Interpreting logistic models

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- Logistic regression is the special case where g is the logit function: $\mathrm{logit}(y) = \mathrm{X}\beta$
- A logistic regression model returns $\mathrm{X}\beta$ on the logit scale
- How can we convert $\mathrm{x}\beta$ to something useful?

Let’s return to the grad school admission example

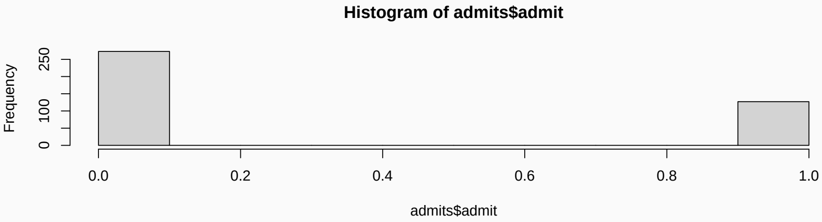
```r
admits<-read_csv(”./data/binary.csv”)
summary(admits)
```

```
##      admit             gre             gpa             rank      
##  Min.   :0.0000   Min.   :220.0   Min.   :2.260   Min.   :1.000  
##  1st Qu.:0.0000   1st Qu.:520.0   1st Qu.:3.130   1st Qu.:2.000  
##  Median :0.0000   Median :580.0   Median :3.395   Median :2.000  
##  Mean   :0.3175   Mean   :587.7   Mean   :3.390   Mean   :2.485  
##  3rd Qu.:1.0000   3rd Qu.:660.0   3rd Qu.:3.670   3rd Qu.:3.000  
##  Max.   :1.0000   Max.   :800.0   Max.   :4.000   Max.   :4.000  
```

Let's explore our outcome

Huh, all this tells us is mean(admits) = 0.3175

```
hist(admits$admit)
```



Let’s look at this as the distribution of the probability of admissions across the data

- First, fit an intercept-only logistic regression model

```
m0<-glm(admit ~ 1, data = admits, family = "binomial")
m0_est<-tidy(m0)
```

- What does this model tell us?

What does this model tell us?

```r
m0_est$estimate ## log odds
```

```
## [1] -0.7652847
```

```r
exp(m0_est$estimate) ## odds
```

```
## [1] 0.4652015
```

```r
invlogit(m0_est$estimate) ## probability
```

```
## [1] 0.3175
```

```r
mean(admits$admit) ## mean admission probability
```

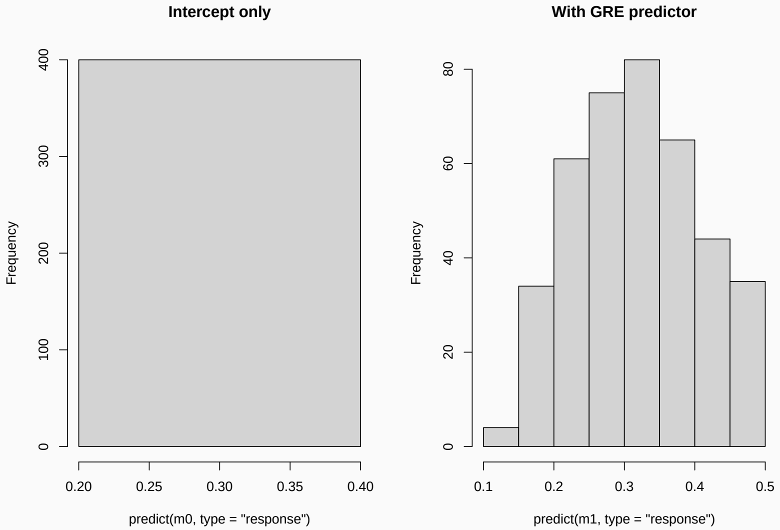
```
## [1] 0.3175
```

Let's add a predictor

```r
m1<-glm(admit ~ 1 + gre, data = admits, family = "binomial")
m1
```

```
##
## Call:  glm(formula = admit ~ 1 + gre, family = "binomial", data
##
## Coefficients:
## (Intercept)          gre
##   -2.901344     0.003582
##
## Degrees of Freedom: 399 Total (i.e. Null);  398 Residual
## Null Deviance:       500
## Residual Deviance: 486.1     AIC: 490.1
```

Before and after - what's going on?



Two linear predictors

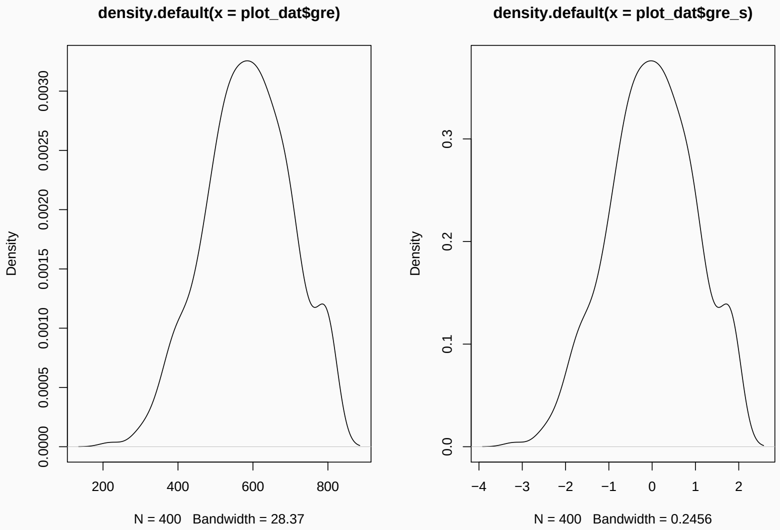
Why do these generate such different predictions?

Intercept only model (m0): $p(admit) = \beta_0$ **With GRE predictor (m1)**: $p(admit) = \beta_0 + \beta_1 GRE_i$

To ease interpretation, let's scale GRE

Scale mean-centers and SD scales variables: $\mathrm{scale}(x_i) = \frac{x_i - \bar{x}}{sd(x)}$

Linear transformations of variables: mean-center and SD scale



Re-estimate the model: much nicer to look at

```r
admits<-admits%>%
  mutate(gre_s = as.numeric(scale(gre)))
m1<-glm(admit ~ 1 + gre_s, data = admits, family = "binomial")
m1_est<-tidy(m1)
m1_est
```

```
## # A tibble: 2 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)   -0.796     0.111     -7.20 6.01e-13
## 2 gre_s          0.414     0.114      3.63 2.80e- 4
```

Interpret the model

```
m1_est
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```

Remember: $\text{logit}(y) = X\beta = \log\left(\frac{y}{1-y}\right)$

So: $y = \text{logit}^{-1}(X\beta) = \frac{\exp(X\beta)}{\exp(X\beta)+1}$

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- What is β_0?

Interpret the model

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So: $y = \mathrm{logit}^{-1}(\mathrm{X}\beta) = \frac{\exp(\mathrm{X}\beta)}{\exp(\mathrm{X}\beta)+1}$

- What is β_0?
- What is β_1?

Refresher on exponentials

$$e^{y_1+y_2} = e^{y_1}e^{y_2}$$

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and

$$e^{y_1-y_2} = \frac{e^{y_1}}{e_2^{y}}$$

Refresher on exponentials

$$e^{y_1+y_2} = e^{y_1}e^{y_2}$$

and

$$e^{y_1-y_2} = \frac{e^{y_1}}{e_2^y}$$

so how can we rewrite:

$$\exp(\mathrm{logit}(y)) = \frac{y}{1-y} = e^{\beta_0+\beta_1 x_1}$$

Non-linear relationships

On the log scale, β_0 and β_1 are related to y multiplicatively because

$$e^{\beta_0+\beta_1 x_1} = e^{\beta_0} e^{\beta_1 x_1}$$

Odds

Odds are defined as the probability of the event occurring divided by the probability of probability of the event not occurring. To obtain odds in a logistic regression, we exponentiate both sides:

$$\frac{y}{1-y} = e^{\beta_0+\beta_1 x_1}$$

Odds

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$$\frac{y}{1-y} = e^{\beta_0+\beta_1 x_1}$$

The odds of $y == 1$ are simply $e^{X\beta}$

The odds ratio is the ratio of two odds - or the proportional change in odds. We can obtain an isolated estimate for the relationship between $\beta_1 x_{1i}$ and y this way:

$$\frac{Odds(y|x_1 = 1)}{Odds(y|x_1 = 0)} = \frac{e^{X\beta+\beta_1}}{e^{X\beta}} = \frac{e^{X\beta} \times e^{\beta_1}}{e^{X\beta}} = e^{\beta_1}$$

The odds ratio can be interpreted as the change in odds of $y == 1$ for a one-unit change in x_1.

Interpreting odds ratios

- Odds ratios appear convenient - e^{β_1} is a percent change in y for a one-unit change in x_1

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How do they work?

In our example: what do these figures mean?

```
new_dat<-c(1,0) # for scale(gre) == 0, mean score
odds_0<-exp(new_dat%*%m1_est$estimate)
odds_0
```

```
##           [,1]
## [1,] 0.4510945
```

```
new_dat1<-c(1,1)
odds_1<-exp(new_dat1%*%m1_est$estimate)
odds_1
```

```
##           [,1]
## [1,] 0.6823082
```

```
odds_1/odds_0 # odds ratio
```

```
##          [,1]
## [1,] 1.512562
```

```
exp(m1_est$estimate[2]) # exp(beta_1)
```

```
## [1] 1.512562
```

Interpreting the odds ratio

The odds of admission are `exp(m1_est$estimate[2])` times higher for a student with a GRE score one standard deviation above the mean than they are for a student with a mean GRE score.

Interpreting the odds ratio

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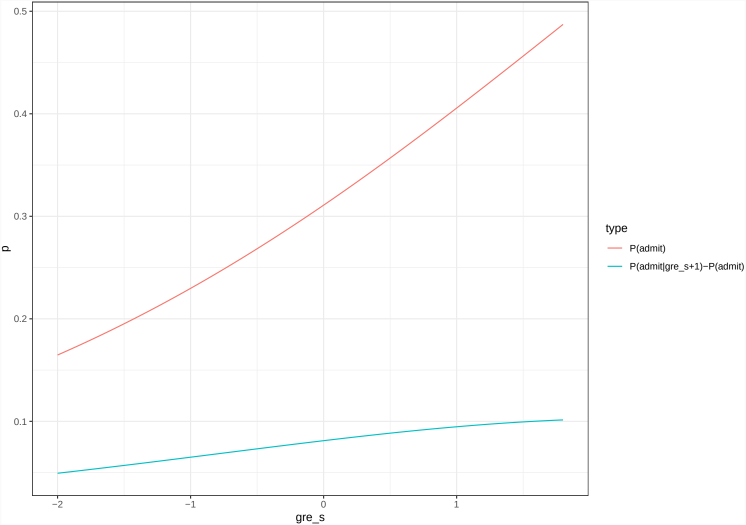
Any trouble you can anticipate here?

The odds of admission are `exp(m1_est$estimate[2])` times higher for a student with a GRE score one standard deviation above the mean than they are for a student with a mean GRE score.

Any trouble you can anticipate here?

A visual example: the "effect" of 1 SD increase in GRE scores on Pr(admit==1)



It is easy enough to work on the probability scale

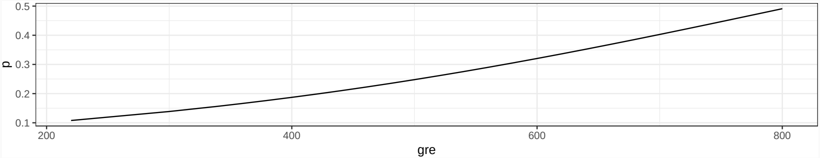
To obtain predicted probabilities of the observed:

- p_hat<-invlogit(predict(m1))
- p_hat<-predict(m1, type = “response”)

On the probability scale

```r
preds<-predict(m1, type = "response")
p_hat<-data.frame(gre = admits$gre,
                  p = preds)

ggplot(p_hat, aes(x = gre, y = p)) +
  geom_line()
```



The basic logic of prediction

1. Choose scenarios of theoretical interest
2. Define these in terms of “counterfactual” (fake) data
3. Plug these fake data into the linear predictor (regression equation)
4. Visualize!

The basic logic of prediction

Reminder: our model is

$$logit(p(admit_i)) = \beta_0 + \beta_1 GRE_i$$

$$admit_i \sim Binomial(1, p)$$

```
m1<-glm(admit ~ gre, data = admits, family = "binomial")
```

1. Choose scenarios of theoretical interest

Low GRE, average GRE, high GRE

Define these scenarios in R

2. Define these in terms of “counterfactual” (fake) data

```
## Look at the distribution of the data to think about scenarios
mean(admits$gre)
```

```
## [1] 587.7
```

```
sd(admits$gre)
```

```
## [1] 115.5165
```

Define these scenarios in R

2. Define these in terms of "counterfactual" (fake) data

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```

Let's define scenarios at the mean, 1 SD below the mean, and 1 SD above the mean

```
fake_data<-data.frame(gre = c(
  mean(admits$gre),
  mean(admits$gre) - sd(admits$gre),
  mean(admits$gre) + sd(admits$gre)
))

fake_data
```

```
##        gre
## 1 587.7000
## 2 472.1835
## 3 703.2165
```

Generating expected probabilities

3. Plug these fake data into the linear predictor (regression equation)

Because $logit(p(admit_i)) = \beta_0 + \beta_1 x_i$, we can compute the expected probability of admission for a student with mean GRE scores as

```
coef(m1)
```

```
##  (Intercept)          gre
## -2.901344270  0.003582212
```

```
### mean GRE scenario: linear predictor
-2.9 + 0.0036 * 587.7
```

```
## [1] -0.78428
```

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```
### mean GRE scenario: linear predictor
-2.9 + 0.0036 * 587.7
```

```
## [1] -0.78428
```

```
### on probability scale
invlogit(-2.9 + 0.0036 * 587.7)
```

```
## [1] 0.3133982
```

3. Plug these fake data into the linear predictor (regression equation)

The `predict()` function makes life very easy here

```
## linear predictor
predict(m1, newdata = fake_data)
```

```
##          1          2          3
## -0.7960785 -1.2098832 -0.3822738
```

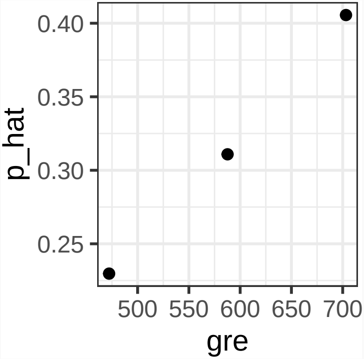
```
## probability scale (inverse logit)
predict(m1, newdata = fake_data, type = "response")
```

```
##         1         2         3
## 0.3108650 0.2297217 0.4055786
```

4. Visualize!

```
### set up our data frame with predictions for plotting
fake_data<-fake_data %>%
  mutate(p_hat = predict(m1, newdata = fake_data, type = "response"))

ggplot(fake_data,
       aes(x = gre, y = p_hat)) +
  geom_point()
```



Break

Fitting a Bayesian logistic regression model

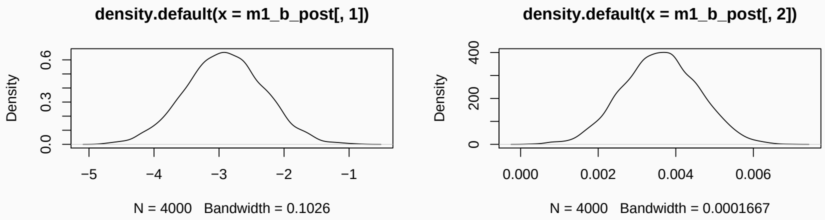
```
m1_b<-stan_glm(admit ~ gre,
               family = "binomial",
               data = admits)
```

Examining our fit

The posterior samples *are* our parameter estimates

```
m1_b_post<-as.data.frame(m1_b)
str(m1_b_post)
```

```
## 'data.frame':    4000 obs. of  2 variables:
##  $ (Intercept): num  -1.68 -1.58 -2.46 -3.17 -2.88 ...
##  $ gre        : num  0.00186 0.0018 0.00271 0.00417 0.00368 ...
```



Posterior parameter estimates and uncertainty

90 percent of parameter values that are compatible with our data and priors fall between

```
quantile(m1_b_post$`(Intercept)`, probs = c(0.05, 0.95))
```

```
##        5%       95%
## -3.900277 -1.936795
```

```
quantile(m1_b_post$gre, probs = c(0.05, 0.95))
```

```
##          5%         95%
## 0.002005807 0.005209586
```

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##        5%       95%
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```

```
quantile(m1_b_post$gre, probs = c(0.05, 0.95))
```

```
##          5%         95%
## 0.002005807 0.005209586
```

How do we summarize uncertainty in `p`?

Uncertainty in the linear predictor

We have two sources of uncertainty in our linear predictor

$logit(p) = \beta_0 + \beta_1 x_1$

Uncertainty in the linear predictor

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$logit(p) = \beta_0 + \beta_1 x_1$

If an applicant had a GRE score of 600, our posterior expected value of $logit(p)$ is

```
### evaluate the linear equation at all draws of the posterior para
logit_p_hat<-m1_b_post$`(Intercept)` + m1_b_post$gre * 600
summary(logit_p_hat)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
## -1.1045 -0.8251 -0.7483 -0.7510 -0.6772 -0.4292
```

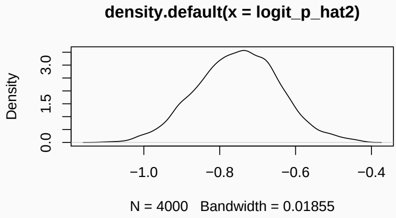
```
### on the probability scale using inverse logit
summary(invlogit(logit_p_hat))
```

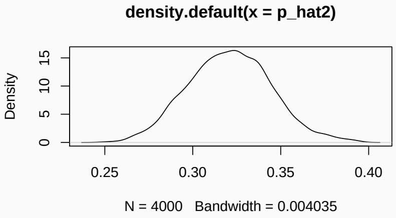
```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##  0.2489  0.3047  0.3212  0.3211  0.3369  0.3943
```

Uncertainty in the linear predictor

The same operation can be performed (more easily!) with `posterior_linpred()` for the linear predictor (logit) scale, and `posterior_epred()` for the original scale (probability)

```
fake_data<-data.frame(gre = 600)
logit_p_hat2<-posterior_linpred(m1_b, newdata = fake_data)
p_hat2<-posterior_epred(m1_b, newdata = fake_data)
```





Generating uncertainty estimates for a series of scenarios

```
fake_data<-data.frame(gre = seq(400,800, by=1))
p_hat<-posterior_epred(m1_b, newdata = fake_data)
### This produces a 4000 row x 401 column matrix, 4000 simulated dr
dim(p_hat)
```

```
## [1] 4000  401
```

Visualizing the uncertainty

Let's compute 90 percent intervals for each scenario, then plot the results

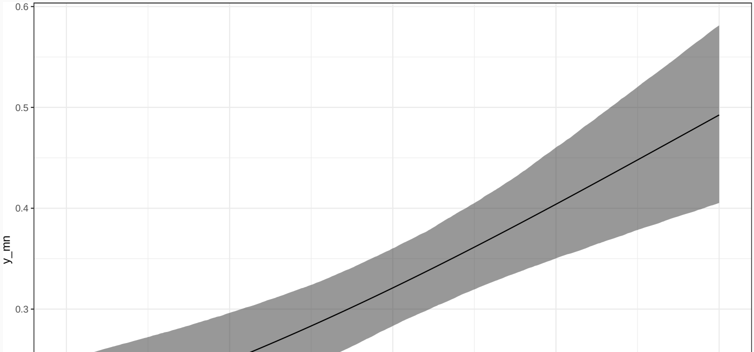
```
### convert to data frame and make it long for plotting
p_hat<-as_tibble(p_hat)
p_hat<-p_hat %>%
  pivot_longer(cols = everything(),
               names_to = "scenario",
               values_to = "p_hat")

## compute the uncertainty interval and posterior mean
p_hat<-p_hat %>%
  mutate(scenario = as.numeric(scenario)) %>%
  group_by(scenario) %>%
  summarise(y_lwr = quantile(p_hat, 0.05),
            y_upr = quantile(p_hat, 0.95),
            y_mn = mean(p_hat))

head(p_hat)
```

Now plot it

```
## attach the GRE scores using scenario number (row number)
fake_data<-fake_data %>%
  mutate(scenario = 1:n())
### join to p_hat
p_hat<-p_hat %>%
  left_join(fake_data)
## plot with uncertainty interval
ggplot(p_hat,
       aes(x = gre, y = y_mn,
           ymin = y_lwr, ymax = y_upr)) +
  geom_ribbon(alpha = 0.5)+
  geom_line()
```



Posterior uncertainty: parameters and predictions

`p_hat` describes our uncertainty in p, driven by our estimated uncertainty in β_0 and β_1.

Does it describe our uncertainty in `admit`?

Posterior uncertainty: parameters and predictions

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Does it describe our uncertainty in `admit`?

Recall that our model is:

$$logit(p(admit_i)) = \beta_0 + \beta_1 GRE_i$$

$$admit_i \sim Binomial(1, p)$$

`p_hat` describes our uncertainty in p, driven by our estimated uncertainty in β_0 and β_1.

Does it describe our uncertainty in `admit`?

Recall that our model is:

$$logit(p(admit_i)) = \beta_0 + \beta_1 GRE_i$$

$$admit_i \sim Binomial(1, p)$$

Uncertainty in `admit` is driven by the binomial distribution

The posterior predictive distribution

We can now take our posterior estimates for p, and draw predictions from the *posterior predictive distribution*. This approach averages over our uncertainty in both the parameters, and in sampling the outcome.

We can use our uncertainty in p to estimate uncertainty in `admit` for new applicants

```
p_hat<-posterior_epred(m1_b, newdata = fake_data)
### first few draws of p for scenario 1, GRE = 400
head(p_hat[,1])
```

```
## [1] 0.2823502 0.2977435 0.2020516 0.1817992 0.1959293 0.1459536
```

```
### simulate admissions for each value of p_hat
admit_hat_scen1<-rbinom(4000, 1, p_hat[,1])
```

The posterior predictive distribution

```
mean(admit_hat_scen1)
```

```
## [1] 0.1885
```

```
sd(admit_hat_scen1)
```

```
## [1] 0.3911598
```

```
### compare admit_hat to p_hat
mean(p_hat[,1])
```

```
## [1] 0.1891432
```

```
sd(p_hat[,1])
```

```
## [1] 0.03495051
```

The posterior predictive distribution

```
admit_hat<-posterior_predict(m1_b, newdata = fake_data)
dim(admit_hat)
```

```
## [1] 4000  401
```

```
admit_hat[1:10, 1:10]
```

```
##       1 2 3 4 5 6 7 8 9 10
##  [1,] 0 0 0 1 0 0 0 0 0  0
##  [2,] 0 1 0 0 1 1 0 0 0  1
##  [3,] 1 0 1 0 0 1 1 0 0  0
##  [4,] 0 1 1 0 0 0 0 0 1  0
##  [5,] 1 0 0 0 0 0 0 0 0  0
##  [6,] 0 0 0 0 1 0 0 0 0  0
##  [7,] 0 0 0 0 0 0 0 0 1  0
##  [8,] 1 0 0 0 0 0 1 0 0  0
##  [9,] 0 0 0 1 0 0 0 0 1  0
## [10,] 0 0 0 0 0 0 0 1 0  0
```

Back to the Titanic

Let’s work through a Bayesian fit

1. Define the model
2. Estimate the model
3. Visualize the model

Prediction and logistic regression

1. Subset the data into training and test data
2. Estimate the model on the training data
3. Predict on the test data
4. Evaluate accuracy