Exercice 1 A student has to answer 10 out of 13 questions in an exam.

1. How many possible choices does he have ?
2. How many possibilities does he have if he has to answer the first two questions ?
3. How many if he has to answer the first or second question, but not both together ?
4. How many if he has to answer exactly 3 of the first 5 questions ?
5. How many if he has to answer at least 3 of the first 5 questions ?

Solution 1

1. The student answers the 10 questions without order and without repetition. So the number of his possible choices is equal to

$$\begin{aligned} |\Omega| &= \binom{13}{10} = C_{13}^{10} \\ &= \frac{13!}{10! \times 3!} \\ &= \frac{13 \times 12 \times 11}{3 \times 2 \times 1} \\ &= 286 \end{aligned} \tag{1}$$

2. A :"He has to answer the first two questions", so the number of his possible choices is equal to

$$\begin{aligned} |A| &= \binom{11}{8} = C_{11}^{8} \\ &= \frac{11!}{8! \times 3!} = \frac{11 \times 10 \times 9}{3 \times 2 \times 1} \\ &= 165 \end{aligned}$$

3. Note by B :"The student has to answer the first or second question, but not both together"

$$\begin{aligned} |B| &= C_2^1 \times C_{11}^9 \\ &= \frac{2!}{1! \times 1!} \times \frac{11!}{9! \times 3!} \\ &= 2 \times \frac{11 \times 10}{2 \times 1} \\ &= 110 \end{aligned}$$

4. Note by C :" he has to answer exactly 3 of the first 5 questions", then

$$\begin{aligned} |C| &= C_5^3 \times C_8^7 \\ &= (\frac{5 \times 4}{2 \times 1}) \times (8) \\ &= 80 \end{aligned}$$

5. Note by D :" he has to answer at least 3 of the first 5 questions"

$$\begin{aligned} |D| &= |C| + C_5^4 \times C_8^6 + C_5^5 \times C_8^5 \\ &= 80 + (5 \times \frac{8 \times 7}{2 \times 1}) + (1 \times \frac{8 \times 7 \times 6}{3 \times 2 \times 1}) \\ &= 80 + 140 + 56 \\ &= 276 \end{aligned}$$

Exercice 2 A coin and a die are tossed in the air, and it is assumed that the sample set S consists of 12 elements : $S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$.

1. Express the following events explicitly :
   — A :"Head and an even number appear",
   — B :"a prime number appear",
   — C :"Tail and an odd number appears".
2. Explicitly express the event :
   (a) A or B is true
   (b) B and C is true
   (c) B only is true
3. Which of the events A, B and C are mutually exclusive ?

Solution 2

$$\Omega = \{xy, x \in \{H, T\} \text{ and } y \in \{1, 2, 3, 4, 5, 6\}\}$$

$$\begin{aligned} \Rightarrow |\Omega| &= 2 \times 6 \\ &= 12 \end{aligned}$$

1. Express the following events explicitly :
   (a) $A = \{Hy,\ y \in \{2, 4, 6\}\}$;
   (b) $B = \{xy,\ x \in \{H, T\} \text{ and } y \in \{2, 3, 5\}\}$;
   (c) $C = \{Ty,\ y \in \{1, 3, 5\}\}$.
2. Explicitly express the event :
   (a) $A \cup B = \{Hy, Tx;\ y \in \{2, 3, 4, 5, 6\} \text{ and } x \in \{2, 3, 5\}\}$;
   (b) $B \cap C = \{Ty;\ y \in \{3, 5\}\}$;
   (c) $B \cap \overline{A} \cap \overline{C}$. We have,

$$\begin{aligned} \overline{A} &= \{Hy, Tx;\ y \in \{1, 3, 5\} \text{ and } x \in \{1, 2, ..., 6\}\} \\ \& \ \overline{C} &= \{Hy, Tx;\ y \in \{1, 2, ..., 6\} \text{ and } x \in \{2, 4, 6\}\} \\ \text{so } \overline{A} \cap \overline{C} &= \{Hy, Tx;\ y \in \{2, 3, 5\} \text{ and } x \in \{2, 3, 5\}\} \\ \text{then } B \cap \overline{A} \cap \overline{C} &= \{Hy, Tx;\ y \in \{3, 5\} \text{ and } x = 2\} \end{aligned}$$

3. Which of the events A, B and C are mutually exclusive ?
   (a) From $|A \cap B| = |A| + |B| - |A \cup B| = 3 + 6 - 2 = 7$, A and B are not mutually exclusive ;

(b) From $B \cap C = \{Ty;\ y \in \{3,5\}\} \neq \emptyset$, B and C are not mutually exclusive ;

(c) From $A \cap C = \emptyset$, A and C are mutually exclusive.

Exercice 3 Three light bulbs are randomly selected from a group of 15 bulbs, 5 of which are defective. Calculate the probability p :

1. no bulb is defective ;
2. only one bulb is defective ;
3. at least one bulb is defective.

Solution 3 We have, $|\Omega| = C_{15}^3 = \frac{15\times14\times13}{3\times2} = 455$.

1. Note by A :"no bulb is defective", then $|A| = C_{10}^3 = \frac{10\times9\times8}{3\times2} = 120$ and $P(A) = \frac{120}{455} = \frac{24}{91}$;
2. Note by B :"only one bulb is defective", then $|B| = C_5^1 \times C_{10}^2 = 5 \times \frac{10\times9}{2} = 225$ and $P(B) = \frac{225}{455} = \frac{45}{91}$;
3. Note by C :"at least one bulb is defective", then $C = \overline{A}$ and $P(C) = 1 - P(A) = \frac{67}{91}$.

Exercice 4 Two numbers from 1 to 9 are drawn at random. Given that the sum obtained is even, calculate the probability p that the two numbers are odd.

Solution 4 We have, $|\Omega| = C_9^2 = \frac{9\times8}{2} = 36$.

1. Note by A :"draw two numbers odd", then $|A| = C_4^2 = 6$ and $P(A) = \frac{6}{36} = \frac{1}{6}$;
2. Note by B :"The sum obtained is even", then $B = A \cup C$ with C :"draw two numbers even" and $|B| = 6 + C_5^2 = 6 + 10 = 16$ and $P(B) = \frac{16}{36} = \frac{4}{9}$;
3. So, $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{3}{8}$ from $P(A \cap B) = P(A)$.

Exercice 5
In a secondary school, 25% of students fail maths, 15% fail chemistry and 10% fail both maths and chemistry. One student is randomly selected.

1. If the student fails chemistry, what is the probability that he/she will also fail maths ?
2. If a student fails maths, what is the probability of failing chemistry as well ?
3. What is the probability of a student failing maths or chemistry ?.

Solution 5 Note by :
— A ;"Student fails maths" ;
— B :"Student fails chemistry".

1. If the student fails chemistry, what is the probability that he/she will also fail maths ?

$$\begin{aligned} P(A|B) &= \frac{P(A\cap B)}{P(B)} \\ &= \frac{0.10}{0.15} \\ &= \frac{2}{3} \end{aligned} \quad (2)$$

2. If a student fails maths, what is the probability of failing chemistry as well ?

$$\begin{aligned} P(B|A) &= \frac{P(B\cap A)}{P(A)} \\ &= \frac{0.10}{0.25} \\ &= \frac{2}{5} \end{aligned} \quad (3)$$

Also by bayes's theorem, we have

$$\begin{aligned} P(B|A) &= \frac{P(A|B)\times P(B)}{P(A)} \\ &= \frac{\frac{2}{3}\times 0.15}{0.25} \\ &= \frac{2}{5} \end{aligned} \quad (4)$$

3. What is the probability of a student failing maths or chemistry ?.

$$\begin{aligned} P(B\cup A) &= P(A) + P(B) - P(A\cap B) \\ &= 0.25 + 0.15 - 0.10 \\ &= 0.30 \end{aligned} \quad (5)$$

Exercice 6
5 balls numbered 2, 7 balls numbered 3 and 10 balls numbered 9 are placed in an urn. We randomly choose a handful of 8 balls, each ball having the same probability of being caught.

1. What is the probability of not having chosen any ball numbered 2 ?
2. What is the probability of having obtained only balls numbered 9 ?
3. What is the probability of having obtained at least one ball of each number ?
4. We start the experiment again by pulling the balls one by one and putting the ball back in the bag after it has been drawn. Calculate the probabilities of the three events above in this new experiment.

Exercice 7

We consider the following game : the player first rolls a non-rigged die. He then draws a token in an urn chosen according to the result of the die. Urn A is chosen when the die gives 1, 2 or 3, urn B when we get 4 or 5 and urn C when we get 6. The urns A, B and C contain the following tokens :

urn A : two red tokens, three blue tokens ;

urn B : two blue tokens, four green tokens ;

urn C : a green token, a red token.

1. What is the probability of getting a red token by this process ?
2. We get a green token. What is the probability that this token comes from urn B ?
3. We get a blue token. What is the probability that the roll of the die yielded 3 ?
4. What is the probability of not getting a green token, knowing that the roll of the die gave 3 or 6 ?
5. Are the event E : "choose from urn C" and the event F :"get a red token" independent ? Justify your answer.