Tests for an Increasing Trend in the Intensity of a Poisson Process: A Power Study

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Introduction

In this study, we investigate two statistical tests for detecting an increasing trend in the intensity of a Nonhomogeneous Poisson Process (NHPP):
1. Laplace Test
2. Boswell's Likelihood Ratio Test

We use Monte Carlo simulations to generate data under different scenarios and estimate the power of these tests.

Theoretical Background

Laplace Test

The core idea of the Laplace test is to analyze the distribution of the arrival times $T_1, T_2, \ldots, T_n$ within the observation interval $[0, T^*]$. The test is based on the Laplace statistic F defined as the average of the arrival times:

$$F = \frac{1}{T^*} \sum_{i=1}^{n} T_i.$$

Under the null hypothesis of a constant intensity, the T_i are uniformly distributed. If the intensity is increasing, the T_i cluster towards the end, making F larger. We reject the null hypothesis if F is larger than a critical value.

Boswell Test

The core idea of the Boswell test is to compare the likelihood of the observed arrival times $T_1, T_2, \ldots, T_n$ under the assumption of a constant intensity (null hypothesis) versus a non-decreasing intensity (alternative hypothesis). The test is based on the likelihood ratio statistic W, which measures the difference between the maximum likelihoods under these two hypotheses:

$$W = 2 \left(\sum_{i=1}^{n} \log(\hat{\lambda}(T_i)) + n \log\left(\frac{T^*}{n}\right) \right),$$

where $\hat{\lambda}(T_i)$ is the optimal non-decreasing estimate of the intensity. Under the null hypothesis, W follows a chi-squared distribution. We reject the null hypothesis if W is larger than a critical value, indicating a better fit for a non-decreasing intensity.

Numerical Simulations

Simulation Methodology

To simulate arrival times $T_1, T_2, \ldots, T_n$ for an NHPP with a given intensity function $\lambda(t)$ over $[0, T^*]$, we use the inverse transform method.

```
# Define the simulation function for a Poisson Process with rate lambda = 1
simulate_hpp <- function(n, T_star) {
  return(sort(runif(n, min = 0, max = T_star)))
}
```

```
# Define the simulation function for a Nonhomogeneous Poisson Process
simulate_nhpp <- function(n, T_star, lambda_func, lambda_inv_func) {
  u = sort(runif(n, min = 0, max = lambda_func(T_star)))
  u = lambda_inv_func(u)
  return(u)
}
```

To compute the power of the Laplace and Boswell tests, we perform a Monte Carlo simulation for different scenarios of increasing intensity functions $\lambda(t)$.

```
# Define the Monte Carlo simulation function
monte_carlo_simulation <- function(n, T_star, lambda_func, lambda_inv_func,
                                   test_func, quantile, n_sim) {
  count_reject = 0
  for (i in 1:n_sim) {
    # Simulate NHPP data
    PPi = simulate_nhpp(n, T_star, lambda_func, lambda_inv_func)

    # Perform the test
    Vobs = test_func(PPi, T_star, n)

    # Check if the test statistic exceeds the critical value
    if (Vobs > quantile) {
      count_reject = count_reject + 1
    }
  }
  return(count_reject / n_sim)
}
```

We simulate data with various increasing intensity functions $\lambda(t)$:

- Exponential trend: $\begin{cases} \lambda(t) = e^{\beta t} \\ \Lambda(t) = \frac{1}{\beta}(e^{\beta t} - 1) \\ \Lambda^{-1}(u) = \frac{1}{\beta}\log(\beta u + 1) \end{cases}$

- Weibull trend: $\begin{cases} \lambda(t) = \beta t^{\beta - 1} \\ \Lambda(t) = t^{\beta} \\ \Lambda^{-1}(u) = u^{1/\beta} \end{cases}$

- Step-function trend: $\begin{cases} \lambda(t) = 1 + 1_{t>\tau} \\ \Lambda(t) = t + \tau \\ \Lambda^{-1}(u) = u - \tau \end{cases}$

```r
# Define cumulative intensity functions and their inverses
# Exponential trend
lambda_exp <- function(beta) {
  return(function(t) { (1 / beta) * (exp(beta * t) - 1) })
}
lambda_inv_exp <- function(beta) {
  return(function(u) { (1 / beta) * log(beta * u + 1) })
}

# Weibull trend
lambda_weibull <- function(beta) {
  return(function(t) { t^beta })
}
lambda_inv_weibull <- function(beta) {
  return(function(u) { u^(1 / beta) })
}

# Step-function trend
```

We define the Laplace and Boswell tests for an increasing intensity trend.

```r
# Define the Laplace test
laplace_test <- function(PPi, T_star, n) {
  # Compute the Laplace statistic
  F = sum(PPi) / T_star

  # Standardize the Laplace statistic
  Z = (F - 0.5) * sqrt(12 * n)

  return(Z)
}

# Define the Boswell test
boswell_test <- function(PPi, T_star, n) {
  # Compute the optimal non-decreasing estimate of the intensity
  lambda_vals = lambda_hat(PPi, T_star, n)

  # Compute the likelihood ratio statistic
  W = 2 * (sum(log(lambda_vals)) + n * log(T_star / n))

  return(W)
}
```

Power Study

Application to Real-World Data

```
b<-2
```