Inverse problem in spatial hydrology: infering rivers bathymetry from water surface measurements

Equations of the mathematical problem

We consider here the *steady-state* moreover *semi-linearized form of the original non linear flow model.*

The equations corresponding to your precise problem has to be written by yourself.
The notes below aim at helping the students work. A few of the equations have been partly written during the lectures.

The direct model

The direct model reads:

$$A(b(x);H(x)) = F(b(x)) \tag{1}$$

with

$$A(b;H) = -\Lambda_{ref}(b)\partial^2_{xx}H(x) + \partial_x H(x) + \frac{\partial_x w(x)}{w(x)}H(x)$$

$$F(b) = \partial_x b(x) + \frac{\partial_x w(x)}{w(x)}b(x)$$

The BCs are non-homogeneous Dirichlet ones.
The effective diffusivity is here given from a reference solution $H_{ref}(x)$: $\Lambda_{ref}(b) = \frac{3}{10}\frac{(H_{ref}(x)-b(x))}{|\partial_x H_{ref}(x)|}$.

Since posed in 1D, the equation is actually an ODE. However, the direct model remains a Boundary Value Problem since the presence of BCs!

Given b, the unique solution of this direct model is denoted by H^b, $H^b = \mathcal{M}(b)$.

The unknown to be identified (the control variable) is $b(x)$.
That is there is a single control variable; however, it is spatially distributed.

The objective function is classically defined as:

$$J_\alpha(b;H) = J_{obs}(H) + \alpha_{reg}J_{reg}(b) \tag{2}$$

with $J_{obs}(H) = \frac{1}{2}\|H - H_{obs}\|_2^2$.

The direct model in weak form For all $z \in V = H^1_0(\Omega)$,

$$\boxed{a(b;H,z) = l(b;z)} \tag{3}$$

with:

$$a(b;H,z) = \int_\Omega \partial_x\left(\Lambda_{ref}(b)z(x)\right)\partial_x H(x)\ dx + \int_\Omega \partial_x H(x)z(x)\ dx + \int_\Omega \frac{\partial w(x)}{w(x)}H(x)z(x)\ dx \tag{4}$$

$$l(b;z) = \int_\Omega \left(\partial_x b(x) + \frac{\partial_x w(x)}{w(x)}b(x)\right)z(x)\ dx \tag{5}$$

Exercise

- Write the TLM of the model.
- Write the adjoint model.
- Write the expression of $j'(b)$.

] **Exercise**

- Write the TLM of the model.
- Write the adjoint model.
- Write the expression of $j'(b)$.

The Tangent Linear Model (TLM)

The TLM is not necessary if adopting the adjoint method to compute the gradient $\nabla j(b)$. Here, the TLM is derived for illustration only.

Given a bathymetry value b_0, given a "direction" of perturbation δb, given the direct model solution H^{b_0}, the TLM reads:

$$\partial_b \left(A(b_0(x); H^{b_0}(x)) \right) \cdot \delta b = \partial_b F(b_0(x)) \cdot \delta b \tag{6}$$

The unknown of the linearized model is the derivative of the state wrt b at b_0 in the direction δb. It is denoted by $W^{\delta b}$, $W^{\delta b} = \partial_b H^{b_0} \cdot \delta b$.

The TLM reads:

$$d_b \left(-\Lambda_{ref}(b_0) \partial^2_{xx} H^{b_0}(x) + \partial_x H^{b_0}(x) + \frac{\partial_x w(x)}{w(x)} H^{b_0}(x) \right) \cdot \delta b = d_b \left(F(\partial_x b(x) + \frac{\partial_x w(x)}{w(x)} b(x) \right) \cdot \delta b \tag{7}$$

$$d_b \left(-\Lambda_{ref}(b_0) \right) \cdot \delta b \; \partial^2_{xx} H^{b_0}(x) - \Lambda_1(b_0) \partial^2_{xx} W^{\delta b}(x) + \partial_x W^{\delta b}(x) + \frac{\partial_x w(x)}{w(x)} W^{\delta b}(x) = ... \tag{8}$$

$$... = \partial_x(\delta b)(x) + \frac{\partial_x w(x)}{w(x)} \; \delta b(x) \tag{9}$$

That is:

$$\boxed{-\Lambda_{ref}(b_0) \; \partial^2_{xx} W^{\delta b}(x) + \partial_x W^{\delta b}(x) + \frac{\partial w(x)}{w(x)} W^{\delta b}(x) = R(H^{b_0}; \delta b)} \tag{10}$$

with: $R(H^{b_0}; \delta b) = \partial_x(\delta b)(x) + \frac{\partial_x w(x)}{w(x)} \; \delta b(x) - \frac{3}{10} \frac{\delta b(x)}{|\partial_x H_{ref}(x)|} \; \partial^2_{xx} H^{b_0}(x)$.

The adjoint model

Given b, given H^b the direct model solution, find the adjoint state P satisfying the following equation.
For all $z \in V$,

$$< \left[\partial_H A(b; H^b) \right]^* \cdot P, z >_{V' \times V} = < \partial_H J_\alpha(b; H^b), z >_{V' \times V} \tag{11}$$

The PDE is linear (the operator $H \mapsto A(b; H)$ is linear), therefore: $\left[\partial_H A(b; H^b) \right]^* \cdot P = A^*(b; P)$.

Noting that $\partial_H J(b; H^b) = J'_{obs}(H^b)$, we get:

$$< A^*(b; P), z >_{V' \times V} = < J'_{obs}(H^b), z >_{V' \times V} \tag{12}$$

with H^b the (unique) solution of the direct model.
NB. The equation is equivalent to: $A^*(b; P) = \partial_H J_\alpha(b; H^b)$ in V'.

The adjoint equation in weak form Observe that for all $z \in V = H_0^1(\Omega)$, $\int_\Omega \partial_x H(x)\ z(x)\ dx = -\int_\Omega \partial_x z(x)\ H(x)\ dx$.

The adjoint equation in weak form reads as follows.
Find $P \in V = H_0^1(\Omega)$ such that for all $z \in V$,

$$\boxed{a^*(b; P, z) = J'_{obs}(H^b) \cdot z} \tag{13}$$

with

$$\boxed{a^*(b; P, z) \equiv a(b; z, P) = \int_\Omega \partial_x \left(\Lambda_{ref}(b)\ P(x)\right) \partial_x z\ dx - \int_\Omega \partial_x P(x)\ z(x)\ dx + \int_\Omega \frac{\partial_x w(x)}{w(x)}\ P(x)\ z(x)\ dx} \tag{14}$$

Classical form of the adjoint model We have $P \in V = H_0^1(\Omega)$ therefore $P = 0$ on $\partial\Omega$.

Moreover for all $z \in V$,

$$\int_\Omega -\partial^2_{xx} \left(\Lambda_{ref}(b)\ P(x)\right) z(x)\ dx - \int_\Omega \partial_x P(x)\ z(x)\ dx + \int_\Omega \frac{\partial_x w(x)}{w(x)}\ P(x)\ z(x)\ dx \tag{15}$$

Therefore the classical form of the adjoint equation reads:

$$-\partial^2_{xx} \left(\Lambda_{ref}(b)\ P(x)\right) - \partial_x P(x) + \frac{\partial_x w(x)}{w(x)}\ P(x) = J'_{obs}(H^b) \tag{16}$$

accompanied with the BC: $P = 0$ on $\partial\Omega$.

For weak solutions i.e. for $P \in V$, the classical form of the equation above is satisfied in the distribution sense, more precisely in V'.

For regular solutions i.e. for P in $C^2(\bar{\Omega})$, this equation is satisfied for all $x \in \Omega$.

The gradient expression

Recall that the objective function is defined as:

$$J(b; H) = J_{obs}(H) + \alpha_{reg} J_{reg}(b) \tag{17}$$

The regularization term $J_{reg}(b)$ may be quadratic or not (in b), depending on its expression...

Recall that: $j(b) = J(b; H^b)$. We have the differential expression:

$$j'(b) \cdot \delta b = \left(\alpha_{reg} J'_{reg}(b) - [\frac{\partial a}{\partial b}(b; H^b, P^b) - \frac{\partial l}{\partial b}(b; P^b)]\right) \cdot \delta b \tag{18}$$

for all $\delta b \in U_{ad}$.

$$\begin{aligned} j'(b) \cdot \delta b = {} & \alpha_{reg} J'_{reg}(b) \cdot \delta b & (19)\\ & - \int_\Omega \partial_x \left(\Lambda'_{ref}(b) P^b(x)\right) \partial_x H^b(x)\ dx & (20)\\ & + \int_\Omega \left(\partial_x(\delta b)(x) + \frac{\partial_x w(x)}{w(x)} \delta b(x)\right) P^b(x)\ dx & (21) \end{aligned}$$

with $\Lambda'_{ref}(b) = -\frac{3}{10} \frac{\delta b(x)}{|\partial_x H_{ref}(x)|}$.

Regularization term expressions We may define for example:

- $J_{reg}(b) = \|\partial_x b - \partial_x H_{ref}\|_2^2$. Therefore: $J'_{reg}(b) \cdot \delta b = 2 < (b' - H'_{ref}), \delta b >$.
- $J_{reg}(b) = \|b - b_b\|_2^2$. Therefore: $J'_{reg}(b) \cdot \delta b = 2 < (b - b_b), \delta b >$.