Regression and uncertainty part 2: stochastic error

Frank Edwards

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Understanding the regression line

```
## # A tibble: 10 x 2
##         x       y
##     <dbl>   <dbl>
##  1  0.849  0.623
##  2  1.03   0.809
##  3  2.09   1.89
##  4 -0.763 -0.315
##  5  1.70   2.23
##  6  0.749  0.514
##  7  0.703  0.305
##  8 -0.145 -0.0992
##  9  1.14   1.40
## 10  2.65   2.64
```

$\hat{\beta}_0 = 0.05$, $\hat{\beta}_1 = 0.95$

- Estimate $\hat{Y}$. Recall that $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$
- Estimate ε. Recall that $\varepsilon = Y - \hat{Y}$

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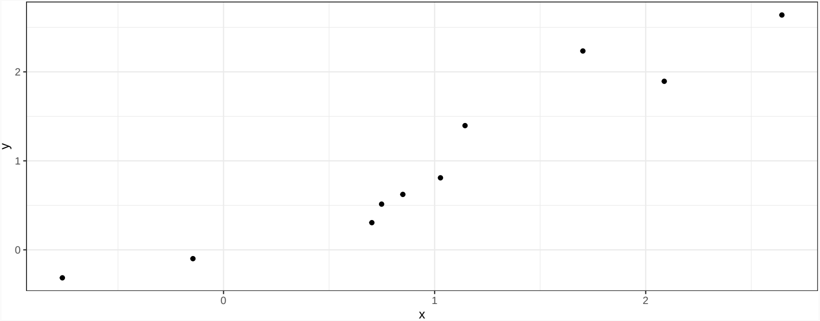
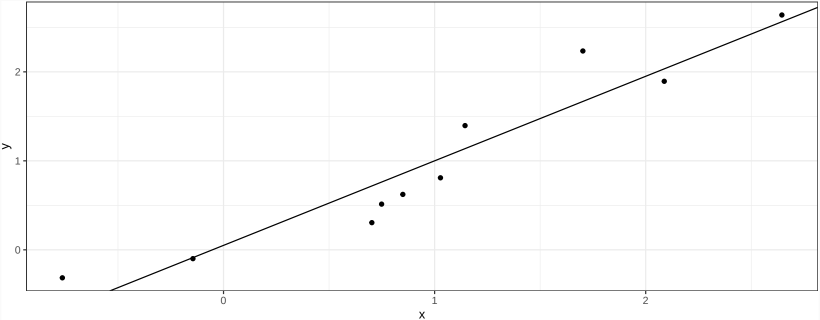


Understanding the regression line: adding the fit

$\hat{\beta}_0 = 0.05$, $\hat{\beta}_1 = 0.95$



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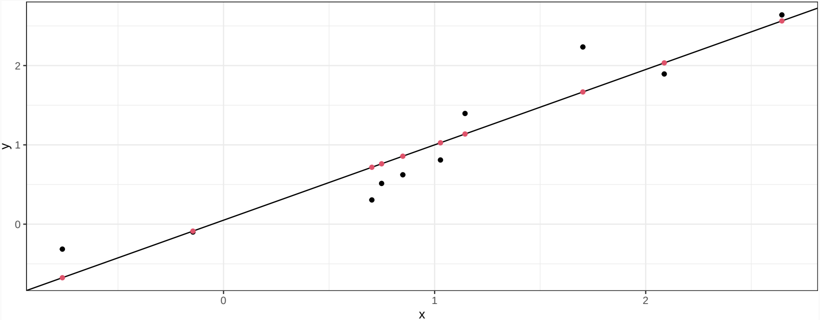

Understanding the regression line: adding ε

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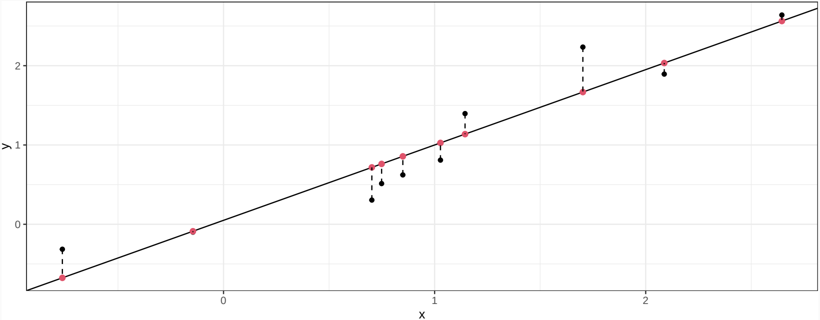

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Ways to express an OLS model

As linear with Normal errors:

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots \varepsilon$$

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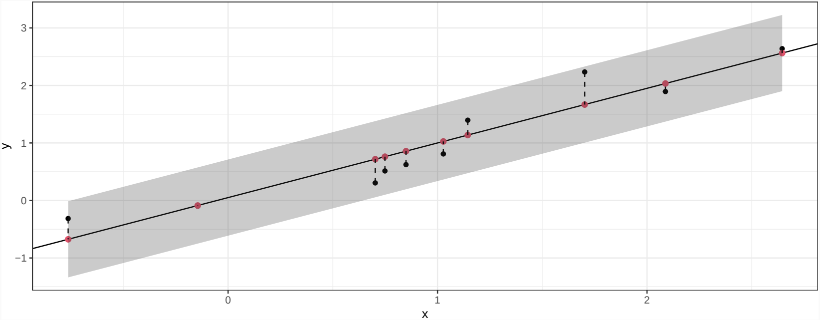
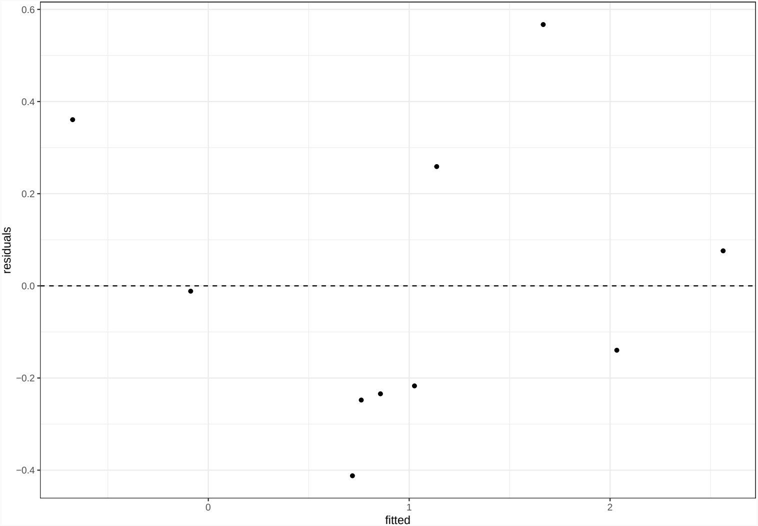
As Normal, with linear vector of means:

$$y \sim N(\beta X, \sigma^2)$$

What this means: 95% of observations should fall in this zone



One way to visualize: residuals vs fitted



Let’s try this with real data

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```
## [1] "4"            "1"            "2"            NA             "3"
## [6] "5+"           "<1"           "do not watch"
```

```
## [1] "8"   "7"   "5"   "6"   "10+" "<5"  NA    "9"
```

```
## # A tibble: 64 x 13
##      age gender grade hispanic race                     height weight helmet_12m
##    <dbl> <chr>  <dbl> <chr>    <chr>                     <dbl>  <dbl> <chr>
##  1    16 female    11 not      Black or African Americ~   1.5    52.6 never
##  2    17 male      11 not      White                      1.78   74.8 rarely
##  3    17 male      11 not      White                      1.75  107.  never
##  4    18 male      12 not      Black or African Americ~   1.7    80.3 never
##  5    16 male      10 not      White                      1.78   81.6 always
##  6    16 male      10 not      Black or African Americ~   1.63   56.7 never
##  7    14 male       9 hispanic White                      1.63   54.4 never
##  8    17 male      11 not      Black or African Americ~   1.83   92.5 never
##  9    15 male       9 not      Black or African Americ~   1.69   67.8 did not r~
## 10    17 male      10 not      White                      1.78   66.7 never
## # i 54 more rows
## # i 5 more variables: text_while_driving_30d <chr>, physically_active_7d <dbl>,
## #   hours_tv_per_school_day <dbl>, strength_training_7d <dbl>,
## #   school_night_hours_sleep <dbl>
```

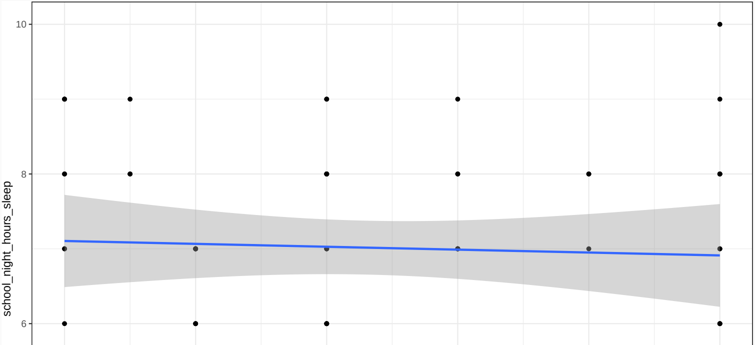
Fit a model for sleep duration predicted by tv watching

```r
m1 <- lm(school_night_hours_sleep ~ hours_tv_per_school_day, data = dat)

tidy(m1)
```

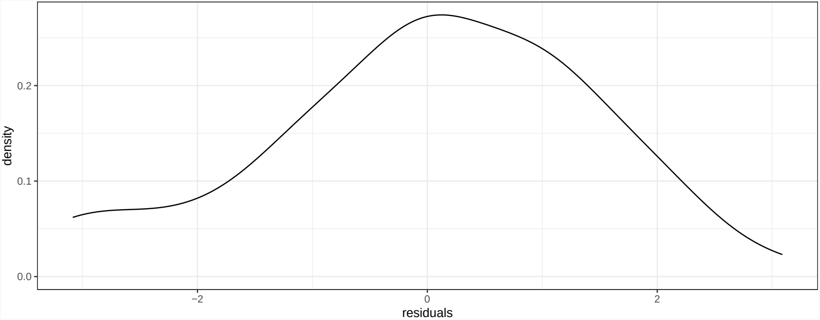
```
## # A tibble: 2 x 5
##   term                    estimate std.error statistic  p.value
##   <chr>                      <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)               7.10       0.308    23.1   4.15e-32
## 2 hours_tv_per_school_day  -0.0386     0.109    -0.356 7.23e- 1
```

```r
ggplot(dat, aes(x = hours_tv_per_school_day, y = school_night_hours_sleep)) + geom_point() +
    geom_smooth(method = "lm")
```



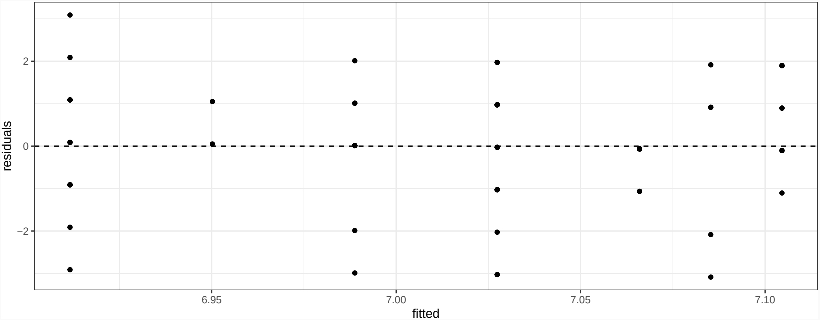
What is the distribution of the residuals? Are they Normal?

```
ggplot(data = data.frame(residuals = resid(m1)), aes(x = residuals)) + geom_density()
```



What about residuals vs fitted?

```r
ggplot(data = data.frame(fitted = fitted(m1), residuals = resid(m1)), aes(x = fitted,
    y = residuals)) + geom_point() + geom_hline(yintercept = 0, lty = 2)
```



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But that's not the only source of uncertainty in our model!

Sources of random variation in our model

$$\hat{\beta}_0 \sim N(\beta_0, s^2_{\beta_0})\ \hat{\beta}_1 \sim N(\beta_1, s^2_{\beta_1})\ y = \beta_1 + \beta_2 + \varepsilon\ \varepsilon \sim N(0, \sigma^2)$$

What does epsilon represent?

Exercise: Make predictions from m1 for y.

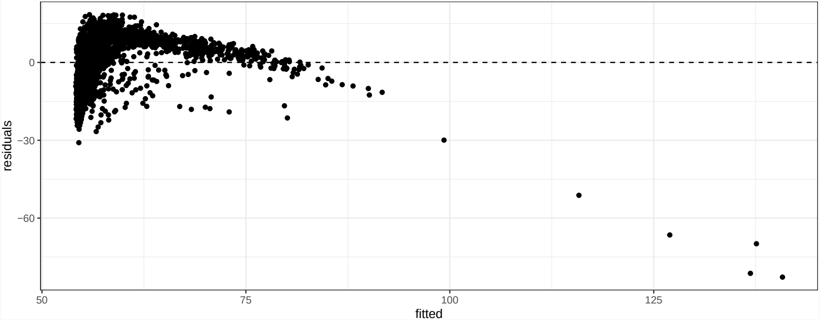
What does non-constant error variance look like?

Data with non-constant error variance will show distinctive patterns in their residuals

```
library(gapminder)
m2 <- lm(lifeExp ~ gdpPercap, data = gapminder)
```

Non-contant error variance

```r
ggplot(data.frame(fitted = fitted(m2), residuals = resid(m2)), aes(x = fitted, y = residuals)) +
    geom_point() + geom_hline(yintercept = 0, lty = 2)
```



Problems

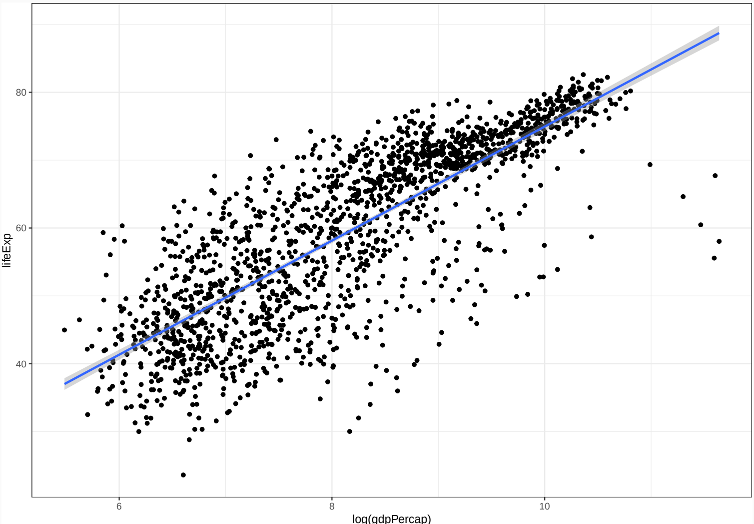
Here there are actually two problems:

1. The linear model does not reflect the data generating process (the relationship between x and y)
2. Error variance is not constant

OK, so log it

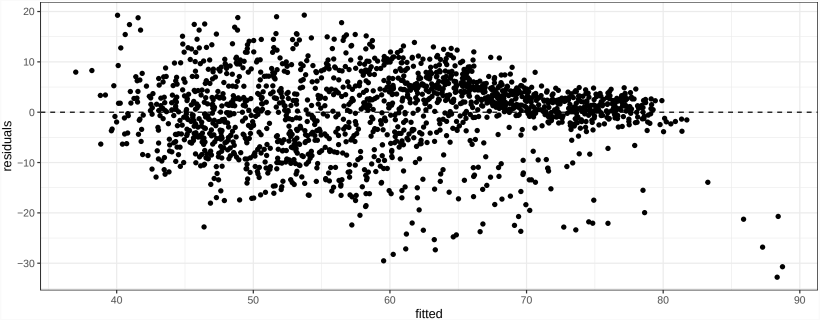
```
library(gapminder)
m3 <- lm(lifeExp ~ log(gdpPercap), data = gapminder)

ggplot(gapminder, aes(x = log(gdpPercap), y = lifeExp)) + geom_point() + geom_smooth(method = "lm")
```



Non-contant error variance

```r
ggplot(data.frame(fitted = fitted(m3), residuals = resid(m3)), aes(x = fitted, y = residuals)) +
    geom_point() + geom_hline(yintercept = 0, lty = 2)
```



In summary

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We can write this two ways (they are equivalent):

1. $y = \beta_0 + \beta_1 X + \varepsilon; \varepsilon \sim N(0, \sigma^2)$
2. $\mu = \beta_0 + \beta_1 X; y \sim N(\mu, \sigma^2)$

In summary

We have multiple sources of random error in our model.

1. We have uncertainty in our estimates of β, that we approximate using the Central Limit Theorem. This is the uncertainty we have about the location of $E(y)$ or μ
2. We have uncertainty in y, which is ordinary sampling error. For linear regression, we are going to assume that conditional on X, this error is Normally distributed