

# Tests for an Increasing Trend in the Intensity of a Poisson Process: A Power Study

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## Introduction

In this study, we investigate two statistical tests for detecting an increasing trend in the intensity of a Nonhomogeneous Poisson Process (NHPP):

1. Laplace Test
2. Boswell's Likelihood Ratio Test

We use Monte Carlo simulations to generate data under different scenarios and estimate the power of these tests.

## Theoretical Background

### Laplace Test

The core idea of the Laplace test is to analyze the distribution of the arrival times  $T_1, T_2, \dots, T_n$  within the observation interval  $[0, T^*]$ . The test is based on the Laplace statistic  $F$  defined as the average of the arrival times:

$$F = \frac{1}{T^*} \sum_{i=1}^n T_i.$$

Under the null hypothesis of a constant intensity, the  $T_i$  are uniformly distributed. If the intensity is increasing, the  $T_i$  cluster towards the end, making  $F$  larger. We reject the null hypothesis if  $F$  is larger than a critical value.

### Boswell Test

The core idea of the Boswell test is to compare the likelihood of the observed arrival times  $T_1, T_2, \dots, T_n$  under the assumption of a constant intensity (null hypothesis) versus a non-decreasing intensity (alternative hypothesis). The test is based on the likelihood ratio statistic  $W$ , which measures the difference between the maximum likelihoods under these two hypotheses:

$$W = 2 \left( \sum_{i=1}^n \log(\hat{\lambda}(T_i)) + n \log \left( \frac{T^*}{n} \right) \right),$$

where  $\hat{\lambda}(T_i)$  is the optimal non-decreasing estimate of the intensity. Under the null hypothesis,  $W$  follows a chi-squared distribution. We reject the null hypothesis if  $W$  is larger than a critical value, indicating a better fit for a non-decreasing intensity.

# Numerical Simulations

## Simulation Methodology

To simulate arrival times  $T_1, T_2, \dots, T_n$  for an NHPP with a given intensity function  $\lambda(t)$  over  $[0, T^*]$ , we use the inverse transform method.

```
# Define the simulation function for a Poisson Process with rate lambda = 1
simulate_hpp <- function(n, T_star) {
  return(sort(runif(n, min = 0, max = T_star)))
}
```

```
# Define the simulation function for a Nonhomogeneous Poisson Process
simulate_nhpp <- function(n, T_star, lambda_func, lambda_inv_func) {
  u <- sort(runif(n, min = 0, max = lambda_func(T_star)))
  u <- lambda_inv_func(u)
  return(u)
}
```

To compute the power of the Laplace and Boswell tests, we perform a Monte Carlo simulation for different scenarios of increasing intensity functions  $\lambda(t)$ .

```
# Define the Monte Carlo simulation function
monte_carlo_simulation <- function(n, T_star, lambda_func, lambda_inv_func,
                                   test_func, threshold, n_sim = 5000) {

  count_reject <- 0
  for (i in 1:n_sim) {
    # Simulate NHPP data
    times <- simulate_nhpp(n, T_star, lambda_func, lambda_inv_func)

    # Perform the test
    test_stat <- test_func(times, T_star, n)

    # Check if the test statistic exceeds the critical value
    if (test_stat > threshold) {
      count_reject <- count_reject + 1
    }
  }
  return(count_reject / n_sim)
}
```

Explain why we reject the null hypothesis if the test statistic observed is larger than the critical value (in term of maths)

We simulate data with various increasing intensity functions  $\lambda(t)$ :

- Exponential trend: 
$$\begin{cases} \lambda(t) = e^{\beta t} \\ \Lambda(t) = \frac{1}{\beta}(e^{\beta t} - 1) \\ \Lambda^{-1}(u) = \frac{1}{\beta} \log(\beta u + 1) \end{cases}$$
- Weibull trend: 
$$\begin{cases} \lambda(t) = \beta t^{\beta-1} \\ \Lambda(t) = t^{\beta} \\ \Lambda^{-1}(u) = u^{1/\beta} \end{cases}$$

```

# Define the intensity functions, the cumulative intensity functions and their inverses
# Exponential trend
lambda_exp <- function(beta) {
  return(function(t) { exp(beta * t) })
}
lambda_cum_exp <- function(beta) {
  return(function(t) { (1 / beta) * (exp(beta * t) - 1) })
}
lambda_inv_exp <- function(beta) {
  return(function(u) { (1 / beta) * log(beta * u + 1) })
}

# Weibull trend
lambda_weibull <- function(beta) {
  return(function(t) { beta * t^(beta - 1) })
}
lambda_cum_weibull <- function(beta) {
  return(function(t) { t^beta })
}
lambda_inv_weibull <- function(beta) {
  return(function(u) { u^(1 / beta) })
}

```

We define the Laplace and Boswell tests for an increasing intensity trend.

```

# Define the Laplace test
laplace_test <- function(times, T_star, n) {
  # Compute the Laplace statistic
  F <- sum(times) / T_star

  # Standardize the Laplace statistic
  Z <- (F - (n / 2)) * sqrt(12 / n)

  return(Z)
}

# Define the Boswell test
boswell_test <- function(times, T_star, n) {
  # Compute the optimal non-decreasing estimate of the intensity
  lambda_vals <- lambda_hat(times, T_star, n)

  # Compute the likelihood ratio statistic
  W <- 2 * (sum(log(lambda_vals)) + n * log(T_star / n))

  return(W)
}

```

We set the critical values for the Laplace and Boswell tests based on the significance level  $\alpha$ .

```

# Define the critical value for the Laplace test
laplace_critical_value <- function(alpha) {
  return(qnorm(1 - alpha))
}

```

```
# Define the critical value for the Boswell test
boswell_critical_value <- function(alpha, n) {
  uniroot(function(w) boswell_survival_func(w, n) - alpha, lower = 0, upper = 100)$root
}
```

Boswell and Brunk (1969) observed empirically that the large sample approximation for the LRT yields a true significance level somewhat larger than the nominal level. We can empirically verify this observation in R using Monte Carlo simulations. The goal is to compare the true significance level of the LRT when using different nominal levels to see which one yields an actual significance level closer to 0.05.

```
# Parameters
n <- 20
T_star <- 2.0
n_sim <- 5000
nominal_levels <- c(0.05, 0.04)

# Run simulations for different nominal levels
results <- data.frame(
  Nominal_Level = nominal_levels,
  True_Significance = sapply(nominal_levels, function(alpha) {
    monte_carlo_significance(n, T_star, alpha, n_sim)
  })
)

# Display the results
print(results)
```

```
##   Nominal_Level True_Significance
## 1          0.05          0.0666
## 2          0.04          0.0538
```

We can see that the true significance level is closer to the nominal level of 0.04 than 0.05.

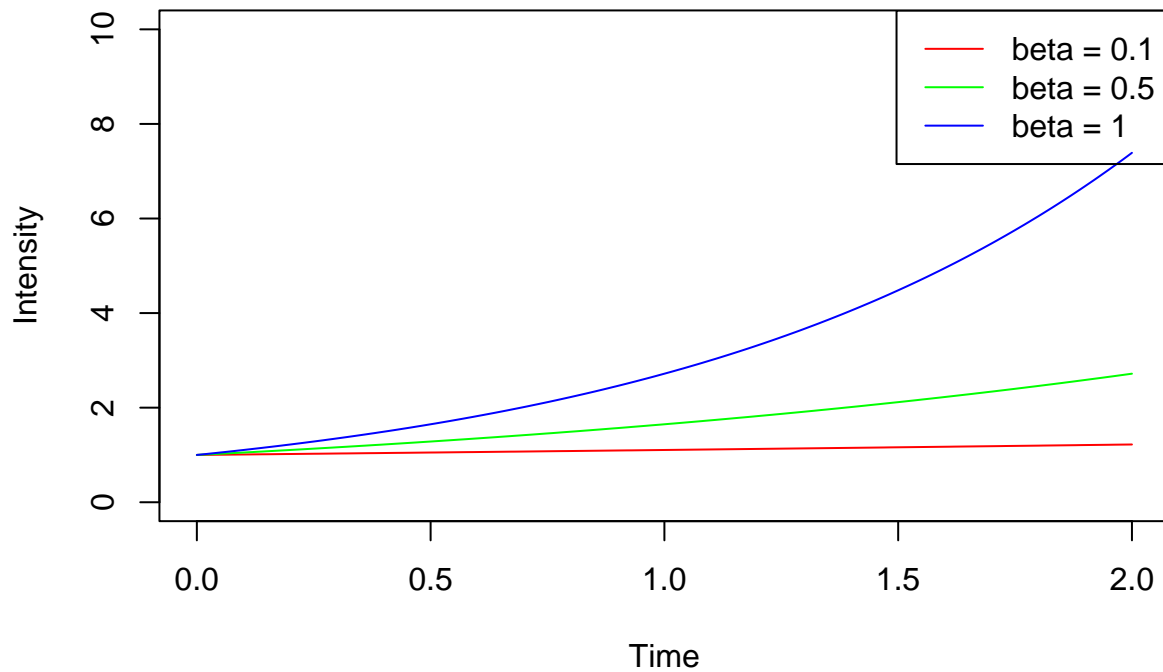
## Power Study

We conduct a power study to evaluate the performance of the Laplace and Boswell tests under different scenarios of increasing intensity functions.

### Exponential Trend

We first consider an exponential trend for different values of the parameter  $\beta$ .

### Exponential Trend: $\lambda(t) = e^{\beta t}$



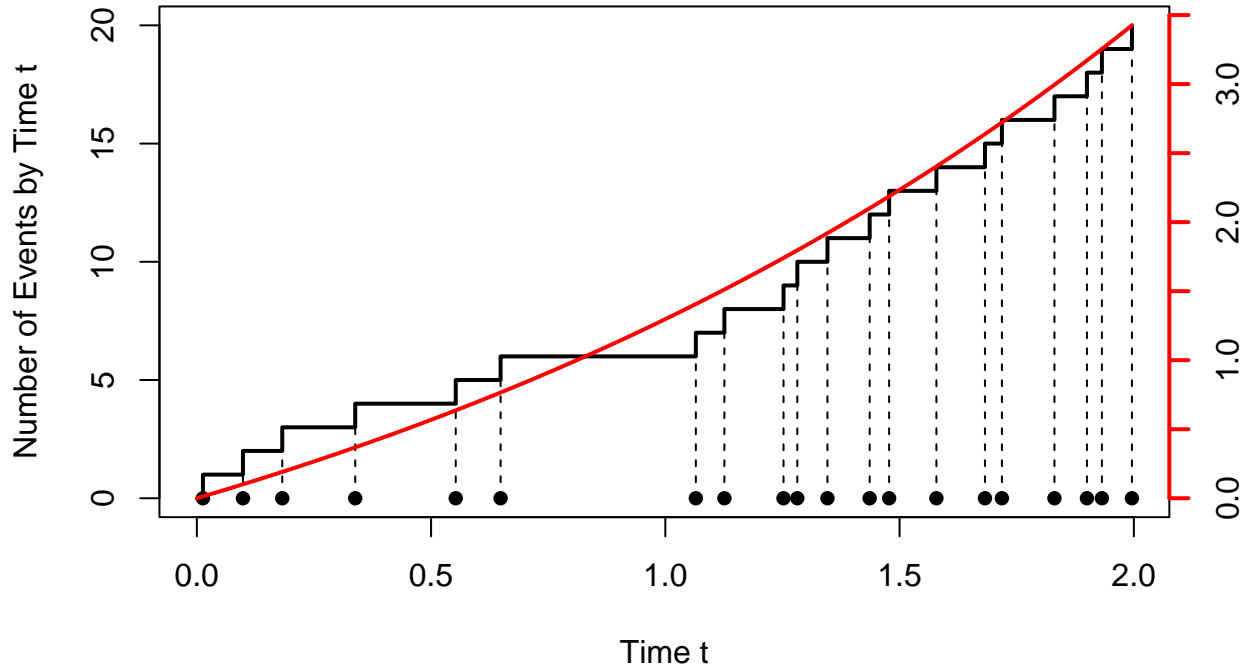
We have simulated a Nonhomogeneous Poisson Process for each scenario (with  $n = 20$ ,  $T^* = 2.0$  and  $\beta = 0.5$ ):

```
# Simulation and plot of a Nonhomogeneous Poisson Process
n <- 20
T_star <- 2.0
beta <- 0.5
lambda_cum_func <- lambda_cum_exp(beta)
lambda_inv_func <- lambda_inv_exp(beta)

# Simulate NHPP data
times <- simulate_nhpp(n, T_star, lambda_cum_func, lambda_inv_func)

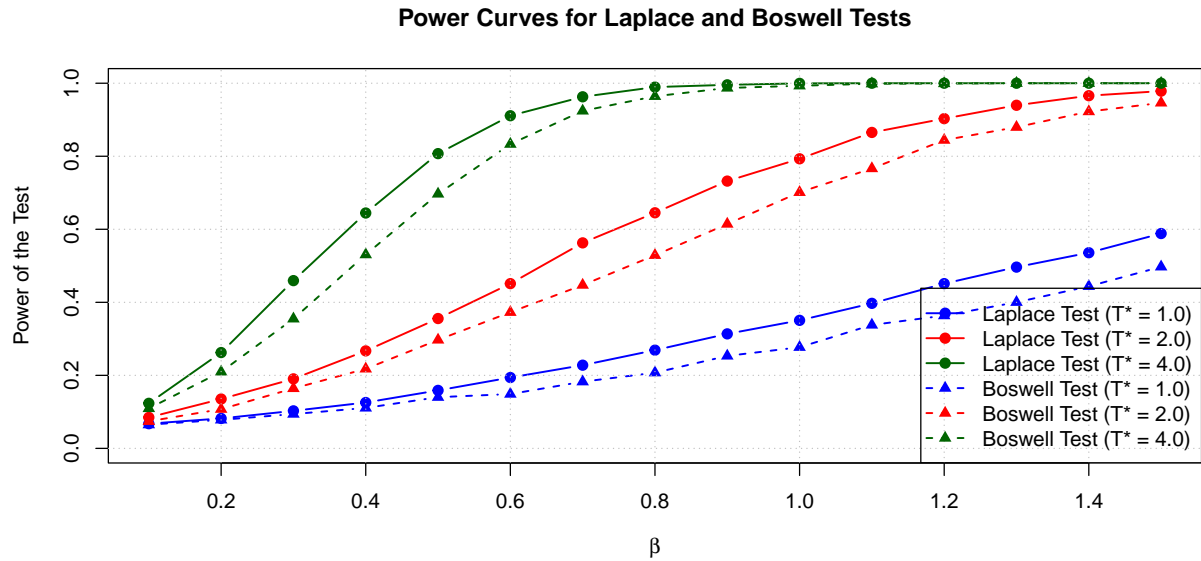
# Plot the NHPP data
plot.PP(times, lambda_cum_func)
```

## Poisson Process



We then compute the power of the Laplace and Boswell tests for each scenario.

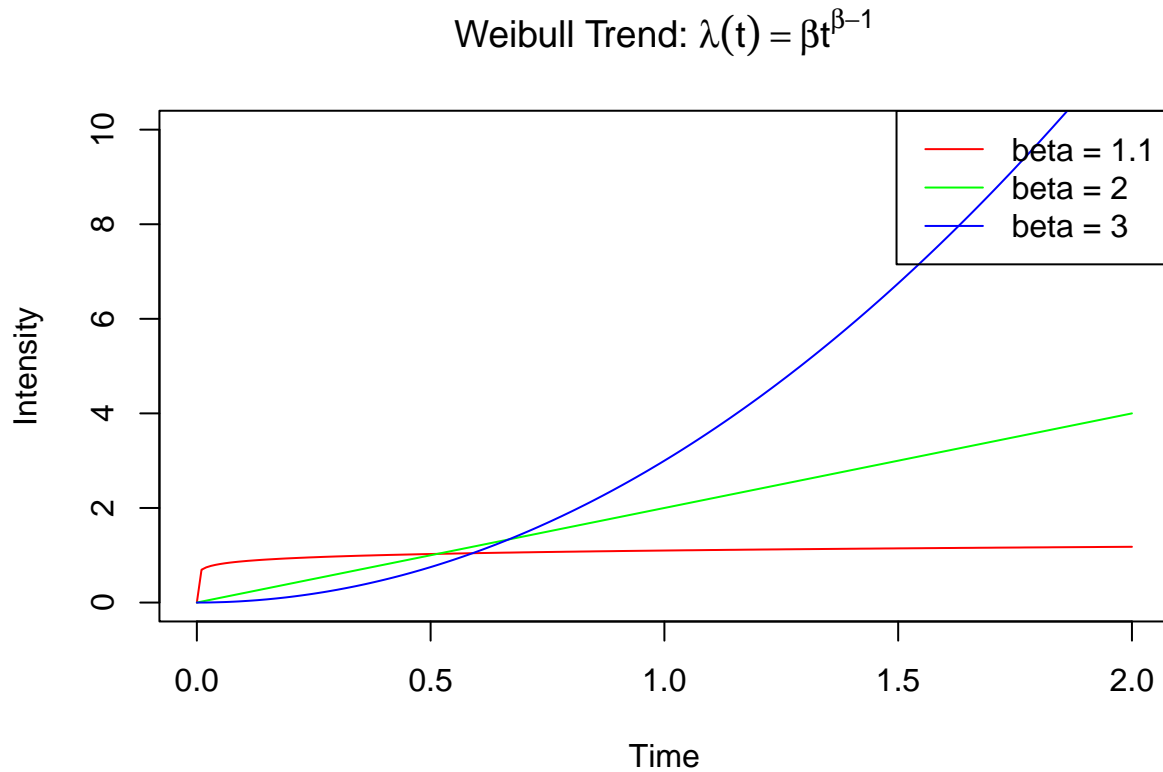
We plot the power curves for the Laplace and Boswell tests as a function of the parameter  $\beta$  with a significance level of  $\alpha = 0.05$ ,  $n = 20$ , and  $T^* = 1.0, 2.0, 4.0$ .



COMMENTARY

## Weibull Trend

We now consider a Weibull trend for different values of the parameter  $\beta$ .



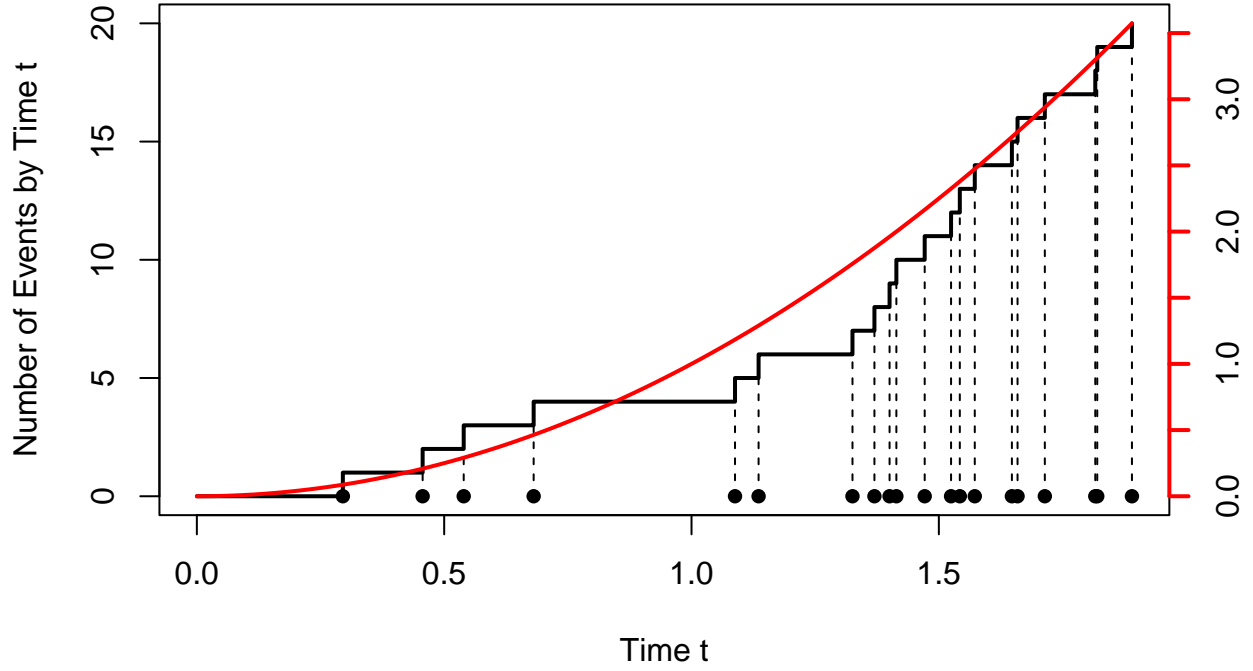
We have simulated a Nonhomogeneous Poisson Process for each scenario (with  $n = 20$ ,  $T^* = 2.0$  and  $\beta = 1.0$ ):

```
# Simulation and plot of a Nonhomogeneous Poisson Process
n <- 20
T_star <- 2.0
beta <- 2.0
lambda_cum_func <- lambda_cum_weibull(beta)
lambda_inv_func <- lambda_inv_weibull(beta)

# Simulate NHPP data
times <- simulate_nhpp(n, T_star, lambda_cum_func, lambda_inv_func)

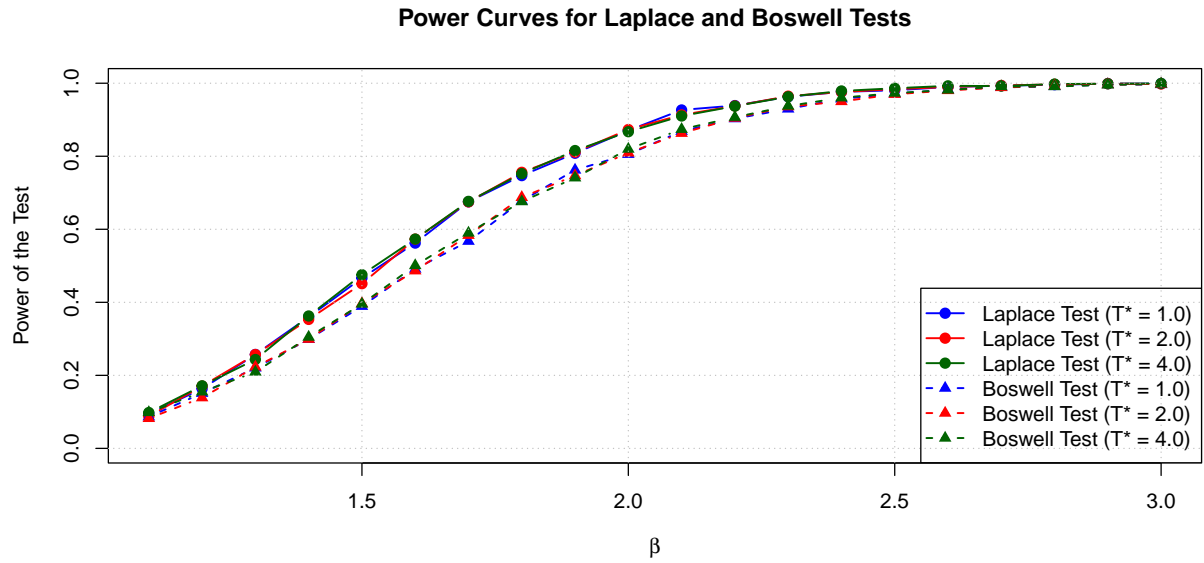
# Plot the NHPP data
plot.PP(times, lambda_cum_func)
```

## Poisson Process



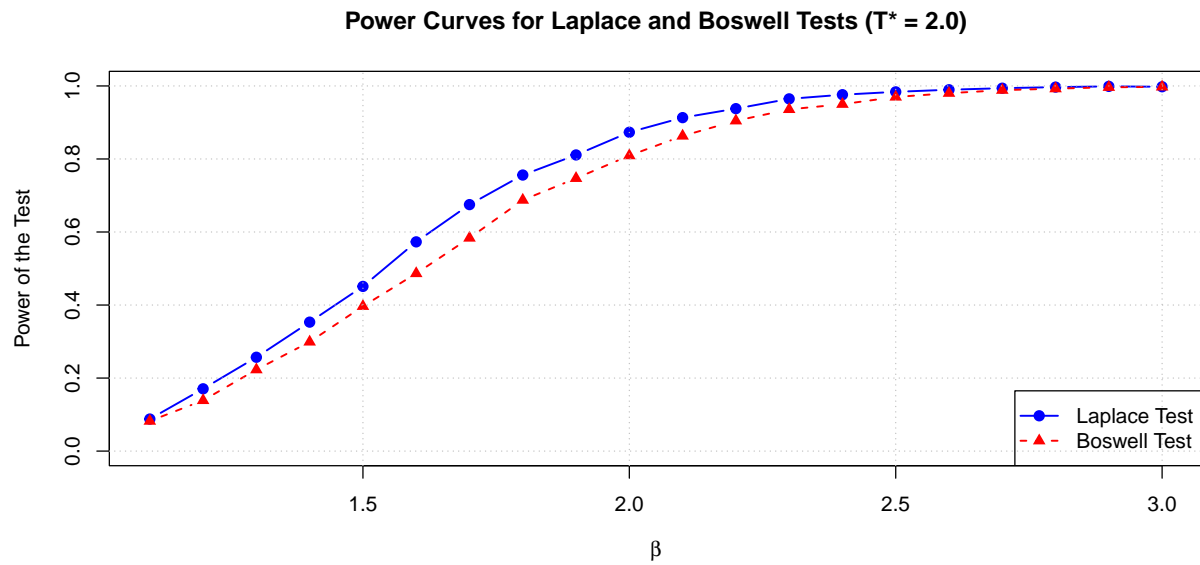
We then compute the power of the Laplace and Boswell tests for each scenario.

We plot the power curves for the Laplace and Boswell tests as a function of the parameter  $\beta$  with a significance level of  $\alpha = 0.05$ ,  $n = 20$ , and  $T^* = 1.0, 2.0, 4.0$ .



Since the power curves for the Weibull trend does not clearly show a difference between the time intervals  $T^* = 1.0, 2.0, 4.0$ , we will focus on the power curves for  $T^* = 2.0$ .





COMMENTARY

## Application to Real-World Data

We apply the Laplace and Boswell tests to a real-world dataset to detect an increasing trend in the intensity of a Poisson process.

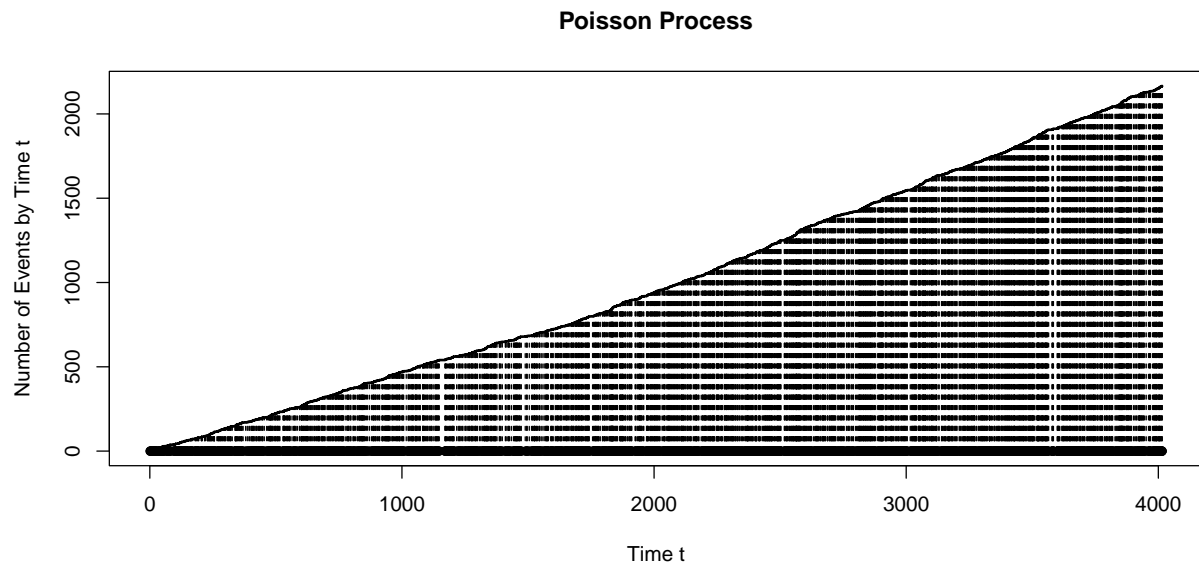
The dataset consists of the number of fire insurance claims in Denmark from 1980 to 1990. We are primarily interested in the arrival times of the claims (the time column) to detect if the rate of claims is increasing over time.

```
library(lubridate)
# Load the Danish dataset
data(danish)
df_data <- data.frame(Times = attr(danish, "times"))
times_intervals = as.numeric(diff(df_data$Times), units="days")

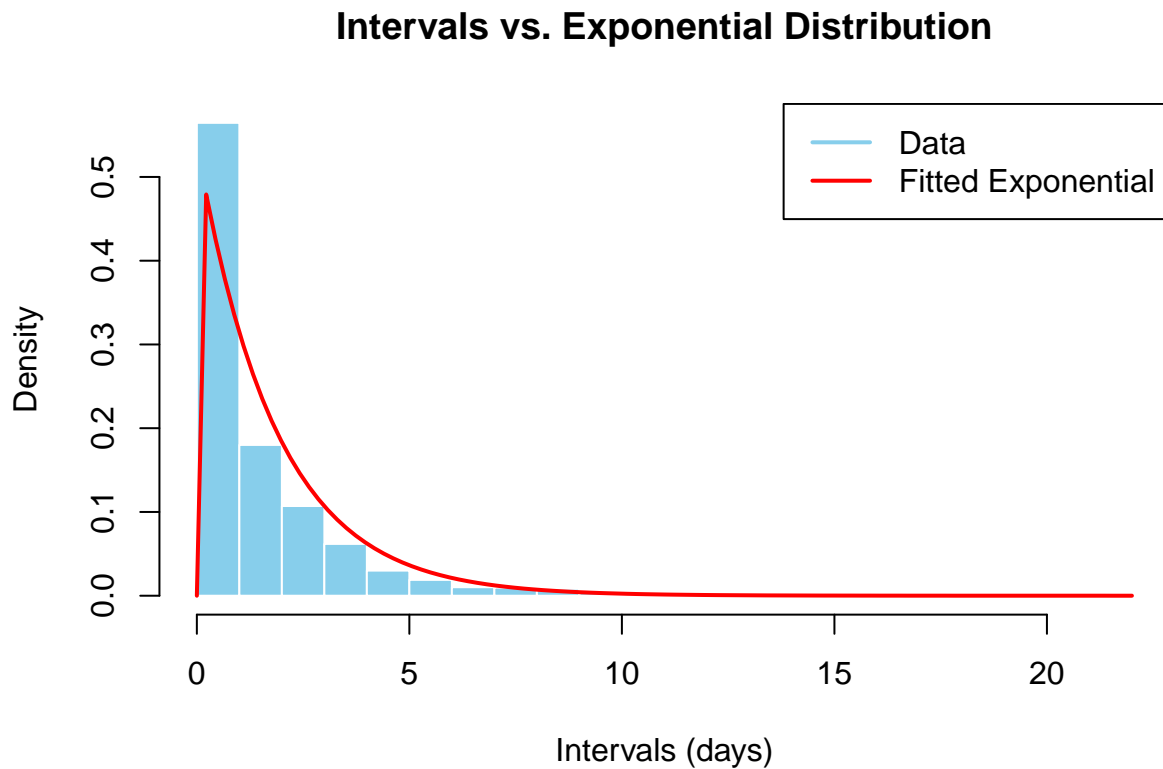
head(df_data)
```

```
##           Times
## 1 1980-01-03 01:00:00
## 2 1980-01-04 01:00:00
## 3 1980-01-05 01:00:00
## 4 1980-01-07 01:00:00
## 5 1980-01-07 01:00:00
## 6 1980-01-10 01:00:00
```

We plot the cumulative number of claims over time to visualize the data.



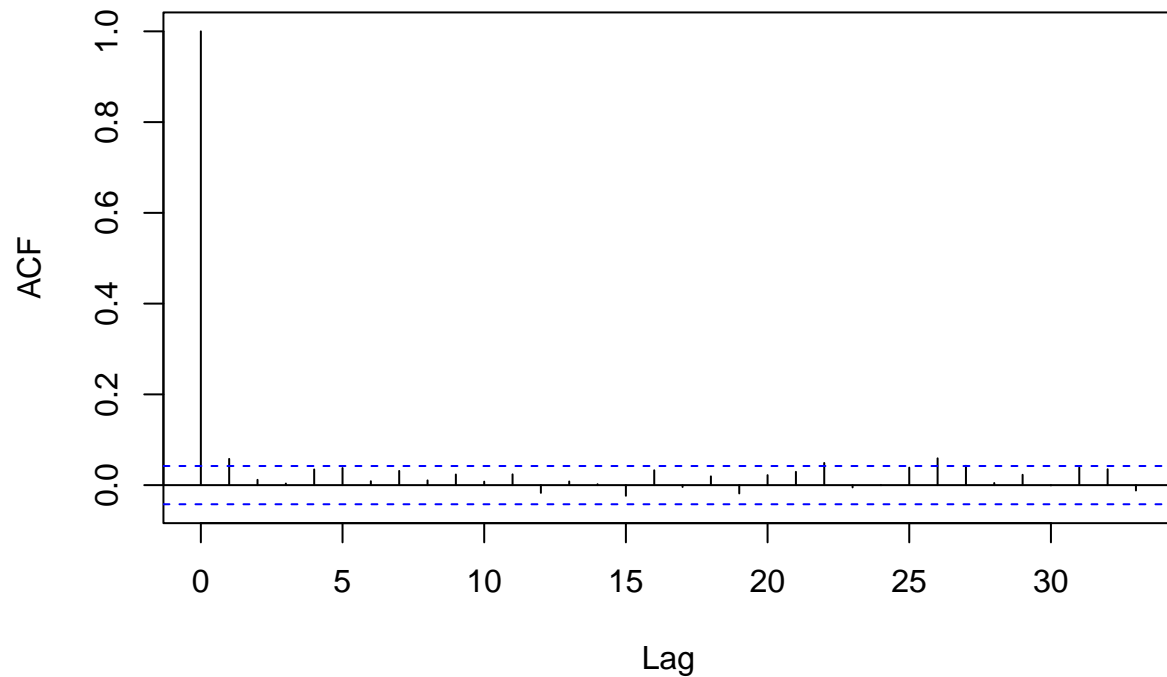
We adjust an exponential distribution to the time intervals to check if the inter-arrival times follow an exponential distribution.



#### COMMENTARY

We then check the autocorrelation of the time intervals to assess the independence of the time intervals.

## Autocorrelation of Time Intervals



### COMMENTARY

We clean the data by removing duplicate time values and computing the new time intervals.

```
# Cleaning the data
new_times_intervals <- times_intervals[times_intervals != 0]

# Variable declarations
times <- cumsum(new_times_intervals)
T_max <- max(times)
n <- length(new_times_intervals)
```

## Application of the tests

We apply the Laplace and Boswell tests to the Danish dataset to detect an increasing trend in the intensity of the Poisson process.

### Laplace Test

```
# Laplace Test
Z_obs <- laplace_test(times, T_max, n)

# Compute the p-value
p_val <- 1 - pnorm(Z_obs)
```

```
print(p_val)
```

```
## [1] 0.0001150628
```

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### Boswell Test

```
# Boswell Test  
W_obs <- boswell_test(times, T_max, n)  
print(W_obs)  
  
# Compute the p-value  
p_val <- boswell_log_survival_func(W_obs, n)  
  
print(p_val)
```