Calculation of probabilities

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1 Introduction

- Combinatorial analysis is a branch of mathematics that studies how to count objects.
- So, This branch of mathematics studies the number of different ways of arranging p things of a set E of cardinality n is called combinatorial analysis.
- It provides counting methods that are particularly useful in probability theory.It provides counting methods that are particularly useful in probability theory.
- The so-called combinatorial probabilities constantly use the formulas of combinatorial analysis developed in this chapter.
- For example, interesting applications of the latter is the demonstration of the development of Newton's binomial used in the calculation of the probabilities of a binomial law.

2 Combinatorial analysis tools

In this section, we give the most important tools of this branch. Given a set E of n objects (n elements),

2.1 Arrangements

Definition 1 *An arrangement of p objects of E is a sequence of p objects taken from among the n objects of E.*

2.2 Number of arrangements

We can distinguish two types of arrangements as following :

2.2.1 Arrangements without repetition

The p objects of E are all different from each other. In this case,

Theorem 1 *The number of arrangements of p objects taken from n (or the p-arrangements of E), denoted by A_n^p, is equal to*

$$A_n^p = n \times (n-1) \times ... \times (n-p+1)$$

Proof: 1 *For $(x_1, x_2, ..., x_p)$ an arrangement of E,*

- *We have n possible choices for x_1;*
- *We have $n-1$ possible choices for x_2;*
- *. . .;*
- *We have $n-p+1$ possible choices for x_p.*

So, we can form $n \times (n-1) \times ... \times (n-p+1)$ arrangements possible.

2.2.2 Arrangements with repetition

The p objects of E are not all different from each other. In this case,

Theorem 2 *The number of arrangements of p objects taken from n (or the p-arrangements of E) is equal to $A_n^p = n^p$.*

Proof: 2 *For $(x_1, x_2, ..., x_p)$ an arrangement of E,*

- *We have n possible choices for x_1;*
- *We have n possible choices for x_2;*
- *...;*
- *We have n possible choices for x_p.*

So, we can form $A_n^p = n \times n \times ... \times n = n^p$ arrangements possible.

Remark 1
1. *We necessarily have $1 \leq p \leq n$ and $n, p \in N$ If $n < p$, then A_n^p;*
2. *Two arrangements of p objects are therefore distinct if they differ in the nature of the objects that compose them or in their order in the sequence.*

Example 1
1. *A DNA sequence consists of a sequence of 4 nucleotides $\{A(Adenine), C(Cytosine), G(Guanine) and T(T$ There are different possible arrangements of two nucleotides or dinucleotides with $p = 2$ and $n = 4$.*
2. *The number of 5-letter words (with or without meaning) formed with the 26 letters of the alphabet corresponds to the number of possible arrangements with p=5 and n=26.*
3. *The third in order during a race of 20 horses constitutes one of the possible arrangements with p=3 and n=20.*
4. *In the previous examples, the order of the elements in the sequence is essential. So for the second example, the word "NICHE" is different from the word "CHIEN".*
5. *But in the first two examples, a base or a letter of the alphabet can be found several times while in the third example, the three horses at the finish are necessarily different. It is therefore necessary to distinguish the number of arrangements with repetition and the number of arrangements without repetition (arrangements in the strict sense).*
6. *Regarding the example of the DNA sequence, the number of dinucleotides expected if we make the assumption that a base can be observed several times in the sequence (which actually corresponds to reality) is therefore: $A_4^2 = 4^2 = 16$ possible dinucleotides. The 16 identifiable dinucleotides in a DNA sequence are :*

AA	AC	AG	AT	CA	**CC**	CG	CT
GA	GC	**GG**	GT	TA	TC	TG	**TT**

7. *Regarding the example of the DNA sequence, the expected number of dinucleotides in a sequence if we make the assumption that a base is observed only once is therefore: $A_4^2 = 4(4-1) = 12$ possible dinucleotides*

   Under this constraint, the 12 possible dinucleotides are :

~~AA~~	AC	AG	AT	CA	~~CC~~	CG	CT
GA	GC	~~GG~~	GT	TA	TC	TG	~~TT~~

   This corresponds to the 16 possible arrangements with repetition ($A_n^p = n^p$) from which the 4 dinucleotides (n) resulting from the association of the same base are subtracted.

2.3 Permutations

2.3.1 Permutations without repetition

Definition 2 *Given a set E of n objects, we call permutations of n distinct objects all ordered sequences of n objects or any arrangement n to n of these objects. The number of permutations of n objects is denoted:* $P_n = A_n^n = n(n-1) \times ... \times 2 \times 1 = n!$

2.3.2 Permutations with repetition

Definition 3 *In the event that there exist several repetitions k of the same object among the n objects, the number of possible permutations of the n objects must be related to the numbers of permutations of the k identical objects.*

The number of permutations of n objects is then: $P_n = \frac{n!}{k!}$

2.4 Combinations

If we take the example of the DNA sequence again, the two arrangements of the AC and CA dinucleotides formed two distinct arrangements, the latter will form only a single combination. For combinations, we no longer speak of a sequence or a series since the notion of the order of the objects is no longer taken into account. We are talking about subsets or parts of objects and then about draws with or without discount.

2.4.1 Combinations without discount

Definition 4 *Given a set E of n objects, combinations of p objects are called any set of p objects taken from among the n objects without discount.*

The number of combinations of p objects taken from n is denoted by C_n^p.

Remark 2 *We necessarily have* $1 \leq p \leq n$ *and* $n, p \in N$ *If* $n < p$, *then* $C_n^p = 0$.

Example 2
1. *The random drawing of 5 cards in a deck of 32 (poker hand) is a combination with p=5 and n=32.*
2. *The formation of a delegation of 5 people from a group of 50 constitutes a combination with p=5 and n=50.*

For these two examples, the objects drawn are clearly distinct.

Proposition 1 *The number of combinations of p objects taken from n and without discount is*

$$C_n^p = \frac{n!}{p! \times (n-p)!} \text{ with } 1 \leq p \leq n$$

Proof: 3 *A combination corresponds to* $p!$ *possible arrangements. So,*

$$C_n^p = \frac{A_n^p}{p!}$$

Example 3 *In the context of the example of the DNA sequence, the number of dinucleotides expected without taking into account the order of the bases in the sequence (hypothesis justified in the case of non-coding DNA) is therefore:*

$$C_4^2 = \frac{4!}{2! \times (4-2)!} = \frac{4 \times 3}{2 \times 1} = 6$$

dinucleotides The 6 possible dinucleotides under this hypothesis are :

AC	AG	AT	CG	CT	GT
CA	GA	TA	GC	TC	TG

The two arrangements of each column represent the same conbination.

2.4.2 Combinations with discount

The number of combinations of p object among our discount is:

$$C_{n+p-1}^{p} = \frac{(n+p-1)!}{p! \times (n-1)!}$$

2.5 Properties on the number of combinations

2.5.1 The symmetry

For, $0 \leq p \leq n$,

$$C_n^p = C_n^{n-p} = \frac{n!}{p! \times (n-p)!}$$

It is up to the same to give the combination of the p objects chosen or else that of the (n-p) which are not.

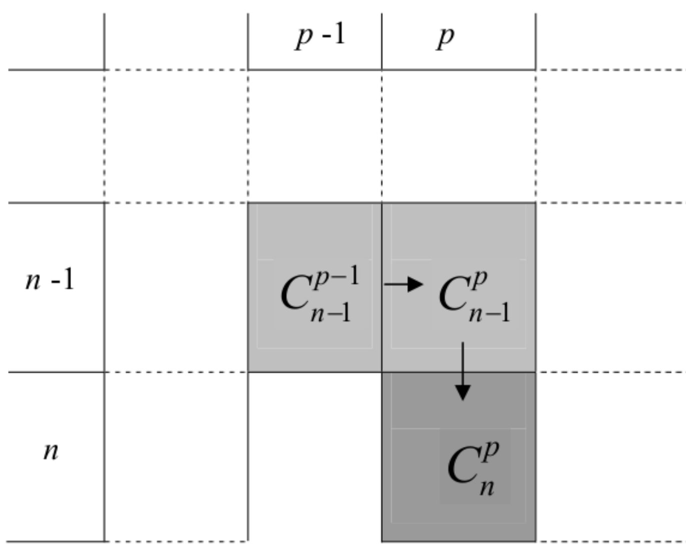
2.5.2 Compound combinations or Pascal's Formula

Among the n elements of a set E, We can partition the set of all the combinations of p elements of E to those containing an element x_0 and those that do not contain it. There are

- C_{n-1}^{p-1} combinations not containing a well-fixed element x_0;
- and C_{n-1}^{p} combinations containing this fixed elemnets x_0.

So, $C_n^p = C_{n-1}^p + C_{n-1}^{p-1}$

The terms of Pascal's triangle result from the direct application of this relation as follows :



2.5.3 Newton's binomial formula

The Pascal triangle makes it possible to obtain by recurrence the numerical coefficients or binomial coefficient of the Newton binomial.

The formula of Newton's binomial corresponds to the decomposition of the different terms of the n^{th} power of the binomial $(a+b)$.

$$\forall (a,b) \in R^2,\ \forall n \in N,\ (a+b)^n = \sum_{p=0}^{n} C_n^p a^{n-p} b^p$$

n	$(a+b)^n$	$\sum_{p=0}^{n} C_n^p a^{n-p} b^p$
0	1	1
1	$(a+b)^1$	a+b
2	$(a+b)^2$	$a^2+2ab+b^2$
3	$(a+b)^3$	$a^3+3a^2b+3ab^2+b^3$
4	$(a+b)^4$	$a^4+4a^3b+6a^2b^2+4ab^3+b^4$
$\vdots$	$\vdots$	$\vdots$

2.6 Partition of a set

Let be E_1, E_2, E_3,..., E_p p subsets of the set E. $\{E_1, E_2, E_3, ..., E_p\}$ is a partition of the set E iff,

1. $\forall i \in \{1,2,3,...,p\};\ E_i \neq \emptyset$;
2. $\forall (i,j) \in \{1,2,3,...,p\}^2;\ E_i \cap E_j = \emptyset$;
3. $\bigcup_{i=1}^{p} E_i = E$.

If $|E_1| = n_1$, $|E_2| = n_2$, $|E_3| = n_3$,..., $|E_p| = n_p$ and $\sum_{i=1}^{p} n_i = n$, then the number of partitions under these conditions is equal to

$$\frac{n!}{n_1! \times n_2! \times n_3! \times ... \times n_p!}$$

3 Terminology of Probability Theory

The following terms in probability theorey help in a better understanding of the concepts of probability.

3.1 Experiment:

A trial or an operation conducted to produce an outcome is called an experiment.

3.2 Random Experiment:

An experiment that has a well-defined set of outcomes but we are not sure which one will appear before the experiment is called a random experiment.
For example, when we toss a coin, we know that we would get a head or a tail, but we are not sure which one will appear.

3.3 Trial:

A trial denotes doing a random experiment.

3.4 Sample Space or Fundamental set:

All the possible outcomes of a random experiment together constitute a sample space noted Ω. For example,

1. the sample space of tossing a coin is $\Omega = \{H(head), T(tail)\}$.
2. the sample space of rolling dice is $\Omega = \{1,2,3,4,5,6\}$.

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4.5 Event:

All subset of Ω is called event.
Every outcome of a random experiment is called an elementary event noted ω or ω_i.

4.6 Favorable Outcome:

An event that has produced the desired result or expected event is called a favorable outcome.
For example, when we roll two dice, $\Omega = \{(x, y);\ x, y \in \{1, 2, 3, 4, 5, 6\}\}$ and the possible/favorable outcomes of getting the sum of numbers on the two dice as 4 are (1,3), (2,2), and (3,1).

4.7 Exhaustive Events:

When the set of all outcomes of an event is equal to the sample space, we call it an exhaustive event.

4.8 Mutually Exclusive Events:

Events that cannot happen simultaneously are called mutually exclusive events (i. e. Events that do not have outcomes in common).
For example, the climate can be either hot or cold. We cannot experience the same weather simultaneously.

5 Events and Probability

In probability theory, an event is a set of outcomes of a random experiment or a subset of the sample space.

5.1 Definition

We call probability each function (map) P from Ω to $[0, 1]$ such that :
If $P(E)$ represents the probability of an event E, then, we have,

1. $P(E) = 0$ if and only if E is an impossible event (i.e. $E = \emptyset$).
2. $P(E) = 1$ if and only if E is a certain event (i.e. $E = \Omega$).
3. $0 \leq P(E) \leq 1$ for every event $E \subset \Omega$.

Two cases of probability arise: Equiprobable case and Non-equiprobable case.

5.2 Equiprobable case

The case where all elementary events have the same chances of being trainedis called equiprobable case otherwise non-equiprobable case.
For example: not faked coin and well balanced dice.
THe probability formula is:
For all event A,

$$P(A) = \frac{Number of favorate outcomes to A}{Total number of possible outcomes} = \frac{|A|}{|\Omega|}$$

Example 4 • *Roll of a well-balanced die.*
$\Omega = \{1, 2, 3, 4, 5, 6\}$ *and* $P(k) = \frac{1}{6}$ *for all* $k \in \Omega$.

- *Launch of a non-faked coin.*
  $\Omega = \{H, T\}$ *and* $P(k) = \frac{1}{2}$ *for all* $k \in \Omega$.

5.3 Non-equiprobable case

For all event A,

$$P(A) = \sum_{\omega \in A} P(\omega)$$

Example 5 • *Roll of a rigged (poorly balanced) die such that even (respectively odd) numbers have the same chance and the chance of having an even number is twice that of having an odd one.*

- $\Omega = \{1, 2, 3, 4, 5, 6\}$ *and* $P(1) = P(3) = P(5)$ *and* $P(2) = P(4) = P(6)$,
- *so,* $\forall k \in \Omega$.

$$P(k) = \begin{cases} \frac{2}{9} & \text{if } k \text{ is even}(i.e. k \in \{2, 4, 6\}) \\ \frac{1}{9} & \text{if } k \text{ is odd}(i.e. k \in \{1, 3, 5\}) \end{cases}$$

6 Different Probability Formulas

6.1 Probability formula with addition rule:

Whenever an event is the union of two other events, say A and B, then

1. $P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$;
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

6.2 Probability formula with the complementary rule:

Whenever an event is the complement of another event, specifically, if A is an event, then

1. $P(not\ A) = P(\neg A) = 1 - P(A)$ or $P(\overline{A}) = 1 - P(A)$;
2. $P(A) + P(\overline{A}) = 1$.

Example 6 *We roll a die twice in a row.*

- $\Omega = \{(x, y);\ x, y \in \{1, 2, 3, 4, 5, 6\}\}$ *and* $|\Omega| = 6^2 = 36$;
- *For the event A:"have the sum of the two results equal to 10",* $A = \{(4, 6), (5, 5), (6, 4)\}$ *and* $P(A) = \frac{|A|}{|\Omega|} = \frac{3}{36} = \frac{1}{12}$;
- *For the event B:"have exactly one of the two results equal to 6",* $B = \{(x, 6), (6, x), x \in \{1, 2, 3, 4, 5, 6\}\}$ *and* $P(B) = \frac{|B|}{|\Omega|} = \frac{10}{36} = \frac{5}{18}$;
- $P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{12} = \frac{11}{12}$ *and* $P(\overline{B}) = 1 - P(B) = 1 - \frac{5}{18} = \frac{13}{18}$;
- $P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{2}{36} = \frac{1}{18}$ *from* $A \cap B = \{(4, 6), (6, 4)\}$;
- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{12} + \frac{5}{18} - \frac{1}{18} = \frac{11}{36}$;

7 Conditional probability

7.1 Probability formula with the conditional rule:

Let be A an event such that $P(A) \neq 0$.

When event A is already known to have occurred, the probability of event B is known as conditional probability or the probability of B knowing that A is realized and is given by:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Example 7 (Continuation of the previous example.) *We roll a die twice in a row.*

- $P(A|B) = \frac{P(B \cap A)}{P(B)} = \frac{\frac{1}{18}}{\frac{5}{18}} = \frac{1}{5}$;
- $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{18}}{\frac{1}{12}} = \frac{12}{18} = \frac{1}{3}$.

8 Independence of events

8.1 Probability formula with multiplication rule:

Theorem 3 *Let be A and B two events of a sample space Ω. The event A and B, denoted $A \cap B$, occur if and only if A and B occur together. Then*

1. **multiplication rule:** $P(A \cap B) = P(A) \times P(B|A) = P(B) \times P(A|B)$ *if A and B are dependent events;*
2. *A and B are independent events if and only if $P(A \cap B) = P(A) \times P(B)$ (i.e. $P(A|B) = P(A)$).*

We can generalize as follows :

Theorem 4 *Let be $E_1, E_2, ..., E_k$ events of a sample space Ω. Then*

1. **multiplication rule:** $P(E_1 \cap E_2 \cap ... \cap E_k) = P(E_1) \times P(E_2|E_1) \times ... \times P(E_k|E_1 \cap E_2 \cap ... \cap E_{k-1})$ *if $E_1, E_2, ..., E_k$ are dependent events;*
2. *$E_1, E_2, ..., E_k$ are independent events if and only if $P(E_1 \cap E_2 \cap ... \cap E_k) = P(E_1) \times ... \times P(E_k)$.*

Example 8 (Continuation of the previous example.) *We roll a die twice in a row.*

$$\begin{cases} P(A \cap B) = \frac{1}{18} \\ P(A) \times P(B) = \frac{1}{12} \times \frac{5}{18} = \frac{5}{216} \end{cases} \Rightarrow P(A \cap B) \neq P(A) \times P(B)$$

So, A and B are no independent (i.e. A and B are dependent).

9 Total Probability

Definition 5 *We call complete system of Ω all partition $A_1, ..., A_k$ of Ω.*

The events $A_1, ..., A_k$ are usually called hypotheses and from their definition follows that

$$P(A_1) + ... + P(A_n) = 1 (= P(\Omega))$$

Theorem 5 *Let be $\{A_1, A_2, ..., A_k\}$ a complete system of Ω. For every event B of Ω we have*

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + ... + P(B \cap A_k)$$

Proof: 4 *If $\{A_1, A_2, ..., A_k\}$ is a complete system of Ω, then $\{(B \cap A_1), (B \cap A_2), ..., (B \cap A_k)\}$ so is for B.*

10 Theorem of bayes

Bayes Formula. Let the event of interest A happens under any of hypotheses H_i with a known (conditional) probability $P(A|H_i)$. Assume, in addition, that the probabilities of hypotheses H_1,...,H_k are known (prior probabilities).

Theorem 6 *Then the conditional (posterior) probability of the hypothesis H_i, $i = 1, 2, ..., k$, given that event A happened, is $P(H_i|A) = \frac{P(A|H_i)P(H_i)}{P(A)}$ where, $P(A) = P(A|H_1)P(H_1) + ... + P(A|H_k)P(H_k)$.*

Example 9

1. *For every event A, $\{A, \overline{A}\}$ is a complete system of Ω.*
2. *Three machines, noted M_i, $i = 1, 2, 3$, manufacture an electronic component with the percentages 40%, 35% and 25% respectively. The percentages of defective electronic component manufactured for each machine are 3%, 4% and 2% respectively. We choose an electronic component at random.*
   (a) *We have, $P(M_1) = 0.40$, $P(M_2) = 0.35$ and $P(M_3) = 0.25$;*
   (b) *Also, $P(D|M_1) = 0.03$, $P(D|M_2) = 0.04$ and $P(D|M_3) = 0.02$;*
   (c) *From $P(M_1) + P(M_2) + P(M_3) = 1$, M_i, $i = 1, 2, 3$ is a complete system of Ω;*
   (d) *$M_i \cap D$, $i = 1, 2, 3$ is a complete system of the event D;*
   (e) *$P(D) = P(D|M_1) \times P(M_1) + P(D|M_2) \times P(M_2) + P(D|M_3) \times P(M_3) = 0.03 \times 0.40 + 0.04 \times 0.35 + 0.02 \times 0.25 = 0.031$;*
   (f) *According to Bayes' theorem, we have*

$$P(M_1|D) = \frac{P(D|M_1) \times P(M_1)}{P(D)} = \frac{0.03 \times 0.40}{0.031} = 0.387$$

11 Calculating Probability

In an random experiment, the probability of an event is the possibility of that event occurring. The probability of any event is a value between (and including) "0" and "1".
Apply the following algorithm by following these steps :

11.1 Algorithm

Step 1: Find the sample space Ω of the random experiment and count its elements. Denote it by $|\Omega|$.

Step 2: Find the number of favorable outcomes of the event A and denote it by $|A|$.

Step 3: To find probability, divide $|A|$ by $|\Omega|$. Then,

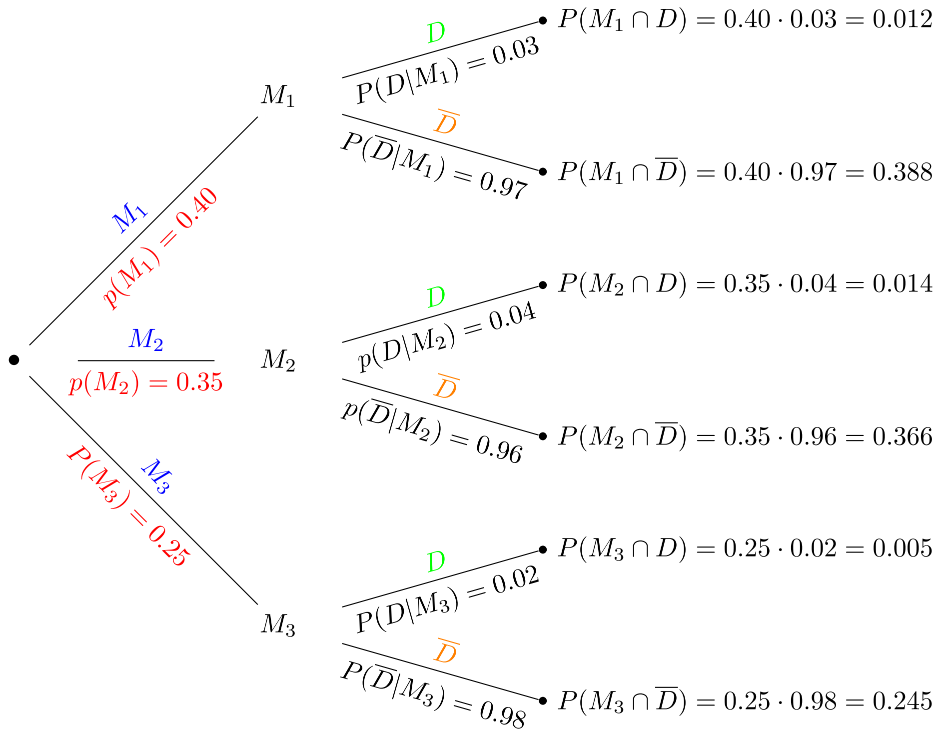
$$P(A) = \frac{|A|}{|\Omega|}$$

.

12 Tree diagram

It is the tree-like representation of the stages of a random experiment.

Example 10 *Three machines, noted M_i, $i = 1, 2, 3$, manufacture an electronic component with the percentages 40%, 35% and 25% respectively. The percentages of defective electronic component manufactured for each machine are 3%, 4% and 2% respectively. We choose an electronic component randomly of the company's stock. The following diagram is the tree diagram of the random experience.*



From $\{M_1, M_2, M_3\}$ *complate system of* Ω*, we have*

$$\begin{aligned} P(D) &= P(D \cap M_1) + P(D \cap M_2) + P(D \cap M_3) \\ &= P(M_1) \times P(D|M_1) + P(M_2) \times P(D|M_2) + P(M_3) \times P(D|M_3) \\ &= 0.012 + 0.014 + 0.005 \\ &= 0.031 \end{aligned} \tag{1}$$

References

[1] Sahraoui Abdelkader. Introduction aux probability. *The annals of statistics*, pages 595–620, 1977.