Course "VDA, optimal control & Model Learning", INSA Toulouse.

Equations of the Programming Practical (PP)

1 On the ill-posedness of the inverse problem

If based on the flow equation only, the problem consisting to infer $b(x)$ is ill-posed !

Indeed, the considered flow equation can be satisfied given an infinity of bathymetry shapes $b(x)$!

Let us show this statement.

1.1 On the non-uniqueness of $b(x)$

Let $(b_1, b_2)(x)$ be two different bathymetry shapes. We have: $\forall x \in [0, L]$,

$$-\Lambda_{ref}(b_k)H"(x) + H'(x) + \frac{\partial_x w}{w}(x)H(x) = b'_k(x) + \frac{\partial_x w}{w}(x)b_k(x) \quad \text{for } k = 1, 2 \tag{1}$$

Plus the non-homogeneous Dirichlet B.C. on H.

We have: $\Lambda_{ref}(b_2) - \Lambda_{ref}(b_1) = -\frac{3}{10}\frac{(b_2-b_1)(x)}{|H'_{ref}(x)|}$.

Let us set:

$$b(x) = (b_2 - b_1)(x) \tag{2}$$

By susbtracting the two equations above, we obtain:

$$b'(x) = \left(\frac{3}{10}\frac{H''(x)}{|H'_{ref}(x)|} - \frac{w'}{w}(x)\right) b(x) \tag{3}$$

We set: $\Phi(x) = \left(\frac{3}{10}\frac{H''(x)}{|H'_{obs}(x)|} - \frac{w'}{w}(x)\right)$. It follows: $b(x) = b(x_0)\exp\left(\int_{x_0}^{x} \Phi(s)\ ds\right)$, that is:

$$b(x) = b(x_0)\exp\left(\frac{3}{10}\int_{x_0}^{x} \frac{H''(x)}{|H'_{obs}(x)|}\ ds - ln|w(x)|\right) \tag{4}$$

with $x_0 > 0$.

<u>Assuming</u> $H(x) = H_{obs}(x)\ \forall x$*, moreover assuming* $H'_{obs}(x) < 0\ \forall x$*, it follows:*

$$b(x) = b(x_0)\exp\left(\frac{3}{10}ln\left(|H'(x)|\right) - ln\left(w(x)\right)\right) \tag{5}$$

$$b(x) = b(x_0)\left(\frac{|H'(x)|^{\frac{3}{10}}}{w(x)}\right) \tag{6}$$

That is:

$$\boxed{b_2(x) = b_1(x) + (b_2 - b_1)(x_0)\left(\frac{|H'(x)|^{\frac{3}{10}}}{w(x)}\right)} \tag{7}$$

In the case where the direct model simply reads as:

$$-\Lambda_{ref}(b_k)H"(x) + H'(x) = b'_k(x) \quad \text{for } k = 1, 2 \tag{8}$$

We obtain:

$$\boxed{b_2(x) = b_1(x) + (b_2 - b_1)(x_0)\,|H'(x)|^{\frac{3}{10}}} \tag{9}$$

This calculation demonstrates that if $b_2(x_0) = b_1(x_0)$ *then the optimal bathymetry shape is unique.*

On the contrary, if $b_2(x_0) \neq b_1(x_0)$ *(and if the flow is a non trivial one with* $H'(x) = 0\ \forall x$*) then an infinity of bathymetry shapes answer the problem !*

These multiple solutions are uniquely determined given one value at one point x_0.

To Do When numerically solving the inverse problem, plot a set of solutions corresponding to different water depth at $x = 0$.

Questions

- Is the optimal contol problem you solve a LQ problem or not ?
- If not, under what additional conditions the optimal control b^* exists and is unique ?
- How do you explain the fact that you numerically obtain one and only one optimal value b^*?

1.2 The equation in variable $b(x)$

Let us consider the original non-linear model:

$$-\Lambda(b)H''(x) + H'(x) + \frac{w'(x)}{w(x)}H(x) = b'(x) + \frac{w'(x)}{w(x)}b(x)$$

Let us re-write the equation in variable $b(x)$. We have:

$$b'(x) + \frac{w'}{w}(x)\,b(x) - \frac{3}{10}\frac{H''(x)}{|H'(x)|}b(x) = -\frac{3}{10}\frac{H(x)}{|H'(x)|}H''(x) + H'(x) + \frac{w'}{w}(x)H(x) \tag{10}$$

In the simplified case (no terms in $\frac{w'}{w}(x)$) and by denoting $S = H'$, this reads:

$$b'(x) - \frac{3}{10}\frac{S'(x)}{|S(x)|}b(x) = -\frac{3}{10}\frac{H(x)}{|S(x)|}S'(x) + S(x) \tag{11}$$

Assuming $S(x) = H'(x) < 0\ \forall x$, it follows:

$$\boxed{b'(x) + \frac{3}{10}\frac{S'}{S}(x)b(x) = L(H,S)(x)} \tag{12}$$

with the RHS $L(S,H)(x) = \frac{3}{10}H(x)\frac{S'}{S}(x) + S(x)$.

The fundamental solution of the homogeneous ODE satisfies:

$$\frac{\bar{b}'}{\bar{b}}(x) = -\frac{3}{10}\frac{S'}{S}(x) \tag{13}$$

Therefore the solution: $\bar{b}(x) = b(x_0)\ |S|^{-3/10}(x)$.

A particular solution reads: $b_*(x) = |S|^{-3/10}(x)\int^x \left(|S|^{-3/10}(s)\ L(S,H)(s)\right) ds$.

Therefore the general solution:

$$b(x) = \ |S|^{-3/10}(x)\left(b(x_0) + \int_{x_0}^{x}\left(|S|^{-3/10}(s)L(S,H)(s)\right) ds\right) \tag{14}$$

$$\boxed{b(x) = S^{-3/10}(x)\left(b(x_0) + \frac{3}{10}\int_{x_0}^{x} S^{-13/10}(s)S'(x)H(s)\ ds + \int_{x_0}^{x} |S|^{+7/10}(s)\ ds\right)} \tag{15}$$

This expression provides the family of solution $b(x)$ in function of $b(x_0)$ given the surface slopes $S(x)$...

Remark

- A slope value highly depends on the considered scale; and even more the curvature S'.
  Setting the value $S(x) = H'(x)$ assumes to have fixed a scale: this defines the flow model scale.
- Given a pdf of $b(x_0)$, e.g. an uniform probability within a bounded intervall $[b_{0,min}, b_{0,max}]$, we can straightforwadly deduce the resulting pdf of $b(x)$ for all x.