FEEG6017 lecture: Relationship between two variables: correlation, covariance and r-squared

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Relationships between variables

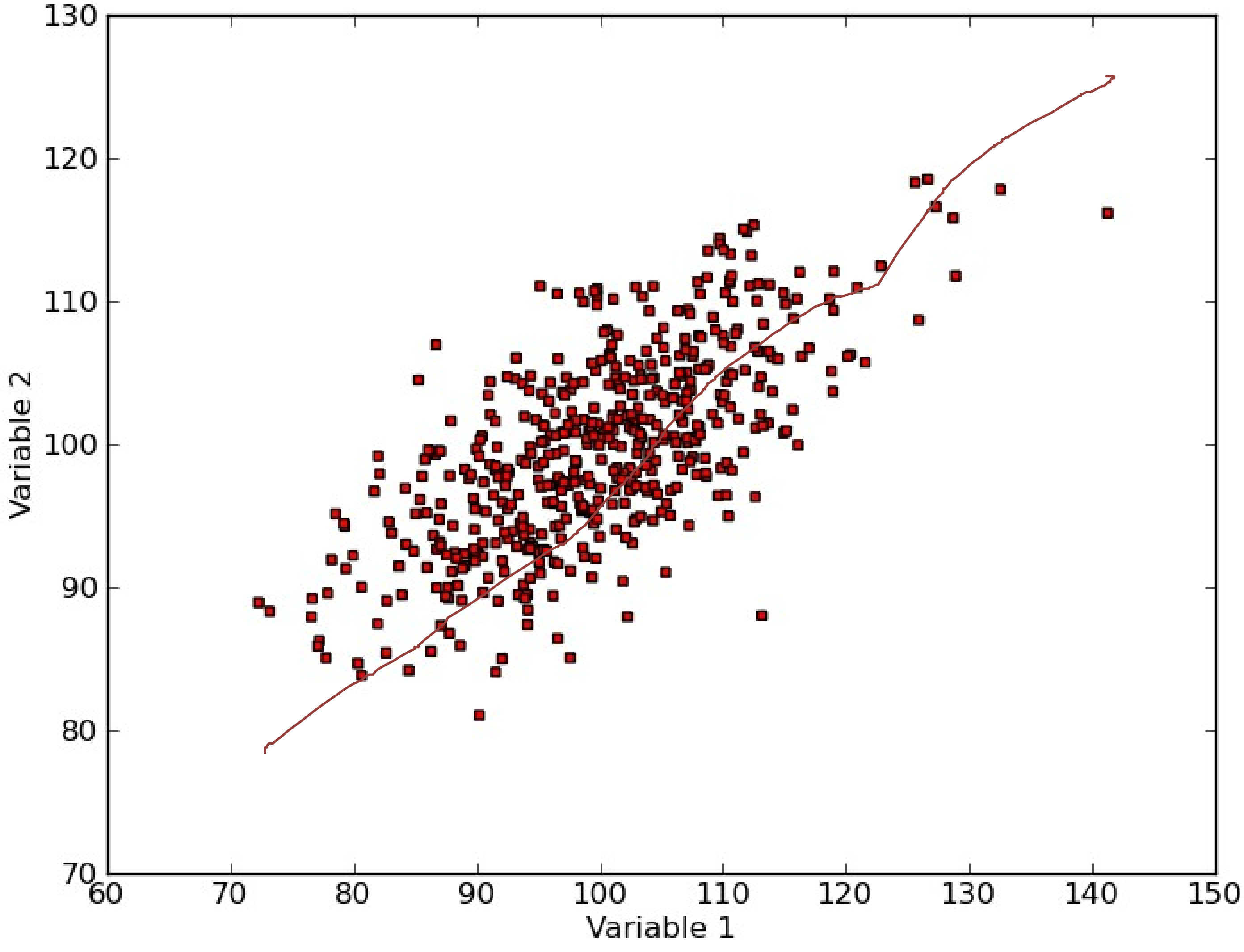
- So far we have looked at ways of characterizing the distribution of a single variable, and testing hypotheses about the population based on a sample.
- We're now moving on to the ways in which two variables can be examined together.
- This comes up a lot in research!

Relationships between variables

- You might want to know:
  - To what extent the change in a patient's blood pressure is linked to the dosage level of a drug they've been given.
  - To what degree the number of plant species in an ecosystem is related to the number of animal species.
  - Whether temperature affects the rate of a chemical reaction.

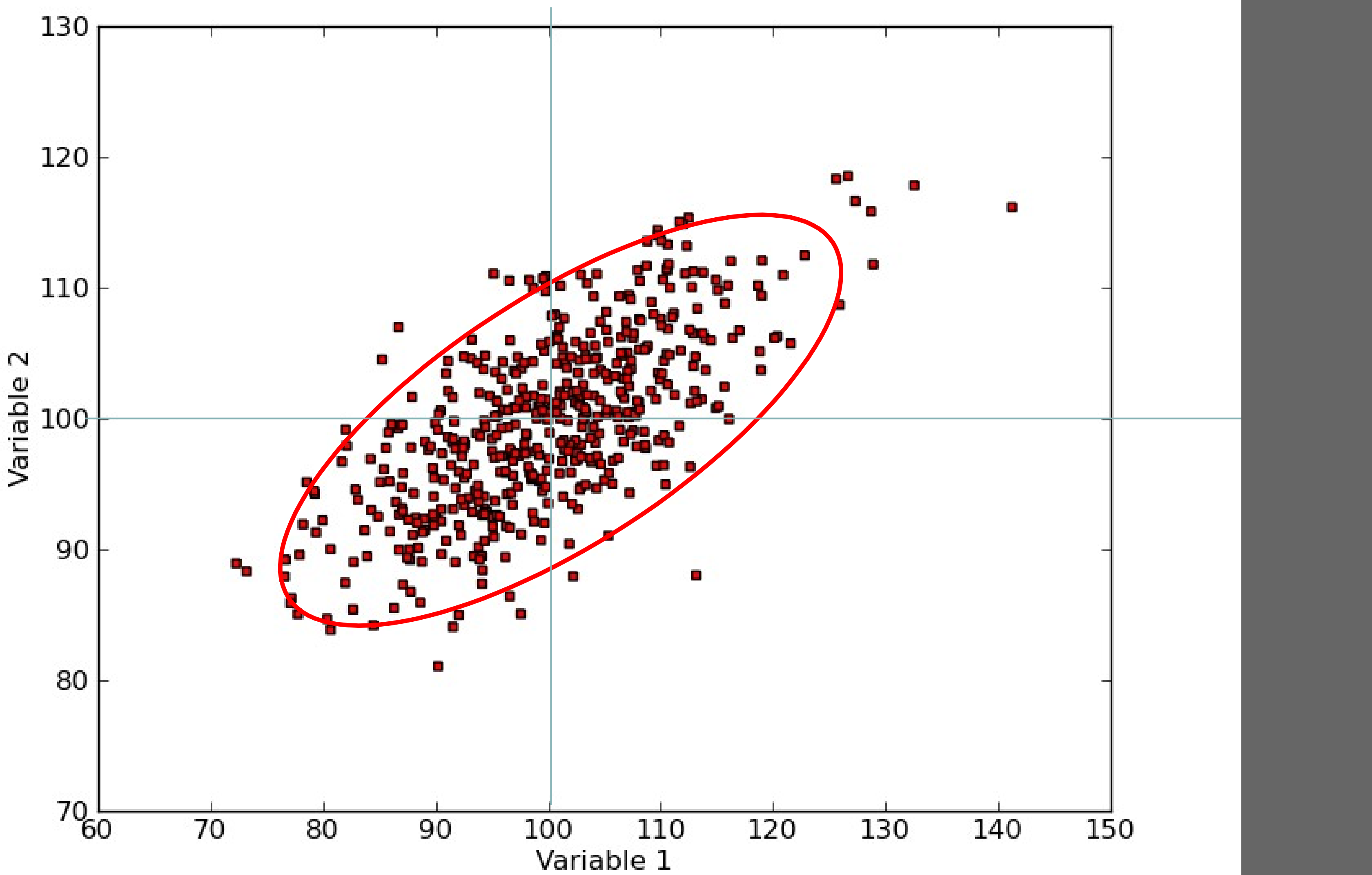
Relationships between variables

- We assume that for each case we have at least two real-valued variables.
- For example: both height (cm) and weight (kg) recorded for a group of people.
- The standard way to display this is using a dot plot or scatterplot.

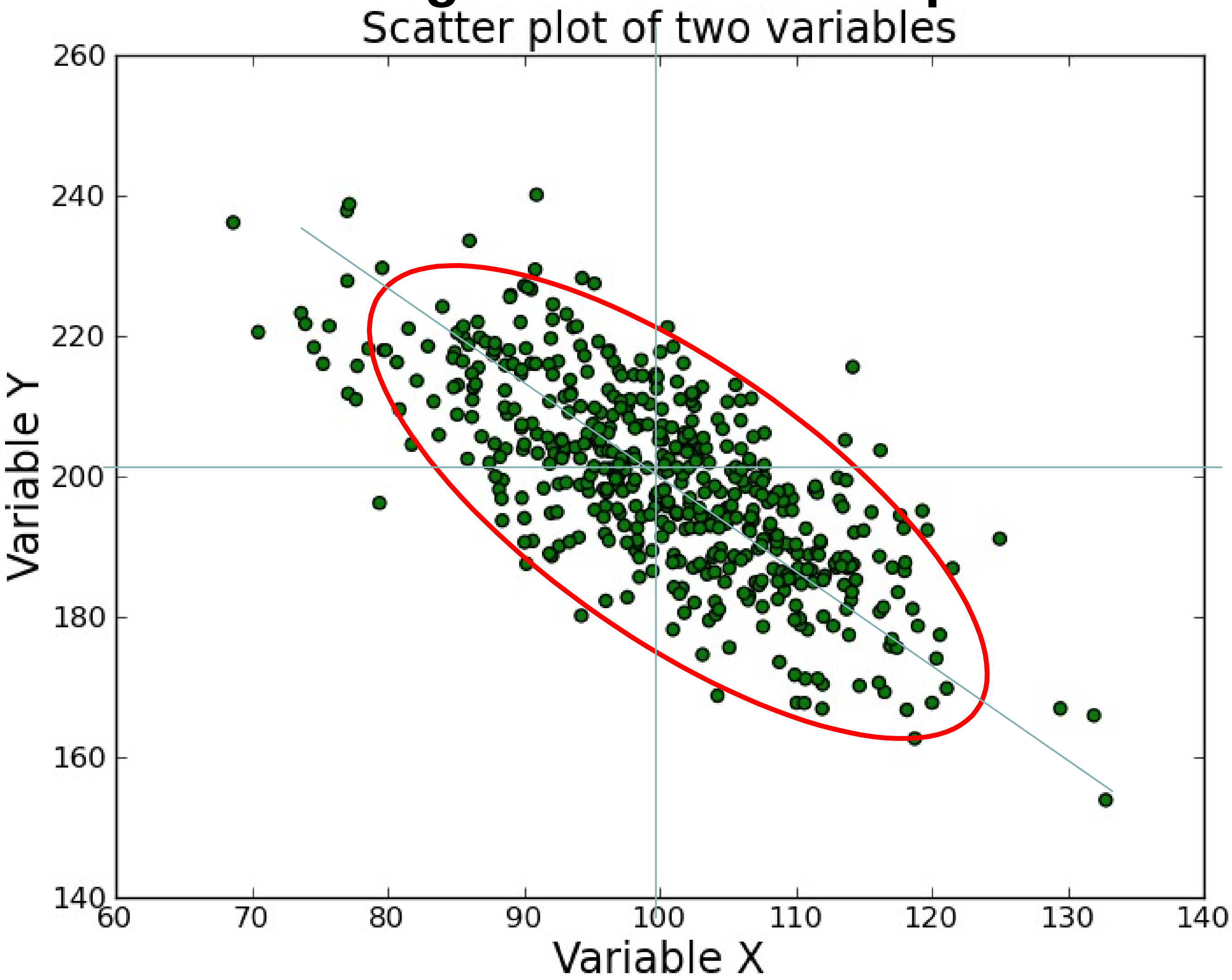


130
120
110
100
90
80
70
60
70
80
90
100
110
120
130
140
150
Variable 1
Variable 2

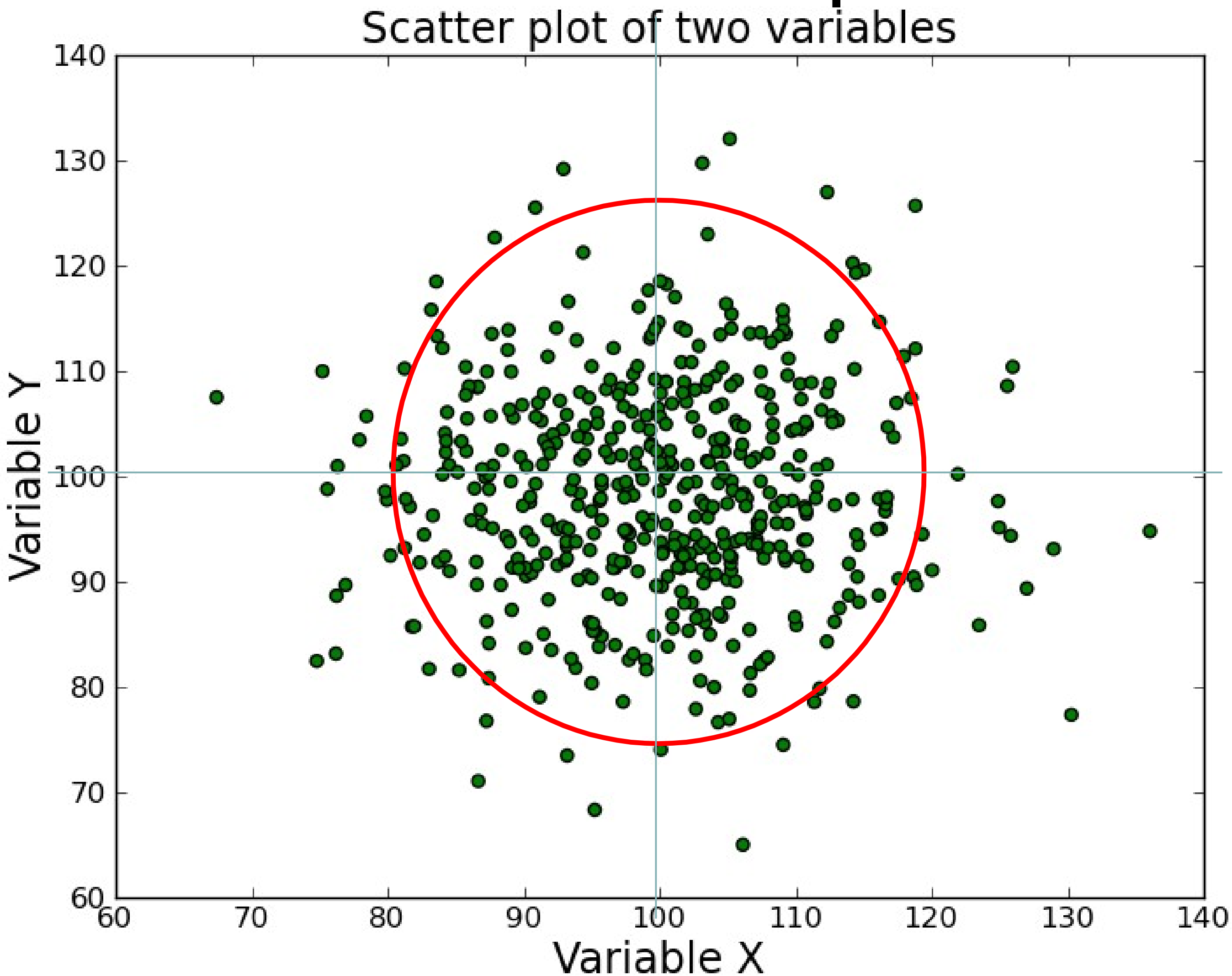
Positive Relationship



Negative Relationship



No Relationship



Measuring relationships?

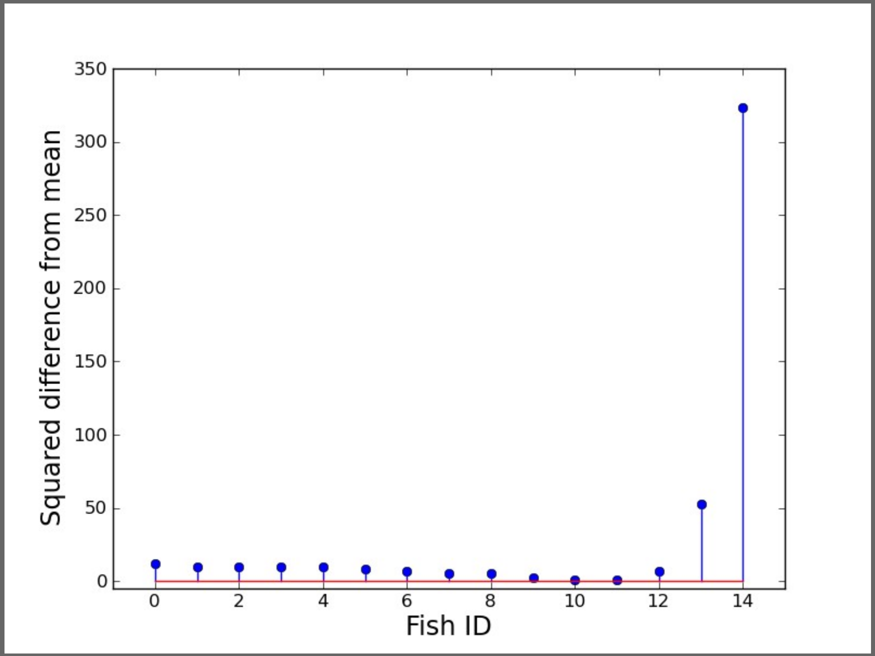
- We're going to need a way of measuring whether one variable changes when another one does.
- Another way of putting it: when we know the value of variable A, how much information do we have about variable B's value?

Recap of the one-variable case

- Perhaps we can borrow some ideas about the way we characterized variation in the single-variable case.
- With one variable, we start out by finding the mean, which is also the expectation of the distribution.

Sum of the squared deviations

- Then find the sum of all the squared deviations from the mean.
- This gives us a measure of the *total* variation: it will be higher for bigger samples.



$$\mathrm{SS} = \sum (x_i - \bar{x})^2$$

Sum of the squared deviations

-
  - We divide this total by N, the sample size...
  - (or N-1 if we are using our sample to estimate the value for a wider population)
  - to get...

$$variance(x) = \frac{\sum(x_i - \bar{x})(x - \bar{x})}{N - 1}$$

The variance

- This is a good measure of how much variation exists in the sample, normalized by sample size.
- It has the nice property of being additive.
- The only problem is that the variance is measured in units squared.
- So we take the square root to get...

The standard deviation

-
  - This is another measure of the "average spread" of the distribution.
  - It is now measured in the original units.
  - The sample standard deviation (division by N-1) is a good estimate for the population standard deviation.

$$s(x) = \sqrt{\frac{\sum(x_i - \bar{x})^2}{N-1}}$$

The standard deviation

- With a good estimate of the population SD, we can reason about the *standard deviation of the distribution of sample means*.
- That's a number that gets smaller as the sample sizes get bigger.
- To calculate this from the sample standard deviation we divide through by the square root of N, the sample size, to get...

The standard error

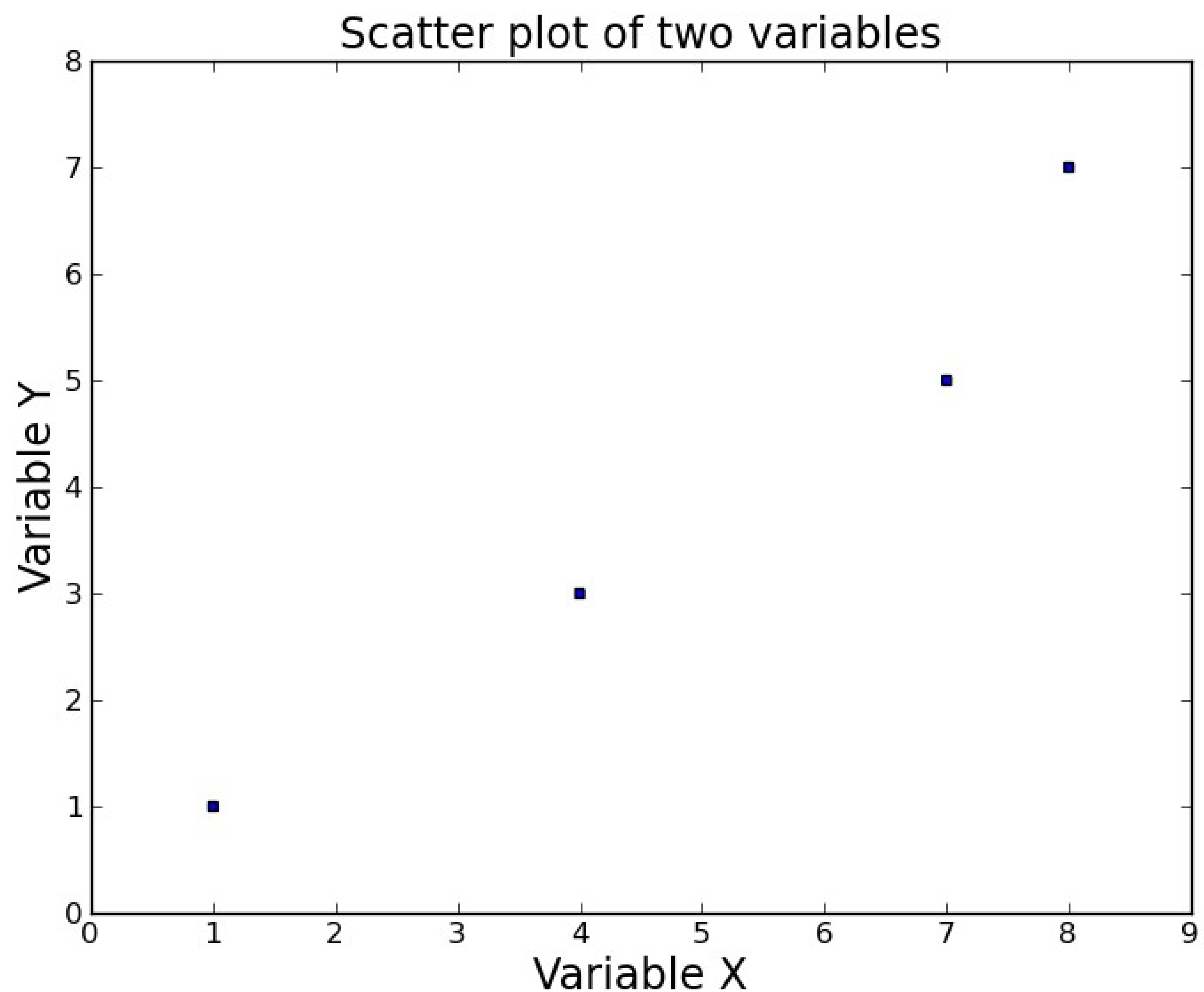
- This measures the precision of our estimation of the true population mean.
- Plus or minus 1.96 standard errors from the sample mean should capture the true population mean 95% of the time.
- The standard error is itself the *standard deviation* of the distribution of the sample means.

Variation in *one* variable

- So, these four measures all describe aspects of the variation in a single variable:
  a. Sum of the squared deviations
  b. Variance
  c. Standard deviation
  d. Standard error
- Can we adapt them for thinking about the way in which two variables might vary together?

Two variable example

- Consider a small sample of four records with two variables recorded, X and Y.
- X and Y could be anything.
- Let's say X is hours spent fishing, Y is number of fish caught.
- Values: (1,1) (4,3) (7,5) (8,7).

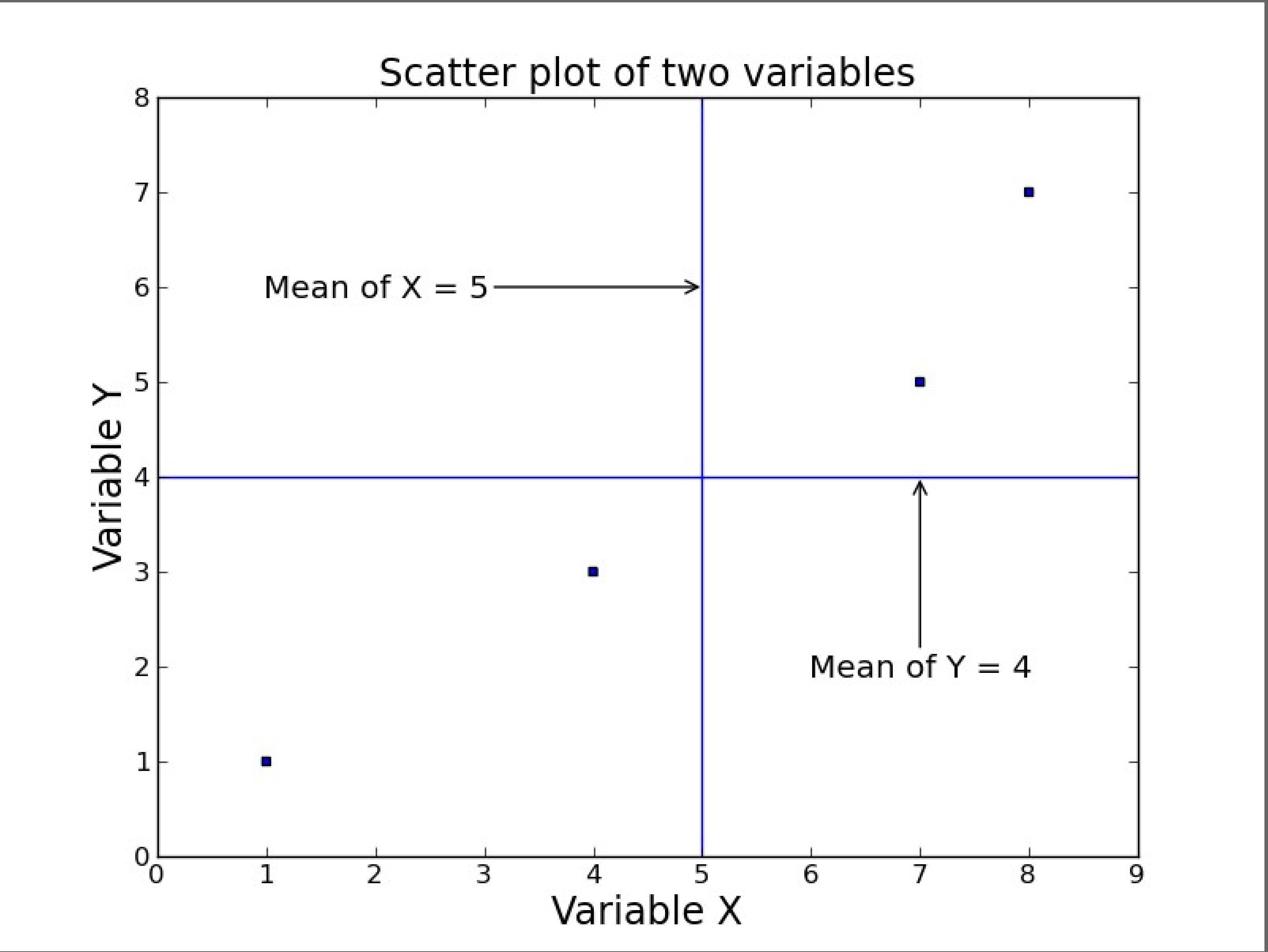


Scatter plot of two variables
Variable Y
Variable X
0
1
2
3
4
5
6
7
8
9

Two variable example

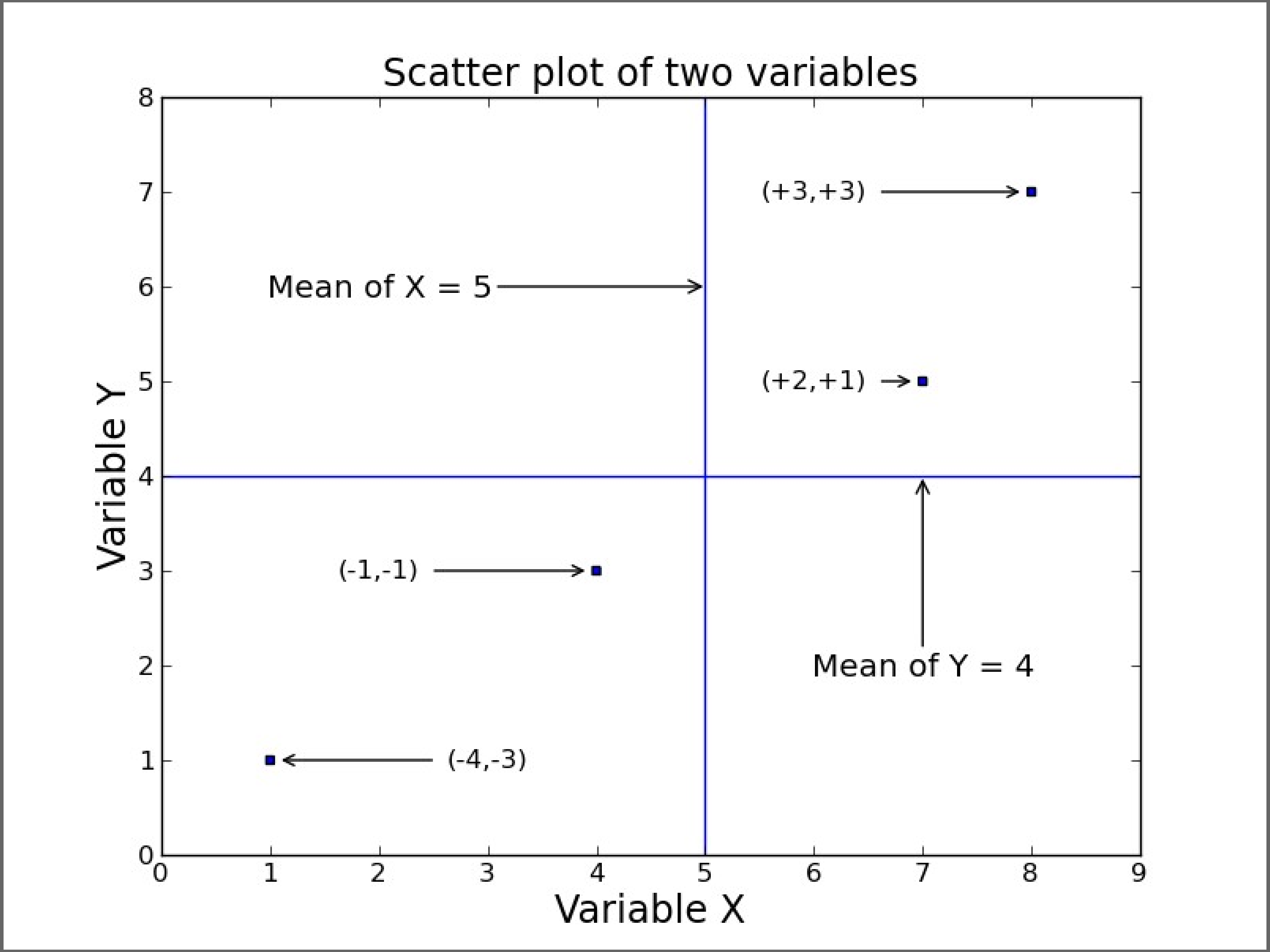
- We can see there's a positive relationship but how should we quantify it?
- We can start by calculating the mean for each variable.

- Mean of X = 5.
- Mean of Y = 4.


Scatter plot of two variables
Mean of X = 5
Mean of Y = 4
Variable Y
Variable X
0
1
2
3
4
5
6
7
8
9

Two variable example

- In the one-variable case, the next step would be to find the deviations from the mean and then square them.
- In the two-variable case, we need to connect the variables.
- We do this by ***multiplying each X-deviation by its associated Y-deviation***


Scatter plot of two variables
Variable Y
Variable X
(+3,+3)
Mean of X = 5
(+2,+1)
(-1,-1)
Mean of Y = 4
(-4,-3)
0
1
2
3
4
5
6
7
8
9

Calculating covariance

- -4 x -3 = 12
- -1 x -1 = 1
- 2 x 1 = 2
- 3 x 3 = 9
- Total of the cross-multiplied deviates = 24.

$$\sum_i \left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)$$

In Formulae

- Variance:

$$V[X] = E[(X - \bar{X})^2]$$

$$V[X] = 1/(N-1)\sum_i (X_i - \bar{X})^2$$

- Covariance:

$$Cov[X,Y] = E[(X - \bar{X})(Y - \bar{Y})]$$

$$Cov[X,Y] = 1/(N-1)\sum_i (X_i - \bar{X})(Y_i - \bar{Y})$$

- Note Bessel's correction in the sample versions ...

Calculating covariance

- Divide by N if this is the population, or divide by N-1 if this is a sample and we're estimating the population.
- If this was the population, we get 24 / 4 = 6.
- If this is a sample and we want to estimate the true population value, we get 24 / 3 = 8.
- Assuming this is a sample, we have a measure of 8 "fish-hours" for the estimated covariance between X and Y.

Properties of covariance

- You might remember the formula for the variance of the sum of two independent random variates. If they are correlated we instead have:

$$V[X+Y]=V[X]+V[Y]+Cov[X,Y]$$

- Also, Cov [.,.] is linear:

$$Cov[X+Y,Z]=Cov[X,Z]+Cov[Y,Z]$$
$$Cov[aX,Y]=a\,Cov[X,Y]$$

Interpreting covariance?

- Covariance has some of the properties we want: positive, negative, and absent relationships can be recognized.
- But "fish-hours" is difficult to interpret.
- Can we scale it in some way? ... Well, the standard deviation of X is in hours, and the standard deviation of Y is in fish...

The correlation coefficient

- So, if we take the covariance and divide by the two standard deviations, we obtain a dimensionless measure:

$$r = \frac{Cov[X, Y]}{\sqrt{V[X]}\sqrt{V[Y]}}$$

- So we obtain a correlation coefficient
- ... or more technically: a Pearson product moment correlation coefficient

The correlation coefficient

- What magnitude will the measure have?
- You can't get anything more strongly related than something with itself (or more strongly anti-related than with minus itself)
- Recall that coveriance of X with itself is just variance

The correlation coefficient

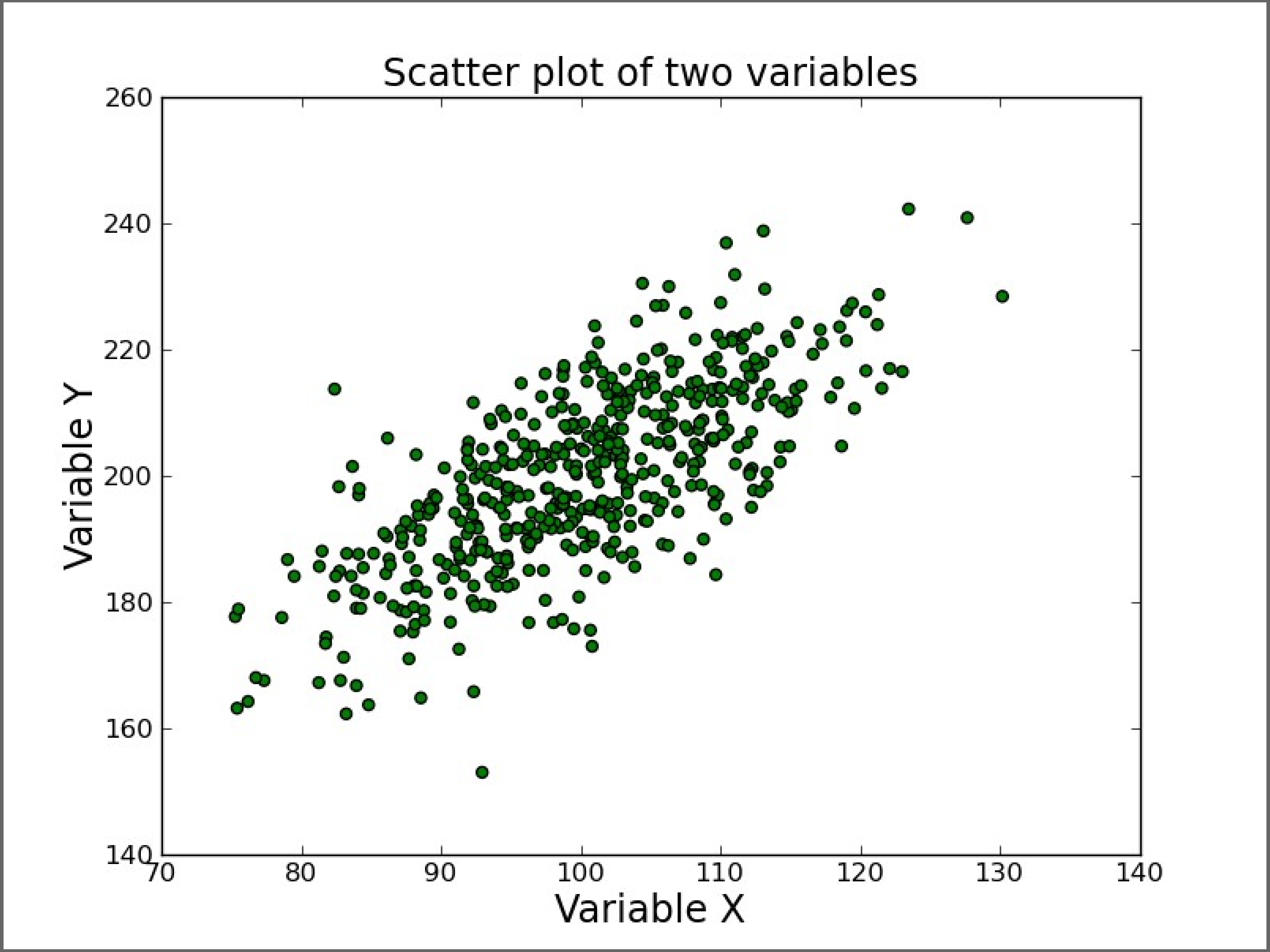
- This measure runs between -1 and 1, and represents negative, absent, and positive relationships.
- It's often referred to as "r".
- It's extremely popular as a way of measuring the strength of a **linear** relationship.

The correlation coefficient

- In our case, the sample standard deviations of X and Y are 3.16 and 2.58 respectively.
- $r = 8 / (3.16 * 2.58) = 0.98$.
- This is a very strong positive relationship, as we can see from the original scatter plot.

Another example

- Invented data set where X is normally distributed, mean = 100, SD = 10.
- For each of 500 cases, Y is equal to X plus a normal variate, mean = 100, SD = 10.
- Y and X are clearly related, but there's also a significant part of the variation in Y that has nothing to do with X.


Scatter plot of two variables
260
240
220
200
180
160
140
70
80
90
100
110
120
130
140
Variable Y
Variable X

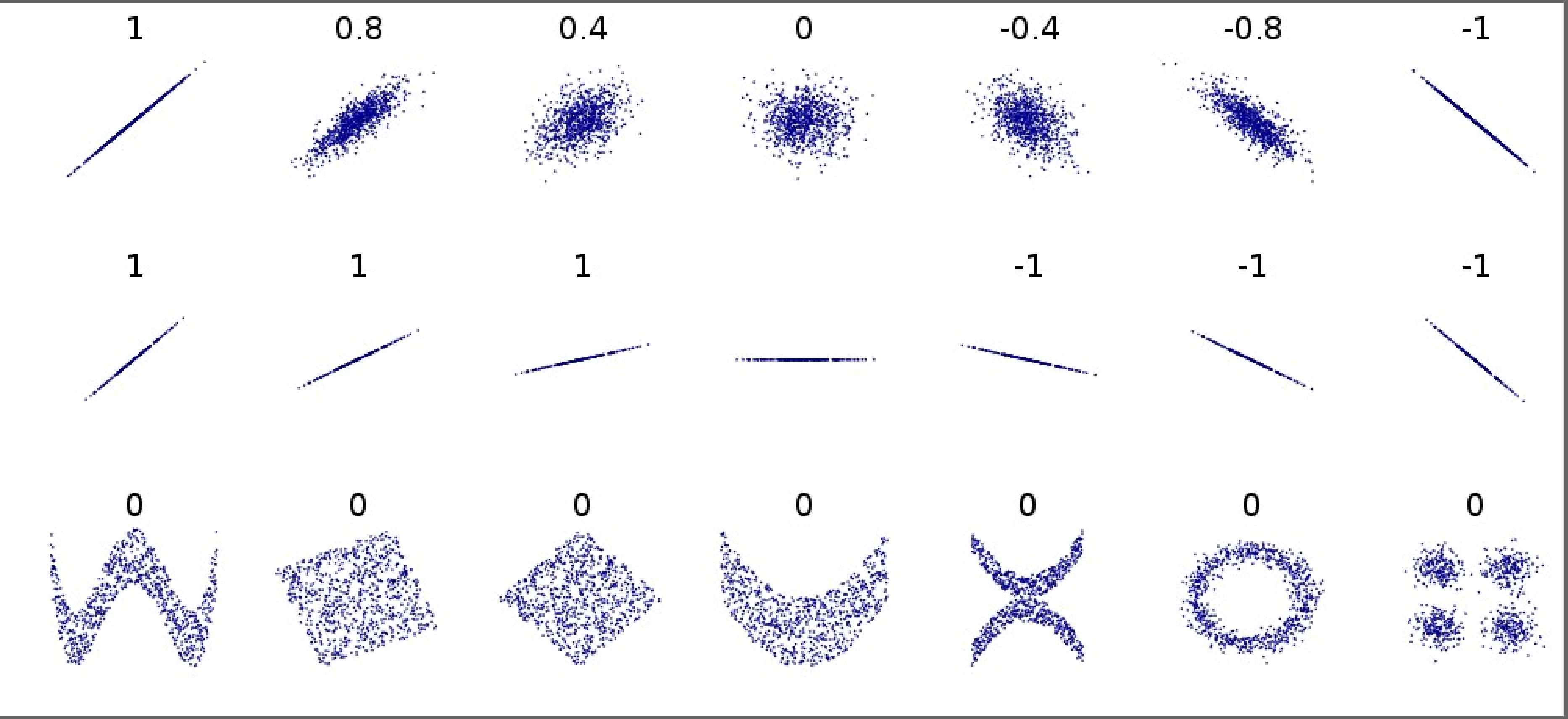
Calculating the correlation coefficient

- In Python, we use `pylab.corrcoef(a,b)` where a and b are lists (returns a matrix).
- In R, it's `cor(a,b)` where a and b are variable names. You can also use `cor(data)` to get a matrix showing the correlation of everything with everything else in the data frame.
- For the previous example, r = 0.72.

Interpreting correlation coefficients

- **0.0 - 0.3:** Weak relationship; may be an artefact of the data set and in fact there is no relationship at all.
- **0.3 - 0.6:** Moderate relationship; you might be on to something, or you might not.
- **0.6 - 0.9:** Strong relationship; you can be confident that these two variables are connected in some way.
- **0.9 - 1.0:** Very strong relationship; variables are almost measuring the same thing.

Correlations measure linear relationships only



Correlation is not causality

- Of course, just because X and Y are correlated does not mean that X causes Y.
- They could both be caused by some other factor Z.
- Y might cause X instead.
- Low correlations might result from no causal linkage, just sampling noise.

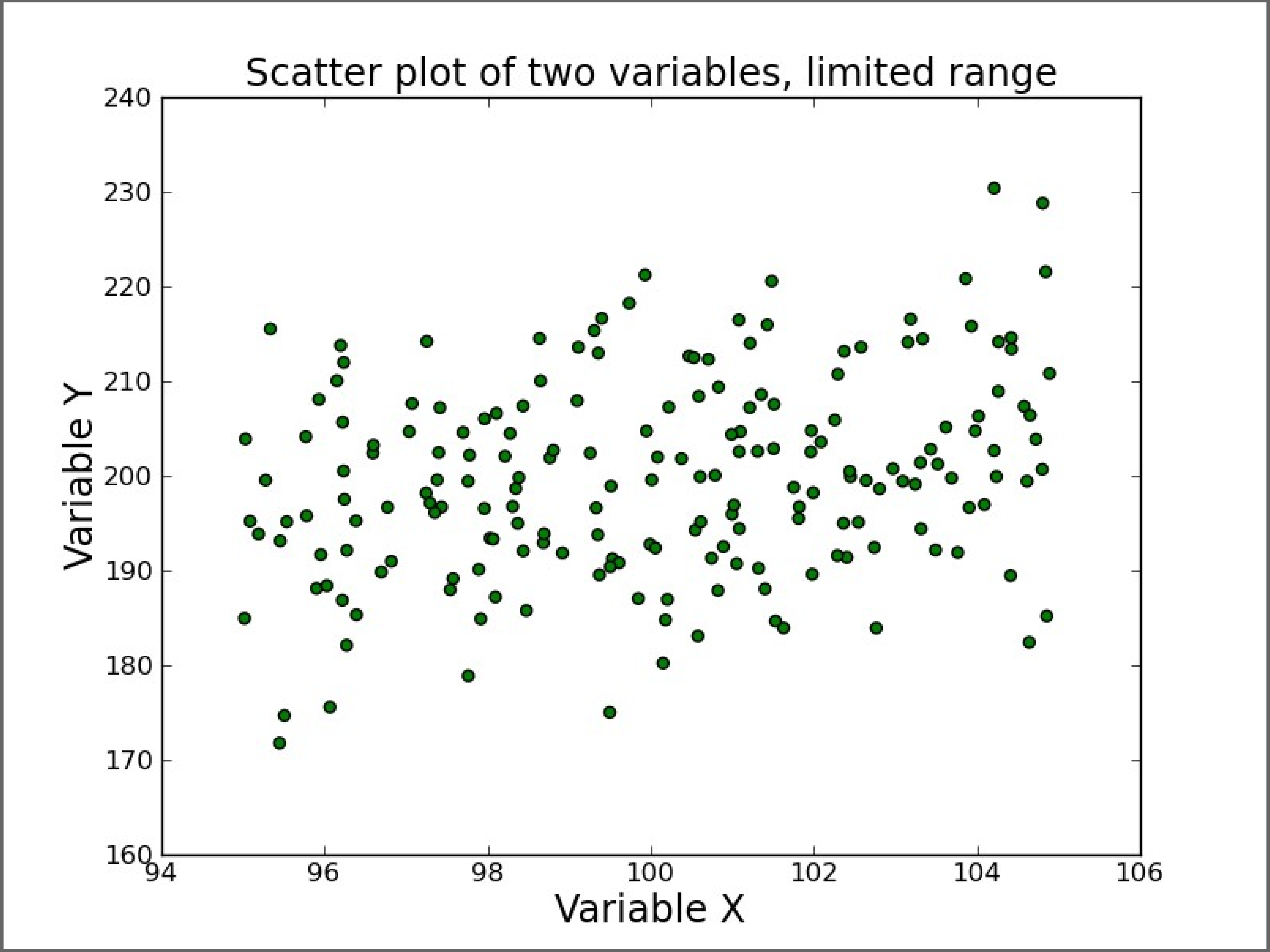
Range effects

- Two variables can be strongly related across the whole of their range, but with no strong relationship in a limited subset of that range.
- Consider the relationship between price and top speed in cars: broadly positive.
- But if we look only at very expensive cars, the two values may be uncorrelated.

Range effects

- Consider the X, Y scatterplot from a few slides back.

- If we limit the range of X to between 95 and 105, the correlation coefficient is only 0.27.


Scatter plot of two variables, limited range
Variable Y
240
230
220
210
200
190
180
170
160
Variable X
94
96
98
100
102
104
106

Confidence intervals

- Confidence intervals for correlation coefficients can be calculated in much the same way as for means.
- As an exercise: using the Python code for this lecture, try drawing samples of size 50 repeatedly from the X, Y distribution and look at the range of values for r you get.

Permutation tests

- Another method is via permutation tests. This is a way to judge noise from small sample sizes.
- Take the data for (X_i, Y_i) and consider permutations $(X_{\pi i}, Y_i)$. You can treat them as a sample which gives you a null hypothesis.
- Last step is to test whether your actual data is likely to have been drawn from the sample.

Information about Y from X

- If I know the correlation between two things, what does knowing one thing tell me about the value of the other?
- Consider the X, Y example. X was a random variable, and Y was equal to X plus another random variable from the same distribution.
- The correlation worked out at about 0.7. Why?

R-squared

- Turns out that if we square the correlation coefficient we get a direct measure of the proportion of the variance explained.

- In our example case we know that X explains exactly 50% of the variance in Y.

- The square root of $0.5 \approx 0.71$.

R-squared

- r = 0.3 explains 9% of the variance.
- r = 0.6 explains 36% of the variance.
- r = 0.9 explains 81% of the variance.
- "R-squared" is a standard way of measuring the proportion of variance we can explain in one variable using one or more other variables. This connects with the next lecture on ANOVA.

Python code

- The Python code used to produce the graphs and correlation coefficients in this lecture is available here.