Solutions to Problem Sheet for Branching Process

1. For a branching process with the family size distribution given by $p_0 = 1/6$, $p_2 = 1/3$, $p_3 = 1/2$. Calculate

   (a) Since we have two individuals at the beginning and they are independent of each other. We need to first find the PGF of a single ancestor. Then we square it to find the PGF of the both ancestors. if $Z_n = W_n + U_n$ is the generation size for the two branches, where W_n and U_n are the generatio sizes for each branch, then

   $$H_{n,Z}(s) = H_{n,W}(s)H_{n,U}(s) = (H_{n,W}(s))^2.$$

   Since, the processes W_n and U_n and identically and independently distributed.
   Now, $H_{2,W}(s) = H_{1,W}(G(s)) = G(G(s))$, where $G(s)$ is PGF of the family size distribution.

   (b) The mean of Z_n is defined as

   $$\mu_2 = \mathrm{E}[Z_2] = H'_{2,Z}(s)|_1 = 2H_{2,W}(s)H'_{2,W}(s)|_1 = 2G'(s)G^2(G(s))|_1$$

   The variance is obtained in the same manner using the formula

   $$\mathrm{var}(Z_2) = [H''_{2,Z}(s) + H'_{2,Z}(s) + (H'_{2,Z}(s))^2]|_{s=1}$$

2. Consider a branching process in which the family size distribution is Poisson with mean λ.

   (a) the probability of extinction will be less than 1 when the mean of the Poisson process is greater than 1, that is, $\lambda > 1$.

   (b) The extinction probability is the smallest solution to the equation

   $$G(q) = q$$

   Here $G(q) = \exp(\lambda(q-1))$, thus we need to verify that

   $$\exp(2(q-1)) = q$$

   for $q = 0.2032$. Since the exponential function is monotonic there is only one solution.

   (c) The expected size of the 10th generation is

   $$\mu_1 0 = \theta^1 0 = \lambda^1 0 = 2^1 0$$

   since the mean family size is $\theta = \lambda$.
   The probability of extinction is calculated using the recurrence formula

   $$q_n = G(q_{n-1})$$

   and the fact that $q_0 = 0$.

3. Let $\{Z_n\}_{n=0}^{\infty}$ be a branching process with $Z_0 = 1$. It is known that $p_0 = p$ and $p_2 = 1 - p$, where $p \in [0, 1]$.

   (a) The probability of extinction is zero when the mean family size is 1, that is

   $$\theta = 0 \times p_0 + 2 \times p_2 = 2(1-p) = 1$$

   that is

   $$p = 1/2$$

   (b) $q < 1$ when $\theta > 1$ that is

   $$2(1-p) > 1$$

   Hence, $0 < p < 1/2$.
   Now the probability of extinction is the smallest solution to the equation

   $$q = G(q),$$

   $$q = p + (1-p)q^2.$$

   (c) Calculate the mean and the variance of Z_n when $p = \frac{1}{3}$. The mean of Z_n is $\mu_n = \theta^n = 2^n(1-p)^n = \frac{4^n}{3^n}$
   The variance is calculated using the formula:

   $$\sigma_n^2 = \frac{\theta^{n-1}(1-\theta^n)}{1-\theta}\sigma^2$$

   where σ^2 is the variance of the family size distribution.