EXERCISES WITH THEIR CORRECTIONS

Exercice 1 Adjoint operator of a symmetrical second-order operator

Let V be a subspace of $H^1(\Omega)$, let A be the linear operator defined from V into V' as follows:

$$A(y) = -div(\lambda \nabla y) + c\ y$$

with λ and c given in $L^\infty(\Omega)$.

Question I) Write the expression of A^* in weak form, in the following cases.

Case 1) With Dirichlet B.C.. Homogeneous Dirichlet conditions are imposed on the whole boundary $\partial\Omega$. In this case, we set $V = H^1_0(\Omega)$.

Show that in this case the operator is self-adjoint, i.e. $A^* = A$.

Case 2) With mixed B.C..

Considering mixed B.C., we set $\partial\Omega = \Gamma_0 \cup \Gamma_1$ *and:*

$$V = H^1_{\Gamma_0}(\Omega) = \{z \in H^1(\Omega),\ z = 0 \text{ on } \Gamma_0\}$$

Question II)
Write the expression of A^* in weak form, next in classical form, in the following cases.
We consider the mixed BC as previously, with:
Case 1) $(-\lambda\ \nabla y \cdot n) = \varphi$ given on Γ_1,
Case 2) $(-\lambda\ \nabla y \cdot n) = \varphi(y)$ given on Γ_1.

***.

Correction. Recall that by definition, given A a *linear* operator, its adjoint operator A^* satisfies:

$$< Ay, z >_{V'\times V} = < y, A^* z >_{V\times V'} \ \ \forall (y,z) \in V \times V$$

Question I)
Case 1)
Let $(y,z) \in V \times V;\ V = H^1_0(\Omega)$.
The operator $A(y) = -div(\lambda\ \nabla y) + c\ y$ is linear.
By definition, we have: $< A(y), z >_{V'\times V} = \int_\Omega [-div(\lambda\nabla y) + c\ y]\ z\ dx$
By applying Green's formulae, we obtain:

$$< A(y), z >_{V'\times V} = \int_\Omega (\lambda \nabla y \nabla z + c\ y\ z)\ dx \ \text{ since } z \in H^1_0(\Omega)$$

By applying Green's formulae again, we obtain:

$$\begin{aligned} < A(y), z >_{V'\times V} &= \int_\Omega [-div(\lambda\nabla z) + c\ z]\ y\ dx + \int_{\partial\Omega} \lambda \nabla z \cdot n\ y\ ds \\ &= \int_\Omega [-div(\lambda\nabla z) + c\ z]\ y\ dx \ \text{ since } y \in H^1_0(\Omega) \\ &= < y, A^* z >_{V\times V'} \end{aligned}$$

Therefore: $A(z) \equiv Az = A^* z$ for all $z \in V$. In other words: $A = A^*$.

Case 2)
Case of mixed BCs with $V = H^1_{\Gamma_0}(\Omega)$.
Let $(y,z) \in V \times V$. By applying Green's formulae, we obtain:

$$< A(y), z >_{V'\times V} = \int_\Omega (\lambda\nabla y \nabla z + c\ y\ z)\ dx - \int_{\Gamma_1} \lambda \nabla y \cdot n\ z\ ds$$

By applying Green's formulae again, it follows:

$$\begin{array}{rcl} < A(y), z >_{V' \times V} & = & \int_\Omega [-div(\lambda \nabla z) + c\ z]\ y\ dx + \int_{\Gamma_1} \lambda \nabla z \cdot n\ y\ ds - \int_{\Gamma_1} \lambda \nabla y \cdot n\ z\ ds \\ & = & < y, A^* z >_{V \times V'} \text{ by definition.} \end{array}$$

We obtained the adjoint operator expression.
For $(p, z) \in V \times V$,

$$< A^* p, z >_{V' \times V} = \int_\Omega (\lambda \nabla p \nabla z + c\ p\ z)\ dx - \int_{\Gamma_1} \lambda \nabla z \cdot n\ p\ ds$$

Therefore:

$$< A^* p, z >_{V' \times V} = \int_\Omega [-div(\lambda \nabla p) + c\ p]\ z\ dx + \int_{\Gamma_1} \lambda\ \nabla p \cdot n\ z\, ds - \int_{\Gamma_1} \lambda\ \nabla z \cdot n\ p\, ds$$

Note that we have:

$$< (Ay - A^* y), z >_{V' \times V} = - \int_{\Gamma_1} \lambda \nabla y \cdot n\ z\ ds + \int_{\Gamma_1} \lambda \nabla z \cdot n\ y\ ds$$

As a consequence, because of the mixed BCs, the operator A is *a-priori* not self-adjoint in $V = H^1_{\Gamma_0}(\Omega)$.
However... see next question!

Question II)

Case 1) We have: $-(\lambda\ \nabla y \cdot n) = \varphi$ given on Γ_1. Therefore, for all $(y, z) \in V \times V$,

$$< Ay, z >_{V' \times V} = \int_\Omega (\lambda \nabla y \nabla z + c\ y\ z)\ dx + \int_{\Gamma_1} \varphi\ z\ ds \tag{0.1}$$

In the case $\varphi \neq 0$, the linear part of the operator is the map $y \mapsto \int_\Omega (\lambda \nabla y \nabla z + c\ y\ z)\ dx$ only.
Indeed the term $\int_{\Gamma_1} \varphi\ z\ ds$ does not depends on y (it is constant wrt y).
As a consequence, the linear part of the operator is self-adjoint again.
In the case $\varphi = 0$, the operator A is self-adjoint.

Case 2) We have: $-(\lambda\ \nabla y \cdot n) = \varphi(u)$.

In the case $\varphi(\cdot)$ not linear, the question does not apply. Indeed, the operator must be linear to define its adjoint.

In the case $\varphi(\cdot)$ linear, we have: $-(\lambda\ \nabla y \cdot n) = \alpha\ y$ on Γ_1, α given.
In this case, we have for all $(y, z) \in V \times V$,

$$< Ay, z >_{V' \times V} = \int_\Omega (\lambda \nabla y \nabla z + c\ y\ z)\ dx + \int_{\Gamma_1} \alpha\ y\ z\ ds \tag{0.2}$$

$$= < y, A^* z >_{V \times V'} \text{ by definition.} \tag{0.3}$$

The (linear) operator A is self-adjoint.
For $(p, z) \in V \times V$,

$$< A^* p, z >_{V' \times V} = \int_\Omega (\lambda \nabla p \nabla z + c\ p\ z)\ dx + \int_{\Gamma_1} \alpha\ p\ z\ ds$$

In the classical form, the operators read in Ω: $Ay = A^* y = -div(\lambda\ \nabla y) + c\ y$.

Recall the mixed BC on y: $y = 0$ on Γ_0, $-(\lambda\ \nabla y \cdot n) = \alpha\ y$ on Γ_1.
These BCs are the same for the adjoint operator: $p = 0$ on Γ_0, $-(\lambda\ \nabla p \cdot n) = \alpha\ p$ on Γ_1.

Exercise 2: Adjoint operator of the advection-diffusion equation

Let V be a subspace of $H^1(\Omega)$. Let $y(x)$ be a scalar function defined in V.
We consider the PDE operator A defined from V into V' as:

$$A(y) = -div(\lambda \nabla y) + \mathbf{w} \cdot \nabla y$$

with λ given in $L^\infty(\Omega)$, $\lambda > 0$, the vector field $\mathbf{w}$ given in $(L^\infty(\Omega))^d$.

Assumption. The vector field $\mathbf{w}$ satisfies the following properties: $div(\mathbf{w}) = 0$ in Ω and $\mathbf{w} \cdot n \geq 0$ on Γ_1.
In the case $\mathbf{w}$ represents a velocity field, the velocity field is incompressible and incoming where the solution value is given (i.e. on Γ_0).

Case 1) Homogeneous Dirichlet conditions. Homogeneous Dirichlet conditions are imposed everywhere on $\partial\Omega$. In this case, we have $V = H_0^1(\Omega)$.

a) Existence and uniqueness.
Let $f \in L^2(\Omega)$ be given. Prove that the BVP,
$A(y) = f$ accompanied with homogeneous Dirichlet B.C. on $\partial\Omega$,
admits an unique weak solution u in V.

b) Adjoint operator.
Write an expression of A^* in weak form, next in its classical form.

Correction.
a) Existence and uniqueness.
The weak form of the model reads as follows.
Find $y \in V = H_0^1(\Omega)$ such that $a(y,z) = l(z)$ with

$$a(y,z) = \int_\Omega \lambda \nabla y \nabla z \; dx + \int_\Omega \mathbf{w} \cdot \nabla y \; z \; dx \text{ and } l(z) = \int_\Omega f \; z \; dx$$

When applying the Lax-Milgram theorem, the most critical property to verify is the coercivity in $V = H_0^1(\Omega)$.
Here, for all $y \in V = H_0^1(\Omega)$,

$$\int_\Omega \mathbf{w} \cdot \nabla y \; y \; dx = \frac{1}{2} \sum_i \int_\Omega \mathbf{w}_i \partial_i (y^2) \; dx = -\frac{1}{2} \sum_i \int_\Omega (y^2) \; \partial_i \mathbf{w}_i \; dx + \int_{\partial\Omega} \mathbf{w}_i n_i \; y^2 \; dx$$

Therefore:

$$\int_\Omega \mathbf{w} \cdot \nabla y \; y \; dx = 0$$

Hence the bilinear form $a(y,z)$ is coercitive in $V = H_0^1(\Omega)$.
The existence and uniqueness of the weak solution follows from the Lax-Milgram theorem.

b) Adjoint operator.
By integrating by part, we get:

$$\int_\Omega \mathbf{w} \cdot \nabla y \; z \; dx = -\int_\Omega div(z\mathbf{w}) \; y \; dx + \int_{\partial\Omega} \mathbf{w} \cdot n \; z \; y \; dx = -\int_\Omega div(z\mathbf{w}) \; y \; dx$$

For A linear, we have by definition:

$$a(y,z) = < Ay, z >_{V' \times V} \underset{def}{=} < A^* z, y >_{V' \times V}$$

with here $V = H_0^1(\Omega)$.
Using div(w)=0 and the Dirichlet B.C., we get:

$$< A^* z, y >_{V' \times V} = \int_\Omega \lambda \nabla z \nabla y \; dx - \int_\Omega \mathbf{w} \cdot \nabla z \; y \; dx$$

Therefore the adjoint of the advection term $(\mathbf{w} \cdot \nabla z)$ is $-(\mathbf{w} \cdot \nabla z)$.
PDE model operators containing the 1st order non symmetrical advective term $(\mathbf{w} \cdot \nabla y)$ are not self-adjoint.
However, the advection term $(\mathbf{w} \cdot \nabla y)$ simply transforms as $-(\mathbf{w} \cdot \nabla p)$.

Case 2) Mixed boundary conditions.

Let us set $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and $V = H^1_{\Gamma_0}(\Omega) = \{z \in H^1(\Omega),\ z = 0 \text{ on } \Gamma_0\}$.

a) Existence and uniqueness.
Let $f \in L^2(\Omega)$ be given. Prove that the BVP
$A(y) = f$ with mixed homogeneous BCs (Dirichlet / Neumann) on Γ_0, Γ_1 respectively, admits an unique weak solution in V.

b) Adjoint operator.
Write an expression of A^* in weak form.
Correction.
a) Existence and uniqueness using the Lax-Milgram theorem.
The analysis is similar to the one in the previous case. We write:

$$\int_\Omega \mathbf{w} \cdot \nabla y\ y\ dx = \frac{1}{2}\sum_i \int_\Omega \mathbf{w}_i \partial_i (y^2)\ dx = -\frac{1}{2}\sum_i \int_\Omega (y^2)\partial_i \mathbf{w}_i\ dx + \int_{\partial\Omega} \mathbf{w}_i n_i\ y^2\ dx$$

Therefore:

$$\int_\Omega \mathbf{w} \cdot \nabla y\ y\ dx = \int_{\Gamma_1} \mathbf{w}_i n_i\ y^2\ dx \geq 0 \quad \text{by assumption.}$$

Therefore the bilinear form $a(y, z)$ is coercitive in V.
The existence and uniqueness of the weak solution follows from the Lax-Milgram theorem.

In other words, in the advection-diffusion equation with an incompressible velocity field $\mathbf{w}$*, the flow must be outgoing on the boundary part where the solution is not imposed.*

b) Adjoint operator.
We have for all (y, z)∈V ×V,

$$a(y, z) =< Ay, z >_{V'\times V}= \int_\Omega \lambda \nabla y \nabla z\ dx - \int_\Omega div(z\mathbf{w})\ y\ dx + \int_{\Gamma_1} \mathbf{w} \cdot n\ z\ y\ ds - \int_{\Gamma_1} \lambda \nabla y \cdot n\ z\ ds$$

Using the incompressibility assumption $div(\mathbf{w}) = 0$, we get: $\int_\Omega div(z\mathbf{w})\ y\ dx = \int_\Omega \mathbf{w} \cdot \nabla z\ y\ dx$.
Recall that by definition: $< Ay, z >_{V'\times V}=< A^* z, y >_{V'\times V}$.

Let us write the adjoint equation LHS term $< A^* p, z >_{V'\times V}$.
For all $(p, z) \in V \times V$,

$$< A^* p, z >_{V'\times V}= \int_\Omega \lambda \nabla p \nabla z\ dx - \int_\Omega \mathbf{w} \cdot \nabla p\ z\ dx + \int_{\Gamma_1} \mathbf{w} \cdot n\ p\ z\ ds - \int_{\Gamma_1} \lambda \nabla z \cdot n\ p\ ds$$

Because of the non-symmetrical 1st order term, A is not self-adjoint.
The boundary terms on Γ_1 have to be clarified from Neumann type BCs.