Solution of Problem Sheet for Exponential and Poisson Processes

1. The time T required to repair a machine is an exponentially distributed random variable with mean $\frac{1}{2}$ hour.
   (a) e^{-1}
   (b) e^{-1},since the exponential distribution has no memory.
2. Let T be the time you spend in the system. Let S_i bethe service time of the i person in queue. Let R be the remaining service time of the person in service. Let S be you service time. Then

$$\mathrm{E}(T) = \mathrm{E}(R + S_1 + S_2 + S_3 + S_4 + S) = 6/u.$$

3. Let T_A, T_B, T_C be the service time for people A, B, C respectively.
   (a) 0
   (b) $P(T_A = 3, T_B = 1, T_C = 1) = 1/27$
   (c) $P(T_A > T_B + T_C) = P(T_A > T_B + T_C | T_A > T_B) P(T_A > T_B) = P(T_A > T_C) P(T_A > T_B) = 1/4$
4. (a)

$$\begin{aligned} P[X_1 < X_2 < X_3] &= P(X_1 = \min(X_1, X_2, X_3)) P(X_2 < X_3 | X_1 = \min(X_1, X_2, X_3)) \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} P(X_2 < X_3 | X_1 = \min(X_1, X_2, X_3)) \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_2}{\lambda_2 + \lambda_3} \end{aligned}$$

   (b)

$$\begin{aligned} P[X_1 < X_2 | \max(X_1, X_2, X_3) = X_3] &= \frac{P[X_1 < X_2, \max(X_1, X_2, X_3) = X_3]}{P[\max(X_1, X_2, X_3) = X_3]}, \text{ definition of conditional probability} \\ &= \frac{P[X_1 < X_2 < X_3]}{P[X_1 < X_2 < X_3] + P[X_2 < X_1 < X_3]}, \text{ Theorem of total probability} \\ &= \left(\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_2}{\lambda_2 + \lambda_3} \right) \left(\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_2}{\lambda_2 + \lambda_3} + \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_1}{\lambda_1 + \lambda_3} \right)^{-1} \end{aligned}$$

   (c) $\mathrm{E}[\max X_i | X_1 < X_2 < X_3] = \mathrm{E}[X_3 | X_1 < X_2 < X_3] = \mathrm{E}[X_3 - X_2 + X_2 - X_1 + X_1 | X_1 < X_2 < X_3]$.
   Conditioning on $X_1 < X_2 < X_3$, the distribution of X_1 is the distribution of $\min(X_1, X_2, X_3)$, the distribution of $X_2 - X_1$ is the distribution of $\min(X_2, X_3)$, the distribution of $X_3 - X_2$ is the distribution of X_3. (using the memoryless property)
   Thus,

$$\mathrm{E}[X_3 - X_2 + X_2 - X_1 + X_1 | X_1 < X_2 < X_3] = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_2 + \lambda_3} + \frac{1}{\lambda_3}$$

   (d) Using the result of the previous question

$$\mathrm{E}[\max X_i] = \sum_{i \neq j \neq k}^{3} = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_j}{\lambda_j + \lambda_k} \left(\frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_2 + \lambda_3} + \frac{1}{\lambda_3} \right)$$

5. (a) $P_A = P[S_1 < S_2] = \frac{\mu_1}{\mu_1 + \mu_2}$
   (b) $P_B = 1 - P[\text{B leaves the system}] = 1 - \left(\frac{\mu_2}{\mu_1 + \mu_2} \right)^2$
   (c) $\mathrm{E}[T] = 1/\mu_1 + 1/\mu_2 + P_A/\mu_2 + P_B/\mu_2$.