PROBABILITY

About Combinatorial Analysis

The aim of Part 1 is to develop a few techniques for determining the number of possible outcomes of a particular experiment without direct enumeration..

The fundamental principle of combinatorial analysis :

Preliminary example: how many number plates can be made up containing **two distinct letters** followed by **three digits**, the **first of which is not 0.**

26 . 25 . 9 . 10 .10 = 585000 plates

The fundamental principle of combinatorial analysis, also called the multiplication principle, states that if there are $\mathbf{n_1}$ **ways** to perform the first task and $\mathbf{n_2}$ **ways** to perform the second task, then there are $\mathbf{n_1 \times n_2}$ ways to perform both tasks in succession. This principle extends to any number of tasks: if there are **n1** ways for the first task, $\mathbf{n_2}$ ways for the second task, and so on up to $\mathbf{n_k}$ ways for the $\mathbf{k^{th}}$ task, then there **are** $\mathbf{n_1 \times n_2 \times \ldots \times n_k}$ ways to perform all **k** tasks in succession. It's fundamental for systematically counting possible outcomes in **permutations**, **combinations**, and **arrangements**.

Permutations and Combinations

- A **permutation** is an arrangement with an order and the order is relevant. The permutationABC is different to the permutation ACB.
- A **combination** is a collection of things without an order or where the order is not relevant. The combination ABC is the same as the combination ACB

Permutation with repitition

If the ordering is relevant, repetitions are allowed and there are n objects to choose from,

then there are n^r different arrangements of r objects possible.

Example

Choose a 4 digit PIN from the digits 0 to 9. Repetition is allowed.

(a) Think of the number of ways you can fill the four places by filling in boxes

Number of ways to do this $\boxed{10}\,\boxed{10}\,\boxed{10}\,\boxed{10}$

$$10 \times 10 \times 10 \times 10 = 10^4 = 10000$$

(b) Use the formula $n^r = 10^4 = 10000$

A_n^r

Permutation without repitition

If the **ordering is relevant**, **repetitions are not allowed** and there are **n** objects to choose from, then there are

$$P(n,r) = {}_nP_r = {}^nP_r = \frac{n!}{(n-r)!}$$

different arrangements of **r** objects possible.

Example

From a group of 10 people in the club choose 3 different people to be president, secretary and treasurer.

Number of ways to do this $\boxed{10}\,\boxed{9}\,\boxed{8} = 10 \times 9 \times 8 = 720$

OR $${}_nP_r = \frac{10!}{7!} = \frac{10X9X8X7X6X5X4X3X2X1}{7X6X5X4X3X2X1} = 10 \times 9 \times 8 = 720$$

COMBINATIONS

If the ordering is not relevant, repetitions are not allowed and

there are **n** objects to choose from, then there are

$$ {}_{n}C_{r} = \frac{n!}{r!(n-r)!} = {}_{n}P_{r} \div r! $$

different combinations of **r** objects possible

Example

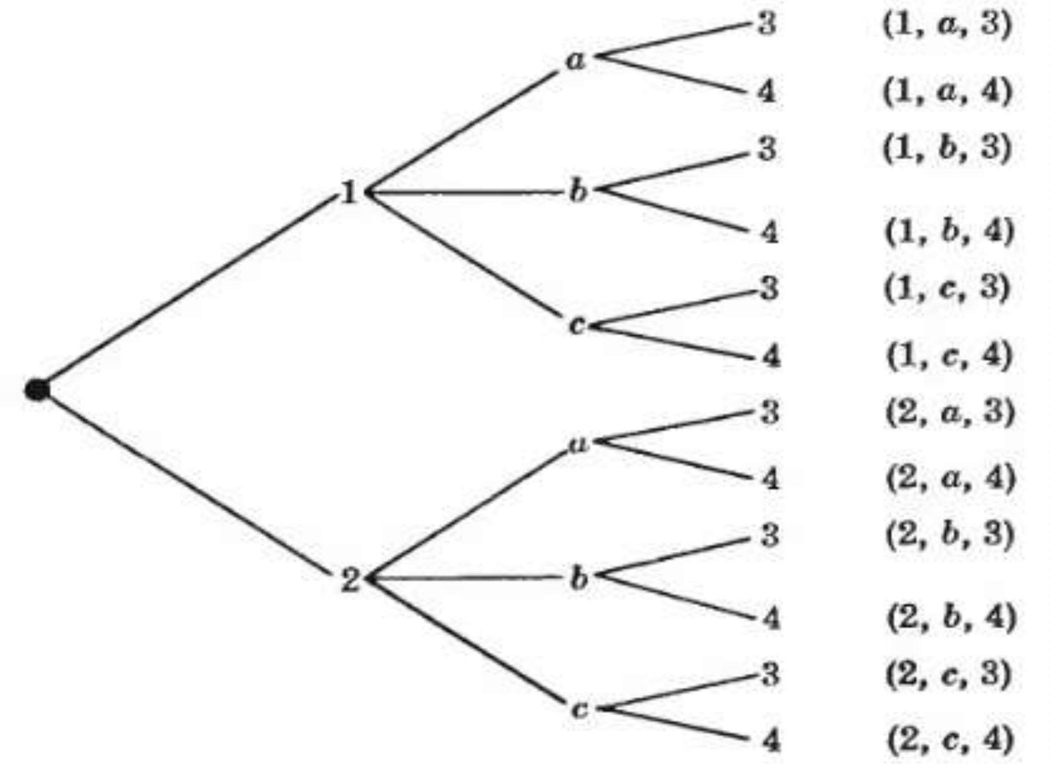
From a group of 10 people, select 3 people to form a committee.

$$_{10}C_3 = \frac{10!}{7! \times 3!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) \times (3 \times 2 \times 1)} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$$

Probability tree diagrams

Probability tree diagrams are a way of organising the information of two or more probability events. Probability tree diagrams show all the possible outcomes of the events and can be used to solve probability questions

Example: A={1, 2}, B={a, b, c} and C={3, 4}. We have the following tree diagram



Probability theory

Probability theory is a branch of mathematics that deals with the interpretation of random events and the likelihood of these events occurring.

Probability theory starts with basic concepts such as random experiments, sample spaces, events, and the probability of events.

A **random experiment** is any process or action that results in one of several possible outcomes, like tossing a coin or rolling a die. The sample space is the set of all possible outcomes of an experiment, and an event is a specific outcome or a set of outcomes.

Terms in Probability

Term	Definition
Sample Space	The set of all possible outcomes in a probability experiment. For instance, in a coin toss, it's "head" and "tail".
Sample Point	One of the possible results in an experiment. For example, in rolling a fair six-sided dice, sample points are 1 to 6.
Experiment	A process or trial with uncertain results. Examples include coin tossing, card selection, or rolling a die.
Event	A subset of the sample space representing certain outcomes. Example: getting "1" when rolling a die.
Favorable Outcome	An outcome that produces the desired or expected consequence.

Probability of an Event

- The probability of an event is a measure of the likelihood that the event will occur, expressed as a number between 0 and 1.

- An event with a probability of 1 is considered certain to happen, while an event with a probability of 0 is certain not to happen.

- Let's discuss different types of events in probability.

Equally Likely Events

Equally likely events are those whose chances or probabilities of happening are equal. Both events are not related to one another.

For example, there are equal possibilities of receiving either a head or a tail when we flip a coin

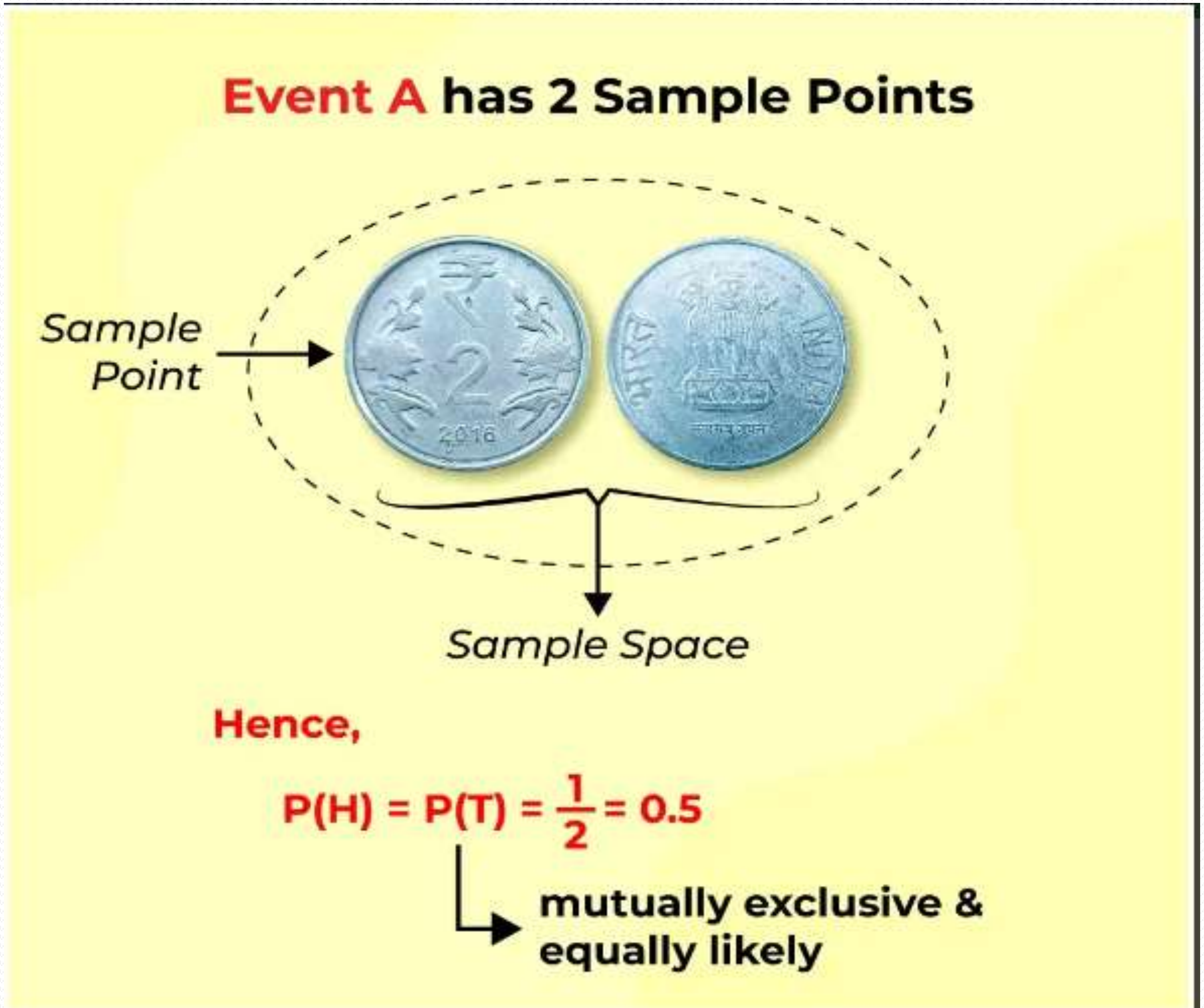
Exhaustive Events

We call an event exhaustive when the set of all experiment results is the same as the sample space.

Mutually Exclusive Events

Events that are mutually exclusive cannot occur at the same time.

For example, the weather may be hot or chilly simultaneously. We can't have the same weather at the same time


Event A has 2 Sample Points
Sample Point
2016
Sample Space
Hence,
P(H) = P(T) = 1/2 = 0.5
mutually exclusive & equally likely

Complimentary events

Two events can be termed as complementary events **if one event can take place only when the other does not occur**. Furthermore, the complement of an event can be defined as the set of outcomes in which it does not occur. Let A be an event. The complementary of an event A is the event denoted $\bar{A}$.

Complementary Events Properties

For two events to be classified as complementary events they must follow certain properties. These are given as follows:

- Complementary events are mutually exclusive. This means that two events that are complementary cannot occur at the same time. In other words, complementary events are disjoint.
- Complementary events are exhaustive. This implies that an event, as well as its complement, must completely fill up the sample space Ω. Thus,

$\Omega = A \cup \overline{A}$

Remark

We can form new events using set operations:

- $A \cup B$ is the event that occurs when A **or** B is realized
- $A \cap B$ is the event that occurs when A **and** B are both realized
- $\overline{A}$ is the event that occurs when A is not realized (complementary event)
- If $A \cap B = \Phi$ we say that A and B are mutually exclusive.

Examples

We throw a die and observe the result obtained.

S = {1, 2, 3, 4, 5, 6}. are the events:

A: appearance of an even number, B: appearance of an odd number, C: appearance of a prime number.

Find: $A \cup C$, $B \cap C$ and $\overline{C}$ (what can we say about A and B?).

Consider an experiment involving a three coin tosses. S contains 8 elements; S = { TTT, TTH, THH, THT, …}

Find the events: A: “consecutive appearance of 2 or more T”; B: “appearance of 3 T or 3 H.

Calculation of Probabilities

Probability associated with a random experiment is a function which associates a real number between 0 and 1 with an event A, and verifying the axioms below:

$$\begin{aligned} P: \quad & P(\Omega) \to [0\,,1] \\ & A \mapsto P(A) \end{aligned}$$

where P(Ω) is the set of all possible parts of the sample space Ω (i.e. the set of all possible events of the random experiment concerned).

Probability Formula

P(E) = Number of favorable outcomes / Total number of outcomes

Note: where P(E) denotes the probability of an event E.

- For all event A, $0 \leq P(A) \leq 1$;
- $P(S) = 1$;
- If A et B mutually exclusive, then $P(A) = P(A) + P(B)$;
- For any sequence of events A_i , $1 \leq i \leq n$, which mutually exclusive, $P(\cup A_i) = \Sigma P(A_i)$

Example

Consider an experiment involving a three coin

tosses. The outcome will be a 3-long string of heads or tails. The sample space is

$\Omega = \{HHH, HHT, HT\,H, HT\,T, T\,HH, T\,HT, T\,T\,H, TTT\}$.

We assume that each possible outcome has the same probability of 1/8. Find the probability of the event

$A = \{\text{exactly 2 heads occur}\} = \{HHT, HT\,H, THH\}$.

Using additivity, the probability of A is the sum of the probabilities of its elements:

$$\begin{aligned}\mathbf{P}\big(\{HHT, HTH, THH\}\big) &= \mathbf{P}\big(\{HHT\}\big) + \mathbf{P}\big(\{HTH\}\big) + \mathbf{P}\big(\{THH\}\big) \\ &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\ &= \frac{3}{8}.\end{aligned}$$

Fundamental properties

- The sum of the probability of an event and its complementary is 1, i.e., $P(A) + P(A) = 1$.
- The probability of a sure event is always 1, i.e, $P(A) = 1$.

For example, if you flip a fair coin and define the event AA as "the coin lands either heads or tails," then the probability of event AA occurring is P(A)=1P(A)=1, because it's certain that the coin will land either heads or tails.

- The probability of any event lies between 0 and 1, i.e, $0 < P(A) < 1$.
- Formula for the union of two events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If A is included in B then $P(A) \leq P(B)$
- For mutually events A and B: $P(A \cup B) = P(A) + P(B)$.

Conditional probability and Independence

Preliminary example: A machine **A** produces **60%** of articles of which **7% have defect D** and a machine **B** produces **40%** of which **5% have defect D**. We take an article at random: What is the probability that:

1. the item comes from machine A?
2. the item comes from machine B?
3. the article comes from A **and** has defect D?
4. the article comes from B **and** does not have defect D?
5. Does the item come from A **or** have defect D?

1. $P(A)=60/100=0.6$ 2. $P(B)=40/100=0.4$

3. $P(A \cap D)=7/100=0.07$

4. $P(B \cap \overline{D})=35/100=0.35$

5. $P(A \cup D) = P(A) + P(D) - P(A \cap D)$

$=0.6+0.12-0.07$

$=0.65$

Now, suppose we take an item from A, what is the probability that it has defect D?

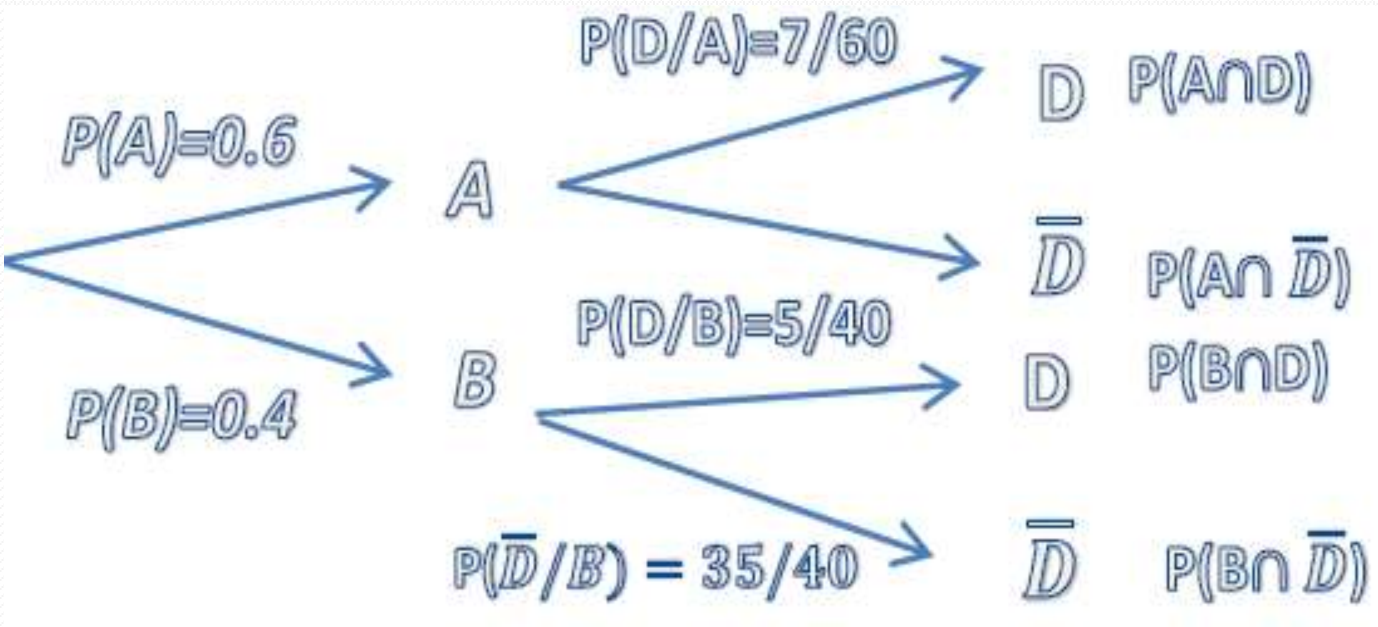
Conditional probability noted:

P(D/A)= number of favorable cases of D/ number of favorable cases of A =7/60,

because the sample space has been reduced to A.

$P(D/A)=P_A(D)= P(A \cap D)/P(A)=0.07/0.6$

Using a tree diagram we obtain:



$P(A\cap D)=P(A)\ P(A/D)$ then $P(A/D)=\frac{P(A\cap D)}{P(A)}$

Definition Let **B** be an event of a sample space **Ω** such that P(B)>o. The probability that an event **A** occurs knowing that B has occurred is defined and noted as follows

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Example

Suppose you roll two dice. What is the probability the sum is 8?

Solution: there are five ways this can happen

{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}, so the prob-

ability is 5/36. Let us call this event **A**.

What is the probability that the sum is 8 given

that the first die shows a 3?

Let **B** be the event that the first die shows a 3 (B={3,1},{3,2}…{3, 6})

Then **P(A∩ B) is the probability that the first die shows a and the sum is 8**, P(A∩ B) =1/36. P(B) = 1/6, so

P(A/B) = (1/36)/(1/6)= 1/6

Partition and Bayes' Theorem

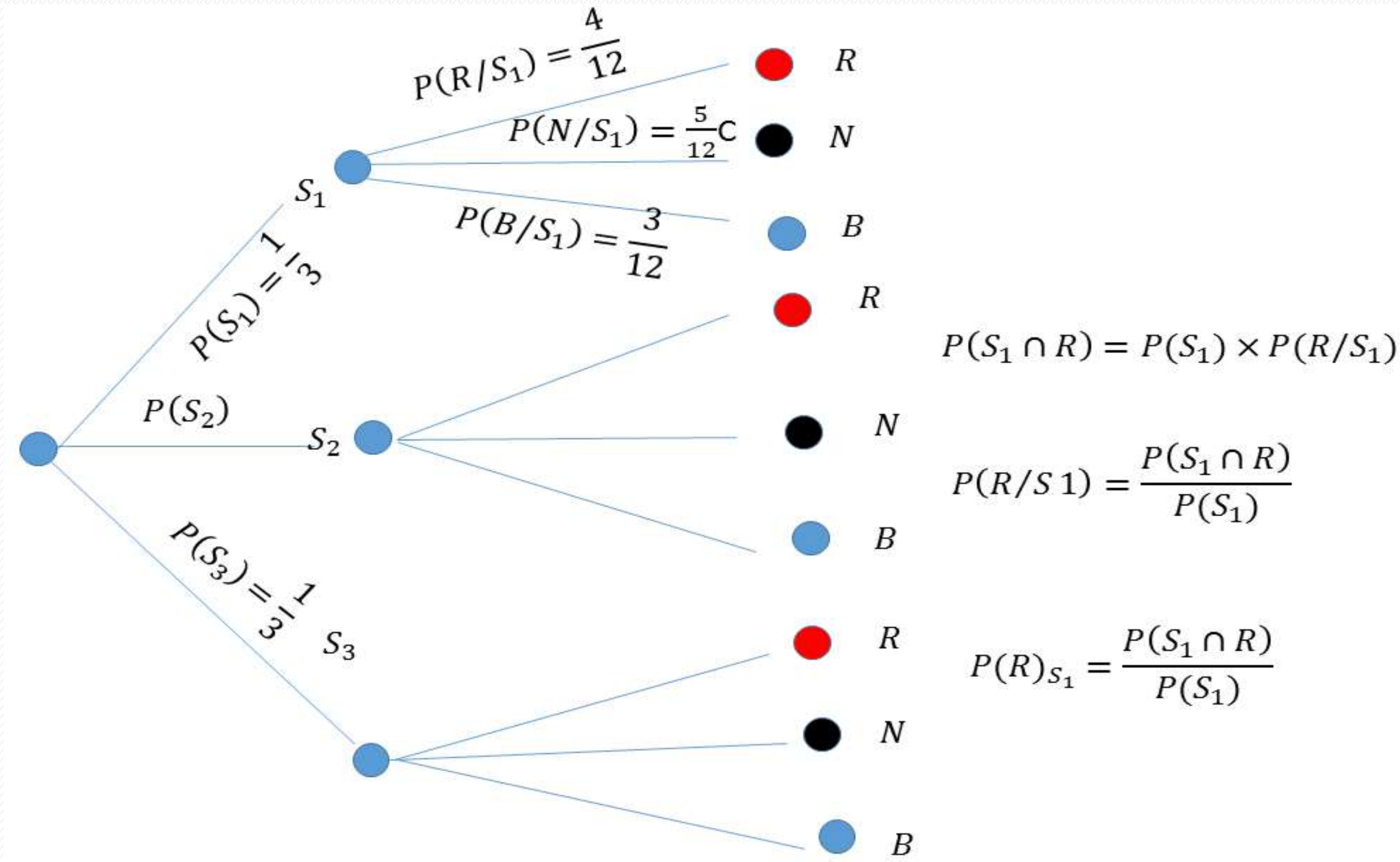
Preliminary example: We have three bags S1, S2 and S3. Bag **S1 contains 4 red balls (R), 3 blue balls (B) and 5 black balls (N)**, **bag S2 contains 6 red balls, 0 blue balls and 4 black balls** and bag **S3 contains 7 red balls, 4 blue balls and 0 black balls**. Let's first calculate the probability of drawing a red ball:

$$P(R) = P(R \cap S_1) + P(R \cap S_2) + P(R \cap S_3)$$

$$= P(S_1)P(R/S_1) + P(S_2)P(R/S_2) + P(S_3)P(R/S_3)$$

$$= \frac{1}{3} \times \frac{4}{12} + \frac{1}{3} \times \frac{6}{10} + \frac{1}{3} \times \frac{7}{11}$$

in fact, the following tree diagram explains this calculation to us:



Same if we want to calculate the probability of drawing a **blue ball** or the probability of drawing a black ball.

$$P(B) = P(B \cap S_1) + P(B \cap S_2) + P(B \cap S_3)\ ;\ P(B) = \frac{1}{3} \times \frac{3}{12} + \frac{1}{3} \times \frac{0}{10} + \frac{1}{3} \times \frac{4}{11}$$

$$P(N) = P(N \cap S_1) + P(N \cap S_2) + P(N \cap S_3)\ ;\ P(N) = \frac{1}{3} \times \frac{5}{12} + \frac{1}{3} \times \frac{4}{10} + \frac{1}{3} \times \frac{0}{11}$$

Now suppose that the drawn **ball is red**, let's calculate the probability that it is drawn from the bag S1

$$P(S1/R) = \frac{P(S_1 \cap R)}{P(R)} = \frac{P(S_1 \cap R)}{P(S_1 \cap R) + P(S_2 \cap R) + P(S_3 \cap R)}$$

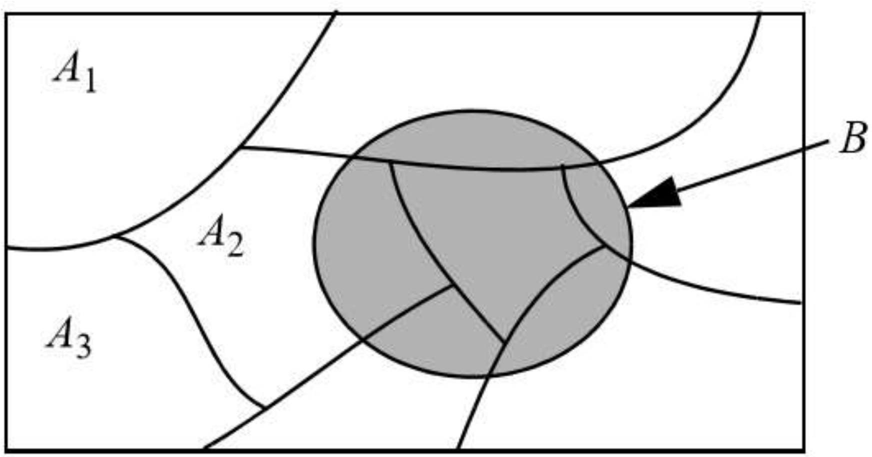
$$= \frac{P(S_1)P(R/S_1)}{P(S_1)P(R/S_1) + P(S_2)P(R/S_2) + P(S_3)P(R/S_3)}$$

$$P(S1/R) = \frac{\frac{1}{3} \times \frac{4}{12}}{\frac{1}{3} \times \frac{4}{12} + \frac{1}{3} \times \frac{6}{10} + \frac{1}{3} \times \frac{7}{11}}$$

Total Probability Theorem

Let $A_1, \ldots, A_n$ be disjoint events that form a partition of the sample space (each possible outcome is included in one and only one of the events $A_1, \ldots, A_n$) and assume that $\mathbf{P}(A_i) > 0$, for all $i = 1, \ldots, n$. Then, for any event B, we have

$$\begin{aligned}\mathbf{P}(B) &= \mathbf{P}(A_1 \cap B) + \cdots + \mathbf{P}(A_n \cap B) \\ &= \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n).\end{aligned}$$



Bayes' theorem:

Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of the sample space, and assume that $\mathbf{P}(A_i) > 0$, for all i. Then, for any event B such that $\mathbf{P}(B) > 0$, we have

$$\begin{aligned}\mathbf{P}(A_i \mid B) &= \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\mathbf{P}(B)} \\ &= \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n)}.\end{aligned}$$

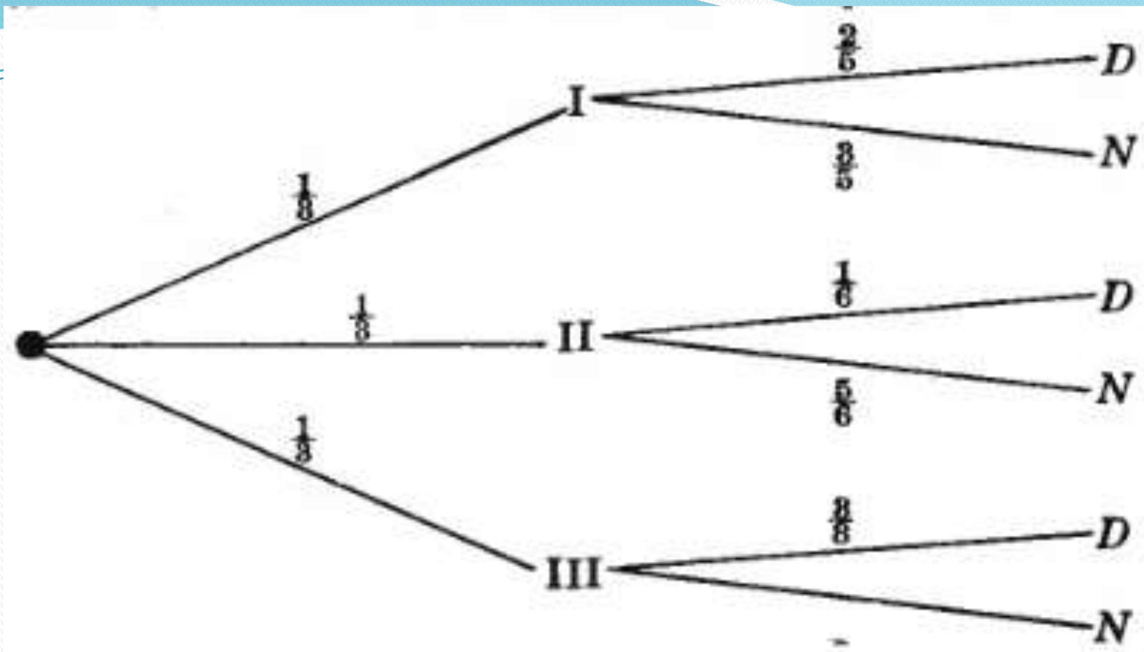
Example: We give three boxes such as:

Box I contains 10 light bulbs, 4 of which are defective.

Box II contains 6 light bulbs, 1 of which is defective

Box III contains 8 electric bulbs, 3 of which are defective We choose a box at random and extract a bulb at random. What is the probability that the bulb is defective?

The result is explained by the following tree diagram:



Since there are three mutually exclusive paths that lead to a defective bulb, the desired probability is equal to the sum of the probabilities of these paths:

$$p = \frac{1}{3}.\frac{2}{5} + \frac{1}{3}.\frac{1}{6} + \frac{1}{3}.\frac{3}{8} = \frac{113}{360}$$

Independence

We say that two events **A and B are independent** if the probability that A will occur is not modified or influenced by the fact that B has occurred or not. This translates to P(A/B)=P(A). In this case:

$P(A) = P(A/B) = P(A \cap B) / P(B)$.

Definition: two events A and B are independent if and only if $P(A \cap B) = P(A).P(B)$.

Consider an example of rolling a die. If A is the event 'the number appearing is odd' and B be the event 'the number appearing is a multiple of 3', then

$$\mathbf{P(A) = 3/6 = 1/2} \text{ and } \mathbf{P(B) = 2/6 = 1/3}$$

Also A and B is the event 'the number appearing is odd and a multiple of 3 so that $\mathbf{P(A \cap B) = 1/6}$

$\mathbf{P(A \mid B) = P(A \cap B)/\ P(B) = (1/6)/(1/3) = 1/2}$

$P(A) = P(A \mid B) = 1/2$, which implies that the occurrence of event B has not affected the probability of occurrence of the event A .

Then A and B are independent events.