Tests for an Increasing Trend in the Intensity of a Poisson Process: A Power Study

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Introduction

A Nonhomogeneous Poisson Process (NHPP) is a stochastic process often used to model phenomena where the rate of occurrence of events changes over time. The rate, or intensity function $\lambda(t)$, represents the expected number of events per unit time at a given time t. Understanding and analyzing the behavior of this rate is crucial in diverse fields such as reliability engineering, healthcare, and environmental studies, as it helps identify patterns, predict future events, and optimize interventions. Detecting increasing trends in the rate can be particularly important, for example, in monitoring system deterioration or identifying escalating risks in processes. Over the years, a variety of statistical methods have been proposed to identify increasing trends in NHPPs. This study aims to extend the work of Bain, Engelhardt, and Wright [1] by providing a detailed theoretical and practical exploration of the Laplace test and Boswell's likelihood ratio test. The paper is structured in two main parts: a presentation of the selected tests, including their theoretical basis and simulated performance comparisons, and a practical application to real-world data.

Changes to be made:

- Ordre de présentation de l'article
- Source sur les exemples (healthcare, ...)

1 Presentation of the Tests

A statistical test evaluates two competing hypotheses: the null hypothesis H_0 (representing the status quo) and the alternative hypothesis H_1 (indicating a deviation from H_0). Based on the sample data, a test statistic is computed and compared to a critical threshold. If the test statistic exceeds this threshold, H_0 is rejected in favor of H_1.
In our case, we test whether the intensity function $\lambda(t)$ of a Poisson process is constant (H_0) or increasing (H_1).

1.1 Laplace Test

1.1.1 Theoretical Basis

Let $(N(t))_{t\geq 0}$ be a Poisson process with intensity function $\lambda(t)$. We observe $N(t)$ in the interval $[0, T^*]$. Let $0 < T_1 < T_2 < \ldots < T_n < T^*$ be the ordered observation times.

Test Statistic

Under H_0, the arrival times $T_1, \ldots, T_n$ (conditioned on $N_{T^*} = n$) behave like order statistics from a uniform distribution. Specifically:
$(T_1, \ldots, T_n)|\{N_{T^*} = n\} \overset{(d)}{=} (U_1, \ldots, U_n)$, where $U_1, \ldots, U_n \underset{i.i.d.}{\sim} \mathcal{U}([0, T^*])$.

Therefore, $(\frac{T_1}{T^*}, \ldots, \frac{T_n}{T^*})|\{N_{T^*} = n\} \overset{(d)}{=} (V_1, \ldots, V_n)$, where $V_1, \ldots, V_n \underset{i.i.d.}{\sim} \mathcal{U}([0, 1])$.
Define the Laplace test statistic as:

$$F = \frac{1}{T^*} \sum_{i=1}^{n} T_i. \tag{1}$$

By the Central Limit Theorem, under H_0, can be standardized as: $Z = \frac{F - \mathbb{E}[F]}{\sqrt{\mathrm{Var}(F)}} = \frac{F - \frac{n}{2}}{\sqrt{\frac{n}{12}}} \sim \mathcal{N}(0, 1)$ for large n.

Decision Rule

If the intensity $\lambda(t)$ is increasing, the arrival times T_i are expected to cluster towards the end of the interval $[0, T^*]$, making F larger. Therefore, we reject H_0 if $F \geq s_\alpha$, where s_α is the critical threshold determined

as: $s_\alpha = \frac{n}{2} + z_{1-\alpha}\sqrt{\frac{n}{12}}$.
with $z_{1-\alpha}$ being the $(1-\alpha)$-th quantile of the standard normal distribution.

Type I Error

$e_1 = \mathbb{P}_{H_0}(F \geq s_\alpha) = \mathbb{P}(Z \geq z_{1-\alpha}) = \alpha$.

P-value

The p-value measures the probability of obtaining a test statistic at least as extreme as the observed F_{obs}, under H_0. It is given by:
$\hat{\alpha} = \mathbb{P}_{H_0}(F \geq F_{\text{obs}}) = 1 - \Phi(\frac{F_{\text{obs}} - \frac{n}{2}}{\sqrt{\frac{n}{12}}})$.

Power

The power of the test is the probability of rejecting H_0 when H_1 is true.
$\pi(\lambda) = \mathbb{P}_{H_1}(F \geq s_\alpha)$.

1.1.2 Implementation

1.2 Boswell's Likelihood Ratio Test

1.2.1 Theoretical Basis

1.2.2 Implementation

2 Application to Real-World Data

2.1 Application Scenario

2.2 Power comparison

2.3 Analysis of the Results

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3 Conclusion

Ceci est une citation [2].

References

[1] Max Engelhardt Lee J. Bain and F. T. Wright. “Tests for an Increasing Trend in the Intensity of a Poisson Process: A Power Study”. In: *Journal of the American Statistical Association* 80.390 (1985), pp. 419–422.

[2] John Smith. “An Example Article”. In: *Journal of Examples* (2020).