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SAMPLE SPACES

DEFINITION :

The *sample space* is the set of all possible outcomes of an experiment.

EXAMPLE : When we *flip a coin* then sample space is

$$\mathcal{S} = \{ H , T \} ,$$

where

$$H \text{ denotes that the coin lands "Heads up"}$$

and

$$T \text{ denotes that the coin lands "Tails up".}$$

For a "*fair coin*" we expect H and T to have the same "*chance*" of occurring, *i.e.*, if we flip the coin many times then about 50 % of the outcomes will be H.

We say that the *probability* of H to occur is 0.5 (or 50 %) .

The probability of T to occur is then also 0.5.

EXAMPLE :

When we *roll a fair die* then the sample space is

$$\mathcal{S} = \{ 1 , 2 , 3 , 4 , 5 , 6 \} .$$

The probability the die lands with k up is $\frac{1}{6}$, $(k = 1, 2, \cdots, 6)$.

When we roll it 1200 times we expect a 5 up about 200 times.

The probability the die lands with an *even number* up is

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} .$$

EXAMPLE :

When we toss a coin 3 times and record the results in the *sequence* that they occur, then the sample space is

$$\mathcal{S} = \{ HHH , HHT , HTH , HTT , THH , THT , TTH , TTT \} .$$

Elements of $\mathcal{S}$ are "*vectors*", "*sequences*", or "*ordered outcomes*".

We may expect each of the 8 outcomes to be equally likely.

Thus the probability of the sequence HTT is $\frac{1}{8}$.

The probability of a sequence to contain precisely two Heads is

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8} .$$

EXAMPLE : When we toss a coin 3 times and record the results without paying attention to the order in which they occur, *e.g.*, if we only record the number of Heads, then the sample space is

$$\mathcal{S} \;=\; \Big\{ \{H,H,H\} \;,\; \{H,H,T\} \;,\; \{H,T,T\} \;,\; \{T,T,T\} \Big\} \;.$$

The outcomes in $\mathcal{S}$ are now *sets* ; *i.e.*, order is not important.

Recall that the ordered outcomes are

$$\{ \; HHH \;,\; HHT \;,\; HTH \;,\; HTT \;,\; THH \;,\; THT \;,\; TTH \;,\; TTT \; \} \;.$$

Note that

$\{H,H,H\}$	corresponds to	*one*	of the ordered outcomes,
$\{H,H,T\}$	,,	*three*	,,
$\{H,T,T\}$	,,	*three*	,,
$\{T,T,T\}$	,,	*one*	,,

Thus $\{H,H,H\}$ and $\{T,T,T\}$ each occur with probability $\frac{1}{8}$,

while $\{H,H,T\}$ and $\{H,T,T\}$ each occur with probability $\frac{3}{8}$.

Events

In Probability Theory subsets of the sample space are called *events*.

EXAMPLE : The set of basic outcomes of rolling a die *once* is

$$\mathcal{S} = \{ 1 , 2 , 3 , 4 , 5 , 6 \} ,$$

so the subset $E = \{ 2 , 4 , 6 \}$ is an example of an event.

If a die is rolled *once* and it lands with a 2 *or* a 4 *or* a 6 up then we say that the event E has *occurred*.

We have already seen that the probability that E occurs is

$$P(E) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} .$$

The Algebra of Events

Since events are *sets*, namely, subsets of the sample space $\mathcal{S}$, we can do the usual *set operations* :

If E and F are events then we can form

E^c	the *complement* of E
$E \cup F$	the *union* of E and F
EF	the *intersection* of E and F

We write $E \subset F$ if E is a *subset* of F.

REMARK : In Probability Theory we use

E^c instead of $\bar{E}$,

EF instead of $E \cap F$,

$E \subset F$ instead of $E \subseteq F$.

If the sample space $\mathcal{S}$ is *finite* then we typically allow any subset of $\mathcal{S}$ to be an event.

EXAMPLE : If we randomly draw *one character* from a box containing the characters a, b, and c, then the sample space is

$$\mathcal{S} \ = \ \{a \ , \ b \ , \ c\} \ ,$$

and there are 8 possible events, namely, those in the set of events

$$\mathcal{E} \ = \ \Big\{ \ \{ \ \} \ , \ \{a\} \ , \ \{b\} \ , \ \{c\} \ , \ \{a,b\} \ , \ \{a,c\} \ , \ \{b,c\} \ , \ \{a,b,c\} \ \Big\} \ .$$

If the outcomes a, b, and c, are equally likely to occur, then

$$P(\{ \ \}) = 0 \quad , \quad P(\{a\}) = \frac{1}{3} \quad , \quad P(\{b\}) = \frac{1}{3} \quad , \quad P(\{c\}) = \frac{1}{3} \ ,$$

$$P(\{a,b\}) = \frac{2}{3} \ , \ P(\{a,c\}) = \frac{2}{3} \ , \ P(\{b,c\}) = \frac{2}{3} \ , \ P(\{a,b,c\}) = 1 \ .$$

For example, $P(\{a,b\})$ is the probability the character is an a *or* a b.

We always assume that the set $\mathcal{E}$ of allowable events *includes the complements, unions, and intersections* of its events.

EXAMPLE : If the sample space is

$$\mathcal{S} = \{a \ , \ b \ , \ c \ , \ d\} \ ,$$

and we start with the events

$$\mathcal{E}_0 = \Big\{ \{a\} \ , \ \{c, d\} \Big\} \ ,$$

then this set of events needs to be extended to (at least)

$$\mathcal{E} = \Big\{ \{\ \} \ , \ \{a\} \ , \ \{c, d\} \ , \ \{b, c, d\} \ , \ \{a, b\} \ , \ \{a, c, d\} \ , \ \{b\} \ , \ \{a, b, c, d\} \Big\} \ .$$

EXERCISE : Verify $\mathcal{E}$ includes complements, unions, intersections.

Axioms of Probability

A *probability function* P assigns a real number (the *probability* of E) to every event E in a sample space $\mathcal{S}$.

$P(\cdot)$ must satisfy the following basic properties :

- $0 \;\leq\; P(E) \;\leq\; 1$,

- $P(\mathcal{S}) \;=\; 1$,

- For any *disjoint events* E_i , $i = 1, 2, \cdots, n$, we have

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) \;=\; P(E_1) + P(E_2) + \cdots P(E_n) \;.$$

Further Properties

PROPERTY 1 :

$$P(E \cup E^c) \;=\; P(E) \;+\; P(E^c) \;=\; 1 \;. \qquad (\text{ Why ? })$$

Thus

$$P(E^c) \;=\; 1 \;-\; P(E) \;.$$

EXAMPLE :

What is the probability of at least one "H" in *four tosses* of a coin?

SOLUTION : The sample space $\mathcal{S}$ will have 16 outcomes. (Which?)

$$P(\text{at least one H}) \;=\; 1 \;-\; P(\text{no H}) \;=\; 1 \;-\; \frac{1}{16} \;=\; \frac{15}{16} \;.$$

PROPERTY 2 :

$$P(E \cup F) \ = \ P(E) \ + \ P(F) \ - \ P(EF) \ .$$

PROOF (using the third axiom) :

$$\begin{aligned} P(E \cup F) \ &= \ P(EF) \ + \ P(EF^c) \ + \ P(E^cF) \\ &= \ [P(EF) \ + \ P(EF^c)] \ + \ [P(EF) \ + \ P(E^cF)] \ - \ P(EF) \\ &= \mathrm{P(E)} \ + \ \mathrm{P(F)} \ - \ \mathrm{P(EF)} \ . \qquad (\text{ Why ? }) \end{aligned}$$

NOTE :

- Draw a Venn diagram with E and F to see this !

- The formula is similar to the one for the number of elements :

$$n(E \cup F) \ = \ n(E) \ + \ n(F) \ - \ n(EF) \ .$$

So far our sample spaces $\mathcal{S}$ have been *finite*.

$\mathcal{S}$ can also be *countably infinite*, *e.g.*, the set $\mathbb{Z}$ of all integers.

$\mathcal{S}$ can also be *uncountable*, *e.g.*, the set $\mathbb{R}$ of all real numbers.

EXAMPLE : Record the low temperature in Montreal on January 8 in each of a large number of years.

We can take $\mathcal{S}$ to be the set of *all real numbers*, *i.e.*, $\mathcal{S} = \mathbb{R}$.

(Are there are other choices of $\mathcal{S}$?)

What probability would you expect for the following *events* to have?

(a) $P(\{\pi\})$ (b) $P(\{x \ : \ -\pi < x < \pi\})$

(How does this differ from finite sample spaces?)

We will encounter such infinite sample spaces many times $\cdots$

Counting Outcomes

We have seen examples where the outcomes in a *finite* sample space $\mathcal{S}$ are *equally likely* , *i.e.*, they have *the same probability* .

Such sample spaces occur quite often.

Computing probabilities then requires counting *all* outcomes and counting *certain types* of outcomes .

The counting has to be done carefully!

We will discuss a number of representative examples in detail.

Concepts that arise include *permutations* and *combinations*.

Permutations

- Here we count of the number of "*words*" that can be formed from a collection of items (*e.g.*, letters).

- (Also called *sequences* , *vectors* , *ordered sets* .)

- The *order* of the items in the word is important;

  e.g., the word *acb* is different from the word *bac* .

- The *word length* is the number of characters in the word.

NOTE :

For *sets* the order is not important. For example, the set $\{a,c,b\}$ is the same as the set $\{b,a,c\}$.

EXAMPLE : Suppose that four-letter words of *lower case* alphabetic characters are generated randomly with equally likely outcomes. (Assume that *letters may appear repeatedly*.)

(a) How many four-letter words are there in the sample space $\mathcal{S}$?

SOLUTION : $26^4 \; = \; 456,976$.

(b) How many four-letter words are there are there in $\mathcal{S}$ that start with the letter "s" ?

SOLUTION : 26^3 .

(c) What is the *probability* of generating a four-letter word that starts with an "s" ?

SOLUTION :

$$\frac{26^3}{26^4} \; = \; \frac{1}{26} \; \cong \; 0.038 \; .$$

Could this have been computed more easily?

EXAMPLE : How many re-orderings (*permutations*) are there of the string *abc* ? (Here *letters may appear only once.*)

SOLUTION : Six, namely, *abc* , *acb* , *bac* , *bca* , *cab* , *cba* .

If these permutations are generated randomly with equal probability then what is the probability the word starts with the letter "*a*" ?

SOLUTION :

$$\frac{2}{6} = \frac{1}{3} .$$

EXAMPLE : In general, if the word length is n and *all characters are distinct* then there are $n!$ permutations of the word. (Why ?)

If these permutations are generated randomly with equal probability then what is the probability the word starts with a particular letter ?

SOLUTION :

$$\frac{(n-1)!}{n!} = \frac{1}{n} . \qquad (\text{ Why ? })$$

EXAMPLE : How many

words of length k

can be formed from

a set of n (distinct) characters ,

(where $k \leq n$) ,

when letters can be used *at most once* ?

SOLUTION :

$$
\begin{aligned}
& n\,(n-1)\,(n-2)\,\cdots\,(n-(k-1)) \\
= \;& n\,(n-1)\,(n-2)\,\cdots\,(n-k+1) \\
= \;& \frac{n!}{(n-k)!} \qquad (\text{ Why ? })
\end{aligned}
$$

EXAMPLE : *Three-letter words* are generated randomly from the *five* characters a , b , c , d , e , where letters can be *used at most once.*

(a) How many three-letter words are there in the sample space $\mathcal{S}$?

SOLUTION : $5 \cdot 4 \cdot 3 \ = \ 60$.

(b) How many words containing a , b are there in $\mathcal{S}$?

SOLUTION : First place the characters

$$a \ , \ b$$

i.e., *select the two indices* of the locations to place them.

This can be done in

$$3 \ \times \ 2 \ = \ 6 \ \text{ ways} . \qquad (\text{ Why ? })$$

There remains one position to be filled with a c , d or an e .

Therefore the number of words is $3 \ \times \ 6 \ = \ 18$.

(c) Suppose the 60 solutions in the sample space are *equally likely* .

What is the *probability* of generating a three-letter word that contains the letters a and b ?

SOLUTION :

$$\frac{18}{60} \quad = \quad 0.3 \ .$$

EXERCISE :

Suppose the sample space $\mathcal{S}$ consists of all *five-letter* words having *distinct alphabetic characters* .

- How many words are there in $\mathcal{S}$?

- How many "special" words are in $\mathcal{S}$ for which *only* the second and the fourth character are vowels, *i.e.*, one of $\{a, e, i, o, u, y\}$?

- Assuming the outcomes in $\mathcal{S}$ to be equally likely, what is the probability of drawing such a special word?

Combinations

Let S be a set containing n (distinct) elements.

Then

a *combination* of k elements from S ,

is

any selection of k elements from S ,

where *order is not important* .

(Thus the selection is a *set* .)

NOTE : By definition a *set* always has *distinct elements* .

EXAMPLE :

There are three *combinations* of 2 elements chosen from the set

$$S = \{a , b , c\} ,$$

namely, the *subsets*

$$\{a, b\} , \{a, c\} , \{b, c\} ,$$

whereas there are six *words* of 2 elements from S ,

namely,

$$ab , ba , ac , ca , bc , cb .$$

In general, given

$$\text{a set } S \text{ of } n \text{ elements ,}$$

the number of possible subsets of k elements from S equals

$$\binom{n}{k} \equiv \frac{n!}{k!\,(n-k)!} \, .$$

REMARK : The notation $\binom{n}{k}$ is referred to as

$$"n \ \textit{choose} \ k".$$

NOTE : $\binom{n}{n} = \frac{n!}{n!\,(n-n)!} = \frac{n!}{n!\ 0!} = 1 \, ,$

since $0! \equiv 1$ (by "convenient definition" !) .

PROOF :

First recall that there are

$$n\ (n-1)\ (n-2)\ \cdots\ (n-k+1) \quad = \quad \frac{n!}{(n-k)!}$$

possible *sequences* of k distinct elements from S .

However, every sequence of length k has $k!$ permutations of itself, and each of these defines the same subset of S.

Thus the total number of subsets is

$$\frac{n!}{k!\ (n-k)!} \quad \equiv \quad \binom{n}{k} \ .$$

EXAMPLE :

In the previous example, with 2 elements chosen from the set

$$\{a \ , \ b \ , \ c\} \ ,$$

we have $n = 3$ and $k = 2$, so that there are

$$\frac{3!}{(3-2)!} \quad = \quad 6 \quad \textit{words} \ ,$$

namely

$$ab \ , \ ba \quad , \quad ac \ , \ ca \quad , \quad bc \ , \ cb \quad ,$$

while there are

$$\binom{3}{2} \quad \equiv \quad \frac{3!}{2! \ (3-2)!} \quad = \quad \frac{6}{2} \quad = \quad 3 \quad \textit{subsets} \ ,$$

namely

$$\{a, b\} \quad , \quad \{a, c\} \quad , \quad \{b, c\} \ .$$

EXAMPLE : If we choose 3 elements from $\{a\ ,\ b\ ,\ c\ ,\ d\}$, then

$$n \ = \ 4 \ \text{ and } \ k \ = \ 3 \ ,$$

so there are

$$\frac{4!}{(4-3)!} \ = \ 24 \quad \text{words, namely} \ :$$

$$\begin{array}{cccccccc}
abc & , & abd & , & acd & , & bcd & , \\
acb & , & adb & , & adc & , & bdc & , \\
bac & , & bad & , & cad & , & cbd & , \\
bca & , & bda & , & cda & , & cdb & , \\
cab & , & dab & , & dac & , & dbc & , \\
cba & , & dba & , & dca & , & dcb & ,
\end{array}$$

while there are

$$\binom{4}{3} \ \equiv \ \frac{4!}{3!\ (4-3)!} \ = \ \frac{24}{6} \ = \ 4 \quad \text{subsets} \ ,$$

namely,

$$\{a, b, c\}\ ,\ \{a, b, d\}\ ,\ \{a, c, d\}\ ,\ \{b, c, d\}\ .$$

EXAMPLE :

(a) How many ways are there to choose a committee of 4 persons from a group of 10 persons, if order is not important?

SOLUTION :

$$\binom{10}{4} \quad = \quad \frac{10!}{4!\,(10-4)!} \quad = \quad 210 \ .$$

(b) If each of these 210 outcomes is equally likely then what is the probability that a particular person is on the committee?

SOLUTION :

$$\binom{9}{3} / \binom{10}{4} \;=\; \frac{84}{210} \;=\; \frac{4}{10} \ . \qquad (\text{ Why ? })$$

Is this result surprising?

(c) What is the probability that a particular person is *not* on the committee?

SOLUTION :

$$\binom{9}{4} / \binom{10}{4} \;=\; \frac{126}{210} \;=\; \frac{6}{10}\,. \qquad (\text{ Why ? })$$

Is this result surprising?

(d) How many ways are there to choose a committee of 4 persons from a group of 10 persons, if one is to be the chairperson?

SOLUTION :

$$\binom{10}{1}\binom{9}{3} \;=\; 10\binom{9}{3} \;=\; 10\,\frac{9!}{3!\,(9-3)!} \;=\; 840\,.$$

QUESTION : Why is this four times the number in (a) ?

EXAMPLE : *Two balls* are selected at random from a bag with *four white* balls and *three black* balls, where order is not important.

What would be an appropriate sample space $\mathcal{S}$?

SOLUTION : Denote the set of balls by

$$B \quad = \quad \{w_1 \ , \ w_2 \ , \ w_3 \ , \ w_4 \ , \ b_1 \ , \ b_2 \ , \ b_3\} \ ,$$

where same color balls are made "distinct" by numbering them.

Then a good choice of the sample space is

$$\mathcal{S} \quad = \quad \text{the set of } \ \textit{all subsets} \ \text{ of } \ \textit{two balls} \ \text{ from } B \ ,$$

because the wording "*selected at random*" suggests that each such subset has the same chance to be selected.

The number of outcomes in $\mathcal{S}$ (which are sets of two balls) is then

$$\binom{7}{2} \quad = \quad 21 \ .$$

EXAMPLE : (continued $\cdots$)

(*Two balls* are selected at random from a bag with *four white* balls and *three black* balls.)

- What is the probability that both balls are white?

SOLUTION :

$$\binom{4}{2} / \binom{7}{2} = \frac{6}{21} = \frac{2}{7}.$$

- What is the probability that both balls are black?

SOLUTION :

$$\binom{3}{2} / \binom{7}{2} = \frac{3}{21} = \frac{1}{7}.$$

- What is the probability that one is white and one is black?

SOLUTION :

$$\binom{4}{1}\binom{3}{1} / \binom{7}{2} = \frac{4 \cdot 3}{21} = \frac{4}{7}.$$

(Could this have been computed differently?)

EXAMPLE : (continued $\cdots$)

In detail, the sample space $\mathcal{S}$ is

$$\left\{ \begin{array}{cccccccc} \{w_1,w_2\}, & \{w_1,w_3\}, & \{w_1,w_4\}, & | & \{w_1,b_1\}, & \{w_1,b_2\}, & \{w_1,b_3\}, & \\ & \{w_2,w_3\}, & \{w_2,w_4\}, & | & \{w_2,b_1\}, & \{w_2,b_2\}, & \{w_2,b_3\}, & \\ & & \{w_3,w_4\}, & | & \{w_3,b_1\}, & \{w_3,b_2\}, & \{w_3,b_3\}, & \\ & & & | & \{w_4,b_1\}, & \{w_4,b_2\}, & \{w_4,b_3\}, & \\ \hline & & & & & \{b_1,b_2\}, & \{b_1,b_3\}, & \\ & & & & & & \{b_2,b_3\} & \end{array} \right\}$$

- $\mathcal{S}$ has 21 outcomes, *each of which is a set* .
- We assumed each outcome of $\mathcal{S}$ has probability $\frac{1}{21}$.
- The *event* "both balls are white" contains 6 outcomes.
- The *event* "both balls are black" contains 3 outcomes.
- The *event* "one is white and one is black" contains 12 outcomes.
- What would be different had we worked with *sequences* ?

EXERCISE :

Three balls are selected at random from a bag containing

2 *red* , 3 *green* , 4 *blue* balls .

What would be an appropriate sample space $\mathcal{S}$?

What is the the number of outcomes in $\mathcal{S}$?

What is the probability that all three balls are *red* ?

What is the probability that all three balls are *green* ?

What is the probability that all three balls are *blue* ?

What is the probability of *one red*, *one green*, and *one blue* ball ?

EXAMPLE : A bag contains 4 *black* balls and 4 *white* balls.

Suppose one draws *two balls at the time*, until the bag is empty.

What is the probability that each drawn pair is *of the same color*?

SOLUTION : An *example of an outcome* in the sample space $\mathcal{S}$ is

$$\Big\{ \ \{w_1, w_3\} \ , \ \{w_2, b_3\} \ , \ \{w_4, b_1\} \ , \ \{b_2, b_4\} \ \Big\} \ .$$

The number of such *doubly unordered* outcomes in $\mathcal{S}$ is

$$\frac{1}{4!}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{1}{4!}\frac{8!}{2!\ 6!}\frac{6!}{2!\ 4!}\frac{4!}{2!\ 2!}\frac{2!}{2!\ 0!} = \frac{1}{4!}\frac{8!}{(2!)^4} = 105 \ \text{(Why?)}$$

The number of such outcomes with *pairwise the same color* is

$$\frac{1}{2!}\binom{4}{2}\binom{2}{2} \ \cdot \ \frac{1}{2!}\binom{4}{2}\binom{2}{2} \quad = \quad 3 \ \cdot \ 3 \quad = \quad 9 \ . \qquad (\ \text{Why ?}\ )$$

Thus the probability each pair is *of the same color* is $9/105 \ = \ 3/35$.

EXAMPLE : (continued $\cdots$)

The 9 outcomes of *pairwise the same color* constitute the *event*

$$
\Big\{ \; \big\{ \{w_1,w_2\} , \{w_3,w_4\} , \{b_1,b_2\} , \{b_3,b_4\} \big\} ,
$$
$$
\big\{ \{w_1,w_3\} , \{w_2,w_4\} , \{b_1,b_2\} , \{b_3,b_4\} \big\} ,
$$
$$
\big\{ \{w_1,w_4\} , \{w_2,w_3\} , \{b_1,b_2\} , \{b_3,b_4\} \big\} ,
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\big\{ \{w_1,w_2\} , \{w_3,w_4\} , \{b_1,b_3\} , \{b_2,b_4\} \big\} ,
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$$
$$
\big\{ \{w_1,w_4\} , \{w_2,w_3\} , \{b_1,b_4\} , \{b_2,b_3\} \big\} \; \Big\} .
$$

EXERCISE :

- How many ways are there to choose a committee of 4 persons from a group of 6 persons, if order is not important?
- Write down the list of all these possible committees of 4 persons.
- If each of these outcomes is equally likely then what is the probability that two particular persons are on the committee?

EXERCISE :

Two balls are selected at random from a bag with three white balls and two black balls.

- Show all elements of a suitable sample space.
- What is the probability that both balls are white?

EXERCISE :

We are interested in *birthdays* in a class of 60 students.

- What is a good sample space $\mathcal{S}$ for this purpose?

- How many outcomes are there in $\mathcal{S}$?

- What is the probability of *no common birthdays* in this class?

- What is the probability of *common birthdays* in this class?

EXAMPLE :

How many *nonnegative* integer solutions are there to

$$x_1 \ + \ x_2 \ + \ x_3 \quad = \quad 17 \ ?$$

SOLUTION :

Consider seventeen 1's separated by bars to indicate the possible values of x_1, x_2, and x_3, *e.g.*,

$$111|111111111|11111 \ .$$

The total number of positions in the "display" is $17 + 2 = 19$.

The total number of *nonnegative* solutions is now seen to be

$$\binom{19}{2} \quad = \quad \frac{19!}{(19-2)!\ 2!} \quad = \quad \frac{19 \times 18}{2} \quad = \quad 171 \ .$$

EXAMPLE :

How many *nonnegative* integer solutions are there to the *inequality*

$$x_1 \ + \ x_2 \ + \ x_3 \ \leq \ 17 \ ?$$

SOLUTION :

Introduce an *auxiliary variable* (or "*slack variable*")

$$x_4 \ \equiv \ 17 \ - \ (x_1 \ + \ x_2 \ + \ x_3) \ .$$

Then

$$x_1 \ + \ x_2 \ + \ x_3 \ + \ x_4 \ = \ 17 \ .$$

Use seventeen 1's separated by 3 bars to indicate the possible values of x_1 , x_2 , x_3 , and x_4 , *e.g.*,

$$111|11111111|1111|11 \ .$$

$$111|11111111|1111|11 \ .$$

The total number of positions is

$$17 \ + \ 3 \ = \ 20 \ .$$

The total number of *nonnegative* solutions is therefore

$$\binom{20}{3} \quad = \quad \frac{20!}{(20-3)!\ 3!} \quad = \quad \frac{20 \times 19 \times 18}{3 \times 2} \quad = \quad 1140 \ .$$

EXAMPLE :

How many *positive* integer solutions are there to the equation

$$x_1 \;+\; x_2 \;+\; x_3 \;=\; 17 \;\;?$$

SOLUTION : Let

$$x_1 \;=\; \tilde{x}_1 \;+\; 1 \quad , \quad x_2 \;=\; \tilde{x}_2 \;+\; 1 \quad , \quad x_3 \;=\; \tilde{x}_3 \;+\; 1 \;.$$

Then the problem becomes :

How many *nonnegative* integer solutions are there to the equation

$$\tilde{x}_1 \;+\; \tilde{x}_2 \;+\; \tilde{x}_3 \;=\; 14 \quad ?$$

$$111|111111111|11$$

The solution is

$$\binom{16}{2} \;=\; \frac{16!}{(16-2)!\;2!} \;=\; \frac{16 \times 15}{2} \;=\; 120 \;.$$

EXAMPLE :

What is the probability the *sum* is 9 in *three rolls of a die* ?

SOLUTION : The number of such *sequences* of three rolls with sum 9 is the number of integer solutions of

$$x_1 + x_2 + x_3 = 9 ,$$

with

$$1 \leq x_1 \leq 6 \quad , \quad 1 \leq x_2 \leq 6 \quad , \quad 1 \leq x_3 \leq 6 .$$

Let

$$x_1 = \tilde{x}_1 + 1 \quad , \quad x_2 = \tilde{x}_2 + 1 \quad , \quad x_3 = \tilde{x}_3 + 1 .$$

Then the problem becomes :

How many *nonnegative* integer solutions are there to the equation

$$\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = 6 ,$$

with

$$0 \leq \tilde{x}_1 , \tilde{x}_2 , \tilde{x}_3 \leq 5 .$$

EXAMPLE : (continued $\cdots$)

Now the equation

$$\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = 6 \ , \qquad (0 \le \tilde{x}_1 , \tilde{x}_2 , \tilde{x}_3 \le 5) \ ,$$

$$1|111|11$$

has

$$\binom{8}{2} = 28 \quad \text{solutions} \ ,$$

from which we must *subtract* the 3 *impossible* solutions

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (6,0,0) \ , \quad (0,6,0) \ , \quad (0,0,6) \ .$$

$$111111|| \ , \quad |111111| \ , \quad ||111111$$

Thus the probability that the *sum* of 3 rolls equals 9 is

$$\frac{28 - 3}{6^3} = \frac{25}{216} \cong 0.116 \ .$$

EXAMPLE : (continued $\cdots$)

The 25 outcomes of the event "*the sum of the rolls is* 9" are

$$
\begin{array}{llllll}
\{ \ 126 \ , & 135 \ , & 144 \ , & 153 \ , & 162 \ , & \\
216 \ , & 225 \ , & 234 \ , & 243 \ , & 252 \ , & 261 \ , \\
315 & 324 \ , & 333 \ , & 342 \ , & 351 \ , & \\
414 \ , & 423 \ , & 432 \ , & 441 \ , & & \\
513 \ , & 522 \ , & 531 \ , & & & \\
612 \ , & 621 & \} \ . & & &
\end{array}
$$

The "lexicographic" ordering of the *outcomes* (which are *sequences*) in this *event* is used for systematic counting.

EXERCISE :

- How many integer solutions are there to the inequality

$$x_1 \;+\; x_2 \;+\; x_3 \quad \leq \quad 17 \quad ,$$

if we require that

$$x_1 \;\geq\; 1 \quad , \quad x_2 \;\geq\; 2 \quad , \quad x_3 \;\geq\; 3 \;\; ?$$

EXERCISE :

What is the probability that the *sum* is *less than or equal to* 9 in *three rolls of a die* ?

CONDITIONAL PROBABILITY

Giving more information can change the probability of an event.

EXAMPLE :

If a coin is tossed two times then what is the probability of two Heads?

ANSWER :

$$\frac{1}{4} .$$

EXAMPLE :

If a coin is tossed two times then what is the probability of two Heads, *given that the first toss gave Heads* ?

ANSWER :

$$\frac{1}{2} .$$

NOTE :

Several examples will be about *playing cards* .

A standard *deck* of *playing cards* consists of 52 cards :

- Four *suits* :

  Hearts , Diamonds (*red*) , and Spades , Clubs (*black*) .

- Each suit has 13 cards, whose *denomination* is

  2 , 3 , $\cdots$, 10 , Jack , Queen , King , Ace .

- The Jack , Queen , and King are called *face cards* .

EXERCISE :

Suppose we draw a card from a shuffled set of 52 playing cards.

- What is the probability of drawing a Queen ?

- What is the probability of drawing a Queen, given that the card drawn is of *suit* Hearts ?

- What is the probability of drawing a Queen, given that the card drawn is a *Face card* ?

What do the answers tell us?

(We'll soon learn the events "Queen" and "Hearts" are *independent* .)

The two preceding questions are examples of *conditional probability* .

Conditional probability is an *important* and *useful* concept.

If E and F are events, *i.e.*, subsets of a sample space $\mathcal{S}$, then

$P(E|F)$ *is the conditional probability of* E *, given* F ,

defined as

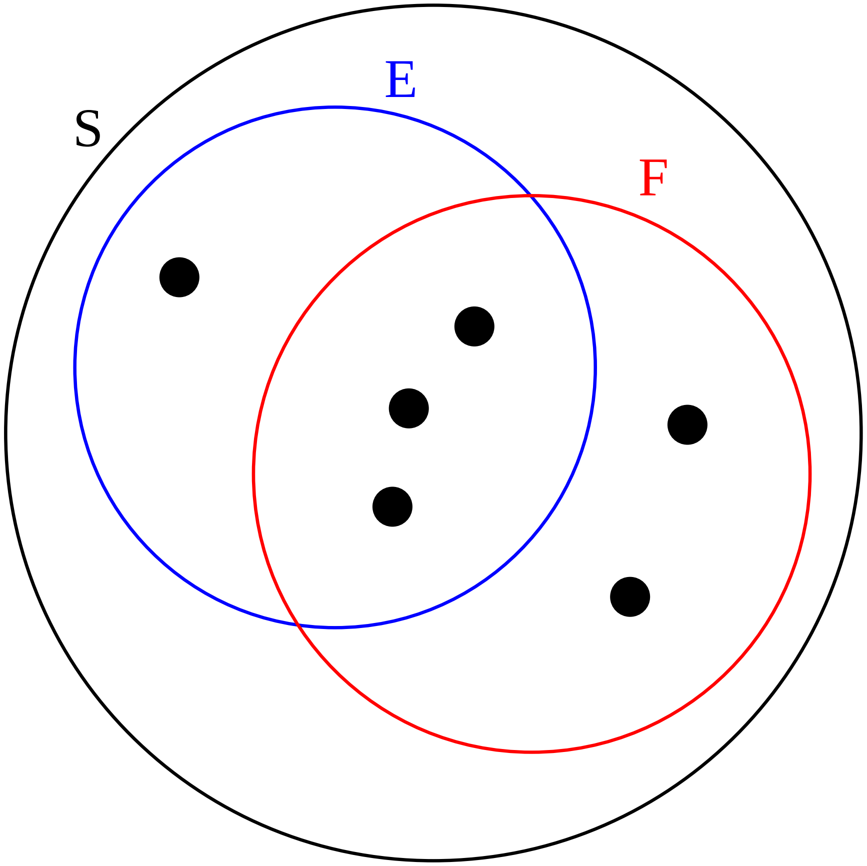
$$P(E|F) \equiv \frac{P(EF)}{P(F)} .$$

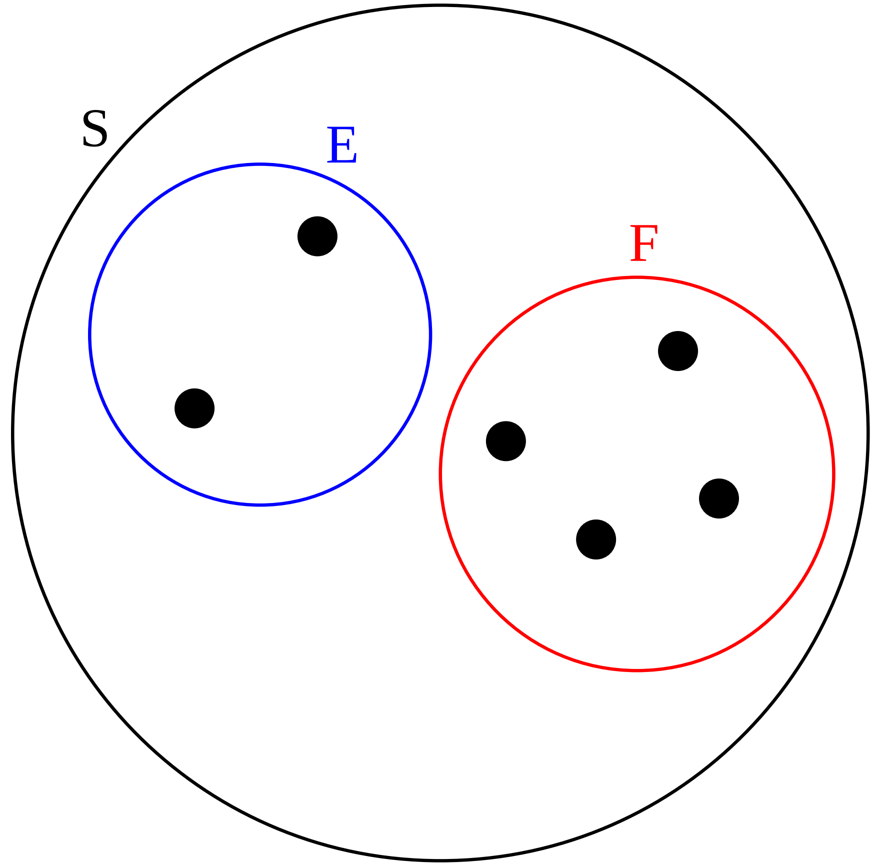
or, equivalently

$$P(EF) = P(E|F)\, P(F) ,$$

(assuming that $P(F)$ is not zero).

$$P(E|F) \equiv \frac{P(EF)}{P(F)}$$

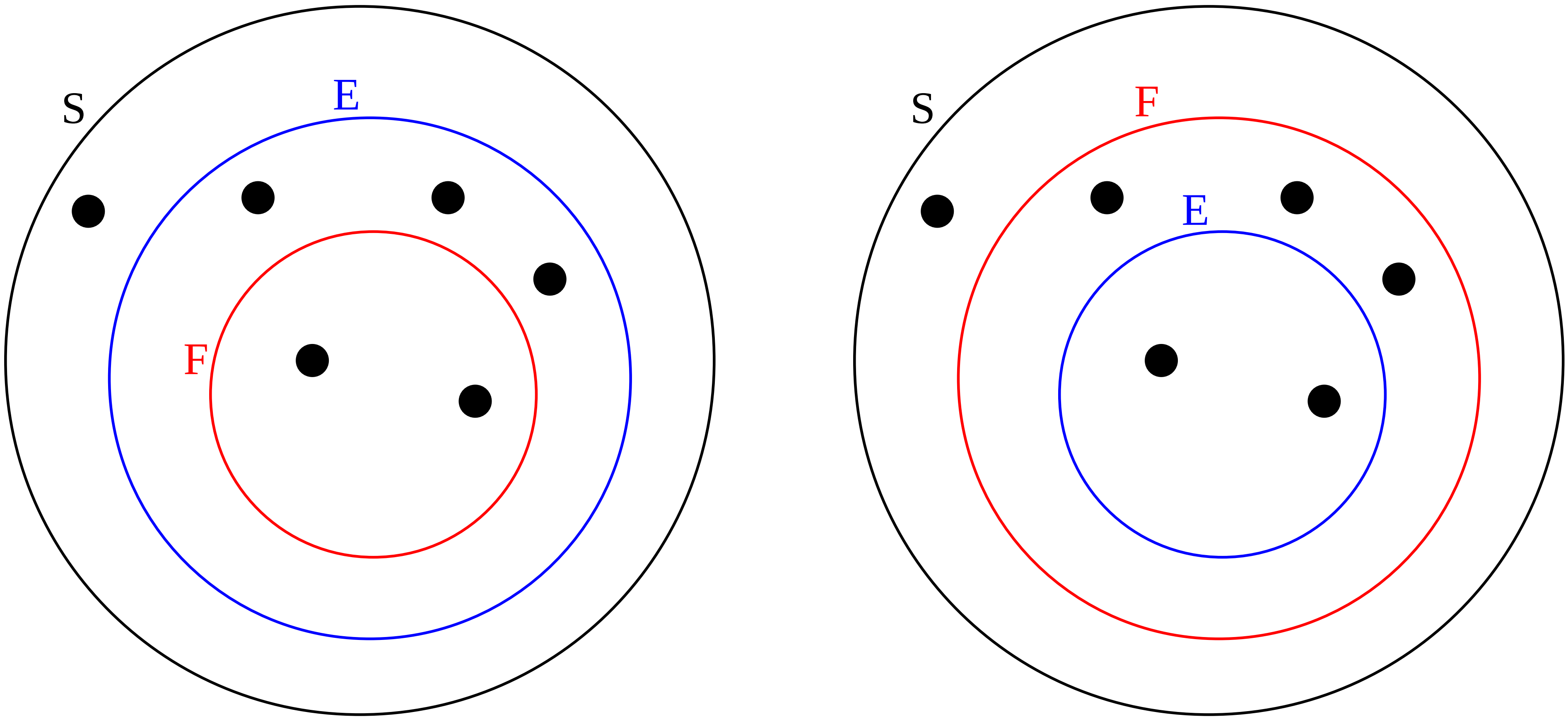




Suppose that the 6 outcomes in $\mathcal{S}$ are equally likely.

What is $P(E|F)$ in each of these two cases ?

$$P(E|F) \equiv \frac{P(EF)}{P(F)}$$



Suppose that the 6 outcomes in $\mathcal{S}$ are equally likely.

What is $P(E|F)$ in each of these two cases ?

EXAMPLE : Suppose a coin is tossed two times.

The sample space is

$$\mathcal{S} \;=\; \{HH\ ,\ HT\ ,\ TH\ ,\ TT\}\ .$$

Let E be the event "*two Heads*" , *i.e.*,

$$E \;=\; \{HH\}\ .$$

Let F be the event "*the first toss gives Heads*" , *i.e.*,

$$F \;=\; \{HH\ ,\ HT\}\ .$$

Then

$$EF \;=\; \{HH\} \;=\; E \qquad (\text{ since } E \subset F\ )\ .$$

We have

$$P(E|F) \;=\; \frac{P(EF)}{P(F)} \;=\; \frac{P(E)}{P(F)} \;=\; \frac{\frac{1}{4}}{\frac{2}{4}} \;=\; \frac{1}{2}\ .$$

EXAMPLE :

Suppose we draw a card from a shuffled set of 52 playing cards.

- What is the probability of drawing a Queen, given that the card drawn is of *suit* Hearts ?

  ANSWER :

$$P(Q|H) \quad = \quad \frac{P(QH)}{P(H)} \quad = \quad \frac{\frac{1}{52}}{\frac{13}{52}} \quad = \quad \frac{1}{13} \; .$$

- What is the probability of drawing a Queen, given that the card drawn is a *Face card* ?

  ANSWER :

$$P(Q|F) \quad = \quad \frac{P(QF)}{P(F)} \quad = \quad \frac{P(Q)}{P(F)} \quad = \quad \frac{\frac{4}{52}}{\frac{12}{52}} \quad = \quad \frac{1}{3} \; .$$

(Here $Q \subset F$, so that $QF = Q$.)

The probability of an event E is sometimes computed more easily

if we condition E on another event F ,

namely, from

$$P(E) \quad = \quad P(\ E(F \cup F^c)\ ) \qquad (\text{ Why ? })$$

$$= \quad P(\ EF \cup EF^c\ ) \quad = \quad P(EF) \ + \ P(EF^c) \qquad (\text{ Why ? })$$

and

$$P(EF) \ = \ P(E|F)\ P(F) \quad , \quad P(EF^c) \ = \ P(E|F^c)\ P(F^c) \ ,$$

we obtain this *basic formula*

$$P(E) \quad = \quad P(E|F) \cdot P(F) \ + \ P(E|F^c) \cdot P(F^c) \ .$$

EXAMPLE :

An insurance company has these data :

The probability of an insurance claim in a period of one year is

4 percent for persons under age 30

2 percent for persons over age 30

and it is known that

30 percent of the targeted population is under age 30.

What is the probability of an insurance claim in a period of one year for a randomly chosen person from the targeted population?

SOLUTION :

Let the sample space $\mathcal{S}$ be all persons under consideration.

Let C be the event (subset of $\mathcal{S}$) of persons filing a claim.

Let U be the event (subset of $\mathcal{S}$) of persons under age 30.

Then U^c is the event (subset of $\mathcal{S}$) of persons over age 30.

Thus

$$\begin{aligned} P(C) &= P(C|U)\ P(U) \ + \ P(C|U^c)\ P(U^c) \\ &= \frac{4}{100}\,\frac{3}{10} + \frac{2}{100}\,\frac{7}{10} \\ &= \frac{26}{1000} \ = \ 2.6\% \ . \end{aligned}$$

EXAMPLE :

Two balls are drawn from a bag with 2 *white* and 3 *black* balls.

There are 20 outcomes (*sequences*) in $\mathcal{S}$. (Why ?)

What is the probability that *the second ball is white* ?

SOLUTION :

Let F be the event that *the first ball is white*.

Let S be the event that *the second second ball is white*.

Then

$$P(S) \ = \ P(S|F)\ P(F) + P(S|F^c)\ P(F^c) \ = \ \frac{1}{4} \cdot \frac{2}{5} \ + \ \frac{2}{4} \cdot \frac{3}{5} \ = \ \frac{2}{5}.$$

QUESTION : Is it surprising that $P(S) \ = \ P(F)$?

EXAMPLE : (continued $\cdots$)

Is it surprising that $P(S) \;=\; P(F)$?

ANSWER : Not really, if one considers the sample space $\mathcal{S}$:

$$
\begin{array}{lllll}
\{ \quad \mathbf{w}_1\mathbf{w}_2 \;, & \mathbf{w}_1 b_1 \;, & \mathbf{w}_1 b_2 \;, & \mathbf{w}_1 b_3 \;, & \\
\mathbf{w}_2\mathbf{w}_1 \;, & \mathbf{w}_2 b_1 \;, & \mathbf{w}_2 b_2 \;, & \mathbf{w}_2 b_3 \;, & \\
b_1\mathbf{w}_1 \;, & b_1\mathbf{w}_2 \;, & b_1 b_2 \;, & b_1 b_3 \;, & \\
b_2\mathbf{w}_1 \;, & b_2\mathbf{w}_2 \;, & b_2 b_1 \;, & b_2 b_3 \;, & \\
b_3\mathbf{w}_1 \;, & b_3\mathbf{w}_2 \;, & b_3 b_1 \;, & b_3 b_2 & \quad \} \;,
\end{array}
$$

where outcomes (*sequences*) are assumed equally likely.

EXAMPLE :

Suppose we draw *two cards* from a shuffled set of 52 playing cards.

What is the probability that the second card is a Queen ?

ANSWER :

$$P(2^{\text{nd}} \text{ card } Q) \quad =$$

$$P(2^{\text{nd}} \text{ card } Q | 1^{\text{st}} \text{ card } Q) \quad \cdot \quad P(1^{\text{st}} \text{ card } Q)$$

$$+ \quad P(2^{\text{nd}} \text{ card } Q | 1^{\text{st}} \text{ card not } Q) \quad \cdot \quad P(1^{\text{st}} \text{ card not } Q)$$

$$= \quad \frac{3}{51} \cdot \frac{4}{52} \quad + \quad \frac{4}{51} \cdot \frac{48}{52} \quad = \quad \frac{204}{51 \cdot 52} \quad = \quad \frac{4}{52} \quad = \quad \frac{1}{13} \, .$$

QUESTION : Is it surprising that $P(2^{\text{nd}} \text{ card } Q) \; = \; P(1^{\text{st}} \text{ card } Q)$?

A useful formula that *"inverts conditioning"* is derived as follows :

Since we have both

$$P(EF) \quad = \quad P(E|F) \; P(F) \; ,$$

and

$$P(EF) \quad = \quad P(F|E) \; P(E) \; .$$

If $P(E) \neq 0$ then it follows that

$$P(F|E) \quad = \quad \frac{P(EF)}{P(E)} \quad = \quad \frac{P(E|F) \cdot P(F)}{P(E)} \; ,$$

and, using the earlier useful formula, we get

$$P(F|E) \quad = \quad \frac{P(E|F) \cdot P(F)}{P(E|F) \cdot P(F) \; + \; P(E|F^c) \cdot P(F^c)} \; ,$$

which is known as *Bayes' formula* .

EXAMPLE : Suppose 1 in 1000 persons has a certain disease.

A test detects the disease in 99 % of diseased persons.

The test also "detects" the disease in 5 % of healthhly persons.

With what probability does a positive test diagnose the disease?

SOLUTION : Let

$$D \sim \text{"diseased"} \quad , \quad H \sim \text{"healthy"} \quad , \quad + \sim \text{"positive"} .$$

We are given that

$$P(D) = 0.001 , \qquad P(+|D) = 0.99 , \qquad P(+|H) = 0.05 .$$

By Bayes' formula

$$P(D|+) = \frac{P(+|D) \cdot P(D)}{P(+|D) \cdot P(D) + P(+|H) \cdot P(H)}$$

$$= \frac{0.99 \cdot 0.001}{0.99 \cdot 0.001 \ + \ 0.05 \cdot 0.999} \cong 0.0194 \quad (\ ! \ )$$

EXERCISE :

Suppose 1 in 100 products has a certain defect.

A test detects the defect in 95 % of defective products.

The test also "detects" the defect in 10 % of non-defective products.

- With what probability does a positive test diagnose a defect?

EXERCISE :

Suppose 1 in 2000 persons has a certain disease.

A test detects the disease in 90 % of diseased persons.

The test also "detects" the disease in 5 % of healthly persons.

- With what probability does a positive test diagnose the disease?

More generally, if the sample space $\mathcal{S}$ is *the union of disjoint events*

$$\mathcal{S} \ = \ F_1 \ \cup \ F_2 \ \cup \ \cdots \ \cup \ F_n \ ,$$

then for any event E

$$P(F_i|E) \ = \ \frac{P(E|F_i) \cdot P(F_i)}{P(E|F_1) \cdot P(F_1) \ + \ P(E|F_2) \cdot P(F_2) \ + \cdots + \ P(E|F_n) \cdot P(F_n)}$$

EXERCISE :

Machines M_1, M_2, M_3 produce these *proportions* of a article

Production : $M_1 \ : \ 10\ \%$, $M_2 \ : \ 30\ \%$, $M_3 \ : \ 60\ \%$.

The probability the machines produce *defective* articles is

Defects : $M_1 \ : \ 4\ \%$, $M_2 \ : \ 3\ \%$, $M_3 \ : \ 2\ \%$.

What is the probability a random article was made by machine M_1, given that it is defective?

Independent Events

Two events E and F are *independent* if

$$P(EF) \;=\; P(E)\,P(F)\;.$$

In this case

$$P(E|F) \;=\; \frac{P(EF)}{P(F)} \;=\; \frac{P(E)\,P(F)}{P(F)} \;=\; P(E)\;,$$

(assuming $P(F)$ is not zero).

Thus

knowing F occurred doesn't change the probability of E .

EXAMPLE : Draw *one* card from a deck of 52 playing cards.

Counting outcomes we find

$$
\begin{aligned}
P(\text{Face Card}) &= \tfrac{12}{52} &&= \tfrac{3}{13} \,, \\
P(\text{Hearts}) &= \tfrac{13}{52} &&= \tfrac{1}{4} \,, \\
P(\text{Face Card and Hearts}) & &&= \tfrac{3}{52} \,, \\
P(\text{Face Card}|\text{Hearts}) & &&= \tfrac{3}{13} \,.
\end{aligned}
$$

We see that

$$P(\text{Face Card and Hearts}) \ = \ P(\text{Face Card}) \cdot P(\text{Hearts}) \quad (= \frac{3}{52}) \, .$$

Thus the events "*Face Card*" and "*Hearts*" are *independent*.

Therefore we also have

$$P(\text{Face Card}|\text{Hearts}) \ = \ P(\text{Face Card}) \quad (= \frac{3}{13}) \, .$$

EXERCISE :

Which of the following pairs of events are independent?

(1) drawing "Hearts" and drawing "Black" ,

(2) drawing "Black" and drawing "Ace" ,

(3) the event $\{2, 3, \cdots, 9\}$ and drawing "Red" .

EXERCISE : *Two* numbers are drawn at random from the set

$$\{ 1 , 2 , 3 , 4 \} .$$

If *order is not important* then what is the sample space $\mathcal{S}$?

Define the following functions on $\mathcal{S}$:

$$X(\{i,j\}) = i + j , \qquad Y(\{i,j\}) = |i - j| .$$

Which of the following pairs of events are independent?

$$(1) \quad X = 5 \quad \text{and} \quad Y = 2 ,$$

$$(2) \quad X = 5 \quad \text{and} \quad Y = 1 .$$

REMARK :

X and Y are examples of *random variables* . (More soon!)

EXAMPLE : If E and F are *independent* then so are E and F^c .

PROOF : $E \;=\; E(F \cup F^c) \;=\; EF \cup EF^c$, where

$$EF \quad \text{and} \quad EF^c \quad \text{are } \textit{disjoint} .$$

Thus

$$P(E) \;=\; P(EF) + P(EF^c) ,$$

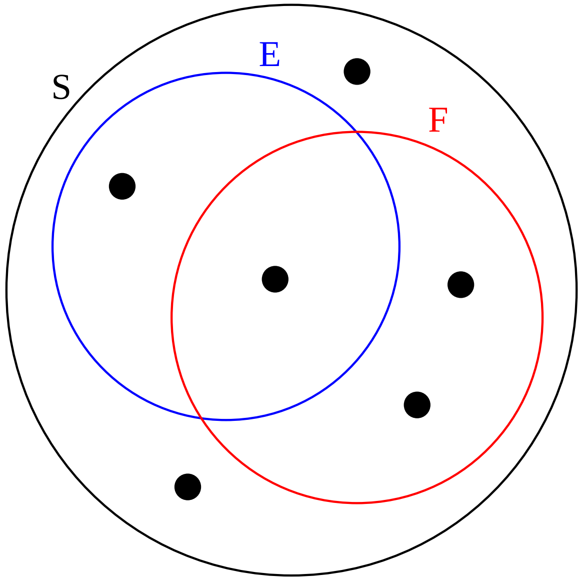
from which

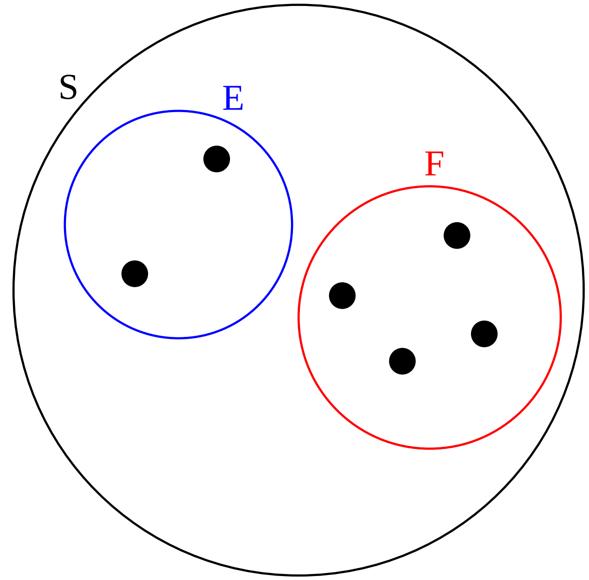
$$\begin{aligned} P(EF^c) &= P(E) - P(EF) \\ &= P(E) - P(E) \cdot P(F) \quad \text{(since } E \text{ and } F \text{ independent)} \\ &= P(E) \cdot (\, 1 - P(F) \,) \\ &= P(E) \cdot P(F^c) \;. \end{aligned}$$

EXERCISE :

Prove that if E and F are *independent* then so are E^c and F^c .

NOTE : *Independence* and *disjointness* *are different things* !





Independent, but not disjoint. Disjoint, but not independent.

(The six outcomes in S are assumed to have equal probability.)

If E and F are *independent* then $P(EF) = P(E)\ P(F)$.

If E and F are *disjoint* then $P(EF) = P(\ \emptyset\ ) = 0$.

If E and F are *independent and disjoint* then one has *zero probability* !

Three events E , F, and G are *independent* if

$$P(EFG) \quad = \quad P(E)\ P(F)\ P(G)\ .$$

and

$$\begin{aligned} P(EF) \quad &= \quad P(E)\ P(F)\ . \\ P(EG) \quad &= \quad P(E)\ P(G)\ . \\ P(FG) \quad &= \quad P(F)\ P(G)\ . \end{aligned}$$

EXERCISE : Are the three events of drawing

(1) a red card ,

(2) a face card ,

(3) a Heart or Spade ,

independent ?

EXERCISE :

A machine M consists of three *independent parts*, M_1, M_2, and M_3 .

Suppose that

$$M_1 \text{ functions properly with probability } \tfrac{9}{10} ,$$

$$M_2 \text{ functions properly with probability } \tfrac{9}{10} ,$$

$$M_3 \text{ functions properly with probability } \tfrac{8}{10} ,$$

and that

the machine M functions if and only if *its three parts function*.

- What is the probability for the machine M to *function* ?
- What is the probability for the machine M to *malfunction* ?