$$

Alors $\sum_{i=1}^{n} 1\!\!1_{x_i \leq y} \sim Bin(n, p = F(x))$

$$P(X = k) = \sum_{k} C_n^k p^k q^{n-k}$$

$$P(1\!\!1_{x_i \leq y} \leq x) = P(\Sigma 1\!\!1_{x_i \leq y} < x)$$

$$= \sum_{k=1}^{n} C_n^k (F(x))^k (1 - F(x))^{n-k}$$

TD 02

Exo 5 :

$F : \mathbb{R} \to \mathbb{R}$ $\quad \theta > 0, \lambda > 0$

$$F(x) = (1 - (1 + x^\theta)^{-\lambda}) 1\!\!1_{\{x > 0\}}$$

$\{ [0, +\infty[$

$\begin{cases} F(.) \text{ est continue} \\ \lim_{x \to 0} F(x) = 0 \ , \lim_{x \to \infty} F(x) = \lim_{x \to \infty} \left(1 - \frac{1}{(1 + x^\theta)^\lambda}\right) = 1 \end{cases}$

point terminal pour sup et inf

$F(x) \in DA(\Phi_\xi)$ avec $\xi > 0$

$x_F = +\infty$

$\bar{F} = 1 - F$

$\bar{F}(x) = 1 - F(x) = x^{-\frac{1}{\xi}} \underbrace{l(x)}$

fct a variations lentes.

$a_n = F^{\leftarrow}(1 - \frac{1}{n})$

$b_n = 0$

2/3

1/ $x_F = F^{\leftarrow}(1)$

$x_F = +\infty$?

2/ $1 - F(x) = x^{-1/\xi} L(x)$?

$\bar{F}(x) = 1 - F(x) = 1 - [1 - (1 + x^{\theta})^{-\lambda}] = (1 + x^{\theta})^{-\lambda}$

$= [x^{\theta}(x^{-\theta} + 1)]^{-\lambda}$

$\bar{F}(x) = x^{-\lambda\theta}(x^{-\theta} + 1)^{-\lambda}$

on prend $\xi = \frac{1}{\theta\lambda}$

$= x^{-\frac{1}{\xi}} \underbrace{(x^{-\theta} + 1)^{-\lambda}}_{l(x)}$

$\lim f(x) = ?\ 1$

$$\lim_{x\to\infty} (x^{-\theta}+1)^{-\lambda} = \lim_{x\to\infty} \frac{1}{(x^{-\theta}+1)^{\lambda}} = \frac{1}{(1)^{\lambda}} = 1$$

4/3 $\Delta(x) = \dfrac{x\,L'(x)}{L(x)}$

$L(x) = (x^{-\theta}+1)^{-\lambda}$

$L'(x) = \lambda\theta x^{-\theta-1}(x^{-\theta}+1)^{-\lambda-1}$

$$a(x) = \frac{x\lambda\theta x^{-\theta-1}(x^{-\theta}+1)^{-\lambda-1}}{(x^{-\theta}+1)^{-\lambda}} = \lambda\theta x^{-\theta}(x^{-\theta}+1)^{-1}$$

$$= x^{-\theta}\underbrace{\lambda\theta\,(x^{-\theta}+1)^{-1}}_{f(x)}$$

5/3 $a_n = F^{\leftarrow}\left(1-\frac{1}{n}\right) = (n^{1/n}-1)^{1/\theta}$

$F(x) = (1-(1+x^{\theta})^{-\lambda})$, $x > 0$

pose $y = 1-(1+x^{+\theta})^{-\lambda}$

$1-y = (1+x^{\theta})^{-\lambda}$

$-\sqrt[\lambda]{1-y} = (1+x^{\theta})$

$-\sqrt[\lambda]{1-y} - 1 = x^{\theta}$

$x = \sqrt[\theta]{-n\sqrt{1-y}-1}$

REDMI NOTE 8 PRO

$$F^{\leftarrow}\left(1-\frac{1}{n}\right) = \left(\left(1-\left(1-\frac{1}{n}\right)\right)^{-1/\lambda} - 1\right)^{1/\theta}$$

$$= \left(\left(\frac{1}{n}\right)^{-1/\lambda} - 1\right)^{1/\theta} = \left(n^{1/\lambda} - 1\right)^{1/\theta}$$

$$\left((1-y)^{-\frac{1}{\lambda}} - 1\right)^{1/\theta}$$

$$\begin{cases} a_n = F^{\leftarrow}\left(1-\frac{1}{n}\right) \\ b_n = 0 \end{cases}$$

$$P(Y_n < x) = F^n(a_n x + b)$$

$$Y_n = \max\left\{\frac{Y_1}{a_n}, \dots, \frac{Y_n}{a_n}\right\}$$

$$F^n(a_n x) = \left[1 - \left[1 + (a_n x)^{\theta}\right]^{-\lambda}\right]^n \qquad , x > 0$$

$$= \exp\left(n \ln\left(1 - \left(1 + (a_n x)^{\theta}\right)^{-\lambda}\right)\right)$$

$$= \exp\left(n\left(1 + a_n^{\theta} x^{\theta}\right)^{-\lambda}\right)$$

$$= \exp\left(n\left(1 + (n^{1/\lambda} - 1)x^{\theta}\right)^{-\lambda}\right)$$

$$=$$

$$\left(1 + (n^{1/\lambda} - 1)x^{\theta}\right)^{-\lambda} = x^{-\lambda\theta}\left(\frac{1}{x^{\theta}} + n^{1/\lambda} - 1\right)^{-\lambda}$$

$$= x^{-\lambda\theta}\left(\frac{1 + n^{1/\lambda}x^{\theta} - \cancel{x^{\theta}}}{x^{\theta}}\right)^{-\lambda}$$

$$= x^{-\lambda\theta}\left(\frac{n^{1/\lambda}\left(\frac{1}{n^{1/\lambda}} + x^{\theta} - \frac{\cancel{x^{\theta}}}{n^{1/\lambda}}\right)}{x^{\theta}}\right)^{-\lambda}$$

REDMI NOTE 8 PRO

$$= x^{-\lambda\theta} \left(\frac{\left(\frac{1}{n^{1/\lambda}} + x^{\theta} - \frac{x^{\theta}}{n^{1/\lambda}} \right)^{-\lambda}}{n\, x^{-\theta\lambda}} \right)$$

$$\Rightarrow \quad \exp\left(- \cancel{n} x^{-\lambda\theta} \left(\frac{\left(\frac{1}{n^{1/\lambda}} + x^{\theta} - \frac{x^{\theta}}{n^{-\lambda}} \right)^{-\lambda}}{\cancel{n}\, x^{-\theta\lambda}} \right) \right)$$

$$\Rightarrow \lim_{n \to \infty} \exp\left(- x^{-\lambda\theta} A \right) = \exp\left(- x^{\lambda\theta} \right)$$

$$\cancel{\left(-1 + \left(n^{1/\lambda} - 1 \right) x^{\theta} \right)^{-\lambda}}$$

REDMI NOTE 8 PRO

Exo 4 :

$$f(x) = \begin{cases} 3x^2 e^{-x^3} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

1) $= \int_{0}^{+\infty} 3x^2 e^{-x^3} dx = 1 ?$

$= \int_{0}^{+\infty} 3x^2 e^{-x^3} dx = 1$ par partie

• continue

2/3 $f'(x) = e^{-x^3}(6x - 9x^4)$

• mode $f'(x) = 0 \Rightarrow e^{-x^3}(6x - 9x^4) = 0$

$6x - 9x^4 = 0 \Rightarrow 9x^3 = 6$

$\Rightarrow x^3 = \frac{6}{9} \Rightarrow x = \left(\frac{2}{3}\right)^{1/3}$

Exo 3 :

1) $= \lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} e^{-e^{-x}} = 0$

$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} e^{-e^{-x}} = 1$

$g(x) = e^{-e^{-x}}$

$g'(x) = e^{-x} e^{-e^{-x}}$

$\lim_{x \to +\infty} e^{-\infty} = 0$

$\lim_{x \to -\infty} e^{-x} = +\infty$

$e^{x} \xrightarrow[x \to -\infty]{} 0$

$e^{x} \xrightarrow[x \to +\infty]{} +\infty$

2/ $\int_{-\infty}^{+\infty} g(x)\,dx = \int_{-\infty}^{+\infty} e^{-x} e^{-e^{-x}} = e^{-e^{-x}} \Big|_{-\infty}^{+\infty}$

$= 1 + 0 = 1$

3/ $F(x) = 1 - e^{-x}$

4/

$X_1 \rightsquigarrow \mathcal{E}xp(1)$, $X_2 \rightsquigarrow \mathcal{E}xp(1)$

$M = \max(X_1, X_2)$

$P(M \leq x) = P(\operatorname{Max}(X_1, X_2) \leq x)$

$= P(X_1 \leq x) \cdot P(X_2 \leq x) = (1 - e^{-x})^2$

5/ $f_n(x) = 2(1 - e^{-x})\, e^{-x}$

$X_1, X_2, \ldots, X_n \rightsquigarrow exp(1)$

$M = \max(X_1, \ldots, X_n)$

$P(M \leq x) = P(\operatorname{Max}(X_1, \ldots, X_n \leq x))$

$= P(X_1 \leq x) \cdot P(X_2 \leq x) \cdots P(X_n \leq x)$

$= (1 - e^{-x})^n$

$\Rightarrow f(x) = n(1 - e^{-x})^{n-1} e^{-x}$

REDMI NOTE 8 PRO

6/ = $\lim_{n\to+\infty} \left(1-\frac{u}{n}\right)^n = e^{-u}$

- $F_n(x) = (1-e^{-x})^n$

- $F_n(x+\ln n) = (1-\exp(-(x+\ln n)))^n$

$= (1-e^{-x-\ln n})^n = (1-e^{-x}n^{-1})^n$

$= (1-e^{-x}\frac{1}{n})^n$

$\lim_{n\to\infty} \left(1-\frac{e^{-x}}{n}\right)^n = e^{-e^{-x}} = g(x)$

Exo1 =

$P(X_n \leq x) = P(n(1-M_n) \leq x)$

$= P(1-M_n \leq \frac{x}{n}) = P(M_n \geq -\frac{x}{n}+1)$

$= 1-P(M_n \leq 1-\frac{x}{n})$

$= 1-F^n(1-\frac{x}{n}) = 1-(1-\frac{x}{n})^n$

La f ct de repartition de (X_n) donner par :

$$F_{X_n}(x) = \begin{cases} 0 & x \leq 0 \\ 1-\left(1-\frac{x}{n}\right)^n & 0 \leq x \leq n \\ 1 & x \geq n \end{cases}$$

Exo5 :-

$$1-\left(\frac{x}{\ldots}\right)^{-k}$$

$$F(x) = \begin{cases} 1 - x^{-\theta} & , x \geq 1 \\ 0 & x < 1 \end{cases} \qquad \theta > 0$$

$$Y = \ln X$$

$$P(Y \leq y) = P(X \leq e^{y}) = F_X(e^{y}) = 1 - (e^{+y})^{-1}$$

$$P(Y \leq y) = 1 - e^{-y} \sim \mathcal{E}(y)$$

Weibul :-

$\xi < 0$, $x_F = +\infty$

$$\bar{F}(x) = 1 - F(x) = (x_F - x)^{1/\xi}\,[P(x_F - x)]^{-1}$$

$$a_n = x_F - F^{\leftarrow}(1 - 1/n) \; ; \; b_n = x_F$$

$$H_\alpha(x) = \Psi_\alpha(x) = \begin{cases} \exp(-(-x)^{\alpha}) & x \leq 0 \quad \alpha < 0 \\ 1 & x > 0 \end{cases}$$

Frechet :-

$\xi > 0$, $x_F = +\infty$

$$\bar{F}(x) = 1 - F(x) = x^{-1/\xi}\,\ell(x)$$

$$a_n = F^{\leftarrow}\left(1 - \frac{1}{n}\right) \; , \; b_n = 0$$

$$H_\alpha(x) = \begin{cases} 0 & x \leq 0 \quad \alpha > 0 \\ \exp(-x^{-\alpha}) & x > 0 \end{cases}$$

REDMI NOTE 8 PRO

Gumbel =

$$\xi = 0, \quad t < x_F \leq +\infty$$

$$\bar{F}(x) = 1 - F(x) = c(x) \exp\left\{ -\int_t^x \frac{1}{a(u)}\, du \right\}$$

$$a_n = q\left(\tfrac{1}{n}\right) \; ; \; b_n = \frac{1}{\bar{F}(a_n)} \int_{a_n}^{x_F} \bar{F}(s)\, ds$$

$$H_0(x) = \Lambda(x) = \exp[-\exp(-x)], \quad x \in \mathbb{R}$$

تشتت ، تمركز ، تموضع

Loi des excès, approche PoT

Distribution de seuil

$$P(X - U \leq y \,/\, X > U) = \frac{P((X - U \leq y) \cap (X > U))}{P(X > U)}$$

$$= \frac{P((X \leq y + U) \cap (X > U))}{P(X > U)} = \frac{P(U < X \leq U + y)}{P(X > U)}$$

$$= \frac{F(U + y) - F(U)}{1 - F(U)}$$

$$\bar{F}_u(y) = 1 - \frac{F(U + y) - F(U)}{1 - F(U)} = \frac{1 - F(U) - F(U + y) + F(U)}{1 - F(U)}$$

$$= \frac{1 - F(U + y)}{1 - F(U)} = \frac{\bar{F}(U + y)}{\bar{F}(U)}$$

Esitimation de vriasemblance (G EV):

$$H_{\xi,\mu,\sigma}(x) = \exp\left\{-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$

$\theta = \xi, \mu, \sigma$

$$h_\theta(y) = \exp\left\{-\left[1+\xi\left(\frac{y_i-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$

$h'_\theta(y_i)$

REDMI NOTE 8 PRO

Exo 3 :-

1/ F fct à variations régulières

$\Longleftrightarrow F$ écrire $F(x) = L(x)\, x^{\alpha}$ tq $\alpha \in \mathbb{R}$

$F^*(x) = 1 - F\left(x_F - \frac{1}{x}\right)$

$= 1 - \left(1 - \left(x_F - x_F + \frac{1}{x}\right)^{-1/\gamma} L\left(\left(x_F - x_F + \frac{1}{x}\right)^{-1}\right)\right)$

$= (x^{-1})^{-1/\gamma} L\left((x^{-1})^{-1}\right) = x^{1/\gamma} L(x)$

d'où F^* est fct à variation régulière d'indice γ

- soit $x_1 \leq x_2 \Rightarrow F(x_1) \geq F(x_2)$ décroissant ??

$F^*(x_1) = x_1^{1/\gamma} L(x_1)$

$F^*(x_2) = x_2^{1/\gamma} L(x_2)$

$x_1 < x_2 \Rightarrow x_1^{1/\gamma} > x_2^{1/\gamma}$ car $\gamma < 0$

$x_1^{1/\gamma} L(x_1) > x_2 L(x_2)$ car $L(x)$ fct positive

$\Rightarrow F(x_1) > F(x_2) \Rightarrow F$ décroissante

REDMI NOTE 8 PRO