Proof (of Doob's Martingale Convergence Theorem)

By the Upcrossings Inequality

$$E\left(U_n[a,b]\right) \le \frac{E\left(\left(\xi_n - a\right)^-\right)}{b-a} \le \frac{M+|a|}{b-a} < \infty,$$

where

$$M = \sup_n E\left(|\xi_n|\right) < \infty.$$

Since $U_n[a,b]$ is a non-decreasing sequence, it follows that

$$E\left(\lim_{n\to\infty} U_n[a,b]\right) = \lim_{n\to\infty} E\left(U_n[a,b]\right) \le \frac{M+|a|}{b-a} < \infty.$$

This implies that

$$P\left\{\lim_{n\to\infty} U_n[a,b] < \infty\right\} = 1.$$

for any $a < b$. Since the set of all pairs of rational numbers $a < b$ is countable, the event

$$A = \bigcap_{a<b \text{ rational}} \left\{\lim_{n\to\infty} U_n[a,b] < \infty\right\} \tag{4.3}$$

has probability 1. (The intersection of countably many events has probability 1 if each of these events has probability 1.)

We claim that the sequence ξ_n converges a.s. to a limit ξ. Consider the set

$$B = \{\liminf_n \xi_n < \limsup_n \xi_n\} \subset \Omega$$

on which the sequence ξ_n fails to converge. Then for any $\omega \in B$ there are rational numbers a, b such that

$$\liminf_n \xi_n(\omega) < a < b < \limsup_n \xi_n(\omega),$$

implying that $\lim_{n\to\infty} U_n[a,b](\omega) = \infty$. This means that B and the event A in (4.3) are disjoint, so $P(B) = 0$, since $P(A) = 1$, which proves the claim.

It remains to show that the limit ξ is an integrable random variable. By Fatou's lemma

$$\begin{aligned} E\left(|\xi|\right) &= E\left(\liminf_n |\xi_n|\right) \\ &\le \liminf_n E\left(|\xi_n|\right) \\ &< \sup_n E\left(|\xi_n|\right) < \infty. \end{aligned}$$

This completes the proof. □