Yahia Fares University of Medea **Academic Year : 2023/2024**
Faculty of Sciences **First Year of Master's Degree in CAPA**
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Worksheet n° 1 : Matrix Numerical Analysis

Exercise 1 :

Let q be the quadratic form defined on the vector space $E = \mathbb{R}^3$ for every element x of E with $x = (x_1,\ x_2,\ x_3)$, on the canonical basis $(e_1,\ e_2,\ e_3)$ by

$$q(x) = 4x_1^2 + 2x_2^2 + 5x_3^2 + 4x_1x_2 + 4x_1x_3 + 2x_2x_3.$$

1. Write the matrix A associated with q.
2. Decompose the quadratic form into a sum of linearly independent squares, using the Gauss method.
3. Find a basis of E and write the matrix B of q on this basis.
4. What is the rank of q ?
5. Give the definition of the S.D.P matrix.
6. Without calculating the determinant of A, is A S.D.P or not ?
7. Let $C \in \mathcal{M}_n(\mathbb{R})$ be symmetric, positive definite. Show that the eigenvalues of C are real and positive.

Exercise 2 :

Let q be the quadratic form on the vector space $E = \mathbb{R}^4$ defined by

$$q(x) = x_1^2 + 3x_2^2 + 4x_3^2 + \alpha x_4^2 + 2\beta x_1x_2,$$

where $x = (x_1, x_2, x_3, x_4)$, α and β are real numbers.

1. Write the polar form of q.
2. Determine the matrix q.
3. Decompose q into a sum of linearly independent squares.
4. Find an orthogonal basis.
5. Calculate the signature of q according to α and β.
6. Deduce the rank of q.

Exercise 3 :

Consider the matrix

$$A = \begin{pmatrix} 1 & \sqrt{2}i & 0 \\ -\sqrt{2}i & 1 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 1 \end{pmatrix}$$

1. Show that A is Hermitian.
2. What can be said about the eigenvalues of A ?
3. Determine the eigenvalues $\lambda_i,\ i = 1 \ldots 3$ of A.
4. Determine the eigenvectors $x_i,\ i = 1 \ldots 3$ of A corresponding to $\lambda_i,\ i = 1 \ldots 3$.
5. Show that the eigenvectors $x_i,\ i = 1 \ldots 3$ are orthogonal.
6. Determine an orthonormal basis.
7. Determine the transition matrix P.
8. Show that the matrix P is unitary.

9. In the new basis $B = \{x_1, x_2, x_3\}$ of $\mathbb{C}$, determine the matrix D.
10. Verify that $A = PDP^{-1}$.
11. Determine the Hermitian form f on $\mathbb{C}^3$ represented by the matrix A in the canonical basis.
12. Show that f is non-degenerate.
13. Determine the quadratic form φ of f. What can be deduced ?
14. Determine the Hermitian form g represented by the matrix D and the quadratic form q. What can be deduced ?

Exercise 4 :

— If the matrix A is Hermitian, show that the eigenvectors associated with different eigenvalues are orthogonal.

— If the matrix A is normal and λ is an eigenvalue of A associated with the eigenvector x, show that x is an eigenvector of $\overline{A}^t$ associated with the eigenvalue $\overline{\lambda}$ of $\overline{A}^t$?

Exercise 5 :

1. Give the definition of an independent matrix.
2. Give the definition of a nilpotent matrix of index p.
3. If A is a nilpotent matrix of index 2, show that $A(I + A)^n = A, \ \forall n \in \mathbb{N}$?