

# Solving linear systems

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P. Berger, M. Daydé, R. Guivarch, E. Simon, D. Ruiz (INP-ENSEEIH- IRIT),  
A. Buttari (CNRS, INP-ENSEEIH-IRIT),  
P. Amestoy, J.-Y. L'Excellent (Mumps Technologies),  
A. Guermouche (Univ. Bordeaux-LaBRI),  
F.-H. Rouet (Ansys, USA),  
Bora Uçar (INRIA-CNRS/LIP-ENS Lyon) and  
L. Giraud (INRIA)

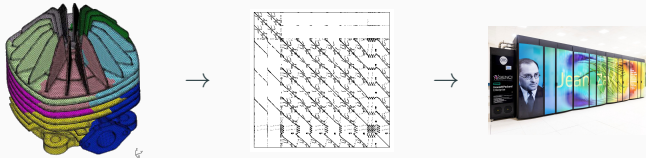
2025 - ModIA

## Ronan Guivarch

Ronan.Guivarch@toulouse-inp.fr

- Enseignant au département Sciences du Numériques de l'INP-ENSEEIH : Calcul Scientifique, Probabilités, Méthodes Itératives et Algèbre Linéaire Creuse, Programmation des cartes FPGA, Calcul Parallèle
- Chercheur au laboratoire IRIT, équipe APO (Algorithmes Parallèles et Optimisation) - Mathématiques Numériques / Calcul Parallèle

- 3 CTD autour des solveurs de Krylov (moi)
- 2 CTD d'introduction à l'Algèbre Linéaire Creuse (ALC) (moi)
- 4 CTD autour des méthodes Multigrilles (Carola Kruse (CERFACS))
- 3 CTD autour des méthodes de Décomposition de Domaine (Alena Kopanicakova (INP-ENSEEIH-IRIT))
- 6 TP sur les 4 sujets précédents
- 1 exam + rendus de TP



## Linear System $Ax = b$

At the foundations of many **scientific computing applications** (discretization of PDEs, step of an optimization method, ...).

## Large-scale computations...

Up to few **billions ( $10^9$ ) of unknowns**, applications asking TeraBytes ( $10^{12}$ ) of memory and Exaflops ( $10^{18}$ ) of computation.

## ...require large-scale computers.

Increasingly **large numbers of cores** available, high **heterogeneity in the computation** (CPU, GPU, FPGA, TPU, etc), and high **heterogeneity in data motions** (RAM to cache, out-of-core, node to node transfer, etc).

# The sparse case

## Matrix sparsity

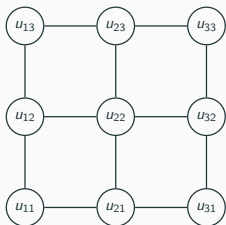
A **sparse matrix** is "any matrix with enough zeros that it pays to take advantage of them" (Wilkinson)

Physical problems are one of the main supplier for sparse linear systems. Let's consider the solution of the Poisson's PDE:

$$-\Delta u(x, y) = f$$

Finite-difference discretization gives the discrete equation:

$$(-\Delta u)_{i,j} \approx \frac{1}{h^2} (-u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + 4u_{i,j}) = f_{i,j}$$



$A$

$x$

$=$

$b$

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$$\left[ \begin{array}{ccc} 4 & -1 & -1 \end{array} \right] \left[ \begin{array}{c} u_{11} \end{array} \right] = \left[ \begin{array}{c} h^2 f_{11} \end{array} \right]$$

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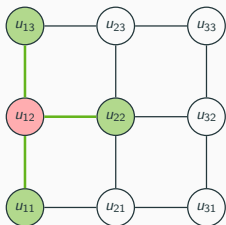
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$$\begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \end{bmatrix} = \begin{bmatrix} h^2 f_{11} \\ h^2 f_{12} \\ h^2 f_{13} \end{bmatrix}$$

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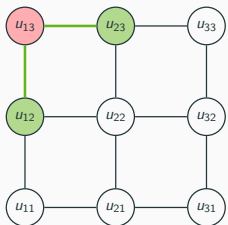
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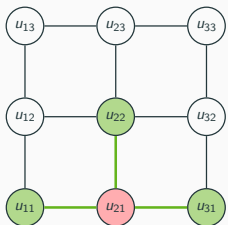
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$$\begin{bmatrix} 4 & -1 & -1 & & \\ -1 & 4 & -1 & & \\ & -1 & 4 & & -1 \\ -1 & & & 4 & -1 \\ & & & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ \textcolor{red}{u}_{21} \\ u_{22} \\ u_{23} \\ u_{31} \\ u_{32} \\ u_{33} \end{bmatrix} = \begin{bmatrix} h^2 f_{11} \\ h^2 f_{12} \\ h^2 f_{13} \\ h^2 f_{21} \\ h^2 f_{22} \\ h^2 f_{23} \\ h^2 f_{31} \\ h^2 f_{32} \\ h^2 f_{33} \end{bmatrix}$$

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$$\begin{bmatrix} 4 & -1 & & -1 & & \\ -1 & 4 & -1 & & -1 & \\ & -1 & 4 & & -1 & \\ -1 & & & 4 & -1 & -1 \\ & -1 & -1 & 4 & -1 & \\ & & & & & -1 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ \textcolor{red}{u_{22}} \\ u_{31} \end{bmatrix} = \begin{bmatrix} h^2 f_{11} \\ h^2 f_{12} \\ h^2 f_{13} \\ h^2 f_{21} \\ h^2 f_{22} \\ h^2 f_{31} \end{bmatrix}$$

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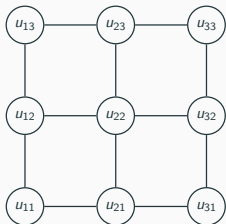
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What are the ways to solve a  $Ax = b \in \mathbb{R}^n$  on computers ?

## Iterative solvers

Compute a sequence of  $x_k$  converging towards  $x$ .

*Examples:* Gauss-Seidel, SOR, Steepest Descent, Conjugate Gradient, Krylov subspace methods, etc.

- Low computational cost and memory consumption if the convergence is quick (about  $\mathcal{O}(n^2)$  (dense case)) operations per iteration), ...
- BUT convergence depends on the matrix properties.

## Direct solvers

Compute a factorization of  $A$  followed by forward and backward substitutions.

*Examples:*  $LDL^T$ , LU, QR, etc.

- High computational cost and memory consumption...
- BUT they are robust and easy to use.

What are the ways to solve a **sparse**  $Ax = b \in \mathbb{R}^n$  on computers ?

## Iterative solvers

Compute a sequence of  $x_k$  converging towards  $x$ .

*Examples:* Gauss-Seidel, Steepest Descent, Conjugate Gradient, SOR, **Krylov subspace methods**, etc.

- **Low computational cost** and **memory consumption** if the convergence is quick (about  $\mathcal{O}(\text{nnz}(A))$  (sparse case)) operations per iteration), ...
- BUT convergence **depends on the matrix properties**.

## Direct solvers: dense/sparse matrix $\Rightarrow$ dense/sparse solver

Compute a factorization of  $A$  followed by forward and backward substitutions.

*Examples:*  $\text{LDL}^T$ , LU, QR, etc.

- **High computational cost** and **memory consumption**...
- BUT they are **robust** and **easy to use**.

Quality of a solution: conditioning and errors

Iterative Methods: Krylov Methods

Iterative Methods: Symmetric Krylov Solvers

Iterative Method: Conjugate Gradient

Iterative Methods: Algebraic preconditioning techniques

## **Quality of a solution: conditioning and errors**

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**Conditionnement**



# Conditionnement - Sensibilité du problème

- Soit  $A$  :

$$\begin{bmatrix} .780 & .563 \\ .913 & .659 \end{bmatrix} \text{ matrice presque singulière}$$

- Soit  $A'$  :

$$\begin{bmatrix} .780 & .563001095 \\ .913 & .659 \end{bmatrix} \text{ matrice singulière}$$

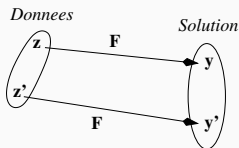
→ une perturbation des entrées de la matrice d'ordre  $O(10^{-6})$  rend le problème insoluble.

- Autre situation si  $A$  est presque singulière : une petite perturbation sur  $A$  et/ou  $b$  (mesure, numérique, ...) → grandes perturbations sur la solution

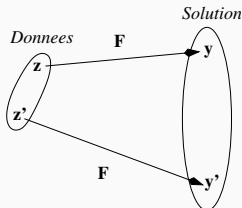
Ceci est propre au problème,  
et est indépendant de l'algorithme de résolution utilisé

# Conditionnement d'un problème

Le conditionnement peut être mesuré par le changement relatif solution /  
changement relatif données  $\frac{\| \frac{F(z') - F(z)}{F(z)} \|}{\| \frac{z' - z}{z} \|}$



Bien conditionné



Mal conditionné

## Définition (Problème bien conditionné)

*Un problème est bien conditionné si une faible variation des données entraîne une faible variation de la solution (c.à.d.  $\|z - z'\|$  petit  $\Rightarrow \|F(z) - F(z')\|$  petit)*

$$Ax = b$$

Les coefficients de  $A$  et  $b$  sont des variables calculées, numériquement incorrectes.



Influence de ces erreurs sur le résultat du système :

$$(A + \Delta A) \cdot (x + \Delta x) = b + \Delta b$$



Recherche d'un majorant de  $\|(x + \Delta x) - x\|/\|x\| = \|\Delta x\|/\|x\|$

Avec une norme induite, résultat généralement sous la forme :

$$K(A) = \|A\| \cdot \|A^{-1}\| \text{ nombre de conditionnement de } A$$

$$\|\Delta x\|/\|x\| \leq K(A) (\|\Delta A\|/\|A\| + \|\Delta b\|/\|b\|)$$

- Plus  $K(A)$  est faible, meilleur est le conditionnement du système
- Valeur minimale de  $K(A)$  ?  $K(A) = \|A\| \|A^{-1}\| \geq \|A \cdot A^{-1}\| = 1$
- Selon la norme,  $K(A)$  prend des valeurs différentes
- Si toutes les valeurs propres de  $A$  sont réelles (par ex.,  $A$  symétrique) :

$$|\lambda_1| \geq \dots \geq |\lambda_n| \quad K(A) = \|A\|_2 \|A^{-1}\|_2 = |\lambda_1|/|\lambda_n|$$

- Si  $A$  est orthogonale,  $K(A) = 1$
- $K(A)$  élevé ne prouve que l'existence d'un **risque** de mauvais comportement lors de la résolution
- **Comment** évaluer  $K(A)$  ? Ne jamais évaluer  $A^{-1}$  et utiliser de préférence des propriétés liées aux valeurs propres
- **Quand** calculer  $K(A)$  ? Si l'on envisage d'utiliser une méthode de résolution connue pour être sensible au conditionnement

Et si  $K(A) \gg 1$ ?? Changer de méthode ou **préconditionner** :

$$A \cdot x = b \rightarrow C \cdot A \cdot x = C \cdot b \text{ avec } K(C \cdot A) < K(A)$$

- Choix idéal de  $C$  :  $A^{-1}$  ????
- Choix réaliste de  $C$  : une approximation de  $A^{-1}$

## **Quality of a solution: conditioning and errors**

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**Forward and Backward Errors**

- Considérons le système linéaire :

$$\begin{bmatrix} .780 & .563 \\ .913 & .659 \end{bmatrix} x = \begin{bmatrix} .217 \\ .254 \end{bmatrix}$$

- Supposons que deux algorithmes différents donnent les deux solutions suivantes :

$$x_1 = \begin{bmatrix} -20.568 \\ 28.881 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} 0.999 \\ -1.00 \end{bmatrix}$$

**De  $x_1$  et  $x_2$ , quelle est la meilleure solution ?**

- Résidus :

$$b - Ax_1 = \begin{bmatrix} -37 \\ -5 \end{bmatrix} \times 10^{-5} \text{ and } b - Ax_2 = \begin{bmatrix} -78 \\ -91 \end{bmatrix} \times 10^{-4}$$

- $x_1$  est la meilleure solution car son résidu est le plus petit
- Solution exacte :

$$x^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

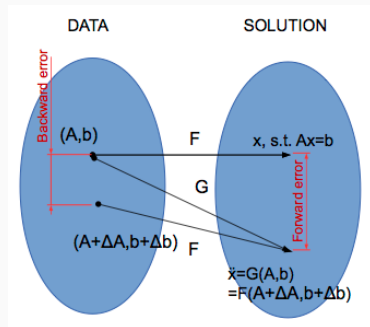
- en fait,  $x_2$  est plus proche de la solution exacte

**La notion de bonne solution est un concept ambigu**



# Erreurs directe et inverse

- $F$  fait correspondre (précision exacte) les données  $(A, b)$  à la solution  $x$  de  $Ax = b$
- $G$  est un algorithme implémentant  $F$  et travaillant en précision finie
- $\tilde{x}$  est telle que  
 $\tilde{x} = G(A, b) = F(A + \Delta A, b + \Delta b)$



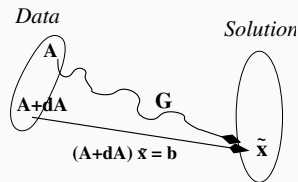
solution calculée (algorithme  $G$ ) = solution exacte d'un problème perturbé

- Amplitude des perturbations (erreur inverse) associée à l'erreur dans la solution (erreur directe)
- La solution calculée sera satisfaisante si elle est la solution exacte d'un problème perturbé avec des perturbations acceptables pour l'utilisateur (ordre des erreurs d'arrondi, incertitude sur les données, ...)

# Erreur inverse d'un algorithme résolvant $Ax = b$

soit  $\tilde{x}$  la solution approchée calculée par l'algorithme  $G$ .

**Objectif :** trouver les plus petites perturbations  $\Delta A$  et  $\Delta b$  telles que  $(A + \Delta A)\tilde{x} = b + \Delta b$



## Théorème (Erreur Inverse (Rigal and Gaches, 1967))

Si nous définissons *l'erreur inverse (backward error)*

$$\eta_{A,b}^N(\tilde{x}) = \min \{ \varepsilon > 0 \text{ tel que } \|\Delta A\|_2 \leq \varepsilon \|A\|_2, \|\Delta b\|_2 \leq \varepsilon \|b\|_2, \\ \text{et } (A + \Delta A)\tilde{x} = b + \Delta b \}$$

Alors on peut prouver que :

$$\eta_{A,b}^N(\tilde{x}) = \frac{\|b - A\tilde{x}\|_2}{\|A\|_2 \|\tilde{x}\|_2 + \|b\|_2}$$

### Erreur Inverse ne faisant intervenir que $b$

$$\begin{aligned}\eta_b^N(\tilde{x}) &= \min \{ \varepsilon > 0 \text{ tel que } \|\Delta b\|_2 \leq \varepsilon \|b\|_2, \\ &\quad \text{et } A\tilde{x} = b + \Delta b \} \\ &= \frac{\|b - A\tilde{x}\|_2}{\|b\|_2} \\ &= \frac{\|r\|_2}{\|b\|_2} \quad (r \text{ est le résidu.})\end{aligned}$$

- $\eta_{A,b}^N$  et  $\eta_b^N$  sont des erreurs inverses "**Normwise**" et sont utilisées dans le critère d'arrêt des méthodes itératives

### Erreur Inverse "*Componentwise*"

en considérant les perturbations au niveau des composantes, nous définissons :

$$\begin{aligned}\eta_{A,b}^C(\tilde{x}) &= \min \{ \varepsilon > 0 \text{ tel que } |\Delta A| \leq \varepsilon |A|, |\Delta b| \leq \varepsilon |b|, \\ &\quad \text{et } (A + \Delta A)\tilde{x} = b + \Delta b \} \\ &= \max_{i=1,\dots,n} \frac{|b - A\tilde{x}|_i}{(|A||\tilde{x}| + |b|)_i}\end{aligned}$$

avec la convention  $\frac{0}{0} = 0$ .

Cette erreur inverse convient bien aux problèmes présentant des différences de magnitude entre les composantes.

## Quality of a solution: conditioning and errors

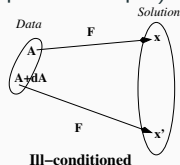
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Synthèse (nombre de conditionnement  
et erreur inverse)

- **Conditionnement** (de manière générale  $\Delta b$  doit être pris en compte) :

Approx. premier ordre  $\frac{\|\Delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\Delta A\|}{\|A\|}$

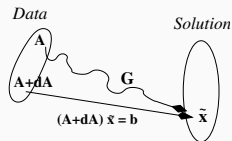
avec  $\kappa(A) = \|A^{-1}\| \|A\|$



- **Erreur inverse :**

$$\eta_{A,b}^N(\tilde{x}) = \frac{\|A\tilde{x} - b\|}{\|A\| \|\tilde{x}\| + \|b\|}$$

(mesure la **qualité** d'un algorithme)



→ à comparer à l' $\varepsilon$ -machine ou l'incertitude sur les données en entrée

- **Prédiction de l'erreur directe :**

**Erreur Directe**  $\leq$  **Nombre de Cond.**  $\times$  **Erreur Inverse**

# Iterative Methods: Krylov Methods

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- Y. Saad, *Iterative Methods for Sparse Linear Systems*, SIAM, 2nd edition
- C.T. Kelley, *Iterative Methods for Linear and Nonlinear Equations*, SIAM
- Luc Giraud and Serge Gratton, "Introduction to Krylov subspace methods for the solution of linear systems" (pdf on Moodle)

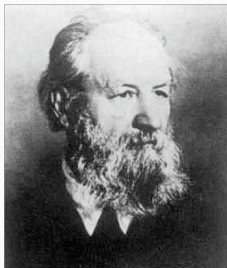


# Iterative Methods: Krylov Methods

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Some background

# Krylov methods: some background



Aleksei Nikolaevich Krylov

1863-1945: Russia, Maritime Engineer

His research spans a wide range of topics, including shipbuilding, magnetism, artillery, mathematics, astronomy, and geodesy. In 1904 he built the first machine in Russia for integrating ODEs.

In 1931 he published a paper on what is now called the "Krylov subspace".

(Most slides of this part are from Luc Giraud / INRIA)

## Definition

Let  $A \in \mathbb{R}^{n \times n}$  and  $r \in \mathbb{R}^n$ ; the space denoted by  $\mathcal{K}_m(A, r)$  (with  $m \leq n$ ) and defined by

$$\mathcal{K}_m(A, r) = \text{Span}\{r, Ar, \dots, A^{m-1}r\}$$

is referred to as the Krylov space of dimension  $m$  associated with  $A$  and  $r$ .

# Assumption

Let us assume  $x_0 = 0$

No loss of generality,

because the situation  $x_0 \neq 0$  can be transformed with a simple shift to the system  $Ay = b - Ax_0 = \bar{b}$ ,

for which obviously  $y_0 = 0$ .

## Why using this search space?

The **minimal polynomial**  $q(t)$  of  $A$  is the unique monic polynomial of minimal degree such that  $q(A) = 0$ .

It is constructed from the eigenvalues of  $A$  as follows:

If the distinct eigenvalues of  $A$  are  $\lambda_1, \dots, \lambda_\ell$  and if  $\lambda_j$  has index  $m_j$  (the size of the largest Jordan block associated with  $\lambda_j$ ), then the sum of all indices is

$$m = \sum_{j=1}^{\ell} m_j, \text{ and } q(t) = \prod_{j=1}^{\ell} (t - \lambda_j)^{m_j}. \quad (1)$$

When  $A$  is diagonalizable,  $m$  is the number of distinct eigenvalues of  $A$ .

When  $A$  is a Jordan block of size  $n$ , then  $m = n$ .

# Why using this search space?

If we write

$$q(t) = \prod_{j=1}^{\ell} (t - \lambda_j)^{m_j} = \sum_{j=0}^m \alpha_j t^j,$$

then the constant term is  $\alpha_0 = \prod_{j=1}^{\ell} (-\lambda_j)^{m_j}$ .

Therefore  $\alpha_0 \neq 0$  iff  $A$  is nonsingular.

Furthermore, from

$$0 = q(A) = \alpha_0 I + \alpha_1 A + \dots + \alpha_m A^m, \quad (2)$$

it follows that

$$A^{-1} = -\frac{1}{\alpha_0} \sum_{j=0}^{m-1} \alpha_{j+1} A^j.$$

This description of  $A^{-1}$  portrays  $x = A^{-1}b$  immediately as a member of the Krylov space of dimension  $m$  associated with  $A$  and  $b$  denoted by  $\mathcal{K}_m(A, b) = \text{Span}\{b, Ab, \dots, A^{m-1}b\}$ .

# Taxonomy of the Krylov subspace approaches

The Krylov methods for identifying  $x_m \in \mathcal{K}_m(A, b)$  can be distinguished in four classes:

- **The Galerkin approach (or Ritz-Galerkin approach) (FOM, CG,...):**  
construct  $x_m$  such that the residual is orthogonal to the current subspace:  $b - Ax_m \perp \mathcal{K}_m(A, b)$ .
- **The minimum norm residual approach (GMRES,...):**  
construct  $x_m \in \mathcal{K}_m(A, b)$  such that  $\|b - Ax_m\|_2$  is minimal that means that the residual is orthogonal to the subspace  $A\mathcal{K}_m(A, b)$  (ie  $b - Ax_m \perp A\mathcal{K}_m(A, b)$ ).
- **The Petrov-Galerkin approach:**  
construct  $x_m$  such that  $b - Ax_m$  is orthogonal to some other  $m$ -dimensional subspace.
- **The minimum norm error approach:**  
construct  $x_m \in A^T\mathcal{K}_m(A, b)$  such that  $\|b - Ax_m\|_2$  is minimal.

## Constructing a basis of $\mathcal{K}_m(A, b)$

- Obvious choice  $b, Ab, \dots, A^{m-1}b$ 
  - not very attractive from the numerical point of view because vectors  $A^j b$  become more and more colinear to the eigenvector associated to the largest eigenvalue.
  - In finite arithmetic, leads to a loss of rank: suppose  $A$  is diagonalizable  $A = VDV^{-1}$ , then  $A^k b = VD^k(V^{-1}b)$ .
  - A better choice is the Arnoldi procedure.



# Iterative Methods: Krylov Methods

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Arnoldi procedure



Walter Edwin Arnoldi

1917-1995: USA.

His main research subjects covered vibration of propellers, engines and aircraft, high speed digital computers, aerodynamics and acoustics of aircraft propellers, lift support in space vehicles and structural materials.

"The principle of minimized iterations in the solution of the eigenvalue problem" in Quart. of Appl. Math., Vol.9 in 1951.

# The Arnoldi procedure

=> builds an orthonormal basis of  $\mathcal{K}_m(A, b)$

```
1  $v_1 = b/\|b\|$ 
2 for  $j = 1, 2, \dots, m$  do
3   for  $i = 1, \dots, j$  do
4      $|$  Compute  $h_{i,j} = v_i^T A v_j$ 
5   end
6    $w_j = A v_j - \sum_{i=1}^j h_{i,j} v_i$ 
7   Compute  $h_{j+1,j} = \|w_j\|$ 
8   exit if ( $h_{j+1,j} = 0$ )
9   Compute  $v_{j+1} = w_j/h_{j+1,j}$ 
10 end
```

**Algorithm 1:** Arnoldi procedure

Steps  $\{3,4,5\}$  and 6 : Classical Gram-Schmidt (CGS)

# The Arnoldi procedure properties

## Proposition

*If the Arnoldi procedure does not stop before the  $m^{\text{th}}$  step, the vectors  $v_1, \dots, v_m$  form an orthonormal basis of the Krylov subspace  $\mathcal{K}_m(A, b)$ .*

## Proof.

The vectors are orthogonal by construction and have a norm equal to 1.

They span  $\mathcal{K}_{m-1}(A, b)$  follows from the fact that each vector  $v_j$  is of the form  $q_{j-1}(A)v_1$ , where  $q_{j-1}$  is a polynomial of degree  $j-1$ .

This can be shown by induction.

For  $j=1$  it is true as  $v_1 = q_0(A)v_1$  with  $q_0 = 1$ .

Assume that it is true for all  $j$  and consider  $v_{j+1}$ .

We have:

$$h_{j+1,j}v_{j+1} = Av_j - \sum_{i=1}^j h_{i,j}v_i = Aq_{j-1}(A)v_1 + \sum_{i=1}^j h_{i,j}q_{i-1}(A)v_1.$$

So  $v_{j+1}$  can be expressed as  $q_j(A)v_1$  where  $q_j$  is of degree  $j$ .

□

# The Arnoldi procedure properties

$$A V_m = V_m H_m + w e_m^T$$

## Proposition

Denote

- $V_m$  the  $n \times m$  matrix with column vector  $v_1, \dots, v_m$
- $\bar{H}_m$  the  $(m+1) \times m$  Hessenberg matrix whose nonzero entries are  $h_{ij}$
- $H_m$  the square matrix obtained from  $\bar{H}_m$  by deleting its last row.

Then the following relations hold:

$$AV_m = V_m H_m + h_{m+1,m} v_{m+1} e_m^T \quad (3)$$

$$AV_m = V_{m+1} \bar{H}_m \quad (4)$$

$$V_m^T A V_m = H_m \quad (5)$$

## Proof.

Equality (3) can be shown for each column. Using Matlab notations:

For  $j \neq m$

$$\begin{aligned} AV_m(:, j) &= V_m H(:, j) + w_m \times 0 \\ \Leftrightarrow Av_j &= \sum_{i=1}^{j+1} v_i h_{i,j} \\ \Leftrightarrow Av_j &= \sum_{i=1}^j h_{i,j} v_i + h_{j+1,j} v_{j+1} \end{aligned}$$

which corresponds to steps 3, 4 and 5 of the algorithm.

For  $j = m$

$$\begin{aligned} AV_m(:, m) &= V_m H(:, m) + w_m \times 1 \\ \Leftrightarrow Av_m &= \sum_{i=1}^m v_i h_{i,m} + w_m \\ \Leftrightarrow w_m &= Av_m - \sum_{i=1}^m h_{i,m} v_i \end{aligned}$$

which corresponds to step 4 of the algorithm.



## Proof.

To show equality (4), we first notice the structure of  $\bar{H}_m$

$$\begin{pmatrix} H_m \\ h_{m+1,m}e_m^T \end{pmatrix}$$

Equality (3) writes

$$\begin{aligned} AV_m &= V_m H_m + h_{m+1,m} v_{m+1} e_m^T \\ &= V_m H_m + v_{m+1} (h_{m+1,m} e_m^T) \\ &= \begin{pmatrix} V_m v_{m+1} \end{pmatrix} \begin{pmatrix} H_m \\ h_{m+1,m} e_m^T \end{pmatrix} \\ &= v_{m+1} \bar{H}_m \end{aligned}$$



## Proof.

Equality (5):

$$V_m^T A V_m = H_m$$

$V_m$  is an orthonormal matrix:  $V_m^T V_m = I_m$  (identity in  $\mathbb{R}^m$ ).

By construction  $v_{m+1}$  is orthogonal to all the previous vectors:

$$V_m^T v_{m+1} = 0.$$

Multiplying Equation (3) by  $V_m^T$  on the left gives

$$V_m^T A V_m = \underbrace{V_m^T V_m}_{I_m} H_m + h_{m+1,m} \underbrace{V_m^T v_{m+1}}_{O_m} e_m^T$$





## When $x_0 \neq 0$

- we assumed  $x_0 = 0$  and we showed that we seek  $x_m$  in the affine subspace  $\mathcal{K}_m(A, b)$ .
- if  $x_0 \neq 0$ , then we can show that we seek  $x_m$  in the affine subspace  $x_0 + \mathcal{K}_m(A, r_0)$  where  $r_0 = b - Ax_0$ .

# Using Arnoldi procedure to solve a linear system

From an initial guess  $x_0$ , we search  $x_m \in x_0 + \mathcal{K}_m = x_0 + \mathcal{K}_m(A, r_0)$ .

How to choose  $x_m$ ?

- Galerkin approach:  $b - Ax_m \perp \mathcal{K}_m$

=> FOM Solver (Full Orthogonalization Method)

- Minimum Norm Residual approach:  $\|b - Ax_m\|_2$  minimal

=> GMRES Solver (Generalised Minimal RESidual)

# **Iterative Methods: Krylov Methods**

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**Galerkin Approach – FOM**

- $x_m \in x_0 + \mathcal{K}_m$  means  $\exists y_m \in \mathbb{R}^m$  such that  $x_m = x_0 + V_m y_m$

## Exercise

*How  $y_m$  can be computed to verify  $b - Ax_m \perp \mathcal{K}_m$ ?*

# FOM Algorithm

```
1 Set the initial guess  $x_0$ 
2  $r_0 = b - Ax_0$ ;  $\beta = \|r_0\|$ ;  $v_1 = r_0/\beta$ ;
3 for  $j = 1, 2, \dots, m$  do
4      $w_j = Av_j$ ;
5     for  $i = 1, \dots, j$  do
6          $h_{i,j} = v_i^T w_j$ ;
7          $w_j = w_j - h_{i,j}v_i$ ;
8     end
9      $h_{j+1,j} = \|w_j\|$ ;
10    if  $h_{j+1,j} = 0$  then
11         $m = j$ ;
12        goto 16
13    end
14     $v_{j+1} = w_j/h_{j+1,j}$ 
15 end
16  $y_m = H_m^{-1}(\beta e_1)$ 
17  $x_m = x_0 + V_m y_m$ 
```

**Algorithm 2:** FOM - MGS variant - Y.Saad - Algo 6.4

steps 4 to 8 : Modified Gram-Schmidt (MGS)

## Some remarks on this algorithm

- with this algorithm presented this way, we compute  $x_m \in x_0 + \mathcal{K}_m$  for a given  $m$ .
- there is no guaranty that  $x_m$  is close to the exact solution ( $\eta_{A,b}^N(\tilde{x}_m) < \varepsilon?$  or  $\eta_b^N(\tilde{x}_m) < \varepsilon?$ ).
- we have to re-formulate the algorithm in order to stop when the wanted precision is achieved (or when we have reach a maximum number of iterations).

### Exercise

*re-write the algorithm*

# Improvements of FOM algorithm

**Question:** is it necessary to compute  $x_j$  and  $r_j$  at each iteration?  
(computational cost: 2 matrix×vector products)

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**Question:** is it necessary to compute  $x_j$  and  $r_j$  at each iteration?  
(computational cost: 2 matrix×vector products)

- a priori, yes:  $\|r_j\|$  is present in the stopping criteria
- we show that

$$b - Ax_j = -h_{j+1,j}e_j^T y_j v_{j+1}$$

and

$$\|b - Ax_j\| = h_{j+1,j}|e_j^T y_j|$$

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## Exercise

*Prove these two relations*

# Improvements of FOM algorithm

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and

$$\|b - Ax_j\| = h_{j+1,j}|e_j^T y_j|$$

## Exercise

*Prove these two relations*

$\Rightarrow$

we don't have to compute  $x_j$  and  $r_j$  at each iteration. We will compute these values once we have converged (or we estimate that we have converged).

# Iterative Methods: Krylov Methods

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Minimum Norm Residual approach –  
GMRES

# Minimum Norm Residual approach

- $x_m \in x_0 + \mathcal{K}_m$  means  $\exists y_m \in \mathbb{R}^m$  such that  $x_m = x_0 + V_m y_m$
- **Question:**  
How  $y_m$  can be computed such that  $\|b - Ax_m\|_2$  is minimal?

## Exercise

*Prove that  $\|b - Ax_m\|_2 = \|\beta e_1 - \bar{H}_m y_m\|_2$*

The GMRES iterate is the vector of  $x_0 + \mathcal{K}_m$  such that

$$\begin{aligned}x_m &= x_0 + V_m y_m \\ y_m &= \arg \min_{y \in \mathbb{R}^m} \|\beta e_1 - \bar{H}_m y\|\end{aligned}$$

# GMRES algorithm

```
1 Set the initial guess  $x_0$ 
2  $r_0 = b - Ax_0$ ;  $\beta = \|r_0\|$ ;  $v_1 = r_0/\beta$ ;
3 for  $j = 1, 2, \dots, m$  do
4      $w_j = Av_j$ ;
5     for  $i = 1, \dots, j$  do
6          $h_{i,j} = v_i^T w_j$ ;
7          $w_j = w_j - h_{i,j}v_i$ ;
8     end
9      $h_{j+1,j} = \|w_j\|$ ;
10    if  $h_{j+1,j} = 0$  then
11         $m = j$ ;
12        goto 16
13    end
14     $v_{j+1} = w_j/h_{j+1,j}$ 
15 end
16 Solve the least-squares problem  $y_m = \arg \min \|\beta e_1 - \bar{H}_m y\|$ 
17  $x_m = x_0 + V_m y_m$ 
```

**Algorithm 3:** GMRES - MGS variant - Y.Saad - Algo 6.9



# Solution of the linear least-square problem

A stable solution technique to solve  $\|\beta e_1 - \bar{H}_m y\|_2$  is to use Givens rotations to transform the Hessenberg matrix into upper triangular form.

$$Q_i = \begin{pmatrix} 1 & & & & & \\ & \dots & & & & \\ & & 1 & & & \\ & & & c_i & s_i & \\ & & & -s_i & c_i & \\ & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix} \begin{array}{l} \leftarrow \text{row } i \\ \leftarrow \text{row } i + 1 \end{array}$$

with  $c_i^2 + s_i^2 = 1$ .

Computing  $B = Q_i A$  copies all the rows but the  $i^{\text{th}}$  and  $(i+1)^{\text{th}}$  from  $A$  in  $B$  and replaces those two rows by linear combinations of them such that  $B_{i+1,i} = 0$  if  $s_i = \frac{a_{i+1,i}}{\sqrt{a_{i,i}^2 + a_{i+1,i}^2}}$  and  $c_i = \frac{a_{i,i}}{\sqrt{a_{i,i}^2 + a_{i+1,i}^2}}$ .

## Application of the first Givens rotation to $\bar{H}_5$

$$\text{Consider } \bar{H}_5 = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} \\ h_{21} & h_{22} & h_{23} & h_{24} & h_{25} \\ & h_{32} & h_{33} & h_{34} & h_{35} \\ & & h_{43} & h_{44} & h_{45} \\ & & & h_{54} & h_{55} \\ & & & & h_{65} \end{pmatrix} \text{ and } \beta e_1 = \begin{pmatrix} \beta \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Multiplying by the left by

$$Q_1 = \begin{pmatrix} c_1 & s_1 & & & \\ -s_1 & c_1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ & & & & & 1 \end{pmatrix} \text{ with } \begin{cases} s_1 = \frac{h_{2,1}}{\sqrt{h_{1,1}^2 + h_{2,1}^2}} \\ c_1 = \frac{h_{1,1}}{\sqrt{h_{1,1}^2 + h_{2,1}^2}} \end{cases}.$$

Clearly  $Q_1^T Q_1 = I$ .

## Result after the first rotation

$$\bar{H}_5^{(1)} = \begin{pmatrix} h_{11}^{(1)} & h_{12}^{(1)} & h_{13}^{(1)} & h_{14}^{(1)} & h_{15}^{(1)} \\ 0 & h_{22}^{(1)} & h_{23}^{(1)} & h_{24}^{(1)} & h_{25}^{(1)} \\ & h_{32} & h_{33} & h_{34} & h_{35} \\ & & h_{43} & h_{44} & h_{45} \\ & & & h_{54} & h_{55} \\ & & & & h_{65} \end{pmatrix} \quad \text{and} \quad \bar{g}_1^{(1)} = \begin{pmatrix} c_1 \beta \\ -s_1 \beta \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

## Result after 5 rotations

This elimination process is continued until the 5<sup>th</sup> rotation is applied which leads to

$$\bar{H}_5^{(5)} = \begin{pmatrix} h_{11}^{(1)} & h_{12}^{(1)} & h_{13}^{(1)} & h_{14}^{(1)} & h_{15}^{(1)} \\ 0 & h_{22}^{(2)} & h_{23}^{(2)} & h_{24}^{(2)} & h_{25}^{(2)} \\ & 0 & h_{33}^{(3)} & h_{34}^{(3)} & h_{35}^{(3)} \\ & & 0 & h_{44}^{(4)} & h_{45}^{(4)} \\ & & & 0 & h_{55}^{(5)} \\ & & & & 0 \end{pmatrix} \quad \text{and} \quad \bar{g}^{(5)} = \begin{pmatrix} \gamma_1^{(1)} \\ \gamma_2^{(2)} \\ \gamma_3^{(3)} \\ \gamma_4^{(4)} \\ \gamma_5^{(5)} \\ \gamma_6^{(5)} \end{pmatrix}$$

## QR Factorisation of $\bar{H}_m$

Generally, the rotation  $Q_i$  is defined by

$$s_i = \frac{h_{i+1,i}}{\sqrt{\left(h_{i,i}^{(i-1)}\right)^2 + h_{i+1,1}^2}} \text{ and } c_i = \frac{h_{i,i}^{(i-1)}}{\sqrt{\left(h_{i,i}^{(i-1)}\right)^2 + h_{i+1,1}^2}}. \quad (6)$$

Define, the unitary matrix

$$Q_m = Q_m \dots Q_1 \in \mathbb{R}^{(m+1) \times (m+1)}$$

and let

$$\begin{aligned} \bar{R}_m &= \bar{H}_m^{(m)} = Q_m \bar{H}_m, \\ \bar{g}_m &= \beta Q_m e_1 = (\gamma_1^{(1)} \dots \gamma_m^{(m)} \gamma_{m+1}^{(m)})^T. \end{aligned}$$

Because  $Q_m$  is unitary we have

$$\min \|\beta e_1 - \bar{H}_m y\| = \min \|Q_m (\beta e_1 - \bar{H}_m y)\| = \min \|\bar{g}_m - \bar{R}_m y\|$$

- The solution of the least-squares problem is obtained by solving the triangular system

$$R_m y = g_m,$$

where  $R_m \in \mathbb{R}^{m \times m}$  is obtained from  $\bar{R}_m$  by deleting its last row and similarly  $g_m \in \mathbb{R}^m$  is obtained from  $\bar{g}_m$  by discarding its last component.

The residual of this least-squares problem is the last component of  $\bar{g}_m$ .

- $QR$  factorisation in the general case
- particular case of  $\bar{H}_m$

# Some properties of GMRES

## Proposition

1. *The rank of  $AV_m$  is equal to the rank of  $R_m$ .  
In particular if  $r_{mm} = 0$ , then  $A$  must be singular.*
2.  $y_m = \operatorname{argmin}_{y \in \mathbb{R}^m} \|\beta e_1 - \bar{H}_m y\|$  is given by

$$y_m = R_m^{-1} g_m.$$

3. *The residual vector is*

$$\begin{aligned} b - Ax_m = V_{m+1}(\beta e_1 - \bar{H}_m y_m) &= V_{m+1} Q_m^T (Q_m \beta e_1 - Q_m \bar{H}_m y_m) \\ &= V_{m+1} Q_m^T (\gamma_{m+1}^{(m)} e_{m+1}) \end{aligned}$$

then

$$\|b - Ax_m\| = |\gamma_{m+1}^{(m)}| \quad (7)$$

## Incremental Factorisation of $\bar{H}_m$

The  $QR$  factorization of  $\bar{H}_{m+1}$  can be cheaply computed from the  $QR$  factorization of  $\bar{H}_m$ .

$$\bar{H}_{m+1} = \left( \begin{array}{c|c} \bar{H}_m & \begin{matrix} h_{1,m+1} \\ \vdots \\ h_{m+1,m+1} \end{matrix} \\ \hline 0 & h_{m+2,m+1} \end{array} \right)$$

It is enough to apply the  $m$  Givens rotations computed to factor  $\bar{H}_m$  to the **last column** of  $\bar{H}_{m+1}$  and build and apply a  $(m+1)^{th}$  rotation to zero the  $h_{m+2,m+1}$  entry.



# Incremental Factorisation of $\bar{H}_m$

$$\bar{H}_6^{(5)} = \left( \begin{array}{ccccc|c} h_{11}^{(1)} & h_{12}^{(1)} & h_{13}^{(1)} & h_{14}^{(1)} & h_{15}^{(1)} & h_{16}^{(1)} \\ 0 & h_{22}^{(2)} & h_{23}^{(2)} & h_{24}^{(2)} & h_{25}^{(2)} & h_{26}^{(2)} \\ & 0 & h_{33}^{(3)} & h_{34}^{(3)} & h_{35}^{(3)} & h_{36}^{(3)} \\ & & 0 & h_{44}^{(4)} & h_{45}^{(4)} & h_{46}^{(4)} \\ & & & 0 & h_{55}^{(5)} & h_{56}^{(5)} \\ & & & & 0 & h_{66} \\ & & & & 0 & h_{76} \end{array} \right) \text{ and } \bar{g}_6^{(5)} = \begin{pmatrix} \gamma_1^{(1)} \\ \gamma_2^{(2)} \\ \gamma_3^{(3)} \\ \gamma_4^{(4)} \\ \gamma_5^{(5)} \\ \gamma_6^{(5)} \\ 0 \end{pmatrix}$$

We then apply both to  $\bar{H}_6^{(5)}$  and  $\bar{g}_5^{(6)}$  a 6<sup>th</sup> Givens rotation defined by

$$s_6 = \frac{h_{76}}{\sqrt{(h_{66}^{(5)})^2 + h_{76}^2}} \text{ and } c_6 = \frac{h_{66}^{(5)}}{\sqrt{(h_{66}^{(5)})^2 + h_{76}^2}}$$

to get  $\bar{R}_6$  and  $\bar{g}_6$ .

# Incremental Factorisation of $\bar{H}_m$

In particular we have  $\bar{g}_6 =$  
$$\begin{pmatrix} \gamma_1^{(1)} \\ \gamma_2^{(2)} \\ \gamma_3^{(3)} \\ \gamma_4^{(4)} \\ \gamma_5^{(5)} \\ c_6 \gamma_6^{(5)} \\ -s_6 \gamma_6^{(5)} = \gamma_7^{(6)} \end{pmatrix}$$
 with

$$\|b - Ax_6\| = |\gamma_7^{(6)}| = |-s_6 \gamma_6^{(5)}| = |s_6| |\gamma_6^{(5)}| = |s_6| \|b - Ax_5\|.$$

By induction it can be shown that

$$\|b - Ax_{m+1}\| = |s_m| \|b - Ax_m\|.$$

Remark:

1. The norm of the residual is monotonically decreasing,
2. If  $s_m = 0$ , the solution is found at step  $m$ .

# "Happy Breakdown" of GMRES

It exists one possible breakdown in the GMRES algorithm if  $h_{k+1,k} = 0$  which prevents to increase the dimension of the search space.

## Proposition

*Let  $A$  be a nonsingular matrix. Then the GMRES algorithm breaks down at step  $k$  (i.e.  $h_{k+1,k} = 0$ ) iff the iterate  $x_k$  is the exact solution of  $Ax = b$ .*

# Iterative Methods: Krylov Methods

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Unsymmetric Krylov solver based on  
Arnoldi procedure – the costs

# Computational cost and memory storage of FOM and GMRES

- $n$ , dimension of the problem,
- $nnz(A)$ , number of non-zeros of  $A$ ,
- $m$ , size of the Krylov subspace where we found a solution  $x_m$  that suits us.

## FOM Algorithm

```
1 Set the initial guess  $x_0$ 
2  $r_0 = b - Ax_0$ ;  $\beta = \|r_0\|$ ;  $v_1 = r_0 / \beta$ ;
3 for  $j = 1, 2, \dots, m$  do
4      $w_j = Av_j$ ;
5     for  $i = 1, \dots, j$  do
6          $h_{i,j} = v_i^T w_j$ ;
7          $w_j = w_j - h_{i,j}v_i$ ;
8     end
9      $h_{j+1,j} = \|w_j\|$ ;
10    if  $h_{j+1,j} = 0$  then
11         $m = j$ ;
12        goto 16
13    end
14     $v_{j+1} = w_j / h_{j+1,j}$ 
15 end
16  $y_m = H_m^{-1}(\beta e_1)$ 
17  $x_m = x_0 + V_m y_m$ 
```

**Algorithm 4:** FOM - MGS variant - Y.Saad - Algo 6.4

## FOM:

### FLoating Point OPerations (Flops)

- $m$  matrix  $\times$  vector products:  $\simeq 2m \times nnz(A)$  operations
- iteration  $j$  :  $\simeq 4 \times j \times n$  operations (GS : ddot :  $\simeq 2n$ , daxpy :  $\simeq 2n$ )  
 $\Rightarrow \simeq 2m^2 n$  for  $m$  iterations
- Total :  $\simeq m(2 \times nnz(A) + 2mn)$   
+ solution  $H_m y = \beta e_1$  and matrix  $\times$  vector product  $V_m \cdot y_m$

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 $\Rightarrow \simeq 2m^2n$  for  $m$  iterations
- Total :  $\simeq m(2 \times nnz(A) + 2mn)$   
+ solution  $H_m y = \beta e_1$  and matrix  $\times$  vector product  $V_m \cdot y_m$

### Memory storage

- base  $V_m$ , vectors  $x_0, x_m, w_j$ , matrix  $\bar{H}_m$
- Total :  $\simeq (m + 3)n + \frac{m^2}{2}$

Computational cost and memory storage very similar for **GMRES**.

**Goal:** Reduce the computational cost and/or the memory storage

- restarted versions:  $\text{FOM}(m)$  and  $\text{GMRES}(m)$
- truncated versions:  $\text{DIOM}(m)$  and  $\text{DQGMRES}(m)$



## Restarted versions

- the idea of these versions is to fix a maximum size for the Krylov subspace  $m$ ,
  - if the solution  $x_m$  in this subspace does not suit us, we start again and build a new subspace taking the solution  $x_m$  as the initial guess  $x_0$ ,
  - and this as many times it is necessary to obtain a solution that verifies the convergence criteria.
  - GMRES( $m$ ) algorithm
- 1 Set the initial guess  $x_0$
  - 2 Call GMRES with stopping criteria ( $\eta_b^N(x_k) \leq \varepsilon$  or  $nbit = m$ )
  - 3 **if**  $\eta_b^N(x_k) > \varepsilon$  **then**
  - 4      $x_0 = x_m$
  - 5     goto 2
  - 6 **end**
- $m$ , maximum size is a parameter of this algorithm and depends of constraints of the target computer.

- the idea of these versions is to maintain orthogonality only among the last  $m$  "Arnoldi's" vectors
- that leads to
  - computational gains during orthogonalisation procedure,
  - sparse structure de  $\bar{H}_m$  that can be exploited
    - incremental factorisation **LU** for DIOM(m) ,
    - simplified incremental factorisation **QR** for DQGMRES(m),
  - memory storage gain of the basis  $V_m$  (we can exhibit a short recurrence between  $x_{m+1}$  and  $x_m$ ) [see the idea when describing symmetric solvers].

# Iterative Methods: Symmetric Krylov Solvers

---

# Arnoldi for symmetric problems: the Lanczos method

## Proposition

*Assume that the Arnoldi's method is applied to a real symmetric matrix  $A$ . The matrix  $H_m$  is tridiagonal and symmetric.*

## Exercise

*Prove the Proposition*

The standard notation used to describe the Lanczos algorithm is to use  $T_m$  rather than  $H_m$  and  $\alpha_j \equiv h_{j,j}$  and  $\beta_j \equiv h_{j-1,j} = h_{j,j-1}$ .

$$T_m = \begin{pmatrix} \alpha_1 & \beta_2 & & & \\ \beta_2 & \alpha_2 & \beta_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{m-1} & \alpha_{m-1} & \beta_m \\ & & & \beta_m & \alpha_m \end{pmatrix}.$$



## Cornelius Lanczos

1893-1974: Hungary.

Lanczos worked on relativity and mathematical physics and invented what is now called the Fast Fourier Transform. He worked with Einstein in Berlin, for the Boeing Aircraft Company, at Purdue University and the Department at the Dublin Institute for Advance Study in Ireland.

# Lanczos algorithm

- 1: Choose an initial vector  $v_1$  of norm equal to one.
- 2:  $\beta_1 = 0$  and  $v_0 = 0$
- 3: **for**  $j = 1, 2, \dots m$  **do**
- 4:    $w_j = Av_j - \beta_j v_{j-1}$
- 5:    $\alpha_j = w_j^T v_j$
- 6:    $w_j = w_j - \alpha_j v_j$
- 7:    $\beta_{j+1} = \|w_j\|$
- 8:   **If**  $\beta_{j+1}=0$  **then** Stop.
- 9:    $v_{j+1} = w_j/\beta_{j+1}$
- 10: **end for**

The following “Arnoldi” relations read:

$$AV_m = V_m T_m + \beta_{m+1} v_{m+1} e_m^T, \quad (8)$$

$$V_m^T AV_m = T_m \quad (9)$$

# Arnoldi relations for a symmetric problem

Arnoldi relations:

$$\begin{aligned}AV_m &= V_m H_m + h_{m+1,m} v_{m+1} e_m^T \\AV_m &= V_{m+1} \bar{H}_m \\V_m^T AV_m &= H_m\end{aligned}$$

Arnoldi relations when the matrix is symmetric:

$$\begin{aligned}AV_m &= V_m T_m + \beta_{m+1} v_{m+1} e_m^T, \\V_m^T AV_m &= T_m\end{aligned}$$

# From Arnoldi procedure to Lanczos algorithm

```
1:  $v_1 = b/\|b\|$ ;  
2: for  $j = 1, 2, \dots, m$  do  
3:    $w_j = Av_j$ ;  
4:   for  $i = 1, \dots, j$  do  
5:      $h_{i,j} = v_i^T w_j$ ;  
6:      $w_j = w_j - h_{i,j}v_i$ ;  
7:   end for  
8:   Compute  $h_{j+1,j} = \|w_j\|$ ;  
9:   exit if ( $h_{j+1,j} = 0$ );  
10:  Compute  $v_{j+1} = w_j/h_{j+1,j}$   
11: end for
```

**Algorithm 5:** Arnoldi (MGS)

```
1: Choose an initial vector  $v_1$  of  
   norm equal to one.  
2:  $\beta_1 = 0$  and  $v_0 = 0$   
3: for  $j = 1, 2, \dots, m$  do  
4:    $w_j = Av_j - \beta_j v_{j-1}$   
5:    $\alpha_j = w_j^T v_j$   
6:    $w_j = w_j - \alpha_j v_j$   
7:    $\beta_{j+1} = \|w_j\|$   
8:   If  $\beta_{j+1}=0$  then Stop.  
9:    $v_{j+1} = w_j/\beta_{j+1}$   
10: end for
```

**Algorithm 6:** Lanczos



# Iterative Methods: Symmetric Krylov Solvers

---

Lanczos algorithm for symmetric linear  
systems

# Lanczos algorithm for symmetric linear systems

- Galerkin approach:  
search  $x_m \in x_0 + \mathcal{K}_m$  such that  $b - Ax_m \perp \mathcal{K}_m$

## Exercise

*how express  $x_m$ ?*

# Lanczos algorithm for symmetric linear systems

- 1:  $r_0 = b - Ax_0$ ;  $\beta = \|r_0\|$ ;  $v_1 = r_0/\beta$
- 2: **for**  $j = 1, 2, \dots, m$  **do**
- 3:    $w_j = Av_j - \beta_j v_{j-1}$  (If  $j = 1$  set  $\beta_1 v_0 = 0$ )
- 4:    $\alpha_j = w_j^T v_j$
- 5:    $w_j = w_j - \alpha_j v_j$
- 6:    $\beta_{j+1} = \|w_j\|$
- 7:   **if**  $\beta_{j+1}=0$  **then**
- 8:      $m = j$ ; goto 12
- 9:   **end if**
- 10:    $v_{j+1} = w_j/\beta_{j+1}$
- 11: **end for**
- 12: Set  $T_m = \text{tridiag}(\beta_i, \alpha_i, \beta_{i+1})$  and  $V_m = (v_1, \dots, v_m)$ .
- 13: Compute  $y_m = T_m^{-1}(\beta e_1)$  and  $x_m = x_0 + V_m y_m$ .

## Exercise

*Show that the residual associated with  $x_m$  is colinear to  $v_{m+1}$  ?*

We have (third time?)

$$\begin{aligned}b - Ax_m &= b - A(x_0 + V_m y_m) \\&= r_0 - AV_m y_m \\&= V_m(\beta e_1) - (V_m T_m + \beta_{m+1} v_{m+1} e_m^T) y_m \text{ (from (8))} \\&= V_m \underbrace{(\beta e_1 - T_m y_m)}_0 - \beta_{m+1} v_{m+1} e_m^T y_m.\end{aligned}$$

1.  $T_m$ , 2 vectors
2. To compute  $v_{j+1}$ , we need  $v_{j-1}$  and  $v_j$ , 3 vectors?  
=> NO, we still need the base  $V_m$  to compute  $x_m$

# Iterative Methods: Symmetric Krylov Solvers

---

The D-Lanczos variant

As  $T_m$  is tridiagonal, we can compute an incremental factorisation **L.U** of  $T_m$  and we will show that:

- factors can be updated at each Lanczos iteration,
- we do not have to store all the basis  $V_m$  to compute  $x_m$ .

## Rappel: factorisation $L.U$

Factorisation  $L.U$  :

For  $k$  from 1 to  $n - 1$  :

1. Compute column  $k$  of  $L$ :

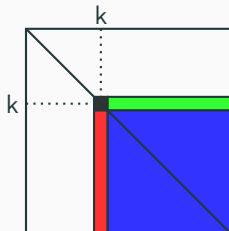
$$A(k+1:n, k) = A(k+1:n, k) / A(k, k)$$

2. Update the sub-block:

$$A(k+1:n, k+1:n)$$

$$= A(k+1:n, k+1:n)$$

$$- A(k+1:n, k) \times A(k, k+1:n)$$





## Factorisation $L.U$ of $T_m = L_m \times U_m$

Exemple :  $T_5 = L_5 \times U_5$

$$T_5 = \begin{pmatrix} \alpha_1 & \beta_2 & & & \\ \beta_2 & \alpha_2 & \beta_3 & & \\ & \beta_3 & \alpha_3 & \beta_4 & \\ & & \beta_4 & \alpha_4 & \beta_5 \\ & & & \beta_5 & \alpha_5 \end{pmatrix}$$

## Factorisation $L.U$ of $T_m = L_m \times U_m$

We show that

- $L$  and  $U$  are bi-diagonal matrices
- the coefficients of the upper diagonal of  $U_m$  are the coefficients  $\beta_i$  of  $T_m$

$$T_5 = \begin{pmatrix} 1 & & & & \\ \lambda_2 & 1 & & & \\ & \lambda_3 & 1 & & \\ & & \lambda_4 & 1 & \\ & & & \lambda_5 & 1 \end{pmatrix} \times \begin{pmatrix} \eta_1 & \beta_2 & & & \\ & \eta_2 & \beta_3 & & \\ & & \eta_3 & \beta_4 & \\ & & & \eta_4 & \beta_5 \\ & & & & \eta_5 \end{pmatrix}$$

## Factorisation $LU$ of $T_m = L_m \times U_m$

Another way to represent the factors  $L$  and  $U$  in the same memory space  $A$

$$\text{memory}(A) := \begin{pmatrix} \eta_1 & \beta_2 & & & \\ \lambda_2 & \eta_2 & \beta_3 & & \\ & \lambda_3 & \eta_3 & \beta_4 & \\ & & \lambda_4 & \eta_4 & \beta_5 \\ & & & \lambda_5 & \eta_5 \end{pmatrix}$$

(remember factorization L.U)

## How compute coefficients of $L_m$ and $U_m$

- Using a recurrence, we show that

$$\lambda_m = \frac{\beta_m}{\eta_{m-1}}, \quad m \in \{2, \dots\}$$

$$\eta_m = \alpha_m - \lambda_m \beta_m, \quad m \in \{1, \dots\}, \beta_1 = 0$$

The approximate solution is given by

$$x_m = x_0 + \underbrace{V_m U_m^{-1}}_{P_m} \underbrace{L_m^{-1}(\beta e_1)}_{z_m},$$

then

$$x_m = x_0 + P_m z_m.$$

Because of the structure of  $U_m$ , we can easily update  $P_m$  and compute its last column using the previous  $p_i$ 's and  $v_m$ .

Equating the last columns of  $P_m U_m = V_m$  gives

$$\begin{aligned}\eta_m p_m + \beta_m p_{m-1} &= v_m \\ \Rightarrow p_m &= \eta_m^{-1} (v_m - \beta_m p_{m-1}).\end{aligned}$$

## Expression of $x_m$

Similarly, we can derive an update for  $z_m$  exploiting the structure of  $L_m$  (bidiagonal matrix)

$$z_m = \begin{pmatrix} z_{m-1} \\ \xi_m \end{pmatrix},$$

where  $\xi_m = -\lambda_m \xi_{m-1}$ .

Consequently we have

$$x_m = x_0 + (P_{m-1}p_m) \begin{pmatrix} z_{m-1} \\ \xi_m \end{pmatrix} = \underbrace{x_0 + P_{m-1}z_{m-1}}_{x_{m-1}} + \xi_m p_m$$

- Short Relation between  $x_m$  et  $x_{m-1}$  :  $x_m = x_{m-1} + \xi_m p_m$   
 $\Rightarrow$  no more storage of  $V_m$
- Memory usage : a vector for  $p_m$  and some scalars

# D-Lanczos algorithm

- 1: Set  $r_0 = b - Ax_0$ ;  $\xi_1 = \beta = \|r_0\|$  and  $v_1 = r_0/\beta$
- 2: Set  $\lambda_1 = \beta_1 = 0$ ,  $p_0 = 0$
- 3: **for**  $j = 1, \dots$  **do**
- 4:   Compute  $w_j = Av_j - \beta_j v_{j-1}$  and  $\alpha_j = w_j^T v_j$
- 5:   **if**  $j > 1$  **then**
- 6:     computed  $\lambda_j = \frac{\beta_j}{\eta_{j-1}}$  and  $\xi_j = -\lambda_j \xi_{j-1}$
- 7:   **end if**
- 8:    $\eta_j = \alpha_j - \lambda_j \beta_j$
- 9:    $p_j = \eta_j^{-1} (v_j - \beta_j p_{j-1})$
- 10:    $x_j = x_{j-1} + \xi_j p_j$
- 11:   **if** converged **then**
- 12:     Stop
- 13:   **end if**
- 14:    $w_j = w_j - \alpha_j v_j$
- 15:    $\beta_{j+1} = \|w_j\|$ ,  $v_{j+1} = w_j/\beta_{j+1}$
- 16: **end for**

## Exercise

Show that:  $\|b - Ax_m\| = \beta_{m+1} \left| \frac{\xi_m}{\eta_m} \right|$



# Some important properties

## Proposition

Let  $r_m = b - Ax_m$  and  $p_m$  the vectors generated by the D-Lanczos algorithm. We have:

1. the residual vector  $r_m$  is colinear to  $v_{m+1}$ . As a consequence, the residual vectors are orthogonal to each other.
2. the vectors  $p_i$  are  $A$ -conjugate, that is:  $\forall i \neq j, p_i^T A p_j = 0$ .

## Iterative Method: Conjugate Gradient

---

# D-Lanczos algorithm

- 1: Set  $r_0 = b - Ax_0$ ;  $\xi_1 = \beta = \|r_0\|$  and  $v_1 = r_0/\beta$
- 2: Set  $\lambda_1 = \beta_1 = 0$ ,  $p_0 = 0$
- 3: **for**  $j = 1, \dots$  **do**
- 4:   Compute  $w_j = Av_j - \beta_j v_{j-1}$  and  $\alpha_j = w_j^T v_j$
- 5:   **if**  $j > 1$  **then**
- 6:     computed  $\lambda_j = \frac{\beta_j}{\eta_{j-1}}$  and  $\xi_j = -\lambda_j \xi_{j-1}$
- 7:   **end if**
- 8:    $\eta_j = \alpha_j - \lambda_j \beta_j$
- 9:    $p_j = \eta_j^{-1} (v_j - \beta_j p_{j-1})$
- 10:    $x_j = x_{j-1} + \xi_j p_j$  // new iterate function of the previous one
- 11:   **if** converged **then**
- 12:     Stop
- 13:   **end if**
- 14:    $w_j = w_j - \alpha_j v_j$
- 15:    $\beta_{j+1} = \|w_j\|$ ,  $v_{j+1} = w_j/\beta_{j+1}$
- 16: **end for**

# D-Lanczos algorithm

- 1: Set  $r_0 = b - Ax_0$ ;  $\xi_1 = \beta = \|r_0\|$  and  $v_1 = r_0/\beta$
- 2: Set  $\lambda_1 = \beta_1 = 0$ ,  $p_0 = 0$
- 3: **for**  $j = 1, \dots$  **do**
- 4:   Compute  $w_j = Av_j - \beta_j v_{j-1}$  and  $\alpha_j = w_j^T v_j$
- 5:   **if**  $j > 1$  **then**
- 6:     computed  $\lambda_j = \frac{\beta_j}{\eta_{j-1}}$  and  $\xi_j = -\lambda_j \xi_{j-1}$
- 7:   **end if**
- 8:    $\eta_j = \alpha_j - \lambda_j \beta_j$
- 9:    $p_j = \eta_j^{-1} (v_j - \beta_j p_{j-1})$  // we need to compute a new direction
- 10:    $x_j = x_{j-1} + \xi_j p_j$  // new iterate function of the previous one
- 11:   **if** converged **then**
- 12:     Stop
- 13:   **end if**
- 14:    $w_j = w_j - \alpha_j v_j$
- 15:    $\beta_{j+1} = \|w_j\|$ ,  $v_{j+1} = w_j/\beta_{j+1}$
- 16: **end for**

# D-Lanczos algorithm

- 1: Set  $r_0 = b - Ax_0$ ;  $\xi_1 = \beta = \|r_0\|$  and  $v_1 = r_0/\beta$
- 2: Set  $\lambda_1 = \beta_1 = 0$ ,  $p_0 = 0$
- 3: **for**  $j = 1, \dots$  **do**
- 4:   Compute  $w_j = Av_j - \beta_j v_{j-1}$  and  $\alpha_j = w_j^T v_j$
- 5:   **if**  $j > 1$  **then**
- 6:     computed  $\lambda_j = \frac{\beta_j}{\eta_{j-1}}$  and  $\xi_j = -\lambda_j \xi_{j-1}$
- 7:   **end if**
- 8:    $\eta_j = \alpha_j - \lambda_j \beta_j$
- 9:    $p_j = \eta_j^{-1}(\textcolor{red}{c}r_{j-1} - \beta_j p_{j-1})$  //  $r_{j-1}$  and  $v_j$  are colinears
- 10:    $x_j = x_{j-1} + \xi_j p_j$
- 11:   **if** converged **then**
- 12:     Stop
- 13:   **end if**
- 14:    $w_j = w_j - \alpha_j v_j$
- 15:    $\beta_{j+1} = \|w_j\|$ ,  $v_{j+1} = w_j/\beta_{j+1}$
- 16: **end for**

## Conjugate Gradient can be derived from D-Lanczos

With a translation of  $p_j$  index to conform with standard notation, we express [step 10](#) of the algorithm:

$$x_{j+1} = x_j + \delta_j p_j \quad (10)$$

and as consequence

$$r_{j+1} = r_j - \delta_j A p_j. \quad (11)$$

Thus a first expression of  $\delta_j$

$$\delta_j = \frac{r_j^T r_j}{r_j^T A p_j}. \quad (12)$$

## Conjugate Gradient can be derived from D-Lanczos (cont.)

We are looking for a new direction  $p_{j+1}$  as a linear combination of  $r_{j+1}$  et  $p_j$  (step 9 with translation of  $p_j$  index)

$$p_{j+1} = r_{j+1} + \gamma_j p_j, \quad (13)$$

We show that we can re-write  $\delta_j$

$$\delta_j = \frac{r_j^T r_j}{p_j^T A p_j}. \quad (14)$$

Then a first expression for  $\gamma_j$  is

$$\gamma_j = -\frac{r_{j+1}^T A p_j}{p_j^T A p_j}, \quad (15)$$

that can be eventually written as

$$\gamma_j = \frac{r_{j+1}^T r_{j+1}}{r_j^T r_j}. \quad (16)$$

# Conjugate Gradient Algorithm

To use the standard notation, let's rename  $\alpha_j = \delta_j$  and  $\beta_j = \gamma_j$  and summarize the two previous slides:

1:  $r_0 = b - Ax_0$ ,  $p_0 = r_0$ .

2: **for**  $j = 0, 1, \dots$  **do**

3:  $\alpha_j = (r_j^T r_j) / (p_j^T A p_j)$

4:  $x_{j+1} = x_j + \alpha_j p_j$

5:  $r_{j+1} = r_j - \alpha_j A p_j$

6:  $\beta_j = (r_{j+1}^T r_{j+1}) / (r_j^T r_j)$

7:  $p_{j+1} = r_{j+1} + \beta_j p_j$

8: **end for**

- Conjugate Gradient is a Krylov solver!
- Beware:  $\alpha_j$  and  $\beta_j$  of GC have nothing to do with those of D-Lanczos algorithm.
- Memory usage: 4 vectors  $(x, p, Ap, r)$  .vs. 5 for D-Lanczos  $(v_m, v_{m-1}, w, p, x)$ .



## Some important remarks

The algorithm relies on a  $LU$  factorization without pivoting of a symmetric tridiagonal matrix. This factorization might not exist and CG can break-down.

1. For symmetric positive definite (SPD) matrices,  $T_m$  is always non singular and a stable  $LU$  decomposition exists. **CG** is then the **method of choice** for this class of matrices.
2. For symmetric indefinite matrices a technique based on a **LQ** decomposition of  $T_m$  exists that is known as **SYMMLQ**.
3. For symmetric indefinite matrices, the solution of the linear system involving  $T_m$  can be replaced by the solution of a linear least-square involving  $\bar{T}_m$ . This method is known as **MINRES** and belong to the minimum norm error class of Krylov solvers.

# Conjugate Gradient properties for SPD case (1)

## Proposition

*Because  $A$  is SPD, the bilinear form  $x^T A y$  define an inner product.*

*The Galerkin condition  $b - Ax_m \perp \mathcal{K}_m$  can be written  $A(x_m - x^*) \perp \mathcal{K}_m$  which also reads  $(x_m - x^*) \perp_A \mathcal{K}_m$ .*

*This latter condition implies that*

$$\|x_m - x^*\|_A$$

*is minimal over  $\mathcal{K}_m$*

## Conjugate Gradient properties for SPD case (2)

We have the following upper-bound for the convergence rate of CG in the SPD case.

### Proposition

*Let  $x_m$  be the  $m^{\text{th}}$  iterate generated by the CG algorithm then*

$$\|x_m - x^*\|_A \leq 2 \cdot \left( \frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1} \right)^m \|x_0 - x^*\|_A.$$

## **Iterative Methods: Algebraic preconditioning techniques**

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# **Iterative Methods: Algebraic preconditioning techniques**

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**Some backgrounds**

# Some properties of Krylov solvers

- CG

$$\|x_m - x^*\|_A = \min_{p \in \mathbb{P}_m, p(0)=1} \|p(A)(x_0 - x^*)\|_A$$

- GMRES, MINRES

$$\|r_m\| = \min_{p \in \mathbb{P}_m, p(0)=1} \|p(A)r_0\|_2$$

- Bound on the rate of convergence of CG:

$$\|x_m - x^*\|_A \leq 2 \cdot \left( \frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1} \right)^m \|x_0 - x^*\|_A.$$

## Some properties of Krylov solvers

- Assumption: if  $A$  diagonalizable with  $m$  distinct eigenvalues then CG, GMRES, MINRES converges in at most  $m$  steps (Cayley-Hamilton theorem - minimal polynomial).
- Assumption: if  $r_0$  has  $k$  components in the eigenbasis then CG, GMRES, MINRES converges in  $k$  steps.

# **Iterative Methods: Algebraic preconditioning techniques**

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**Driving principles**



# Driving principles to design preconditioners

Find a non-singular matrix  $M$  such that  $M.A$  has "better" properties v.s. the convergence behaviour of the selected Krylov solver:

- $M.A$  has less distinct eigenvalues
- $M.A \approx I$  in some sense

The preconditioner should

- be cheap to compute and to store,
- be cheap to apply,
- ensure a fast convergence.

With a good preconditioner the solution time for the preconditioned system should be significantly less than for the unpreconditioned system.

# **Iterative Methods: Algebraic preconditioning techniques**

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**Preconditioned CG**

## The particular case of CG

- for CG,  $M$  must be symmetric (to keep a symmetric linear system)
- $M$  be given in a factorized form  $CC^T$
- CG can be applied to  $\tilde{A}\tilde{x} = \tilde{b}$   
 $\tilde{A} = C^TAC$ ,  $C\tilde{x} = x$  and  $\tilde{b} = C^Tb$  (Saad 9.4 with different notations).

# Notations

- Let  $x_k = C\tilde{x}_k$  ;  $p_k = C\tilde{p}_k$  ;  $\tilde{r}_k = C^T r_k$  ;  $z_k = CC^T r_k = Mr_k$
- Conjugate Gradient algorithm for preconditioned system:

```
1:  $\tilde{r}_0 = \tilde{b} - \tilde{A}\tilde{x}_0$ ,  $\tilde{p}_0 = \tilde{r}_0$ .  
2: for  $k = 0, 1, \dots$  do  
3:    $\alpha_k = (\tilde{r}_k^T \tilde{r}_k) / (\tilde{p}_k^T \tilde{A}\tilde{p}_k)$   
4:    $\tilde{x}_{k+1} = \tilde{x}_k + \alpha_k \tilde{p}_k$   
5:    $\tilde{r}_{k+1} = \tilde{r}_k - \alpha_k \tilde{A}\tilde{p}_k$   
6:    $\beta_k = (\tilde{r}_{k+1}^T \tilde{r}_{k+1}) / (\tilde{r}_k^T \tilde{r}_k)$   
7:    $\tilde{p}_{k+1} = \tilde{r}_{k+1} + \beta_k \tilde{p}_k$   
8:   if Convergence Stop  
9: end for
```

- Let's re-arrange the algorithm  $\Rightarrow$  PCG algorithm



Writing the algorithm only using the unpreconditioned variables leads to:

## Preconditioned Conjugate Gradient algorithm

1. Compute  $r_0 = b - Ax_0$ ,  $z_0 = Mr_0$  and  $p_0 = z_0$
2. For  $k=0, 2, \dots$  Do
3.    $\alpha_k = r_k^T z_k / p_k^T A p_k$
4.    $x_{k+1} = x_k + \alpha_k p_k$
5.    $r_{k+1} = r_k - \alpha_k A p_k$
6.    $z_{k+1} = M r_{k+1}$
7.    $\beta_k = r_{k+1}^T z_{k+1} / r_k^T z_k$
8.    $p_{k+1} = z_{k+1} + \beta_k p_k$
9.   if  $x_k$  accurate enough then stop
10. EndDo

## Exercise

### 1. *A*-norm minimization

- *A*-norm of preconditioned system  $\|\tilde{x}_k - \tilde{x}^*\|_{\tilde{A}}$  is minimal over  $\mathcal{K}(\tilde{A}, \tilde{b}, k)$
- $\|x_k - x^*\|_A$ ?

### 2. Orthogonality

- $p_k$ ?
- $r_k$ ?

# **Iterative Methods: Algebraic preconditioning techniques**

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**Preconditioner taxonomy and examples**



# Preconditioner taxonomy and examples

There are two main classes of preconditioners

- **Implicit preconditioners:**

approximate  $A$  with a matrix  $M$  such that solving the linear system  $Mz = r$  is easy.

- **Explicit preconditioners:**

approximate  $A^{-1}$  with a matrix  $M$  and just perform  $z = Mr$ .

The governing ideas in the design of the preconditioners are very similar to those followed to define iterative stationary schemes. Consequently, all the stationary methods can be used to define preconditioners.

# Stationary methods

Let  $x_0$  be given and  $M \in \mathbb{R}^{n \times n}$  a nonsingular matrix, compute

$$x_k = x_{k-1} + M(b - Ax_{k-1}).$$

Note that  $b - Ax_{k-1} = A(x^* - x_{k-1}) \Rightarrow$  the best  $M$  is  $A^{-1}$ .

The stationary scheme converges to  $x^* = A^{-1}b$  for any  $x_0$  iff  $\rho(I - MA) < 1$ , where  $\rho(\cdot)$  denotes the spectral radius.

Let  $A = L + D + U$

- $M = I$  : Richardson method,
- $M = D^{-1}$  : Jacobi method,
- $M = (L + D)^{-1}$  : Gauss-Seidel method.

Notice that  $M$  has always a special structure and the inverse must never be explicitly computed ( $z = B^{-1}y$  reads *solve the linear system  $Bz = y$* ).

# Preconditioner location

Several possibilities exist to solve  $Ax = b$ :

- Left preconditioner

$$MAx = Mb.$$

- Right preconditioner

$$AMy = b \text{ with } x = My.$$

- Split preconditioner if  $M = M_1 M_2$

$$M_2 A M_1 y = M_2 b \text{ with } x = M_1 y.$$

Notice that the spectrum of  $MA$ ,  $AM$  and  $M_2 A M_1$  are identical (for any matrices  $B$  and  $C$ , the eigenvalues of  $BC$  are the same as those of  $CB$ )

## Preconditioner location v.s. stopping criterion

The stopping criterion are based on backward error

$$\eta_{A,b}^N = \frac{\|b - Ax\|}{\|A\|\|x\| + \|b\|} < \varepsilon \text{ or } \eta_b^N = \frac{\|b - Ax\|}{\|b\|} < \varepsilon.$$

In PCG we can still compute  $\eta$ .

For GMRES, using a preconditioner means running GMRES on

|                    |                     |                     |
|--------------------|---------------------|---------------------|
| Left precondition. | Right precondition. | Split precondition. |
| $MAx = Mb$         | $AMy = b$           | $M_2AM_1y = M_2b$   |

The free estimate of the residual norm of GMRES is associated with the preconditioned system.

## Preconditioner location v.s. stopping criterion (cont)

|                | Left precondition.                           | Right precondition.                        | Split precondition.  |
|----------------|--|--|--|
| $\eta_M(A, b)$ | $\frac{\ MAx - Mb\ }{\ MA\  \ x\  + \ Mb\ }$ | $\frac{\ AMy - b\ }{\ AM\  \ x\  + \ b\ }$ | $\frac{\ M_2AM_1y - M_2b\ }{\ M_2AM_1\  \ y\  + \ M_2b\ }$ |
| $\eta_M(b)$    | $\frac{\ MAx - Mb\ }{\ Mb\ }$                | $\frac{\ AMy - b\ }{\ b\ }$                | $\frac{\ M_2AM_1y - M_2b\ }{\ M_2b\ }$                     |

Using  $\eta_M(b)$  for right preconditioned linear system will monitor the convergence as if no preconditioner was used.

# **Iterative Methods: Algebraic preconditioning techniques**

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**Some classical algebraic preconditioners**

## Some classical algebraic preconditioner

- Incomplete factorization :  $IC$ ,  $ILU(p)$ ,  $ILU(p, \tau)$
- SPAI (Sparse Approximate Inverse): compute the sparse approximate inverse by minimizing the Frobenius norm  $\|MA - I\|_F$
- FSAI (Factorized Sparse Approximate inverse): compute the sparse approximate inverse of the Cholesky factor by minimizing the Frobenius norm  $\|I - GL\|_F$
- AINV (Approximate Inverse): compute the sparse approximate inverse of the  $LDU$  or  $LDL^T$  factors using an incomplete biconjugation process

# Incomplete factorizations

One variant of the  $LU$  factorization writes:

IKJ variant - Top looking variant

```
1.  for  $i = 2, \dots, n$  do
2.    for  $k = 1, \dots, i - 1$  do
3.       $a_{i,k} = a_{i,k} / a_{k,k}$ 
4.    for  $j = k + 1, \dots, n$  do
5.       $a_{i,j} = a_{i,j} - a_{i,k} * a_{k,j}$ 
6.    end for
7.  end for
```



## Zero fill-in ILU - $ILU(0)$

Let denote  $NZ(A)$  the set of (row,column) index of the nonzero entries of  $A$ .

$ILU(0)$

1. **for**  $i = 2, \dots, n$  **do**
2.     **for**  $k = 1, \dots, i - 1$  and  $(i, k) \in NZ(A)$  **do**
3.          $a_{i,k} = a_{i,k} / a_{k,k}$
4.     **for**  $j = k + 1, \dots, n$  and  $(i, j) \in NZ(A)$  **do**
5.          $a_{i,j} = a_{i,j} - a_{i,k} * a_{k,j}$
6.     **end for**
7. **end for**
8. **end for**

### Definition

The initial level of fill of an entry  $a_{i,j}$  is defined by:

$$lev(i,j) = \begin{cases} 0 & \text{if } a_{i,j} \neq 0 \text{ or } i = j, \\ \infty & \text{otherwise.} \end{cases}$$

Each time this entry is modified in line 5 of the  $LU$  top looking algorithm, its level fill is updated by

$$lev(i,j) = \min\{lev(i,j), lev(i,k) + lev(k,j) + 1\}. \quad (17)$$

## A first example

$$A = \begin{pmatrix} x & x & x & x \\ x & x & & \\ x & & x & \\ x & & & x \end{pmatrix} \quad lev = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \infty & \infty \\ 0 & \infty & 0 & \infty \\ 0 & \infty & \infty & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

ILU(p)

1. **for** all nonzero entries  $a_{i,j}$  set  $lev_{i,j} = 0$
2. **for**  $i = 2, \dots, n$  **do**
3.   **for**  $k = 1, \dots, i - 1$  and  $lev_{i,k} \leq p$  **do**
4.      $a_{i,k} = a_{i,k} / a_{k,k}$
5.     **for**  $j = k + 1, \dots, n$  **do**
6.        $a_{i,j} = a_{i,j} - a_{i,k} * a_{k,j}$
7.        $lev(i, j) = \min\{lev(i, j), lev(i, k) + lev(k, j) + 1\}$
8.       **if**  $lev(i, j) > p$  **then**  $a_{i,j} = 0$
9.     **end for**
10.   **end for**
11. **end for**

- Fix a drop tolerance  $\tau$  and a number of fill  $p$  to be allowed in each row of the incomplete  $LU$  factors. At each step of the elimination process, drop all fill-ins that are smaller than  $\tau$  times the 2-norm of the current row; for all the remaining ones keep only the  $p$  largest.
- Trade-off between amount of fill-in (construction time and application time for the preconditioner) and decrease of number of iterations.

# **Iterative Methods: Algebraic preconditioning techniques**

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**Spectral preconditioners**

## Motivations

- The convergence of Krylov methods for solving the linear system often depends to a large extent on the eigenvalue distribution. In many cases, it is observed that “removing” the smallest eigenvalues can greatly improve the convergence
- Many preconditioners are able to cluster most of the eigenvalues close to one but still leave a few close to the origin
- “Moving” these eigenvalues by tuning the parameters that control these preconditioners is often difficult and might lead to very expensive preconditioners to set-up and to apply
- Performing a low rank update might enable to shift the smallest eigenvalues.

# **Iterative Methods: Algebraic preconditioning techniques**

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**Multigrid preconditioners**

Part 2 with Carola Kruse



# **Iterative Methods: Algebraic preconditioning techniques**

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**Preconditioners built from a Domain  
Decomposition Method**

Part 3 with Alena Kopanicakova