Norms

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Plan

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1. Vector Norms
2. Matrix Norms

Definition

A vector norm $\| \,.\, \|$ is a function from $\mathbb{C}^n$ to $\mathbb{R}$ with three properties:

- N1: $\| x \| \geq 0$, for all $x \in \mathbb{C}^n$, and $\| x \| = 0$ if and only if $x = 0$.
- N2: $\| x + y \| \leq \| x \| + \| y \|$ for all $x, y \in \mathbb{C}^n$ (Triangle inequality)
- N3: $\| \alpha x \| = | \alpha | \| x \|$ for all $\alpha \in \mathbb{C}$, $x \in \mathbb{C}^n$.

The vector $p - norms$ below are useful for computational purposes, as well as analysis.

Fact

Vector p-norms Let $x \in \mathbb{C}^n$ with elements $x = (x_1, x_2, \dots, x_n)^T$ the $p - norm$

$$\| x \|_p = \left(\sum_{j=1}^{n} | x_j |^p \right)^{\frac{1}{p}}, p \geq 1.$$

is a vector norm.

Example

If e_j is a canonical vector, then $\parallel e_j \parallel_p = 1$ for $p \geq 1$.
If $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$,then

$$\parallel e \parallel_1 = n, \parallel e \parallel_\infty = 1, \ \parallel e \parallel = n^{\frac{1}{p}}, 1 < p < \infty$$

The three $p - norms$ below are the most popular, because they are easy to compute.

- One norm $\parallel x \parallel_1 = \sum_{j=1}^{n} \mid x_j \mid$
- Two(or Euclidean)norm: $\parallel x \parallel_2 = \sqrt{\sum_{j=1}^{n} \mid x_j \mid^2} = \sqrt{x^* x}$
- Infinity (or maximum)norm: $\parallel x \parallel_\infty = \max_{1 \leq j \leq n} \mid x_j \mid$

Example

If $x = (1, 2, \ldots, n)^T \in \mathbb{R}^n$, then

$$\parallel x \parallel_1 = \frac{1}{2} n(n+1), \ \parallel x \parallel_2 = \sqrt{\frac{1}{2} n(n+1)(2n+1)}, \ \parallel x \parallel_\infty = n$$

Fact

Let $x, y \in \mathbb{C}^n$. Then

- *Holder inequality:* $| x^* y | \leq \| x \|_1 \| y \|_\infty$
- *Cauchy-Schwarz inequality:* $| x^* y | \leq \| x \|_2 \| y \|_2$

Example

Let $x \in \mathbb{C}^n$ with elements $x = (x_1, \ldots, x_n)^T$. The Holder inequality and Cauchy-Schwartz inequality imply respectively

$$| \sum_{i=1}^{n} x_i | \leq n \max | x_i |, \; | \sum_{i=1}^{n} x_i | \leq \sqrt{n} \| x \|_2$$

Matrix Norms

We need to separate matrices from vectors inside the norms. To see this let $Ax = b$ be a nonsingular linear system and let $A\overline{x} = \overline{b}$ be a perturbed system.
The normwise absolute error is $\| x - \overline{x} \| = \| A^{-1}(b - \overline{b}) \|$. In order to isolate the perturbation and derive a bound of the form $\| A^{-1} \|, \ \| b - \overline{b} \|$, we have to define a norm for matrices.

Definition

A matrix norm $\| \,.\, \|$ is a function from $\mathbb{C}^{m\times n}$ to $\mathbb{R}$ with three properties:

- N1: $\| A \| \geq 0$, for all $A \in \mathbb{C}^{m\times n}$, and $\| A \| = 0$ if and only if $A = 0$.
- N2: $\| A + B \| \leq \| A \| + \| B \|$ for all $A, B \in \mathbb{C}^{m\times n}$ (Triangle inequality)
- N3: $\| \alpha A \| = | \alpha | \| A \|$ for all $\alpha \in \mathbb{C}, \ A \in \mathbb{C}^{m\times n}$.

Because of the triangle inequality, matrix norms are well-conditioned in the absolute sens and in the relative sens.

Fact

If $A, E \in \mathbb{C}^{m\times n}$, *then* $|\| A+E \| - \| A \||\leq \| E \|$.

Proof.

The triangle inequality implies $\| A+E \|\leq \| A \| + \| E \|$, hence $\| A+E \| - \| A \|\leq \| E \|$. Similarly $\| A \| = \| (A+E)-E \|\leq \| A+E \| + \| E \|$, so that $-\| E \|\leq \| A+E \| - \| A \|$.
The result follows from

$$-\| E \|\leq \| A+E \| - A \|\leq \| E \| .$$

$\square$

The matrix p-norms below are based on the vector p-norms and measure how much a matrix can stretch a unit-norm vector

Fact (Matrix p-Norms)

Let $A \in \mathbb{C}^{n\times m}$. *the p-norm*

$$\| A \|_p = \max_{x\neq 0} \frac{\| Ax \|_p}{\| x \|_p} = \max_{\|x\|_p=1} \| Ax \|_p$$

is a matrix norm.

Remarque

The matrix p-norms are extremely useful because they satisfy the following submultiplicative inequality

Let $A \in \mathbb{C}^{m\times n}$ and $y \in \mathbb{C}^n$. Then

$$\| Ay \|_p \leq \| A \|_p \| y \|_p$$

This is clearly true for $y = 0$, and for $y \neq 0$ it follow from

$$\| A \|_p = \max_{x\neq 0} \frac{\| Ax \|_p}{\| x \|_p} \geq \frac{\| Ay \|_p}{\| y \|_p}$$

The matrix one norm is equal to the maximal absolute column sum.

Fact (Infinity Norm)

Let $A \in \mathbb{C}^{m\times m}$. Then

$$\| A \|_\infty = \max_{1\le i\le m} \| A^* e_i \|_1 = \max_{1\le i\le m} \sum_{j=1}^{n} | \alpha_{ij} |$$

Proof.

Denote the rows of A by $r_i^* = e_i^* A$, and let r_k have the largest one norm, $\| r_k \|_1 = \max_{1\le i\le m} \| r_i \|_1$.

- Let y be a vector with $\| A \|_\infty = \| Ay \|_\infty$ and $\| y \|_\infty = 1$. Then

  $$\| A \|_\infty = \| Ay \|_\infty = \max_{1\le i\le m} \| r_i \|_1 \| y \|_\infty = \| r_k \|_1,$$

  where the inequality follows from Fact. Hence $\| A \|_\infty \le \max_{1\le i\le n} \| r_i \|_1$
- For any vector y with $\| y \|_\infty = 1$ we have $\| A \|_\infty \ge \| Ay \|_\infty \ge | r_k^* y |$. Now we show how to choose the elements of r_k^*. Choose the elements of y such that $\rho_j y_j = | \rho_j |$.
  Then $\| y \|_\infty = 1$ and $| r_k^* y | = \sum_{j=1}^n \rho j y_j = \sum_{j=1}^n | \rho_j | = \| r_k \|_1$. Hence

  $$\| A \|_\infty \ge | r_k^* y | = \| r_k \|_1 = \max_{1\le i\le m} \| r_i \|_1 .$$

The p-norms satisfy the following submultiplicative inequality.

Fact (Norm of Product)

If $A \in \mathbb{C}^{m \times n}$ *and* $B \in \mathbb{C}^{n \times p}$, *then*

$$\| AB \| \leq \| A \|_p \| B \|_p$$

Proof.

Let $x \in \mathbb{C}^p$ such that $\| AB \|_p = \| ABx \|_p$ and $\| x \|_p = 1$. Applying Remark... twice gives

$$\| AB \|_p = \| ABx \|_p \leq \| A \|_p \| Bx \|_p \leq \| A \|_p \| B \|_p \| x \|_p = \| A \|_p \| B \|_p \,.$$

$\square$