Exercice corrigé ACP-ACP normée

Présenté par : M. HAMMAD

soit :

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 2 \\ 3 & 3 \\ 4 & 2 \end{pmatrix}$$

Le tableau des donné correspondant à des mesures effectuée sur 5 individus de poids statistique égaux pour les deux variables X^1, et X^2.

solution

$$X = A' \Longleftrightarrow X = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 2 \end{pmatrix}.$$

1. Centrée le tableau
   (a) Le vecteur moyen est $\overline{X} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.
   (b) La matrice centrée X_c est :
   $$X_c = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ -1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$
2. La matrice variance covarience V est :
   $V = X_c D_p X_c^{'}$, tel que $D_p = \frac{1}{5} I_2$,
   alors $V = \frac{1}{5} \begin{pmatrix} 10 & 3 \\ 3 & 2 \end{pmatrix}$.
3. Les éléments propres sont :

   (a) Pour u_1
   $V u_1 = \lambda_1 u_1$ tel que $u_1 = \frac{\sqrt{10}}{10} \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \lambda_1 = \frac{11}{5}$.
   (b) Pour u_2
   $V u_2 = \lambda_2 u_2$ tel que $u_2 = \frac{\sqrt{10}}{10} \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \lambda_2 = \frac{1}{5}$.
   $\lambda_1 = \frac{11}{5} > \lambda_2 = \frac{1}{5}$.

4. La part d'inertie P tel que :
   $P = \Delta u_1 + \Delta u_2$
   $$part(\Delta u_1) = \frac{\lambda_1}{Tr(VM)} = \frac{\frac{11}{5}}{\frac{12}{5}} \simeq 0.92 = 92\%.$$
   $$part(\Delta u_2) = \frac{\lambda_2}{Tr(VM)} = \frac{\frac{1}{5}}{\frac{12}{5}} \simeq 0.08 = 8\%.$$
   donc $part(P) \simeq 100\%$ la première droite contien plus d'informations.

5. Les composants principales :
   On a : $\|u_1\| = \|u_2\| = 1$
   $$C^1 = X'_c M u_{1n} = \frac{\sqrt{10}}{10}\begin{pmatrix} -2 & -1 \\ -1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ -1 \end{pmatrix} = \frac{\sqrt{10}}{10}\begin{pmatrix} 7 \\ 3 \\ 0 \\ -4 \\ -6 \end{pmatrix}.$$
   $$C^2 = X'_c M u_{2n} = \frac{\sqrt{10}}{10}\begin{pmatrix} -2 & -1 \\ -1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \frac{\sqrt{10}}{10}\begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 2 \end{pmatrix}.$$
   Par la suite les individus sont :
   $A = \left(\frac{7\sqrt{10}}{10}, \frac{\sqrt{10}}{10}\right), B = \left(\frac{3\sqrt{10}}{10}, \frac{-\sqrt{10}}{10}\right), C = (0, 0\ , D = \left(\frac{-4\sqrt{10}}{10}, \frac{-2\sqrt{10}}{10}\right),$
   $E = \left(\frac{-6\sqrt{10}}{10}, \frac{2\sqrt{10}}{10}\right).$

   Les coordonnés des variables :
   $$\frac{D_p(x^1, C^1)}{\|C^1\|_{Dp}} = \sqrt{\lambda_1}[u_{1n}]_1 = \sqrt{\frac{11}{5}}\frac{\sqrt{10}}{10}\begin{pmatrix} -3 \\ -1 \end{pmatrix}_1 = \frac{-3\sqrt{22}}{10}.$$
   $$\frac{D_p(x^1, C^2)}{\|C^2\|_{Dp}} = \sqrt{\lambda_2}[u_{2n}]_1 = \sqrt{\frac{1}{5}}\frac{\sqrt{10}}{10}\begin{pmatrix} 1 \\ -3 \end{pmatrix}_1 = \frac{\sqrt{2}}{10}.$$

$$\frac{D_p(x^2, C^1)}{\|C^1\|_{Dp}} = \sqrt{\lambda_1}[u_{1n}]_2 = \sqrt{\frac{11}{5}}\frac{\sqrt{10}}{10}\begin{pmatrix} -3 \\ -1 \end{pmatrix}_2 = \frac{-\sqrt{22}}{10}.$$

$$\frac{D_p(x^2, C^2)}{\|C^2\|_{Dp}} = \sqrt{\lambda_2}[u_{2n}]_2 = \sqrt{\frac{1}{5}}\frac{\sqrt{10}}{10}\begin{pmatrix} 1 \\ -3 \end{pmatrix}_2 = \frac{-3\sqrt{2}}{10}.$$

$$x^1 = \left(\frac{-3\sqrt{22}}{10}, \frac{\sqrt{2}}{10}\right), x^2 = \left(\frac{-\sqrt{22}}{10}, \frac{-3\sqrt{2}}{10}\right).$$

Les coordonnés des variables sur le plans principal :

$$M(e_1, u_{1n}) =< e_1, u_{1n} >= \frac{\sqrt{10}}{10}(1,0)\begin{pmatrix} -3 \\ -1 \end{pmatrix} = \frac{-3\sqrt{10}}{10}.$$

$$M(e_1, u_{2n}) =< e_1, u_{2n} >= \frac{\sqrt{10}}{10}(1,0)\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \frac{\sqrt{10}}{10}.$$

$$M(e_2, u_{1n}) =< e_2, u_{1n} >= \frac{\sqrt{10}}{10}(0,1)\begin{pmatrix} -3 \\ -1 \end{pmatrix} = \frac{-\sqrt{10}}{10}.$$

$$M(e_2, u_{1n}) =< e_2, u_{2n} >= \frac{\sqrt{10}}{10}(0,1)\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \frac{-3\sqrt{10}}{10}.$$

$$I = \left(\frac{-3\sqrt{10}}{10}, \frac{\sqrt{10}}{10}\right), II = \left(\frac{-\sqrt{10}}{10}, \frac{-3\sqrt{10}}{10}\right).$$

ACP normée

1. Calcul de R la matrice corrélation
   $R = ZD_pZ'$ tel que $D_p = \frac{1}{5}I_2$ et $Z = D_{\frac{1}{\sigma}}X_c$

$$D_{\frac{1}{\sigma}} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ & \frac{\sqrt{10}}{2} \end{pmatrix}$$

alors
$$\begin{aligned} R &= D_{\frac{1}{\sigma}} X_c D_p X_c^{'} D_{\frac{1}{\sigma}} \\ &= D_{\frac{1}{\sigma}} V D_{\frac{1}{\sigma}} \\ &= \begin{pmatrix} 1 & \frac{3\sqrt{5}}{10} \\ \frac{3\sqrt{5}}{10} & 1 \end{pmatrix} \end{aligned}$$

2. Les élément propres :

(a) Pour u_1

$Ru_1 = \mu_1 u_1$ tel que $u_1 = \dfrac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mu_1 = \dfrac{10+3\sqrt{5}}{10}$.

(b) Pour u_2

$Ru_2 = \mu_2 u_2$ tel que $u_2 = \dfrac{\sqrt{2}}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \mu_2 = \dfrac{10-3\sqrt{5}}{10}$

$\mu_1 = \dfrac{10+3\sqrt{5}}{10} > \mu_2 = \dfrac{10-3\sqrt{5}}{10}$.

La part d'inertie de plan P tel que :
$P = \Delta u_1 + \Delta u_2$

$$part(\Delta u_1) = \frac{\mu_1}{Tr(VM)} = \frac{\frac{10+3\sqrt{5}}{10}}{2} \simeq 0.84 = 84\%$$

$$part(\Delta u_2) = \frac{\mu_2}{Tr(VM)} = \frac{\frac{10-3\sqrt{5}}{10}}{2} \simeq 0.16 = 16\%$$

donc $part(P) = 84\%$ la première droite contien plus d'informations.

Les composants principales :
On a : $\|u_1\| = \|u_2\| = 1$

$$C^1 = X_c^{'} M u_{1n} = \frac{\sqrt{2}}{2} \begin{pmatrix} -\sqrt{2} & \frac{-10}{2} \\ \frac{-\sqrt{2}}{2} & 0 \\ 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{10}}{2} \\ \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\sqrt{10}}{10} \begin{pmatrix} \frac{-\sqrt{10}-2\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{10}+\sqrt{2}}{2} \\ \sqrt{2} \end{pmatrix}.$$

$$C^2 = X_c' M u_{2n} = \frac{\sqrt{2}}{2}\begin{pmatrix} -\sqrt{2} & \frac{-10}{2} \\ \frac{-\sqrt{2}}{2} & 0 \\ 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{10}}{2} \\ \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} \frac{-\sqrt{10}+2\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{10}-\sqrt{2}}{2} \\ -\sqrt{2} \end{pmatrix}.$$

Par la suite les individus sont :

$$A = \left(\frac{-2-\sqrt{5}}{2}, \frac{2-\sqrt{5}}{2}\right), B = \left(\frac{-1}{2}, \frac{1}{2}\right), C = (0,0), D = \left(\frac{1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right),$$

$E = (1, -1)$.

Les coordonnés des variables :

$$\frac{D_p(x^1, C^1)}{\|C^1\|_{Dp}} = \sqrt{\mu_1}[u_{1n}]_1 = \sqrt{\frac{10+3\sqrt{5}}{10}}\frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_1 = \sqrt{\frac{10+3\sqrt{5}}{10}}\frac{\sqrt{2}}{2}.$$

$$\frac{D_p(x^1, C^2)}{\|C^2\|_{Dp}} = \sqrt{\mu_2}[u_{2n}]_1 = \sqrt{\frac{10-3\sqrt{5}}{10}}\frac{\sqrt{2}}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}_1 = -\sqrt{\frac{10-3\sqrt{5}}{10}}\frac{\sqrt{2}}{2}.$$

$$\frac{D_p(x^2, C^1)}{\|C^1\|_{Dp}} = \sqrt{\mu_1}[u_{1n}]_2 = \sqrt{\frac{10+3\sqrt{5}}{10}}\frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_2 = \sqrt{\frac{10+3\sqrt{5}}{10}}\frac{\sqrt{2}}{2}.$$

$$\frac{D_p(x^2, C^2)}{\|C^2\|_{Dp}} = \sqrt{\mu_2}[u_{2n}]_2 = \sqrt{\frac{10-3\sqrt{5}}{10}}\frac{\sqrt{2}}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}_2 = \sqrt{\frac{10-3\sqrt{5}}{10}}\frac{\sqrt{2}}{2}.$$

$$x^1 = \left(\sqrt{\frac{10+3\sqrt{5}}{10}}\frac{\sqrt{2}}{2}, -\sqrt{\frac{10-3\sqrt{5}}{10}}\frac{\sqrt{2}}{2}\right), x^2 = \left(\sqrt{\frac{10+3\sqrt{5}}{10}}\frac{\sqrt{2}}{2}, \sqrt{\frac{10-3\sqrt{5}}{10}}\frac{\sqrt{2}}{2}\right)$$

Les coordonnés des variables sur$\triangleright le plans principal$:

$$\mathrm{M}(\mathrm{e}_1, u_{1n}) =< e_1, u_{1n} >= \frac{\sqrt{2}}{2}(1,0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}.$$

$$M(e_1, u_{2n}) =< e_1, u_{2n} >= \frac{\sqrt{2}}{2}(1,0)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{-\sqrt{2}}{2}$$

$$M(e_2, u_{1n}) =< e_2, u_{1n} >= \frac{\sqrt{2}}{2}(0,1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}.$$

$$M(e_2, u_{1n}) =< e_2, u_{2n} >= \frac{\sqrt{2}}{2}(0,1)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}. \quad I = \left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right), II = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$

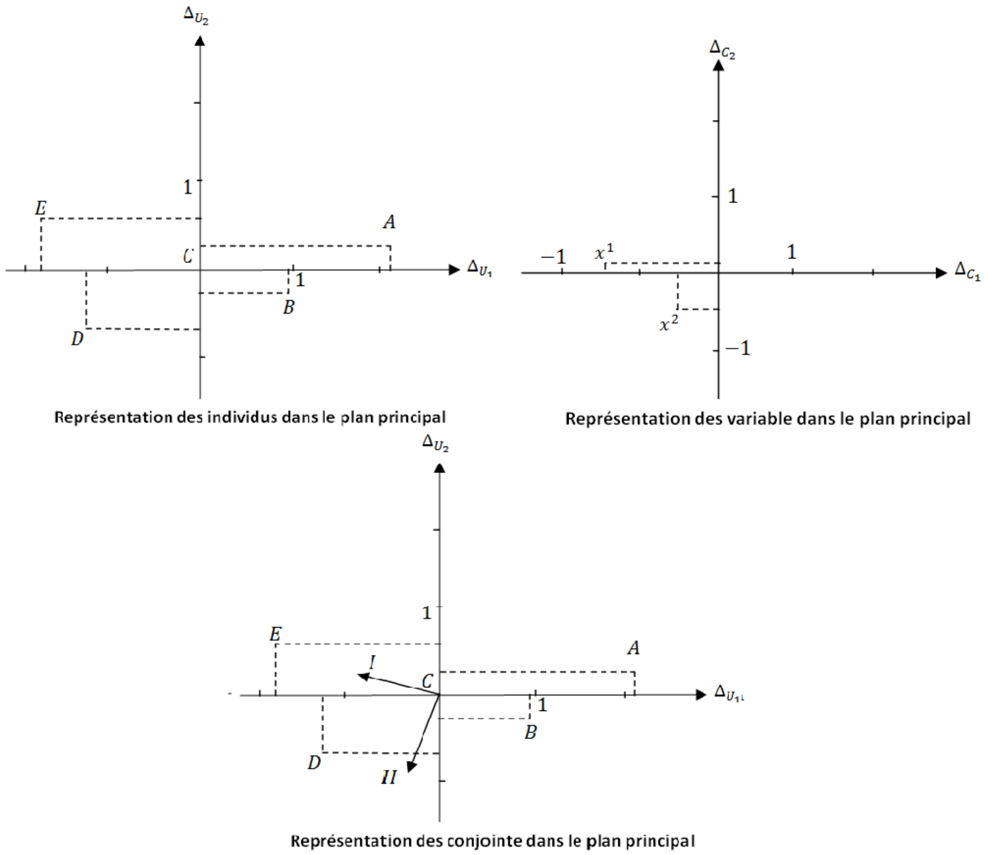

FIGURE 1 – Représentation ACP.

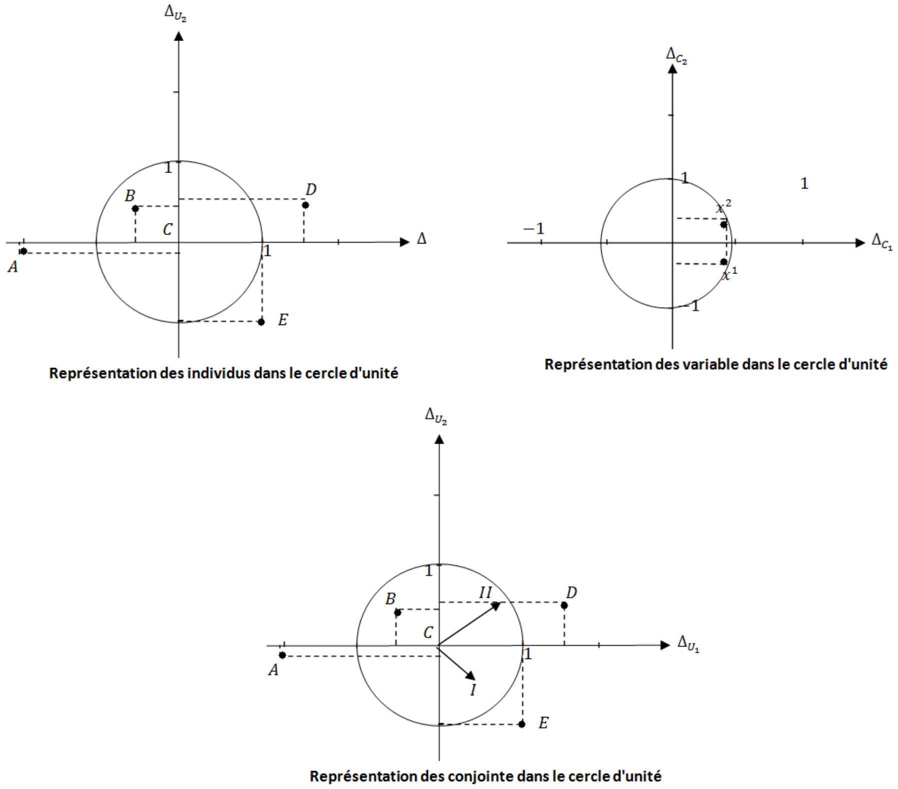

FIGURE 2 – Représentation ACP normé.