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Année Universitaire: 2019/2020
Level: 2^{nd} Master/ Option: A.M.S.
Module: Stochastic Processes 3

FINAL EXAM

1. $(B_t)_{t\geq 0}$ is a Brownian motion on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

   (a) Show that If f in $L^2([0,T])$ then

   $$\mathbb{E}\left[\exp\left(\int_0^t f(s)dB_s\right)\right] = \exp\left(\frac{1}{2}\int_0^t f^2(s)ds\right);\ for\ every\ t \in [0,T].$$

   (b) Let τ to be a stopping time on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $X \in L^2_{ad}([0,T]\times\Omega)$.Show that

   $$\mathbb{E}\left(\int_0^{t\wedge\tau} X_s dB_s\right) = 0;\ for\ every\ t \in [0,T].$$

   (c) Prove that, for every $t > 0$:

   $$\int_0^t e^{B_s}ds \overset{\mathcal{L}}{\sim} t\int_0^1 e^{\sqrt{t}B_s}ds.(use\ scaling\ propertiy\ of\ Brownian\ motion)$$

2. $(B_t)_{t\geq 0}$ is a Brownian motion on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

   (a) Compute $Z = \int_0^1 1_{\{B_t=0\}}dB_t$.

   (b) Let $Z = \int_0^1 1_{\{B_t\geq 0\}}dB_t$. Compute $\mathbb{E}(Z)$ *and* $\mathbb{V}ar(Z)$.

3. Let X be the solution (*geometric Brownian motion*) of the SDE:

   $$\begin{cases} dX_t = bX_t dt + \sigma X_t dB_t \\ X_0 = 1. \end{cases}$$

   (a) In this part we will determine, by two ways, a real number α such that $(X_t^\alpha)_{t\geq 0}$ is a martingale:

      i. The first one by using Ito's formula to calculate stochastic differential of X_t^α.

      ii. The second one by resolving the above SDE.

   (b) Let α takes the value found in (a) and Let τ the exit time of X out of the interval $]\frac{1}{2},\ 2[$, that is $\tau = \inf\left\{t \geq 0;\ X_t = \frac{1}{2}\ or\ X_t = 2\right\}$.

      i. Express $\mathbb{E}(X_\tau^\alpha)$ in terms of α and $\mathbb{P}(X_\tau = 2)$.

      ii. Show that $\mathbb{E}(X_{\tau\wedge t}^\alpha) = 1$.

      iii. Prove that $t \mapsto X_{\tau\wedge t}^\alpha$ is bounded and deduce that $\lim_{t\to\infty}\mathbb{E}(X_{\tau\wedge t}^\alpha) = \mathbb{E}(X_\tau^\alpha)$.(*use Monotone Convergence Theorem*).

      iv. Deduce that $\mathbb{P}(X_\tau = 2) = \dfrac{1-2^{-\alpha}}{2^\alpha - 2^{-\alpha}}$.

4. Let us consider the SDE which is assumed adimitting a unique solution.

   $$\begin{cases} dX_t = \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right)dt + \sqrt{1+X_t^2}dB_t. \\ X_0 = x \in \mathbb{R}. \end{cases} \tag{1}$$

(a) Give the stochastic differential of $Y_t = \ln\left(\sqrt{1+X_t^2} + X_t\right)$.

(b) Deduce an explicit solution of (1).

Hint: $z \mapsto \ln\left(\sqrt{1+z^2}+z\right)$ *is the inverse function of* $y \mapsto sh(y)$.

5. Prove by two ways (by definition and by Ito's formula) that $X_t = B_t^3 - 3tB_t$ is a martingale.

Hint:
- $\mathbb{E}(B_t^4) = 3t$
- $\mathbb{E}(\,|XY|\,) \leq \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}$