Stat 520 Hwk1 Solution:
Courtesy to Alexandra Chronopoulou, Ben Fanson, Thomas L. McLean, Andreana Robertson !

## Q1.4

1. $\mathbf{X_t} = \alpha + \mathbf{bZ_t} + \mathbf{cZ_{t-2}}$.

$$\begin{aligned} E(X_t) &= a + bE(Z_t) + cE(Z_{t-2}) = a \\ Cov(X_{t+h}, X_t) &= b^2Cov(Z_{t+h}, Z_t) + bcCov(Z_{t+h}, Z_{t-2}) \\ &+ bcCov(Z_{t+h-2}, Z_t) + c^2Cov(Z_{t+h-2}, Z_{t-2}), \end{aligned}$$

thus

$$Cov(X_{t+h}, X_t) = \begin{cases} (b^2 + c^2)\sigma^2 \ , & \text{if } h = 0 \\ bc\sigma^2 \ , & \text{if } h = \pm 2 \\ 0 \ , & \text{otherwise} \end{cases}$$

As the mean function and autocovariance function doesn't depend on $t$, we conclude that the above process is stationary.

2. $\mathbf{X_t} = \mathbf{Z_1} \cos(\mathbf{ct}) + \mathbf{Z_2} \sin(\mathbf{ct})$.

$$\begin{aligned} E(X_t) &= E(Z_1)\cos(ct) + E(Z_2)\sin(ct) = 0. \\ Cov(X_{t+h}, X_t) &= \{\cos(ct)\cos(ct + ch) + \sin(ct)\sin(ct + ch)\}\sigma^2 \\ &= \cos(ch)\sigma^2 \end{aligned}$$

stationary

3. $\mathbf{X_t} = \mathbf{Z_t} \cos(\mathbf{ct}) + \mathbf{Z_{t-1}} \sin(\mathbf{ct})$.

$$\begin{aligned} E(X_t) &= E(Z_t)\cos(ct) + E(Z_{t-1})\sin(ct) = 0 \\ Cov(X_{t+h}, X_t) &= Cov\{Z_{t+h}\cos(ct + ch) + Z_{t+h-1}\sin(ct + ch), \\ &\quad Z_t\cos(ct) + Z_{t-1}\sin(ct)\} \end{aligned}$$

Thus,

$$Cov(X_{t+h}, X_t) = \begin{cases} \sigma^2, & \text{if } h = 0 \\ \sin(ct + c)\cos(ct)\sigma^2 \ , & \text{if } h = 1 \\ \cos(ct - c)\sin(ct)\sigma^2 \ , & \text{if } h = -1 \\ 0 \ , & \text{otherwise} \end{cases}$$

Since the autocovariance function is a function of $t$, the above process is nonstationary.

4. $\mathbf{X_t} = \alpha + \mathbf{bZ_0}$

$$\begin{aligned} E(X_t) &= \alpha + E(Z_0) = 0 \\ Cov(X_{t+h}, X_t) &= b^2\sigma^2 \end{aligned}$$

<u>stationary</u>

5. $\mathbf{X_t} = \mathbf{Z_0}\cos(\mathbf{ct})$

$$\begin{aligned} E(X_t) &= E(Z_0)\cos(ct) = 0 \\ Cov(X_{t+h}, X_t) &= Cov(Z_0\cos(ct+ch), Z_0\cos(ct)) = \cos(ct+ch)\cos(ct)\sigma^2 \end{aligned}$$

<u>nonstationary</u>

6. $\mathbf{X_t} = \mathbf{Z_t Z_{t-1}}$

$$\begin{aligned} E(X_t) &= E(Z_t)E(Z_{t-1}) = 0 \\ Cov(X_{t+h}, X_t) &= Cov(Z_{t+h}\ Z_{t+h-1}, Z_t\ Z_{t-1}) \\ &= E(Z_{t+h}Z_{t+h-1}Z_tZ_{t-1}) \end{aligned}$$

Thus,

$$Cov(X_{t+h}, X_t) = \begin{cases} \sigma^4, & \text{if } h = 0 \\ 0, & \text{if } h \neq 0 \end{cases}$$

<u>stationary</u>

## <u>Q1.5</u>

(a)

$$Cov(X_{t+h}, X_t) = Cov(Z_{t+h} + \theta Z_{t+h-2}, Z_t + \theta Z_{t-2})$$

Thus

$$Cov(X_{t+h}, X_t) = \begin{cases} 1+\theta^2, & \text{if } h = 0 \\ \theta\ , & \text{if } h = \pm 2 \\ 0\ , & \text{otherwise} \end{cases}$$

Take $\theta = 0.8$, then

$$\gamma_x(h) = Cov(X_{t+h}, X_t) = \begin{cases} 1.64 & \text{if } h = 0 \\ 0.8, & \text{if } h = \pm 2 \\ 0\ , & \text{otherwise} \end{cases}$$

Note $\gamma_x(0) = 1.64$, then

$$\rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)} = \begin{cases} 1 & \text{if } h = 0 \\ 0.4878, & \text{if } h = \pm 2 \\ 0\ , & \text{otherwise} \end{cases}$$

(b) Plug $X_t = Z_t + \theta Z_{t-2}$ in $Var(\frac{X_1+X_2+X_3+X_4}{4})$

$$\begin{aligned} Var\left(\frac{X_1+X_2+X_3+X_4}{4}\right) &= \frac{1}{16} Var\left\{\theta Z_{-1} + \theta Z_0 + (1+\theta)Z_1 + (1+\theta)Z_2 + Z_3 + Z_4\right\}. \\ &= \frac{1}{4}(1+\theta+\theta^2) \end{aligned}$$

thus

$$Var\left(\frac{X_1+X_2+X_3+X_4}{4}\right) = \begin{cases} 0.61\ , & \text{if } \theta = 0.8 \\ 0.21\ , & \text{if } \theta = -0.8 \end{cases}$$

**Q1.7**

$$\begin{aligned} E(X_t+Y_t) &= E(X_t) + E(Y_t) = \mu_X(t) + \mu_Y(t) \\ \gamma_{X+Y}(h) &= Cov(X_t+Y_t, X_{t+h}+Y_{t+h}) \\ &= Cov(X_t, X_{t+h}) + Cov(X_t, Y_{t+h}) + Cov(Y_t, X_{t+h}) + Cov(Y_t, Y_{t+h}) \\ &= \gamma_X(h) + \gamma_Y(h). \end{aligned}$$

Since $\{X_t\}, \{Y_t\}$ are stationary, $\mu_X(t) + \mu_Y(t)$, $\gamma_X(h) + \gamma_Y(h)$ are independent of t. Thus $\{X_t + Y_t\}$ is also stationary.

**Q1.10**

By definition of $\nabla m_t$ we have:

$$\begin{aligned} \nabla m_t &= m_t - m_{t-1} \\ &= \sum_{k=0}^{p} c_k t^k - \sum_{k=0}^{p} c_k (t-1)^k \\ &= \sum_{k=0}^{p} c_k \left\{t^k - (t-1)^k\right\} \end{aligned}$$

Clearly, this is a polynomial of degree $(p-1)$ in t. Hence, by deduction, $\nabla^{p+1} m_t = 0$.

**Q1.18**

Figure 1 shows the raw data. Figure 2 shows the deseasonalized data. ACF/PACF are also listed as below. Here is the result from ITSM of testing randomness on residuals.

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Ljung - Box statistic = 55.384 Chi-Square ( 20 ), p-value = .00004
McLeod - Li statistic = 15.829 Chi-Square ( 20 ), p-value = .72716
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Turning points = 43.000 ~ AN(46.667,sd = 3.5324), p-value = .29926
Diff sign points = 35.000~AN(35.500,sd = 2.4664), p-value = .83935
Rank test statistic = .12450E+04 ~ AN(.12780E+04,sd = .10285E+03),
p-value = .74833
Jarque-Bera test statistic (for normality) = .13695
Chi-Square (2), p-value = .93382
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By Ljung-Box test, we shall definitely conclude alternative hypothesis at any reasonable confidence level, ie., the residuals are not independent. We also plot the predictions for the following 24 and 48 months together with 95% prediction bounds (Figure 5). An upward trend is indicated by the prediction plots.

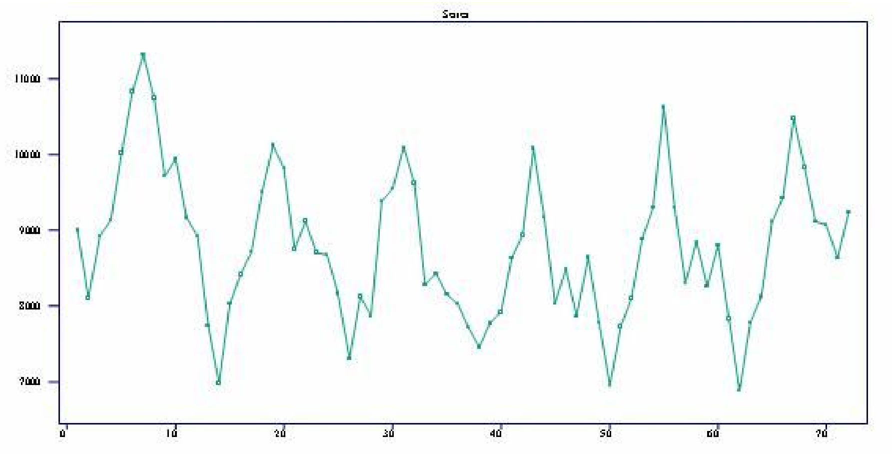


Figure 1: Raw Data

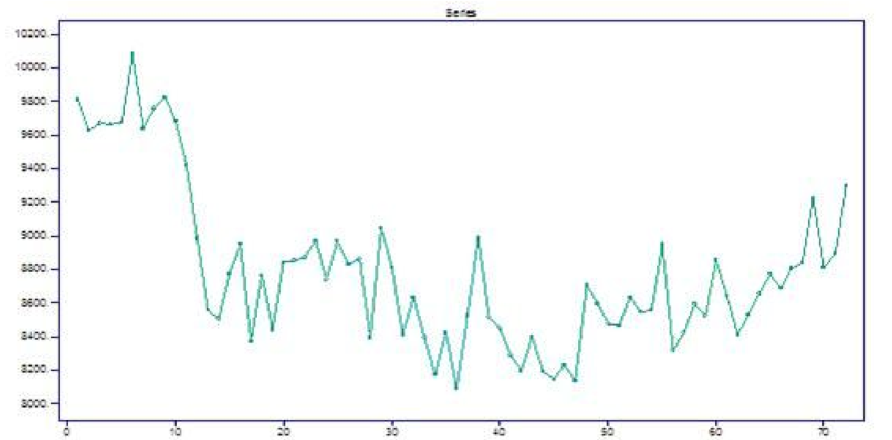


Figure 2: Deseasonalized Data

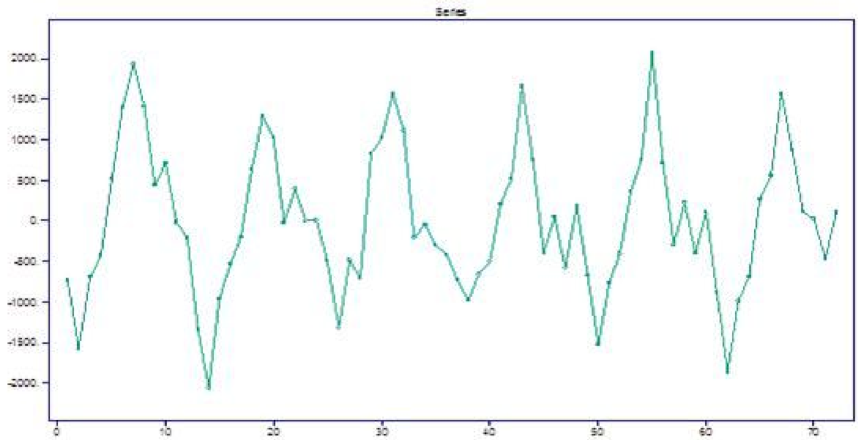


Figure 3: Data with Quadratic Fit

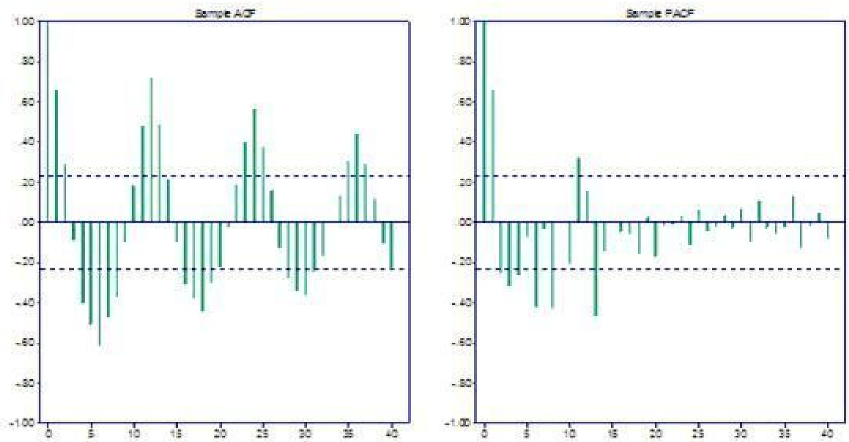


Figure 4: ACF/ PACF

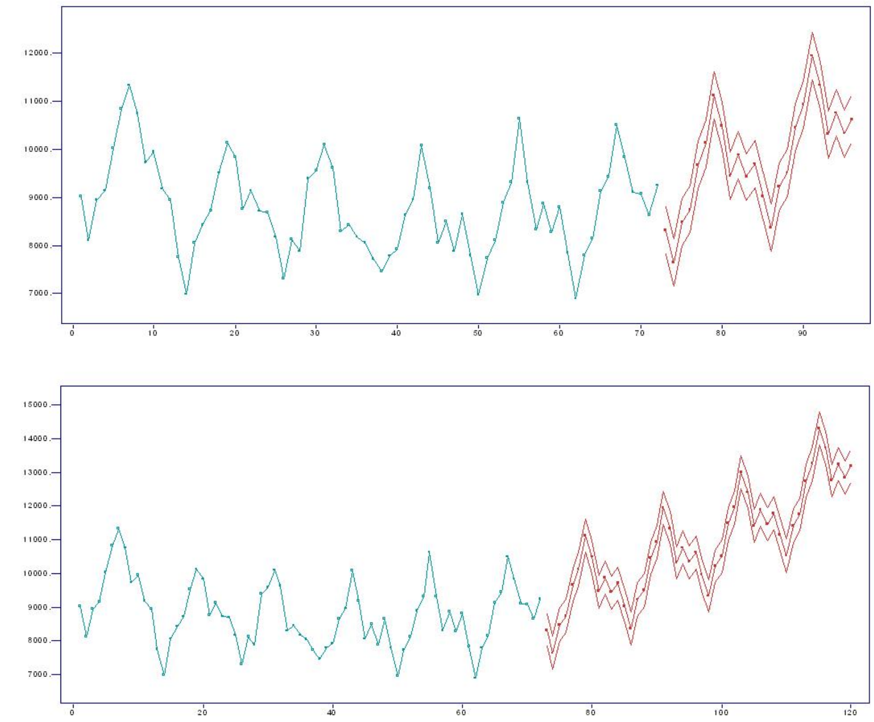

Figure 5: Forecast for 24 and 48 months