Regression and uncertainty

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Large sample (asymptotic) theorems, point estimates, and uncertainty

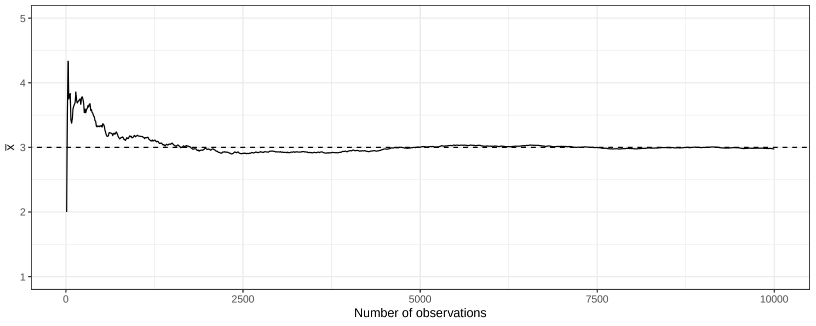
The law of large numbers

As a sample of draws from a random variable increases, the sample mean converges to the population mean $E(X)$

$$\bar{x}_n \to E(X)$$

The law of large numbers: point estimates converge to population parameters as n increases

A Monte Carlo simulation where we draw from Binomial(10, 0.3) 1 time up to 1000 times, then compute $\overline{x}$



The Central Limit Theorem

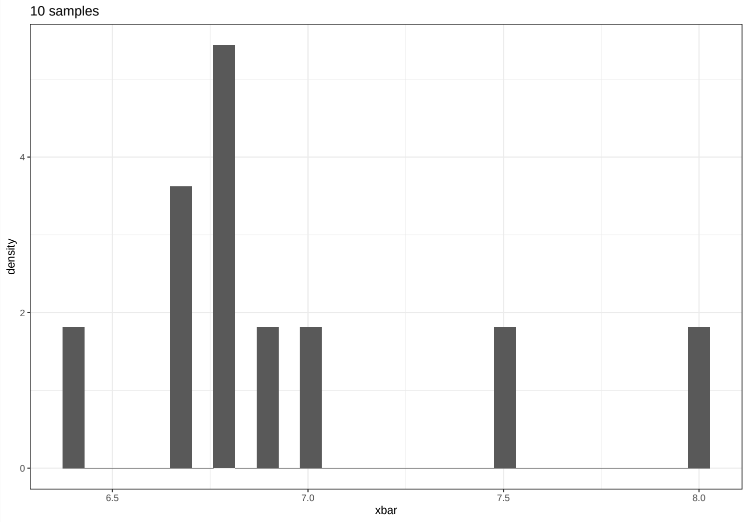
- If we draw independent random samples, as sample size n increases, the distribution of the sample mean $\bar{x}$ approaches a Normal distribution.

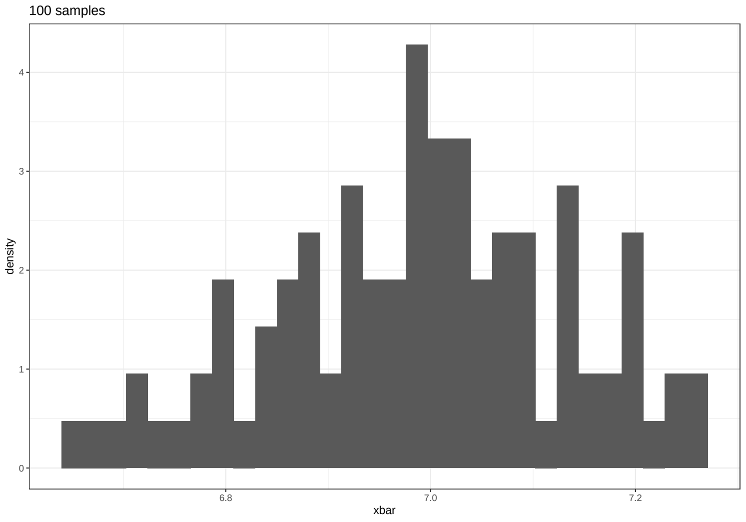
How many students eat pizza in a week?

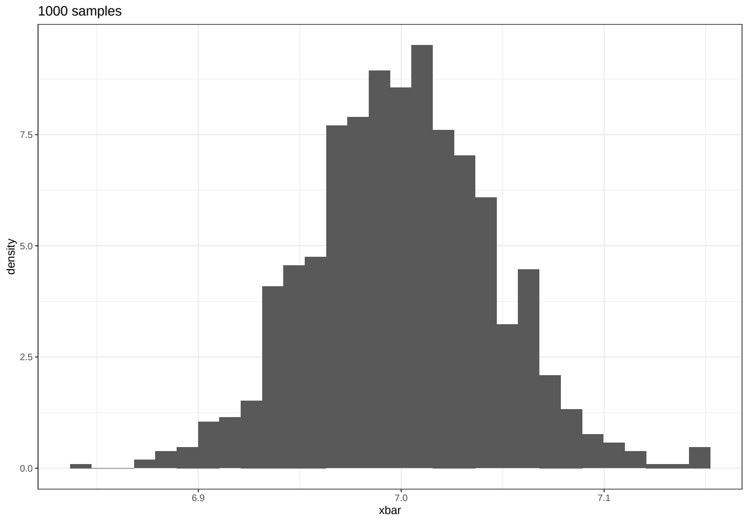
We want to estimate $\overline{pizza}$, the proportion of students who eat pizza per week on campus. Our approach: randomly select 10 classrooms, then randomly select 10 students from each class. Count the number who ate pizza within the prior week (0 = No pizza, 1 = pizza)

Monte Carlo simulations of a binomial variable p=0.7, n=10

1. Take 10 draws from $pizza \sim Binomial(10, 0.7)$
2. Compute $\overline{pizza}$
3. Repeat many times!


10 samples
density
xbar
0
2
4
6.5
7.0
7.5
8.0


100 samples
density
xbar
0
1
2
3
4
6.8
7.0
7.2


1000 samples
density
7.5
5.0
2.5
0.0
6.9
7.0
7.1
xbar

Of course, we generally don't replicate 1000 times.

Let's take 1 draw from the pizza study

```
n<-10 # 10 students in a class
p<-0.7 # 70% chance of a 1
pizza<-rbinom(n = n, p = p, size = 10)
```

What can we say about the proportion of students who eat pizza at Rutgers?

Our point estimate for the proportion is

```
mean(pizza)
```

```
## [1] 7.1
```

Our standard deviation for the study is

```
sd(pizza)
```

```
## [1] 1.449138
```

We can describe our estimate for the *sampling distribution* of $\overline{pizza}$ as a Normal distribution centered at 7.1 with a standard error of 0.46.

We can construct a 95% confidence interval to describe how uncertain we are about the location of $\overline{pizza}$. Here, that interval is:

```
## bounds = +/- 1.96 (Normal PDF for 95% mass)
se<-sd(pizza)/sqrt(10)
mean(pizza) + 1.96 * se
```

```
## [1] 7.998185
```

```
mean(pizza) - 1.96 * se
```

```
## [1] 6.201815
```

How should we interpret this interval?

Hypothesis testing

The problem

- I think that the true proportion of students who eat pizza is 0.5
- How can I use my data to evaluate this claim?

The basic approach

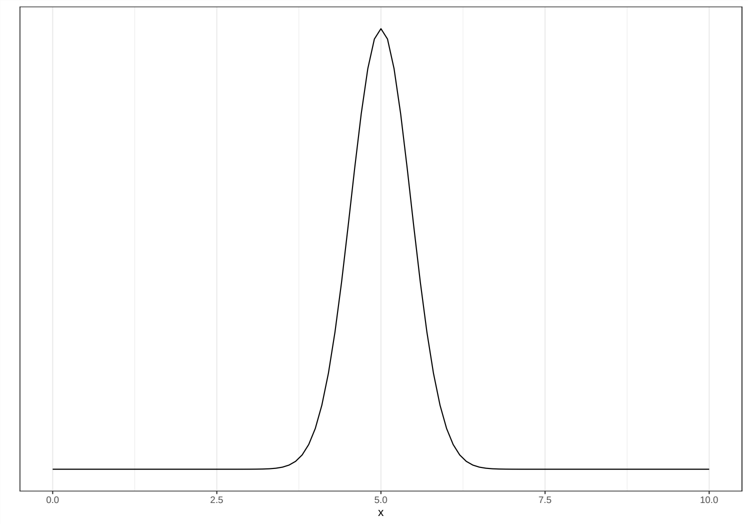
- Establish a hypothesis

1. $H_1 : E[pizza] = 5$

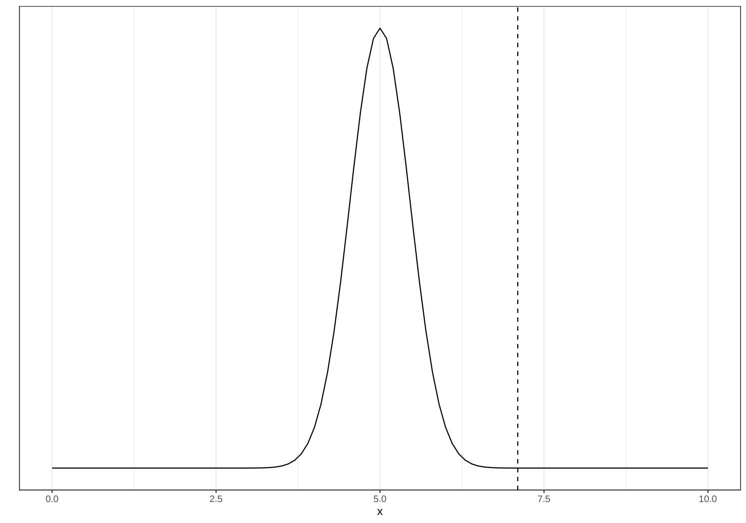
- Evaluate how likely our observations are under the hypothesis

1. We know via CLT that $\overline{pizza} \sim N(\mu, \sigma^2)$
2. How likely is 7.1 under a distribution with mean 5 and SE 0.46?

Our hypothesis for the sampling distribution of pizza habits



What we observed



How likely was our observation if H1 were true?

Use the Normal PDF to estimate

```
pnorm(q = mean(pizza), mean = 5, sd = sd(pizza)/sqrt(10))
```

```
## [1] 0.9999977
```

That's the proportion of observations that fall below our observation *if* H_1 is true. To convert this to how likely we are to observe our data *if* H_1 is true, subtract from 1

```
1 - pnorm(mean(pizza), 5, sd(pizza)/sqrt(10))
```

```
## [1] 2.296417e-06
```

What can we conclude?

Back to regression

The linear regression model

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

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β_0: The value of y when x is equal to zero

β_1: The average increase in y when x increases by one unit

ε: The distance between the line $y = \beta_0 + \beta_1 X$ and the actual observed values of y.

The line $E(y_i) = \beta_0 + \beta_1 x_i$ provides an expected value for y_i based on the values of x_i.

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The standard error of β is calculated as:

$$SE_{\beta} = \sqrt{\frac{\sum \varepsilon_i^2}{(n-2)\sum(x_i - \bar{x})^2}}$$

Note that the numerator captures variance in y and the denominator captures variance in x

Uncertainty and OLS

The Mark of a Criminal Record

```r
### read and format Pager data
cr<-read_csv("https://raw.githubusercontent.com/f-edwards/intro_stats/master/data/criminalrecord.csv")
cr<-cr %>%
  select(callback, crimrec)

head(cr)
```

```
## # A tibble: 6 x 2
##   callback crimrec
##      <dbl>   <dbl>
## 1        1       1
## 2        0       0
## 3        1       0
## 4        1       0
## 5        0       1
## 6        0       1
```

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- Does a criminal record make a callback less likely?
- Implied null hypothesis: No difference in callback rates

$$H_0 : E[Callback|Crimrec = T] - E[Callback|Crimrec = F] = 0$$

Written differently:

$$H_0 : E[Callback|Crimrec = T] = E[Callback|Crimrec = F]$$

Let’s estimate the model for the effect of crimrec on callback

```r
library(broom)
m0<-lm(callback ~ crimrec, data = cr)
tidy(m0)
```

```
## # A tibble: 2 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)    0.226    0.0196     11.6  1.82e-28
## 2 crimrec       -0.125    0.0277     -4.53 6.87e- 6
```

Write this out as a regression equation.

- What does β_0 mean?
- What does β_1 mean?

Setting up our hypothesis test

H_0 : No effect of crimrec on callback.

What does this imply in terms of β?

Recall that our model says $E[callback] = \beta_0 + \beta_1 Crimrec$

Our null hypothesis for the central research question

$$H_0 : \beta_1 \sim N(0, SE^2_{\beta_1})$$

What do we observe?

```r
tidy(m0)
```

```
## # A tibble: 2 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
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```

Computing the null hypothesis test manually

How likely is -0.125 if H_0 is true?

Let's check our data against the Normal PDF for H_0

```
pnorm(-0.125, 0, 0.0277)
```

```
## [1] 3.201352e-06
```

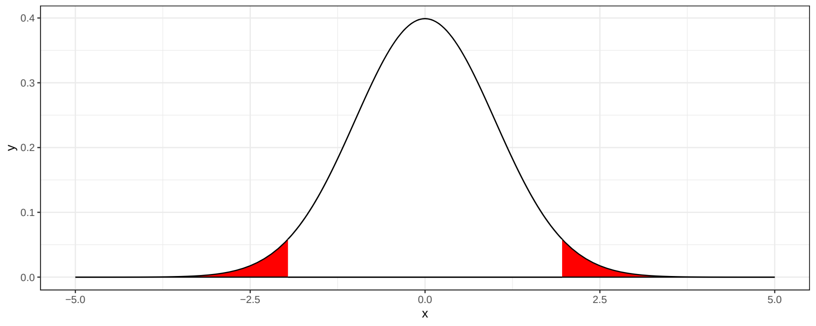
What do we think?

The Normal PDF and hypothesis testing

The logic of a hypothesis test

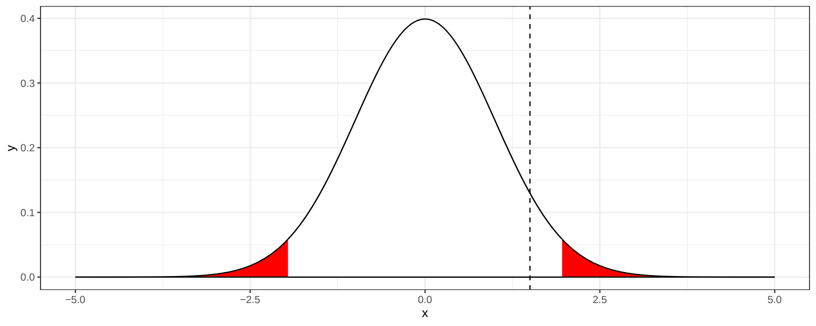
Assume $H_0 : Z(\beta) \sim N(0, 1)$ (standardized Beta follows a Z distribution)

We decide *a priori* that anything outside of the central 95% of the Normal PDF is inconsistent with H_0



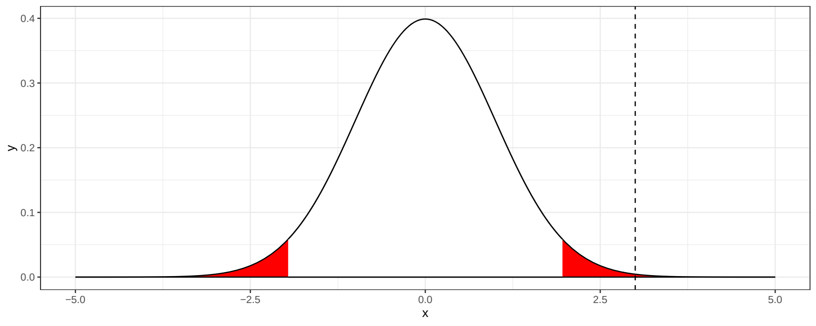
The logic of a hypothesis test

Now we observe our data and estimate our model. We find $\beta = 1.2$ and $SE = 0.8$. We convert that into a z-score $z = 1.2/0.8 = 1.5$ and check where it falls



The logic of a hypothesis test

Now we observe our data and estimate our model. We find $\beta = 1.2$ and $SE = 0.4$. We convert that into a z-score $z = 1.2/0.4 = 3$ and check where it falls



Null hypothesis testing: a recipe

1. Specify your null hypothesis (typically $Z(\beta) \sim N(0, 1)$)
2. Specify your critical value (sometimes called α), the threshold for finding 'statistical significance'
3. Estimate your model, compute a z statistic for β
4. Compute whether your $Z(\beta)$ falls in the critical region of the z distribution

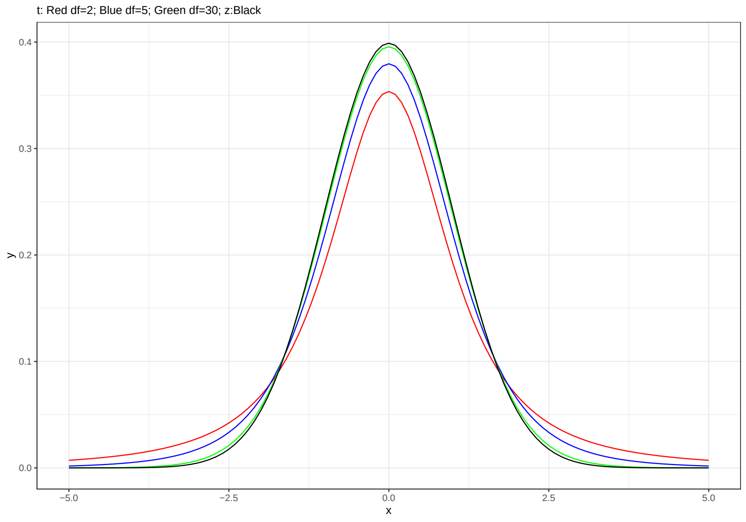
A note on t, z, and sample size

Technically, R will perform t tests, not z tests on our regression models.

When our *degrees of freedom* are large, the t distribution converges to the z distribution (Normal(0,1)).

Degrees of freedom for regression are defined as $n - k$, where n is sample size, and k is the number of parameters we are estimating in our model.

The t and the z: convergence for large DF



Using the central limit theorem to calculate confidence intervals, compute p-values

If the sampling distribution for β is defined as:

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Then we can construct a 95 percent CI for β

$$\hat{\beta} \pm 1.96 \times SE_{\beta}$$

Using the central limit theorem to calculate confidence intervals, compute p-values

If the sampling distribution for β is defined as:

$\hat{\beta} \sim N(\beta, SE_{\beta}^2)$

Then we can construct a 95 percent CI for β

$$\hat{\beta} \pm 1.96 \times SE_{\beta}$$

And conduct a z test for $\hat{\beta}$ by evaluating how likely our estimated $\hat{\beta}$ is under the null hypothesis

$H_0 : \beta \sim N(0, SE_{\beta}^2)$

Using OLS to estimate the SATE

```r
cr_ols<-lm(callback ~
             crimrec,
           data = cr)

tidy(cr_ols)
```

```
## # A tibble: 2 x 5
##   term        estimate std.error statistic  p.value
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```

- What is the implied null hypothesis here?
- How do we compute ‘statistic’ (z-statistic)?
- How do we compute ‘p.value’?

Interpretation

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tidy(cr_ols)
```

```
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```

- What can we conclude about β_1?
- What does this tell us about our focal research question?

Statistical significance give us information about whether differences we observe for some outcome across levels of a predictor in our data are likely to occur if there were no underlying differences in the data-generating process.

Statistical significance: an interpretation guide

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Put simply: if there were no difference across levels, how likely would I be to observe what I did observe?

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Put simply: if there were no difference across levels, how likely would I be to observe what I did observe?

What *t* tests and *z* tests do is provide a quick signal vs noise check for our data.

Statistical significance: CAUTIONS

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DO NOT USE SIGNIFICANCE TESTING ALONE TO DECIDE WHAT YOUR MODEL SHOULD BE