STAT 758, Spring 2012
Key solution for Home Work 2
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ACF, stationarity

2.1 Let $\{X_t\}$ be a sequence of uncorrelated random variables, each with mean 0 and variance σ^2. For each of the following processes, find its representation in terms of lagged X-values (*i.e.* $X_{t\pm k}$, $k = 0, 1, 2, ...$), in terms of powers of the backshift operator B, find weights of a corresponding linear filter $W_t = \sum w_j X_{t-j}$, find the mean and variance of W_t

a) $W_t = \nabla X_t$:
$W_t = (1 - B)X_t = X_t - X_{t-1}$
$W_t = \sum_{j=0}^{1} w_j X_{t-j}$, where $\{w_k\} = \{1, -1\}$.

b) $W_t = \nabla^2 X_t$:
$W_t = (1 - B)^2 X_t = (1 - 2B + B^2)X_t$
$W_t = X_t - 2X_{t-1} + X_{t-2}$
$W_t = \sum_{j=0}^{2} w_j X_{t-j}$, where $\{w_k\} = \{1, -2, 1\}$.

c) $W_t = \nabla_3 X_t$:
$W_t = (1 - B^3)X_t = (1 - B^3)X_t = X_t - X_{t-3}$
$W_t = \sum_{j=0}^{3} w_j X_{t-j}$, where $\{w_k\} = \{1, 0, 0, -1\}$.

d) $W_t = \nabla^2 \nabla_3 X_t$:
$W_t = (1 - B)^2(1 - B^3)X_t$
$W_t = (1 - 2B + B^2 - B^3 + 2B^4 - B^5)X_t$
$W_t = X_t - 2X_{t-1} + X_{t-2} - X_{t-3} + 2X_{t-4} - X_{t-5}$
$W_t = \sum_{j=0}^{5} w_j X_{t-j}$,
where $\{w_k\} = \{1, -2, 1, -1, 2, -1\}$.

2.2 Find all non-trivial (not equal to identity) linear filters of the form $\alpha + \beta B + \gamma B^2$ (*i.e.* find α, β, and γ) that do not change the variance and mean of uncorrelated sequences.

Solution: Let X_t be an uncorrelated sequence of random variables. Applying a linear filter of the form $\alpha + \beta B + \gamma B^2$ yields a new time series Y_t:

$$Y_t = (\alpha + \beta B + \gamma B^2)X_t = \alpha X_t + \beta X_{t-1} + \gamma X_{t-2}$$

We begin by examining the requirement that the mean of Y_t is unchanged

$$\begin{aligned} E(X_t) &= E(Y_t) \\ &= E(\alpha X_t + \beta X_{t-1} + \gamma X_{t-2}) \\ &= E(\alpha X_t) + E(\beta X_{t-1}) + E(\gamma X_{t-2}) \\ &= \alpha E(X_t) + \beta E(X_{t-1}) + \gamma E(X_{t-2}) \\ &= (\alpha + \beta + \gamma) E(X_t) \end{aligned}$$

In which case, the coefficients of the linear filter must satisfy the following relationship $\boxed{\alpha + \beta + \gamma = 1}$. Next we consider the variance, which also remains unchanged, such that

$$\begin{aligned} \mathrm{Var}(X_t) &= \mathrm{Var}(Y_t) \\ &= \mathrm{Var}(\alpha X_t + \beta X_{t-1} + \gamma X_{t-2}) \\ &= \mathrm{Var}(\alpha X_t) + \mathrm{Var}(\beta X_{t-1}) + \mathrm{Var}(\gamma X_{t-2}) \\ &= \alpha^2 \, \mathrm{Var}(X_t) + \beta^2 \, \mathrm{Var}(X_{t-1}) + \gamma^2 \mathrm{Var}(X_{t-2}) \\ &= (\alpha^2 + \beta^2 + \gamma^2) \, \mathrm{Var}(X_t) \end{aligned}$$

Thus the coefficients of the linear filter must also satisfy the condition $\boxed{\alpha^2 + \beta^2 + \gamma^2 = 1}$.

Finally, the solution is the set of all triplets (α, β, γ), excluding $(\alpha = 1, \beta = \gamma = 0)$, obtained as the intersection of a three-dimensional unit sphere and a plane $\alpha + \beta + \gamma = 1$. For example, the following triplet is a valid solutions: $(\alpha = \gamma = 0, \beta = 1)$.

2.3 Let $\{Z_t\}$ be a sequence of independent normal random variables, each with mean 0 and variance σ^2, and let a, b, c be constants. For each process below decide whether it is stationary or not. For each stationary process find the mean, autocovariance function, and autocorrelation function:

Solution:
1) $X_t = a + bZ_t + cZ_{t+1}$

(i) $E(X_t) = \mu$:

$$\begin{aligned} E(X_t) &= E(a + bZ_t + cZ_{t+1}) \\ &= a + bE(Z_t) + cE(Z_{t+1}) \\ &= a \qquad \checkmark \end{aligned}$$

(ii) $\gamma_X(h) = f(h)$:

$$\begin{aligned}
\gamma_X(h) &= E\{[X_t - \mu][X_{t+h} - \mu]\} \\
&= E\{[a + bZ_t + cZ_{t+1} - a][a + bZ_{t+h} + cZ_{t+1+h} - a]\} \\
&= E\{b^2 Z_t Z_{t+h} + bc[Z_t Z_{t+h+1} + Z_{t+1} Z_{t+h}] + c^2 Z_{t+1} Z_{t+h+1}\} \\
&= b^2 E(Z_t Z_{t+h}) + bcE(Z_{t+1} Z_{t+h}) + c^2 E(Z_{t+1} Z_{t+h+1}) \\
&= \begin{cases} (b^2 + c^2)\sigma^2, & h = 0 \\ bc\sigma^2, & h = 1 \qquad \checkmark \\ 0, & h \geq 2 \end{cases}
\end{aligned}$$

This time series meets the criteria for a weakly stationary process, with mean and autocovariance as shown above and the following ACF:

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} 1, & h = 0 \\ \frac{bc}{b^2+c^2}, & h = 1 \\ 0, & h \geq 2 \end{cases}$$

2) $X_t = a + bZ_0$

(i) $E(X_t) = \mu$:

$$\begin{aligned}
E(X_t) &= E(a + bZ_0) \\
&= a + bE(Z_0) \\
&= a \qquad \checkmark
\end{aligned}$$

(ii) $\gamma_X(h) = f(h)$:

$$\begin{aligned}
\gamma_X(h) &= E\{[X_t - \mu][X_{t+h} - \mu]\} \\
&= E\{[a + bZ_0 - a][a + bZ_0 - a]\} \\
&= E(b^2 Z_0^2) \\
&= b^2\sigma^2 \qquad \checkmark
\end{aligned}$$

This time series meets the criteria for a weakly stationary process, with mean a, covariance $b^2\sigma^2$ and an ACF of 1 at lag $h = 0$.

3) $X_t = Z_t Z_{t-1}$

(i) $E(X_t) = \mu$:

$$\begin{aligned}
E(X_t) &= E(Z_t Z_{t-1}) \\
&= 0 \qquad \checkmark
\end{aligned}$$

(ii) $\gamma_X(h) = f(h)$:

$$\begin{aligned}
\gamma_X(h) &= E\{[X_t - \mu][X_{t+h} - \mu]\} \\
&= E\{[Z_t Z_{t-1} - 0][Z_{t+h} Z_{t+h-1} - 0]\} \\
&= E(Z_t Z_{t-1} Z_{t+h} Z_{t+h-1}) \\
&= \begin{cases} \sigma^4, & h = 0 \\ 0, & h \neq 0 \end{cases} \qquad \checkmark
\end{aligned}$$

This time series meets the criteria for a weakly stationary process, with mean and autocovariance as shown above and the following ACF:

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} 1, & h = 0 \\ 0, & h \neq 0 \end{cases}$$

2.4 Let $\{Z_t\}$ be a stationary process with mean zero and let a and b be constants. If $X_t = a + bt + S_t + Z_t$, where S_t is a seasonal component with period d, show that process $\nabla\nabla_d X_t$ is stationary and express its autocovariance function in terms of that of $\{Z_t\}$.

Solution: We can express the process $Y_t = \nabla\nabla_d X_t$ in terms of the backshift operator B:

$$\begin{aligned}
Y_t = \nabla\nabla_d X_t &= (1 - B)(1 - B^d)X_t \\
&= (1 - B - B^d + B^{d+1})X_t
\end{aligned}$$

We can then rewrite this equation in terms of process Z_t

$$\begin{aligned}
Y_t &= a + bt + S_t + Z_t \\
&\quad - a - b(t-1) - S_{t-1} - Z_{t-1} \\
&\quad - a - b(t-d) - S_{t-d} - Z_{t-d} \\
&\quad + a + b(t-d-1) + S_{t-d-1} + Z_{t-d-1} \\
&= Z_t - Z_{t-1} - Z_{t-d} + Z_{t-d-1}.
\end{aligned}$$

We are now ready to consider the two necessary conditions for a weakly stationary process.

(i) $E(Y_t) = \mu$:

$$\begin{aligned} E(Y_t) &= E(Z_t - Z_{t-1} - Z_{t-d} + Z_{t-d-1}) \\ &= E(Z_t) - E(Z_{t-1}) - E(Z_{t-d}) + E(Z_{t-d-1}) \\ &= 4E(Z_t) \\ &= 0 \quad \checkmark \end{aligned}$$

(ii) $\gamma_{_Y}(h) = f(h)$:

$$\begin{aligned} \gamma_{_Y}(h) &= E\{[Y_t - \mu][Y_{t+h} - \mu]\} \\ &= E\{[Y_t - 0][Y_{t+h} - 0]\} \\ &= E(Y_t Y_{t+h}) \\ &= E\{[Z_t - Z_{t-1} - Z_{t-d} + Z_{t-d-1}][Z_{t+h} - Z_{t+h-1} - Z_{t+h-d} + Z_{t+h-d-1}]\} \\ &= E\{Z_t Z_{t+h} - Z_t Z_{t+h-1} - Z_t Z_{t+h-d} + Z_t Z_{t+h-d-1} \\ &- Z_{t-1} Z_{t+h} + Z_{t-1} Z_{t+h-1} + Z_{t-1} Z_{t+h-d} - Z_{t-1} Z_{t+h-d-1} \\ &- Z_{t-d} Z_{t+h} + Z_{t-d} Z_{t+h-1} + Z_{t-d} Z_{t+h-d} - Z_{t-d} Z_{t+h-d-1} \\ &+ Z_{t-d-1} Z_{t+h} - Z_{t-d-1} Z_{t+h-1} - Z_{t-d-1} Z_{t+h-d} + Z_{t-d-1} Z_{t+h-d-1}\}. \end{aligned}$$

Assuming $d \geq 1$, this simplifies to

$$\boxed{\gamma_{_Y}(h) = \begin{cases} \gamma_{_Y}(-h), & h <= -1 \\ 4\gamma_{_Z}(0), & h = 0, d \neq 1 \\ 5\gamma_{_Z}(0), & h = 0, d = 1 \\ -4\gamma_{_Z}(0), & h = d = 1 \\ -2\gamma_{_Z}(0), & h = 1 \text{ or } h = d, d \neq 1 \\ \gamma_{_Z}(0), & h = d \pm 1 \\ 0, & \text{o.w.} \end{cases}} \qquad \checkmark$$