Chapter 03:

Probability

- Probability and statistics are related in an important way. Probability is used as a tool; it allows you to evaluate the reliability of your conclusions about the population when you have only sample information.

① Combinatorial Analysis

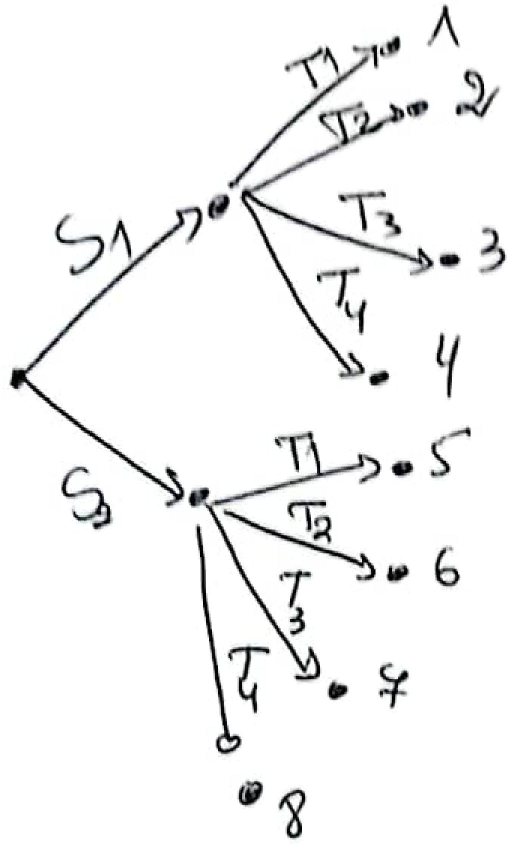
this section deals with finding effective methods for counting the number of ways that things can occur. In fact, many problems in probability theory can be solved simply by counting the number of different ways that a certain event can occur. the mathematical theory of counting is formally known as " Combinatorial analysis"

1-1/ Fundamental Principle of counting : Tree Diagrams

* If one thing can be accomplished in n_1 different ways and after this a second thing can be accomplished in n_2 different ways, ..., and finally a kth thing can be accomplished in n_k different ways, then all k things can be accomplished in the specified order in $n_1 n_2 \dots n_k$ different ways.

Example: - If a man has 2 shirts and 4 ties, then he has $2 \cdot 4 = 8$ ways of choosing a shirt and then a tie.

- A diagram, called a "tree diagram" because of its appearance, is often used in connection with the above principle.
- Letting the shirts be represented by S_1, S_2 and the ties by T_1, T_2, T_3, T_4 the various ways of choosing a shirts and then a tie are indicated in the tree diagram



Tree diagram

1-3) Permutations

Number of ways to order n distinct elements:

Based on the reasoning mentioned earlier, if we have n distinct elements, there would be $n \times (n-1) \times (n-2) \times \dots \times 2 \times 1$ ways of ordering them. We denote this quantity by $n!$ and in words by n factorial.

Number of ways to order n elements (some of which are not distinct): The number of distinct orderings of n objects, n_1 of which are type 1, n_2 of which are of type 2, ..., and n_r of which are of type r, equal to

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}, \quad n_1 + n_2 + \cdots + n_r = n.$$

* Number of ways to select r elements from n elements (order is important): The total number of ways of ordering r elements, chosen from n distinct elements, is equal to: $n(n-1)(n-2)\cdots(n-r+1)$

this quantity can be also expressed as $n!/(n-r)!$

$\frac{n!}{(n-r)!}$ - this is denoted by P_r^n the number of permutations of n things taken r at a time.

1-3/ Combinations

* Number of ways to select r elements from n distinct elements (order is not important)

It is possible to choose, without regard of order r elements from n distinct elements in

$$\frac{n!}{r!(n-r)!}$$

This is an important quantity in combinatorics and will be denoted C_n^n.

Note that $C_n^0 = C_n^n = 1$

Example: ① the number of different arrangements or permutations, consisting of 3 letters each that can be formed from the 7 letters A, B, C, D, E, F, G is:

$$P_r^n = \frac{7!}{4!} = 7.6.5 = 210.$$

suppose that a set consists of n objects of which n_1 are of one type

② the number of different permutations of the 11 letters of the word M I S S I S S I P P I, which consists of 1 M, 4 I's, 4 S's and 2 P's, is:

$$\frac{11!}{1!4!4!2!} = 34,650.$$

③ the number of ways in which 3 cards can be chosen selected from a total of 8 different cards is

$$C_3^8 = \frac{8.7.6}{3!} = 56$$

C_r^n it is also called a "Binomial Coefficient" and is sometimes written a $\binom{n}{r}$, because they arise in the "binomial expansion"

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \dots + \binom{n}{n} y^n$$

or:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k}.$$

<u>Exercice</u>: Twenty books are to be arranged on a shelf, eleven on travel, five on cooking, and four on gardening. the books in each category are to be grouped together. How many arrangements are possible?.

<u>Solution</u>: We have 11! arrangements for the travel 5! arrangements for the cooking, and 4! arrangements for the gardening books. We can also permute the three different classes of books in 3! ways. Thus:

$$\text{total} = (11!)(5!)(4!)(3!)$$

(2) Axiomes of Probability

In this section we introduce the concept of the probability of an event and then show how these probabilities can be computed in certain situations.

2.1/ Sample Space and Events

Experiment: An experiment is the process by which an observation (or measurement) is obtained.

* the observation or measurement generated by an experiment may or may not produce a numerical value. Here are some examples of experiments:

- Recording a test grade.
- Measuring daily rainfall.
- Interviewing a householder to obtain his or her opinion on a greenbelt zoning ordinance.

When an experiment is performed, What we observe is an outcome called a simple event, often denoted by the capital E with a subscript.

<u>Simple event</u>: A simple event is the outcome that is observed on a single repetition of the experiment.

<u>Example</u>: Experiment: Toss a die and observe the number that appears on the upper face. List the simple events in the experiment.

<u>Solution</u>: When the die is tossed once, there are six possible outcomes. There are the simple events, listed below:

Event E_1: observe a 1
" E_2: " a 2.
" E_3: " a 3

Event E_4: ~~Obs~~ observe a 4
" E_5: " a 5.
" E_6: " a 6.

* We can now define an event as a collection of simple events, often denoted by a capital letter.

<u>Event</u>: An event is a collection of simple events.

Example continued: We can define the events A and B for the die tossing experiment:

A: observe an odd number.

B: Observe a number less than 4.

Since event A occurs if the upper face is 1, 3, or 5, it is a collection of three simple events and we write: $A = \{E_1, E_2, E_3\}$. Similarly, the event B occurs if the upper face is 1, 2, 3 and is defined as a collection of set of these three simple events: $B = \{E_1, E_2, E_3\}$.

* Sometimes when one event occurs, it means that another event cannot.

Mutually Exclusive: Two events are mutually exclusive if, when one event occurs, the others cannot, and vice versa.

Example continued: the events A and B are not mutually exclusive, because they have two outcomes in common (1;3)

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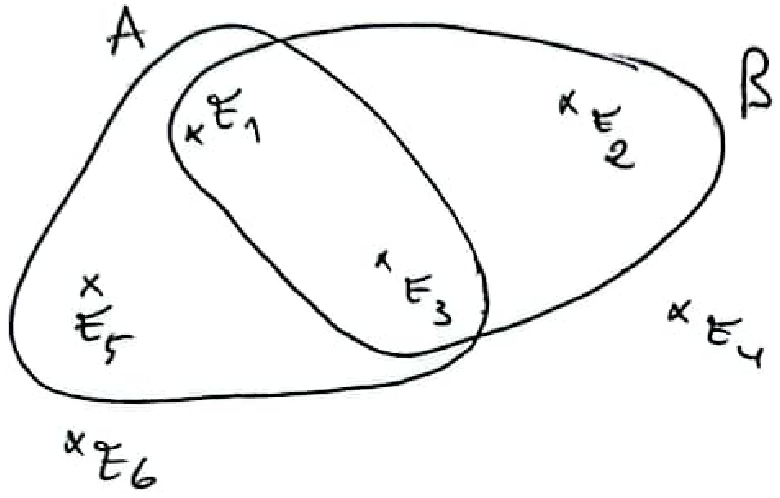
Definition: the set of all siple events is called the "sample space" S.

* Sometimes it helps to visualize an experiment using a picture called a Venn diagram.



Venn digram

Example: Experiment: Toss a single coin and observe the result. these are the siple events:

E_1: Observe a head (H).

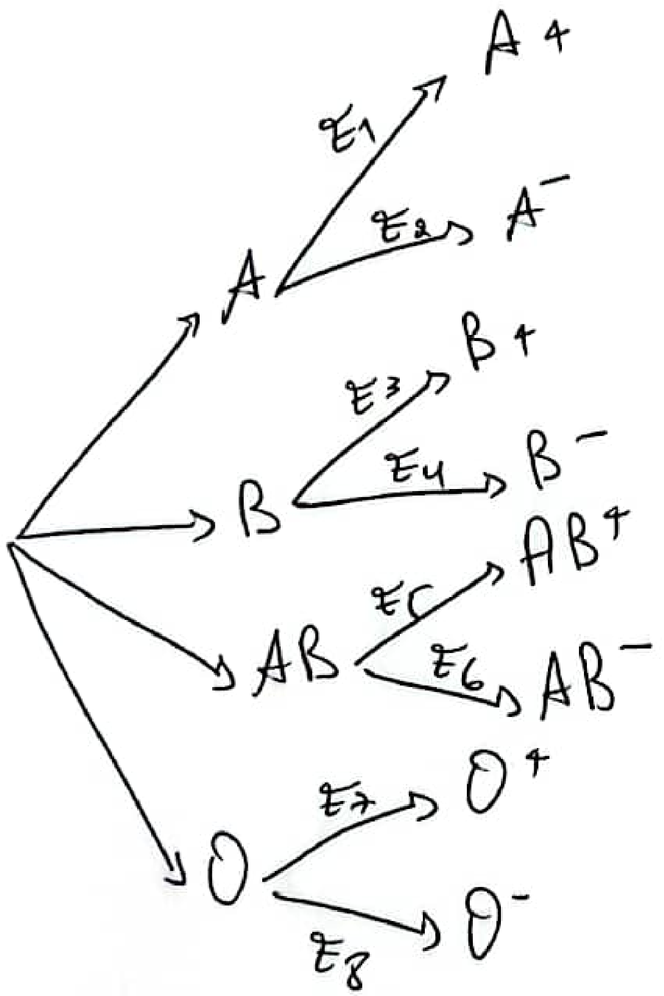
E_2: Observe a tail (T).

the sample space is $S = \{E_1, E_2\}$ or more siply $S = \{H, T\}$

* Some experiments can be generated in stages, and the sample space can be displayed in a "Tree diagram"

Example: A medical technician records a person's blood type and Rh factor. List the simple events in the experiment.

Solution: For each person, a two-stage procedure is needed to record the two variables of interest. The Tree diagram is :



* Experiment: Record a person's blood type. The four mutually exclusive possible outcomes are these simple events: E_1: Blood type A ; E_2: Blood type B.
E_3: " " AB; E_4: " " O

The sample space is $S = \{E_1; E_2; E_3; E_4\}$

2-21 Event Relations

Sometimes the event of interest can be formed as a combination of several other events.

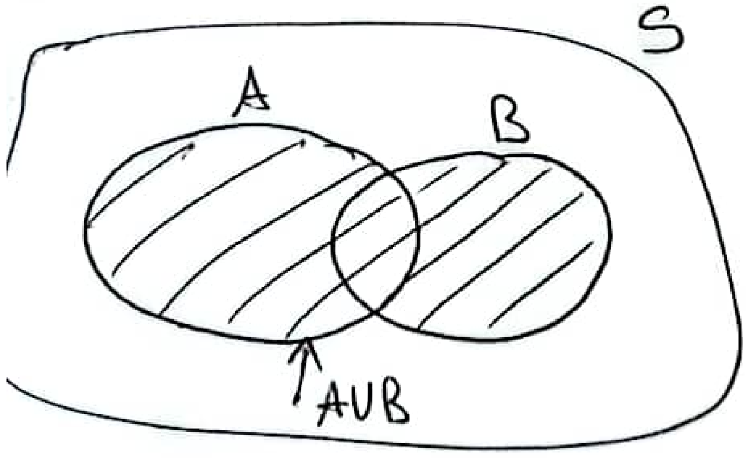
let A and B be two events defined on the sample space S

Here are three important relationships between events:

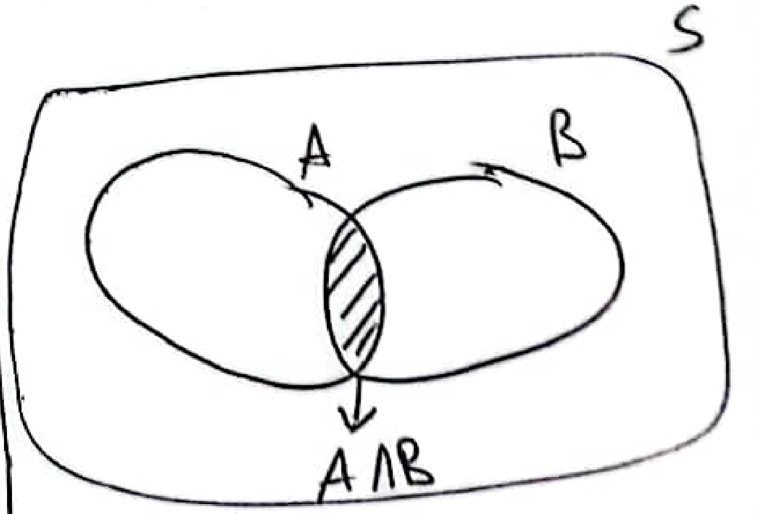
Def ①: the "union" of events A and B, denoted by $A \cup B$, is the event that either A or B or both occur.

Def ②: the "intersection" of events A and B, denoted by $A \cap B$, is the event that both A and B occur.

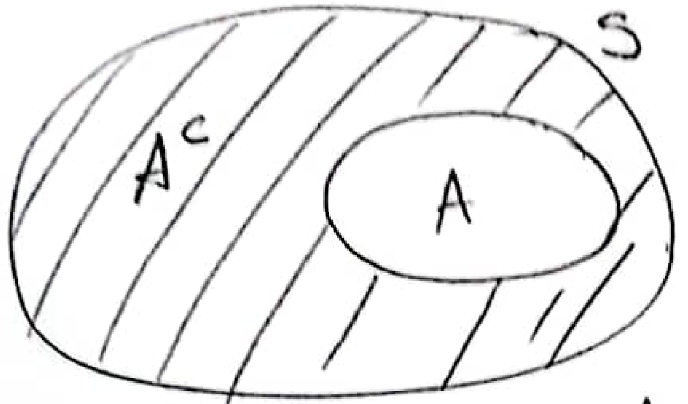
Def ③: the "complement" of an event A, denoted by A^c, is the event that does not occur.



Venn diagram of $A \cup B$



Venn diagram of $A \cap B$



Venn diagram of the complement of an event.

Key point: * Intersection $\Leftrightarrow$ "both" or just "and".

* Union $\Leftrightarrow$ "either" or just "or"

Example: Two fair coins are tossed, and the outcome is recorded. These are the events of interest:

A: Observe at least one head.

B: Observe at least one tail.

* Define the events A, B, $A \cap B$, $A \cup B$ and A^c as collections of simple events ~~and find~~

Solution: ~~Recall~~ E_1: HH (head on first coin, head on second)

E_2: HT.

E_3: TH

E_4: TT

$A = \{E_1, E_2, E_3\}$. and $A^c = \{E_4\}$.

Similarly:

$B = \{E_2, E_3, E_4\}$ $\quad A \cap B = \{E_2, E_3\}$.

$A \cup B = \{E_1, E_2, E_3, E_4\} = S$.

Some Properties of union and Intersection

* Commutative laws: $E \cup F = F \cup E$
* Associative laws: $(E \cup F) \cup G = E \cup (F \cup G)$.
* Distributive laws: $(E \cup F) \cup G = (E \cup G) \cup (F \cup G)$.
* Note that $E \cup S = S$, $E \cap S = E$

$$E \cup \emptyset = E.$$

3) Calculating Probabilities using simple events

- the probability of an event A is a measure of our ~~belief~~ belief that the event A will occur.
- One pratical way to interpret this measure is with the concept of relative frequency. If an experiment is performed n times, then relative frequency of a particular occurrence A is:

$$\text{Relative frequency} = \frac{\text{Frequency}}{n}$$

- Where the frequency is the number of times the event A occured.
- In this population, the relative frequency of the event A is defined as the probability of event A ; that is

$$P(A) = \lim_{n \to \infty} \frac{\text{Frequency}}{n}$$

Rq: $P(A)$ must be a proportion lying between 0 and 1

$P(A) = 0$ if the event A never occurs,

$P(A) = 1$ if the event A always occurs.

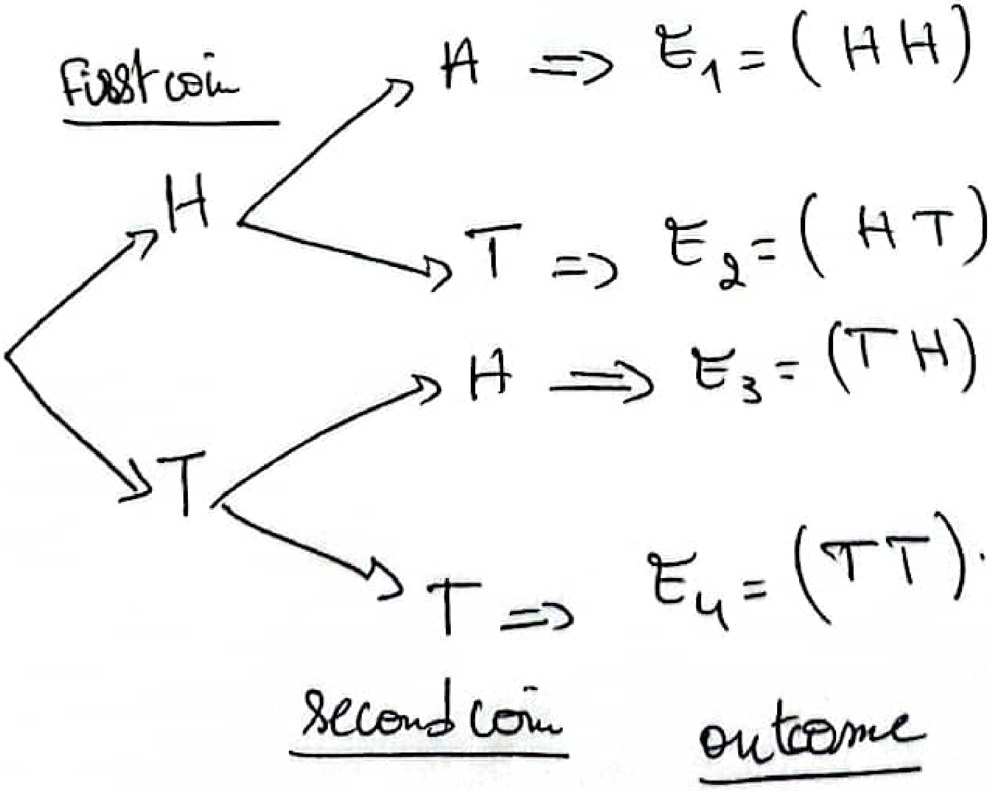
* **Requirements for simple event probabilities**
    - Each probability must lie between 0 and 1.
    - the sum of the probabilities for all simple events in S equals 1.

Def: the probability of an event A is equal to the sum of the probabilities of the simple events contained in A.

Example: Toss two fair coins and record the outcome. Find the probability of observing exactly one head in the two tosses.

Solution: the letters H and T mean that you observed a head or a tail, respectively, on a particular toss. To assign probabilities to each of the four simple events, you need to remember that the coins are fair. therefore, any of the four simple events is as likely as any other.

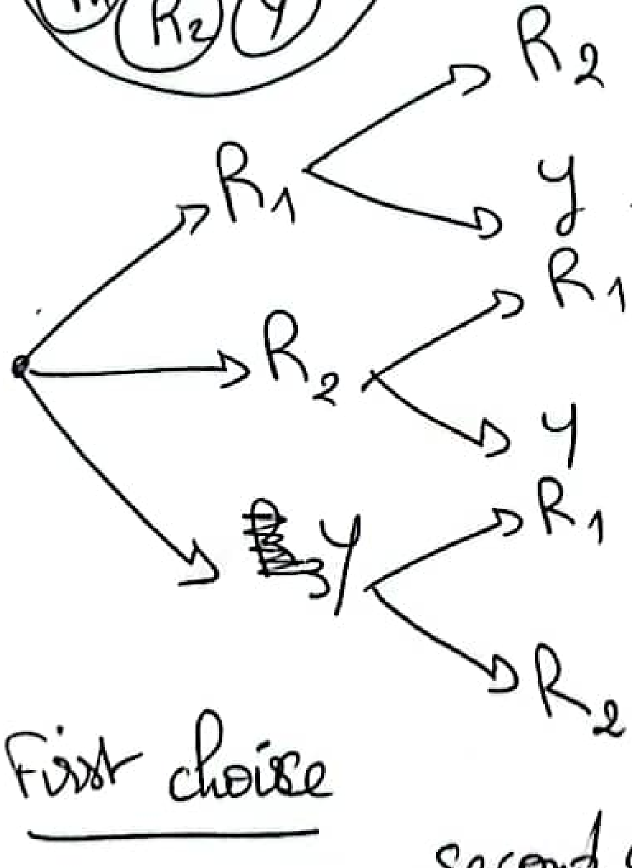
Since the sum of the four simple events must be 1, each must have probability ($P(E_i) = \frac{1}{4}$).



To find $P(A) = P$ (observed exactly one head), you need to find all the simple events that result in event A, E_2 and E_3: $P(A) = P(E_2) + P(E_3) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

<u>Example</u>: A candy dish contains one yellow and two red candies, you close your eyes, choose two candies one at a time from the dish, and record their colors. What is the probability that both candies are red?

<u>Solution</u>:



simple event	Proba
$(R_1\ R_2)$	1/6
$(R_1\ Y)$	1/6
$(R_2\ R_1)$	1/6
$(R_2\ Y)$	1/6
$(Y\ R_1)$	1/6
$(Y\ R_2)$	1/6

* all six choises should be equally likely

then $A = \{R_1R_2; R_2R_1\}$.

thus : $P(A) = P(R_1R_2) + P(R_2R_1) = 1/6 + 1/6 = 1/3$.

Key point: A tree diagram helps to find simple events.

* Calculating the Probability of an event

① List all the simple events in the sample space.

② Assign an appropriate probability to each simple event.

③ Determine which simple events results in the event of interest.

④ Sum the probabilities of the simple events that result in the event of interest.

④ Calculating Probabilities for unions and Complements

* Given two events A and B, the probability of their union $A \cup B$ is equal to:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

* When two events A and B are "mutually exclusive" or "disjoint"; it means that when A occurs, B cannot and vice versa. this means that the probability that they both occur $P(A \cap B) = 0$. and

$$P(A \cup B) = P(A) + P(B)$$

* the complements of events A and A^c are disjoint $A \cup A^c = S$ then: $P(A) + P(A^c) = 1 \Rightarrow P(A^c) = 1 - P(A)$.

Key point

For a Sample Space having N equally likely outcomes:

$$P(\{1\}) = P(\{2\}) = \cdots = P(\{N\}) = \frac{1}{N}$$

Probability of any event E is

$$P(E) = \frac{\text{number of points in } E}{\text{number of points in } S}$$

Example: In a telephone survey of 1000 adults, respondents were asked about the expense of a college education and the relative necessity of some form of financial assistance. The respondents were classified according to whether they currently had a child in college and whether they thought the loan burden for most college students is too high, the right amount or too little. The proportions responding in each category are shown in the probability table. Suppose one respondent is chosen at random from this group.

	Too high (A)	Right Amount (B)	Too little (C)
Child in College (D)	0,35	0,08	0,01
No child in colleg (E)	0,25	0,20	0,11

1/ What is the probability that the respondent has a child in college?

2/ What is the probability that the respondent doesn't not have a child in college?

3/ What is the probability that the respondent has a child in college or thinks that the loan burden is too high?

Solution : 1/ The event that a respondent has a child in college will occur regardless of his or her response to the question about loan burden. that is, event D consist of the simple events in the first row:

$$P(D) = 0{,}35 + 0{,}08 + 0{,}01 = 0{,}44$$

In general, the probabilities of marginal events such as D and A are found by summing the probabilities in the appropriate row or column.

2/ the event that the respondent does not have a child in college is the complement of the event D denoted D^c:

$$P(D^c) = 1 - P(D) = 1 - 0{,}44 = 0{,}56.$$

3/ the event of interest is $P(A \cup B)$.

$$P(A \cup D) = P(A) + P(D) - P(A \cap D)$$
$$= 0{,}60 + 0{,}44 - 0{,}35 = 0{,}69$$

⑤ Independence, Conditional Probability

there is a probability rule that can be used to calculate the probability of the intersection of several events. However, this rule depends on the important statistical concept of independent or dependent events.

<u>Definition</u>: Two events A and B are said to be independent if and only if the probability of event B is not influenced or changed by the occurrence of event A or vice versa.

<u>Example</u>: Consider tossing a single die two times, and define two events: A: Observe a 2 on the first toss
B: Observe a 2 on the second toss

If the die is fair, the probability of event A is $P(A) = 1/6$
Consider the probability of event B. Regardless of whether event A has or has not occurred, the probability of observing a 2 on the second toss is $1/6$. We could write

$P(B$ given that A occurred$) = 1/6$

$P(B$ " " A did not occur$) = 1/6$.

Since the probability of event B is not changed by the occurrence of event A, we say that A and B are "independent events".

- the probability of an event A, given that the event B has occurred, is called the "Conditional probability of A, given that B has occurred", denoted by $P(A/B)$. the vertical bar is read "given" and the events appearing to the right of the bar are those that you know have occurred.

- the probability that both A and B occur when the experiment is performed is:

$$P(A \cap B) = P(A) \cdot P(B/A).$$

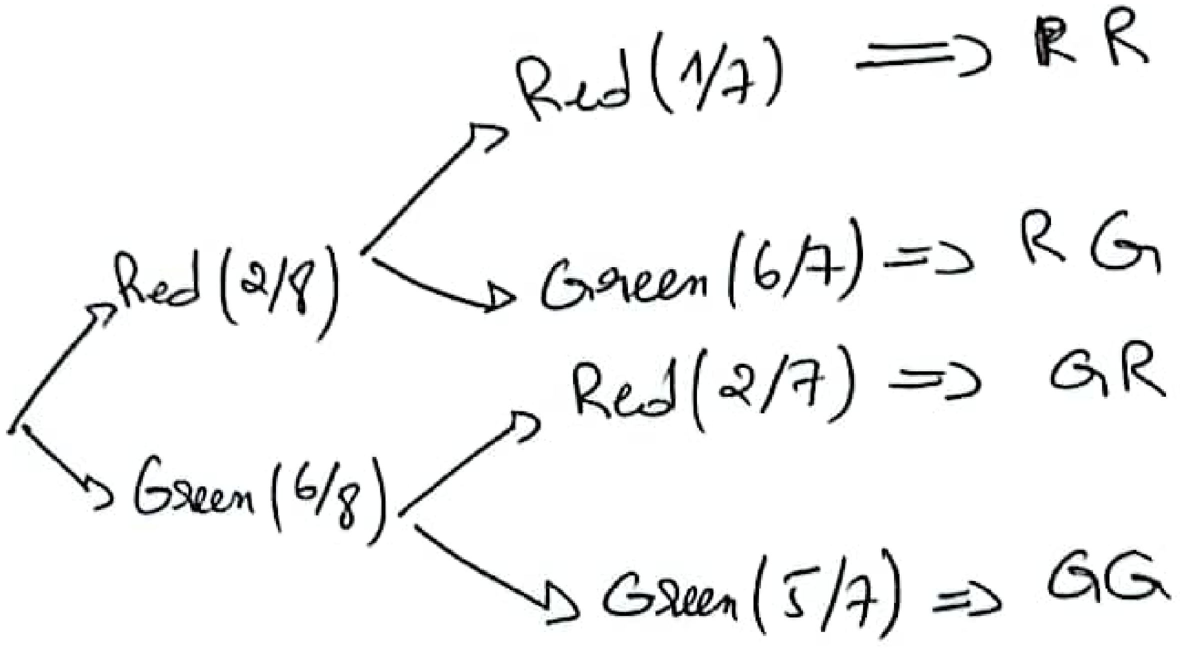
or

$$P(A \cap B) = P(B) \cdot P(A|B).$$

Example: In a color preference experiment, eight toys are placed in a container. The toys are identical except for color two are red, and six are green. A child is asked to choose two toys at random. What is the probability that the child chooses the two red toys?

Solution: You can visualize the experiment using a tree diagram as shown in:

R: Red toy is chosen
G: Green toy is chosen



A: both toys ar red, can be constructed as the intersection of two events:

$A = (R \text{ on first choice}) \cap (R \text{ on second choice})$.

$P(A) = P(R \text{ on first choice } \cap R \text{ on second choice})$

$= P(R \text{ on first choice}) . P(R \text{ on second choice} / R \text{ on first})$

$$= (2/8) . (1/7) = \frac{2}{56} = \boxed{\frac{1}{28}}$$

* ## Conditional Probabilities

- the conditional Probability of event A, given that event B has occurred is:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) \neq 0.$$

- the conditional Probability of event B, give that event A has occurred is:

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad \text{if } P(A) \neq 0.$$

Example: (Colorblindness, continued) Suppose that in general population, there are 51% men and 49% women, and that the proportions of colorblind men and women are shown in the probability Table below:

	Men (B)	Women (B^c)	Total
Colorblind (A)	0,04	0,002	0,042
Not colorblind (A^c)	0,47	0,488	0,958
Total	0,51	0,49	1,00

If a person is drawn at random from this population and is found to be a man (event B), What is the probability that the man is colorblind (event A)? If we know that the event B has occurred, we must restrict our focus to only 51% of the population that is male. The probability of being colorblind, given that the person is male, is 4% of the 51% or:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0,04}{0,51} = 0,78.$$

- What is the probability of being colorblind give that the person is female? Now we are restricted to only the 49% of the population that is female, and.

$$P(A/B^c) = \frac{P(A \cap B^c)}{P(B^c)} = \frac{0,02}{0,49} = 0,004$$

Notice that the probability of event A changes depending on whether event B occurred.

This indicates that these two events are dependent

- When two events are independent that is, if the probability of event B is the same, whether or not event A has occurred, then event A does not affect event B

and $P(B/A) = P(B)$

- If two events A and B are independent, the probability that both A and B occur is

$$P(A \cap B) = P(A) . P(B).$$

or

$$P(B/A) . P(B).$$

otherwise, the events are said to be dependent.

<u>Example</u>: From previous examples, you know that

$$S = \{HH, HT, TH, TT\}.$$

(Toss two coins and observe the outcome. Define these event (A: Head on the first coin) (B: Tail on the second coin)

Are events A and B independent?

$P(A) = \frac{1}{2}$ $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{4}$

Since $P(A).P(B) = \frac{1}{4}$ and $P(A \cap B) = \frac{1}{4}$

we have $P(A).P(B) = P(A \cap B)$.

and the two events must be independent.

⑥ Bayes's Rule

Example: Let us reconsider the experiment involving colorblindness example: B: the person selected is a man.

B^C: " " " is a Woman

Taken together make up the sample space S, consisting of both men and women. Since colorbling people can be either male or female, the event A, which is that is person is colorblind consist of both those siple events that are in A and B and those siple events that are in A and B^C. Since these two intersections are mutually exclusive, you can write the event A as:

$A = (A \cap B) \cup (A \cap B^C)$ and

$P(A) = P(A \cap B) + P(A \cap B^C) = 0{,}04 + 0{,}002 = 0{,}042$

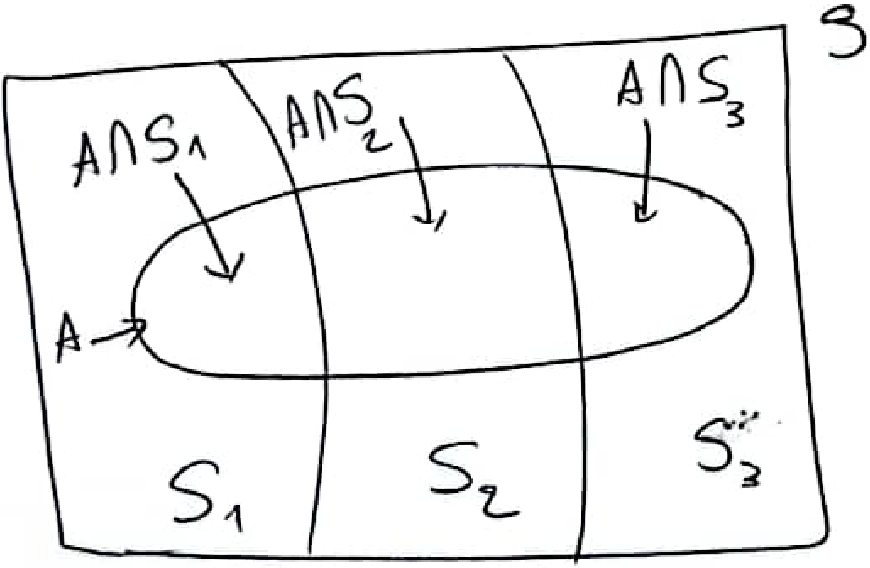
Suppose now that the sample space can be partitioned into k subpopulations, $S_1, S_2, S_3, \dots, S_k$ that as in the colorblindness example are mutually exclusive and exhaustive, that is taken together they make up the entire sample space. In a similar way, you can express an event A as:

$$A = (A \cap S_1) \cup (A \cap S_2) \cup (A \cap S_3) \cup \dots \cup (A \cap S_k)$$

then

$$P(A) = P(A \cap S_1) + P(A \cap S_2) + \dots + P(A \cap S_k).$$

this is illustrated for $k = 3$.



Decomposition of event A.

Law of Total probability

Given a set of events $S_1, S_2, S_3, \ldots, S_k$ that are mutually exclusive and exhaustive and an event A, the probability of event A can be expressed as:

$$P(A) = P(S_1).P(A|S_1) + P(S_2).P(A|S_2) + \ldots + P(S_k).P(A|S_k)$$

Bayes' Rule

$$P(S_i|A) = \frac{P(S_i).P(A|S_i)}{\sum_{j=1}^{k} P(S_j).P(A|S_j)}.$$

for $i = 1, \ldots, k$.