Probability, 2

Frank Edwards

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- Each value is mutually exclusive
- The set of all values is exhaustive (the sample space Ω)
- Discrete random variables take a finite number of values (e.g. TRUE, FALSE)
- Continuous random variables are real numbers, and take on an infinite number of values

The simplest random variable: the binary Bernoulli

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Bernoulli (binary) variables are typically represented as $[0, 1]$ or $[T, F]$. They can also be two-level character variables, like $[\text{pass, fail}]$ or $[\text{plaid, stripes}]$.

A coin flip as a Bernoulli random variable

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$P(X = 1) = p$

Random variable (probability distribution) notation

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In this notation, we name the variable X, note that it is randomly distributed $\sim$, name the distribution it follows *Bernoulli*, and list the parameters for that distribution p.

Let's flip some coins

```
sample_of_flips <- rbinom(5, 1, 0.5)
pander(table(sample_of_flips))
```

0	1
2	3

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This is the result of taking 5 draws from a Bernoulli random variable with probability 0.5.

Describing a probability distribution: probability mass

We use a probability mass function to show how likely each value is in a random variable

The probability mass function (PMF) of a variable X is defined as the probability that a variable takes on a particular value x.

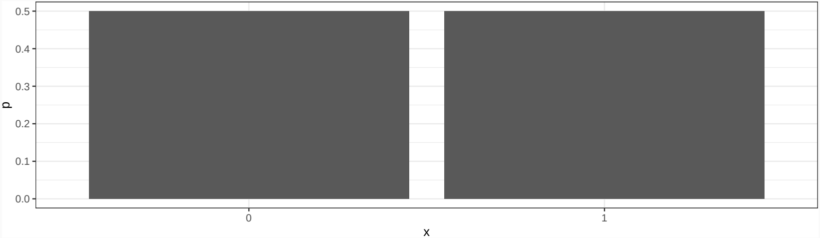
$$PMF(x) = P(X = x)$$

For a Bernoulli variable, $PMF(X = 1) = p$ and $PMF(X = 0) = 1 - p$

The probability mass function for our coin flip

$$PMF\left(X = 1\right) = p = 0.5$$

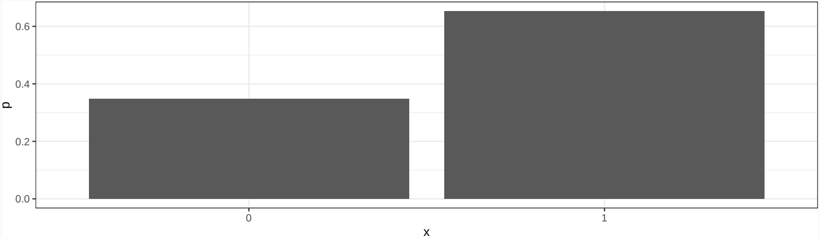
$$PMF\left(X = 0\right) = 1 - p = 0.5$$



The probability mass function for passing the bar in NJ (p=0.653)

$$PMF(X = 1) = p = 0.653$$

$$PMF(X = 0) = 1 - p = 0.347$$



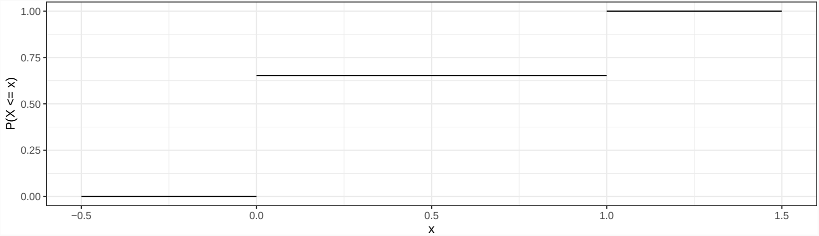
Describing a probability distribution: cumulative probability

How likely is a variable to take a value less than or equal to a specified value?

We define the cumulative distribution function as the sum of all probabilities up to a value x

$$CDF(X) = P(X \leq x) = \sum_{k \leq x} PMF(k)$$

The CDF always ranges from 0 to 1, and never decreases as x increases.



Uniform random variables

Uniform random variables have an equal probability of taking any real value within a given interval $[a, b]$.

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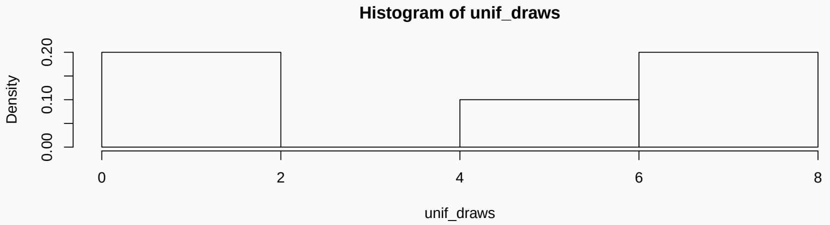
Uniform random variables

Let's simulate it! 10 draws

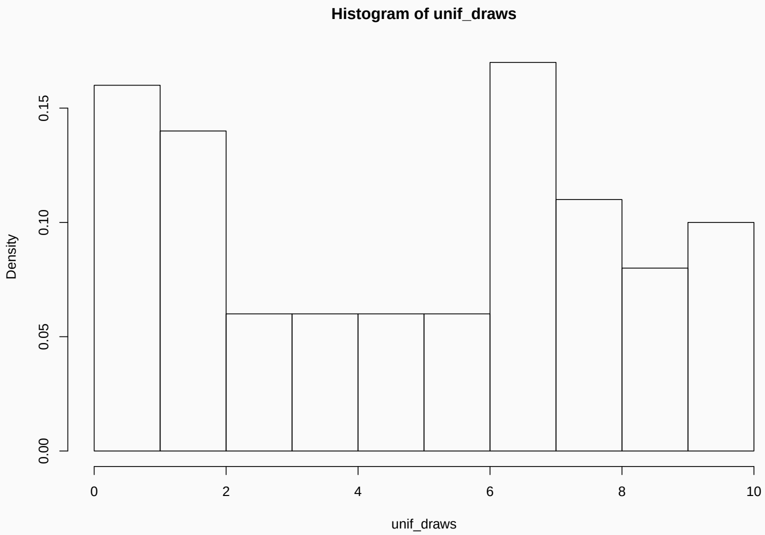
```r
unif_draws <- runif(10, min = 0, max = 10)
unif_draws
```

```
##  [1] 6.5493983 7.8063308 6.3490301 1.6410698 1.2223043 0.7049035 1.7576788
##  [8] 6.8328609 5.3540054 5.7468302
```

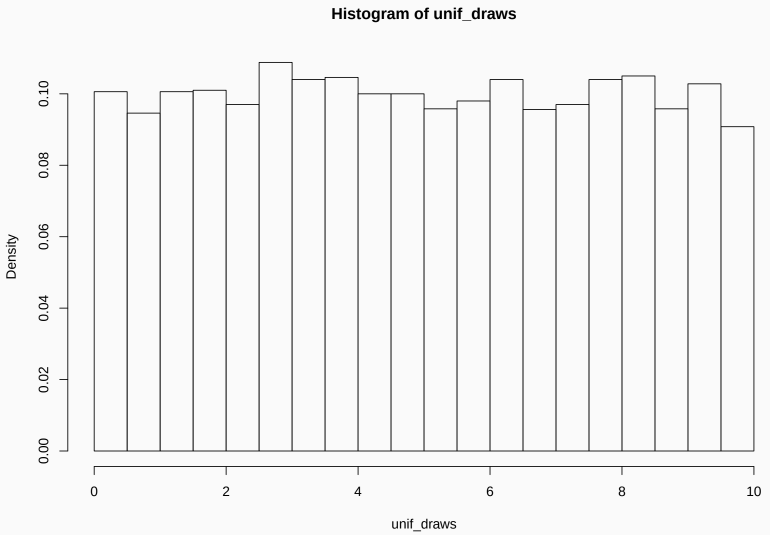
```r
hist(unif_draws, freq = F)
```



Uniform random variable: 100 draws



Uniform random variable: 10000 draws



Properties of uniform random variables

For a uniform random variable on the interval $[a, b]$, the probability of drawing any value between a and b is

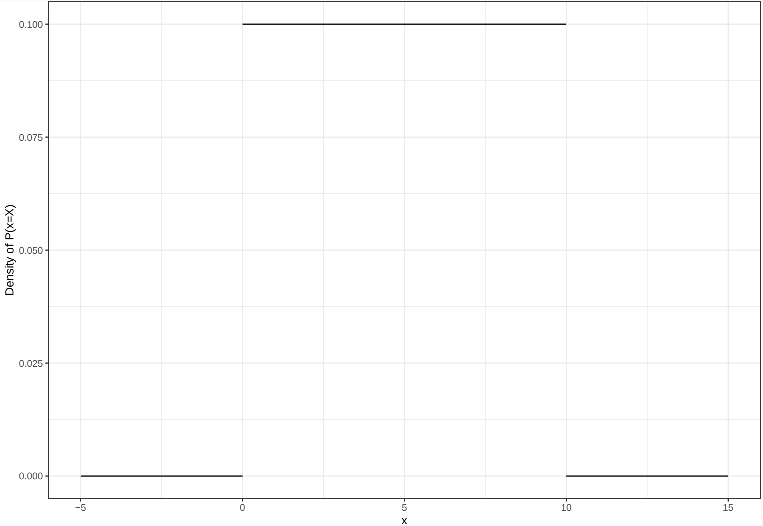
$$\frac{1}{b-a}$$

Formally, the PDF (density, not mass for continuous) and CDF are defined as

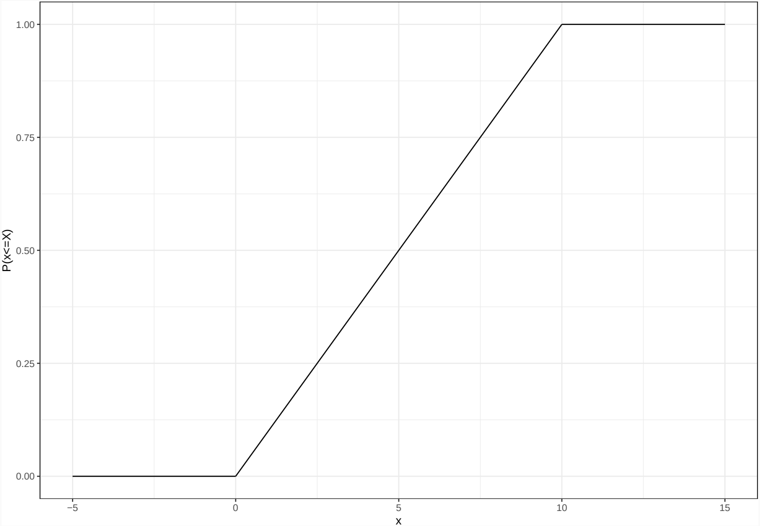
$$\text{PDF: } \begin{cases} \frac{1}{b-a} & \text{for } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

$$\text{CDF: } \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } x \in [a, b) \\ 1 & \text{for } x \geq b \end{cases}$$

Probability Density function for $X \sim Uniform(0, 10)$



Cumulative Distribution Function for $X \sim Uniform(0, 10)$



Recall that a CDF for a discrete variable is the sum of all probabilities for values $x \leq X$

We can't sum over each value when X is continuous. Instead, we'll take the integral

$$CDF(x) = P(x \leq X) = \int_{-\infty}^{x} PDF(x)dx$$

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Binomial random variables represent the count of successes in a fixed number of trials of a Bernoulli experiment.

Formally:

A binomial random variable is the sum of n independently and identically distributed (i.i.d) Bernoulli random variables.

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Binomial variables take on integer values between 0 and n

Back to flipping coins

Imagine we flipped a coin 5 times, and then repeated the exercise twice more

```
## [1] 0 0 0 0 1
```

```
## [1] 1 0 1 0 0
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Each of these trials is a sample from $X \sim Binomial(n, p)$ where $n = 5$ and $p = 0.5$

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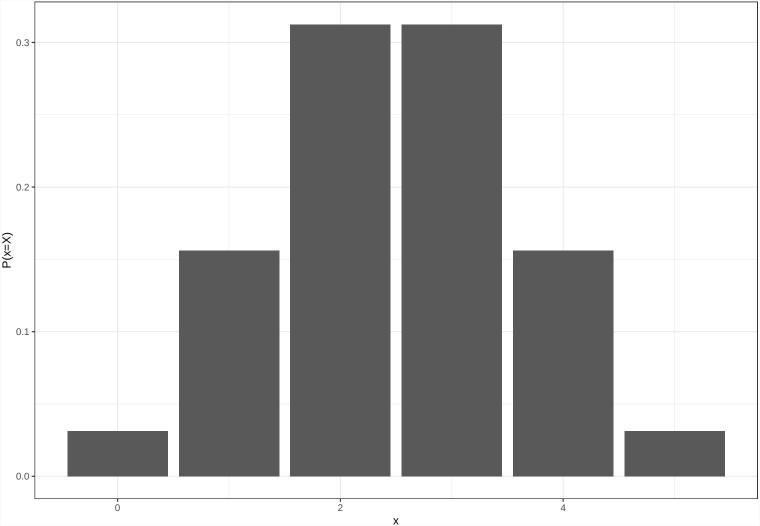
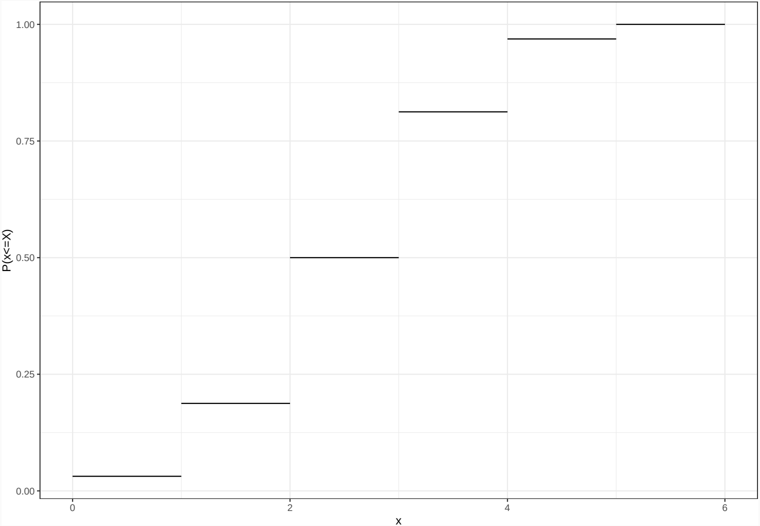
What is x for each trial?

Probability Mass Function for $X \sim Binomial(5, 0.5)$



Cumulative Distribution Function for $X \sim Binomial(5, 0.5)$

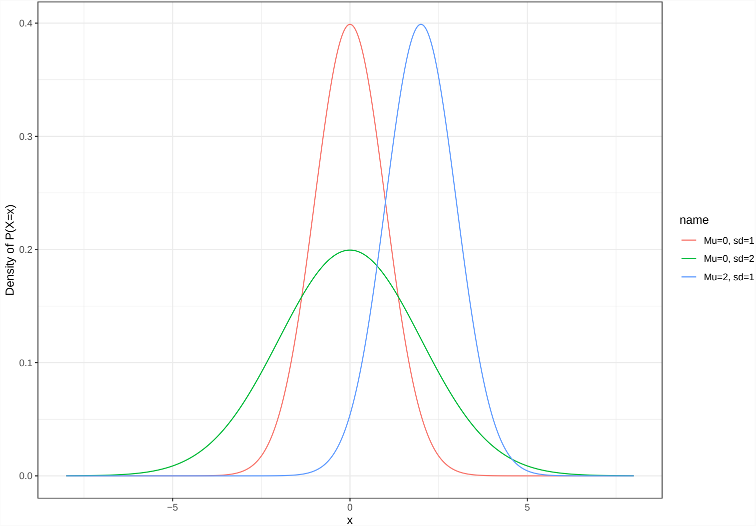


The Normal Distribution

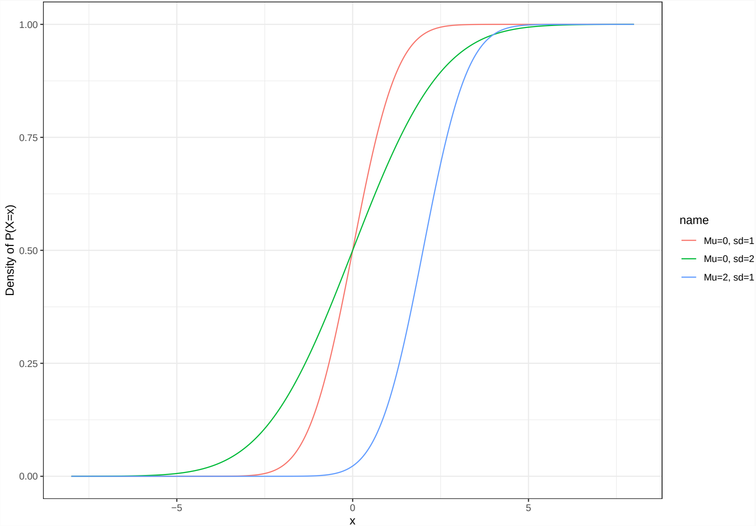
The Normal (Gaussian) distribution is continuous, and takes on values from $[-\infty, \infty]$. It has two parameters, the mean μ and standard deviation σ (or variance σ^2).

- μ determines the location of the distribution
- σ determines the spread of the distribution

The Normal PDF



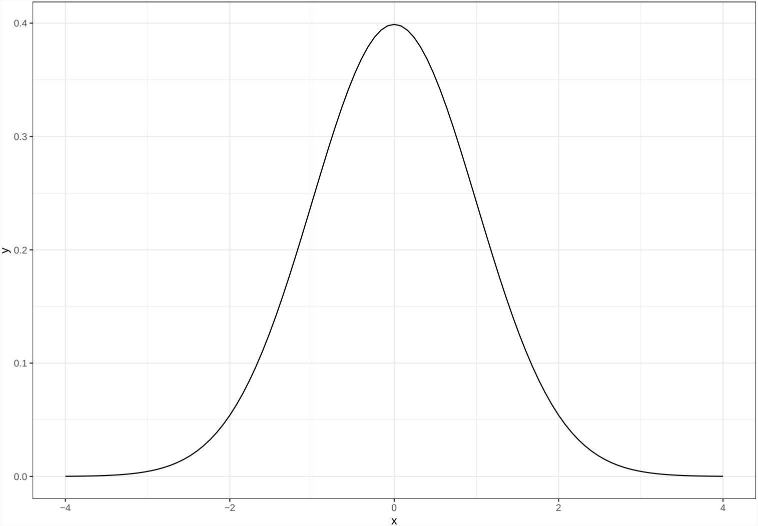
The Normal CDF



Special features of Normal distributions:

- The sum of many random variables from other distributions are often Normal
- For $X \sim N(\mu, \sigma^2)$, $Z = X + c$ is also Normal: $Z \sim (\mu + c, \sigma^2)$
- $Z = cX$ is distributed $Z \sim N(c\mu, (c\sigma)^2)$
- Z-scores of a Normal random variable are $N(0, 1)$

Area under the curve: interpreting the PDF and CDF



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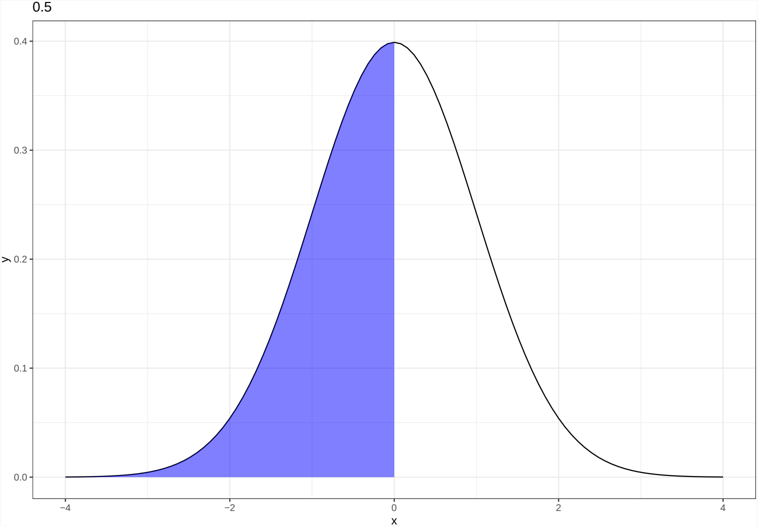

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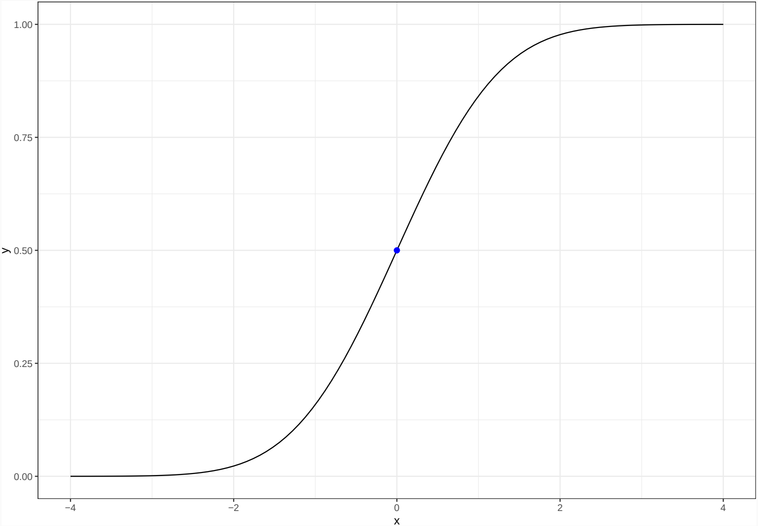

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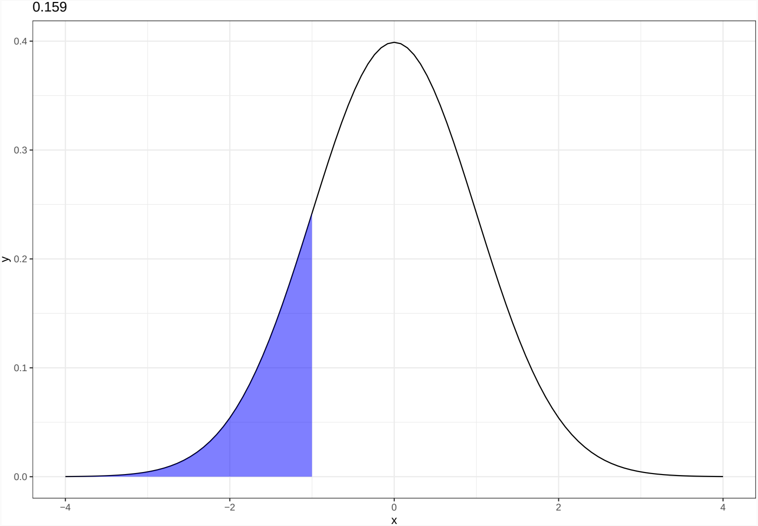

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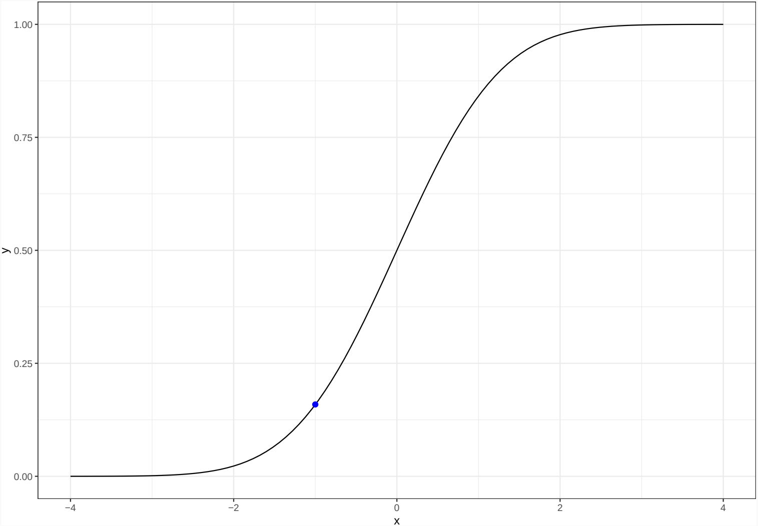

Z-scores and area under the curve

Recall that to obtain a z-score, we subtract the mean and divide by the standard deviation:

$$\text{z-score} = \frac{X - \mu}{\sigma}$$

For a Normal variable, z-scores are distributed $z \sim N(0, 1)$

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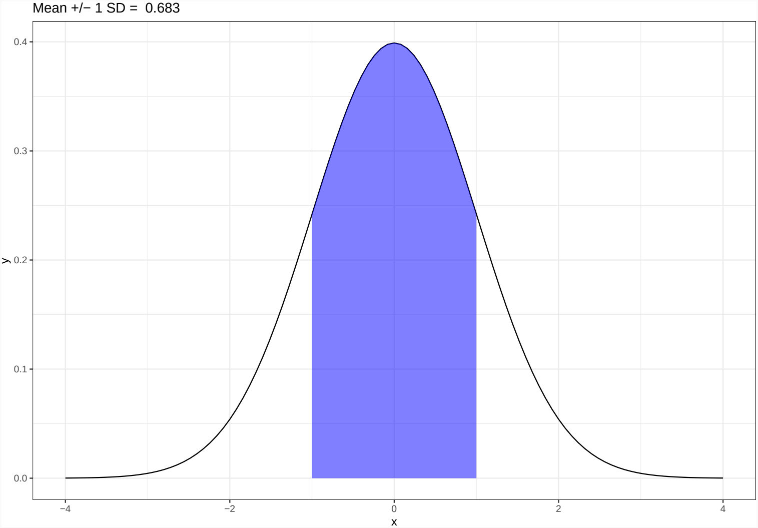
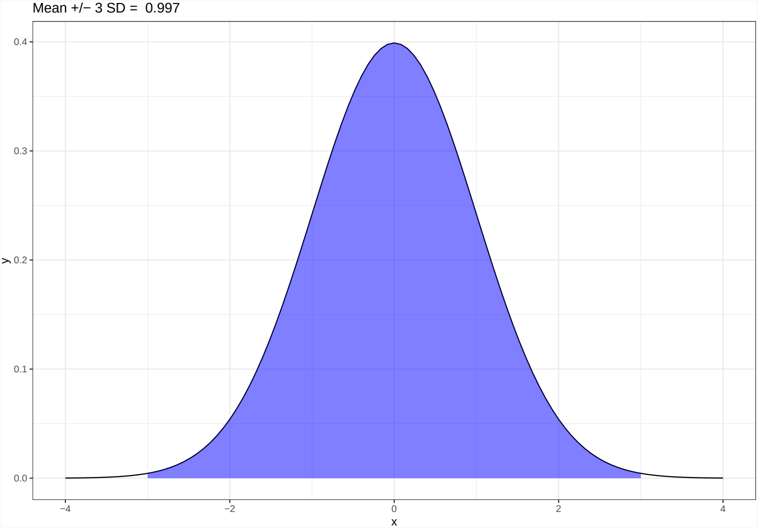


Mean +/− 1 SD = 0.683
0.4
0.3
0.2
0.1
0.0
y
−4
−2
0
2
4
x


Mean +/− 2 SD = 0.954
0.4
0.3
0.2
0.1
0.0
y
−4
−2
0
2
4
x


Mean +/− 3 SD = 0.997
0.4
0.3
0.2
0.1
0.0
y
−4
−2
0
2
4
x

Useful probability distribution functions

```r
### Normal(0,1) probability density function
dnorm(x = 0, mean = 0, sd = 1)
```

```
## [1] 0.3989423
```

```r
### Normal(0,1) cumulative distribution function
pnorm(q = 0, mean = 0, sd = 1)
```

```
## [1] 0.5
```

```r
### Random draw from a normal(0,1) distribution
rnorm(n = 1, mean = 0, sd = 1)
```

```
## [1] -0.6976793
```

```r
### CDF position for a given probability (quantile)
qnorm(p = 0.75, mean = 0, sd = 1)
```

```
## [1] 0.6744898
```

```r
### You can also use dbinom(), pbinom(), rbinom(), qbinom()
```

The expectation of a random variable

The expectation of a random variable $E(X)$ is the mean of a random variable.

Be careful not to confuse $E(X)$ and $\bar{x}$.

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For a discrete variable, the expectation is the sum of all values of x weighted by their probability, given by the PDF $f(x)$.

$$E(X) = \sum_{x} x \times f(x)$$

Because continuous variables take on an infinite number of values, we compute the expectation with an integral

$$\int x \times f(x) dx$$

Variance and standard deviation of a random variable

Recall that for a sample, the standard deviation sd

$$sd = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

And the sample variance is sd^2

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For a random variable X, the variance is defined via the expectation instead of sample mean

$$V(X) = E\left[\{X - E(X)\}^2\right]$$

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$$V(X) = E\left[\left\{X - E(X)\right\}^2\right]$$

Note the similarities in the two equations

Large sample (asymptotic) theorems

As a sample of draws from a random variable increases, the sample mean converges to the population mean $E(X)$

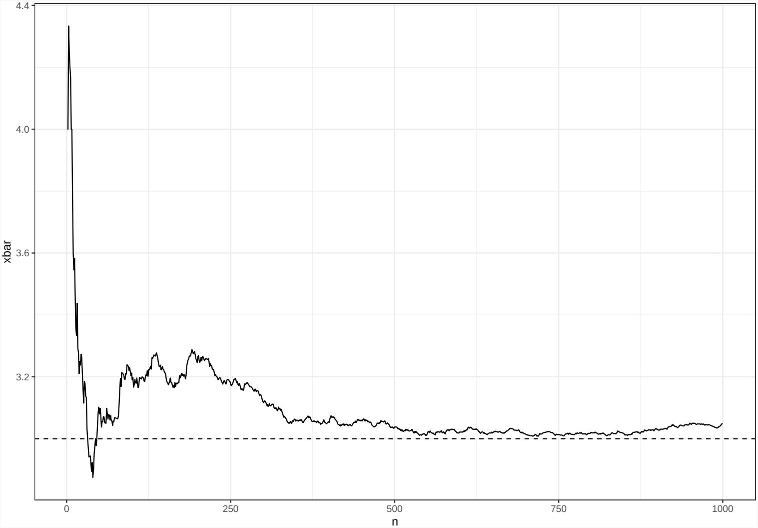
$$\bar{x}_n \longrightarrow E(X)$$

Monte Carlo simulation for the mean of a binomial variable

To test the law of large numbers, let's draw from a binomial variable with varying sample sizes.

We expect that $\bar{x}$ will converge to $E(X)$ as the sample size n increases

```
## MC simulation, 1000 reps
sims <- 1000
## Take 1000 draws from binomial(0.3, 10)
x <- rbinom(sims, p = 0.3, size = 10)
### output df
out <- data.frame(n = 1:sims, xbar = NA)
for (i in 1:sims) {
    out$xbar[i] <- sum(x[1:i])/i
}
## or use xbar<-cumsum(x)/1:sims
```


4.4
4.0
3.6
3.2
xbar
0
250
500
750
1000
n

The Central Limit Theorem

- As n increases, the distribution of the sample mean $\bar{x}$ approaches a Normal distribution.

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- If our samples are independent, and each observation within the sample is iid, the distribution of z-scores of sample means converges to a $Normal(0, 1)$ distribution

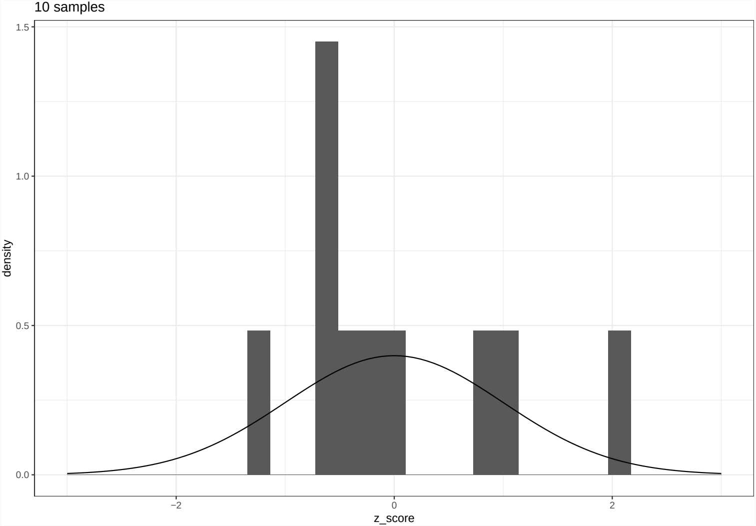
- As n increases, the distribution of the sample mean $\bar{x}$ approaches a Normal distribution.
- This relationship holds for many distributions (Bernoulli, Binomial, Normal, others we'll discuss later)
- If our samples are independent, and each observation within the sample is iid, the distribution of z-scores of sample means converges to a $Normal(0, 1)$ distribution
- The Central Limit Theorem allows us to make statements about uncertainty when we haven't observed the population mean or variance

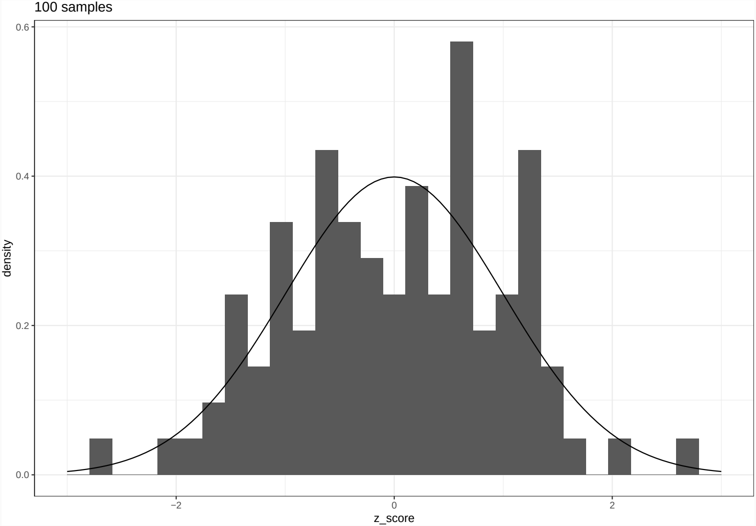
Monte Carlo simulations of a binomial variable p=0.7, n=10

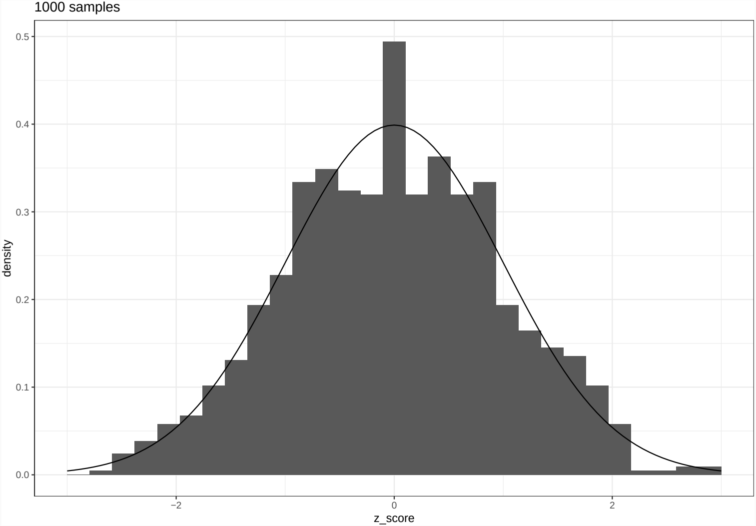
```
### Binomial random variable, 10 observations, probability of success = 0.7
### E(X) = np = 0.7 * 10 pretend this is classes with size 10, probability of
### passing = 0.7 sample size = 10 classes of 10 students, size = 100 classes,
### size = 1000 classes simulate each sample size for 1000 replications
sims <- 1000
n <- 10
p <- 0.7
xbar_10 <- rep(NA, 10)
xbar_100 <- rep(NA, 100)
xbar_1000 <- rep(NA, 1000)
for (i in 1:10) {
    x_10 <- rbinom(10, p = p, size = n)
    xbar_10[i] <- (mean(x_10))
}

for (i in 1:100) {
    x_100 <- rbinom(100, p = p, size = n)
    xbar_100[i] <- (mean(x_100))
}

for (i in 1:1000) {
    x_1000 <- rbinom(1000, p = p, size = n)
    xbar_1000[i] <- (mean(x_1000))
}
```


10 samples
1.5
1.0
0.5
0.0
density
−2
0
2
z_score


100 samples
density
0.6
0.4
0.2
0.0
−2
0
2
z_score



1000 samples
0.5
0.4
0.3
0.2
0.1
0.0
density
−2
0
2
z_score

Lab, HW

- Struggling with probability? Try Khan Academy probability and distributions.
- Homework (late work): Any late assignments (including this week's HW) are due without penalty by Friday. After that, I'm going to start deducting points for late work.
- Homework: Question 6.6.3. Due 12/4. Start early on this to give yourself time to ask questions
- Lab: Obama vote share example, and some hints for the homework