Probability, 2

Frank Edwards

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- Each value is mutually exclusive
- The set of all values is exhaustive (the sample space Ω)
- Discrete random variables take a finite number of values (e.g. TRUE, FALSE)
- Continuous random variables are real numbers, and take on an infinite number of values

The simplest random variable: the binary Bernoulli

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Bernoulli (binary) variables are typically represented as $[0, 1]$ or $[T, F]$. They can also be two-level character variables, like $[\text{pass, fail}]$ or $[\text{plaid, stripes}]$.

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$P(X = 1) = p$

Random variable (probability distribution) notation

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In this notation, we name the variable X, note that it is randomly distributed $\sim$, name the distribution it follows *Bernoulli*, and list the parameters for that distribution p.

Let’s flip some coins

```
set.seed(12345)
sample_of_flips<-rbinom(5, 1, 0.5)
table(sample_of_flips)
```

```
## sample_of_flips
## 0 1
## 1 4
```

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```

```
## sample_of_flips
## 0 1
## 1 4
```

This is the result of taking 5 draws from a Bernoulli random variable with probability 0.5.

Describing a probability distribution: probability mass

We use a probability mass function to show how likely each value is in a random variable

The probability mass function (PMF) of a variable X is defined as the probability that a variable takes on a particular value x.

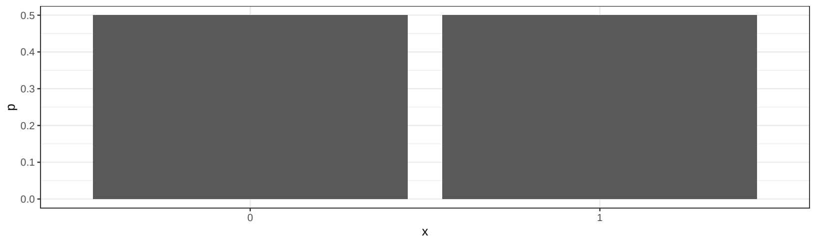
$$PMF(x) = P(X = x)$$

For a Bernoulli variable, $PMF(X = 1) = p$ and $PMF(X = 0) = 1 - p$

The probability mass function for our coin flip

$$PMF(X = 1) = p = 0.5$$

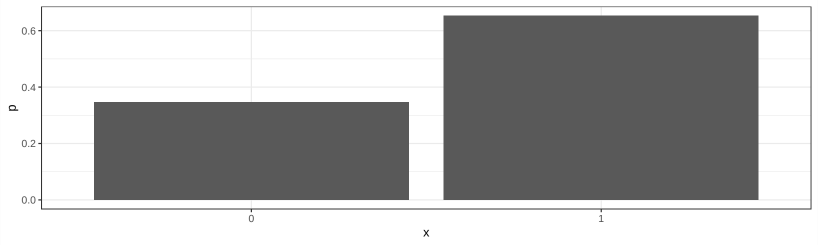
$$PMF(X = 0) = 1 - p = 0.5$$



The probability mass function for passing the bar in NJ (p=0.653)

$$PMF(X = 1) = p = 0.653$$

$$PMF(X = 0) = 1 - p = 0.347$$



How likely is a variable to take a value less than or equal to a specified value?

We define the cumulative distribution function as the sum of all probabilities up to a value x

$$CDF(X) = P(X \leq x) = \sum_{k \leq x} PMF(k)$$

The CDF always ranges from 0 to 1, and never decreases as x increases.

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What does $X \sim \mathit{Uniform}(0, 10)$ look like?

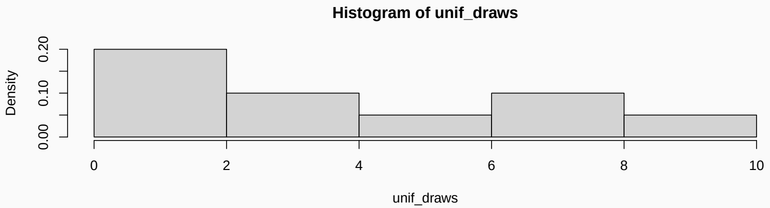
Uniform random variables

Let's simulate it! 10 draws

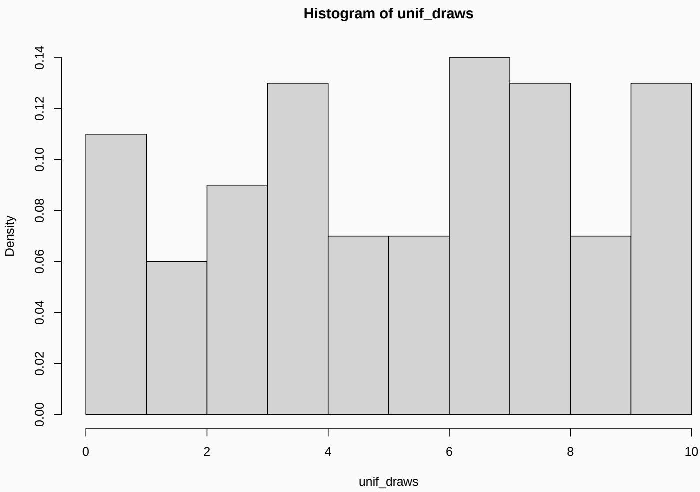
```
unif_draws<-runif(10, min=0, max=10)
unif_draws
```

```
##  [1] 1.66371785 3.25095387 5.09224336 7.27705254 9.89736938 0.34535435
##  [7] 1.52373490 7.35684952 0.01136587 3.91203335
```

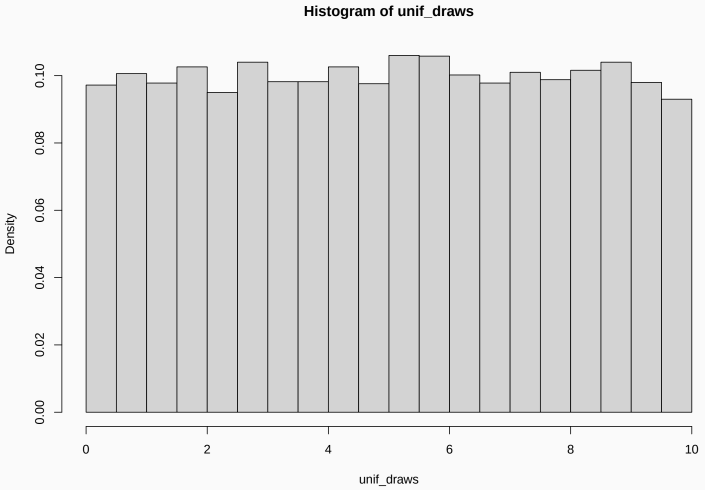
```
hist(unif_draws, freq=F)
```



Uniform random variable: 100 draws



Uniform random variable: 10000 draws



Properties of uniform random variables

For a uniform random variable on the interval $[a, b]$, the probability of drawing any value between a and b is

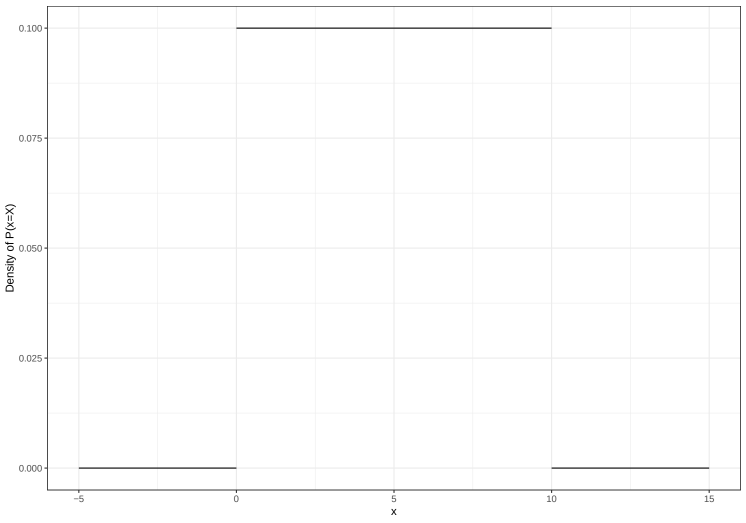
$$\frac{1}{b-a}$$

Formally, the PDF (density, not mass for continuous) and CDF are defined as

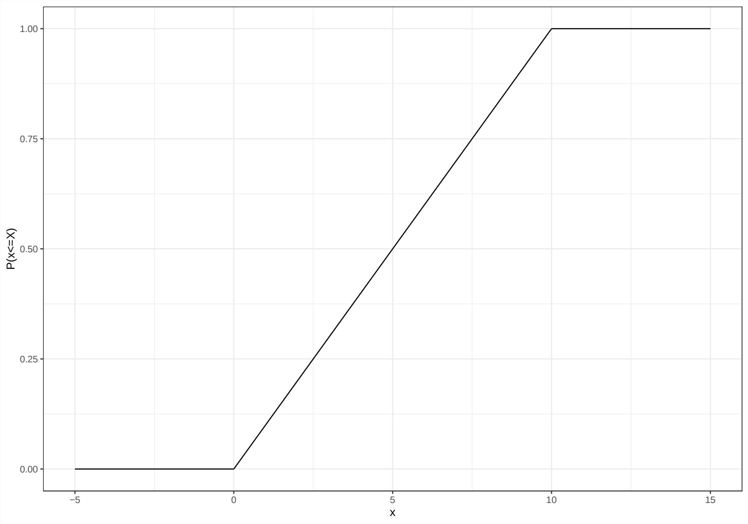
$$\text{PDF:} \begin{cases} \frac{1}{b-a} & \text{for } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

$$\text{CDF:} \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } x \in [a, b) \\ 1 & \text{for } x \geq b \end{cases}$$

Probability Density function for $X \sim Uniform(0, 10)$



Cumulative Distribution Function for $X \sim Uniform(0, 10)$



A note on CDF for continuous variables

Recall that a CDF for a discrete variable is the sum of all probabilities for values $x \leq X$

We can't sum over each value when X is continuous. Instead, we'll take the integral

$$CDF(x) = P(x \leq X) = \int_{-\infty}^{x} PDF(x)dx$$

The binomial distribution

When we repeat Bernoulli trials many times, we get a binomial random variable.

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Binomial random variables represent the count of successes in a fixed number of trials of a Bernoulli experiment.

Formally:

A binomial random variable is the sum of n independently and identically distributed (i.i.d) Bernoulli random variables.

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Formally:

A binomial random variable is the sum of n independently and identically distributed (i.i.d) Bernoulli random variables.

Binomial variables take on integer values between 0 and n

Imagine we flipped a coin 5 times, and then repeated the exercise twice more

```
## [1] 0 1 0 0 0
```

```
## [1] 0 0 0 0 1
```

```
## [1] 1 1 1 1 0
```

Each of these trials is a sample from $X \sim Binomial(n, p)$ where $n = 5$ and $p = 0.5$

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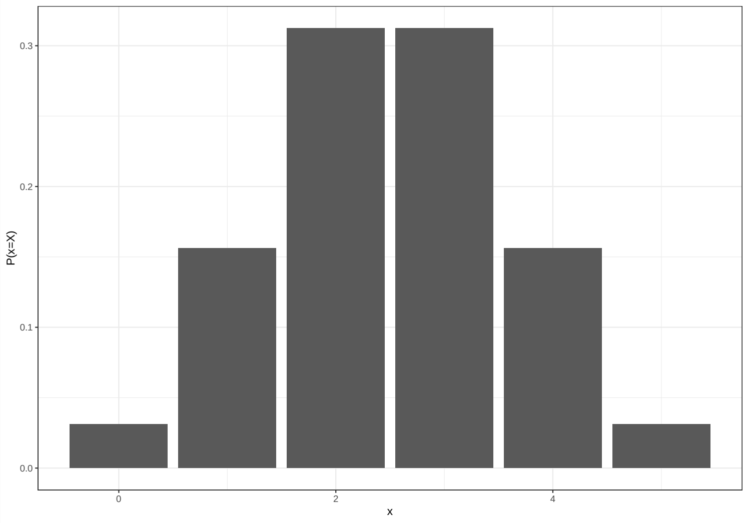
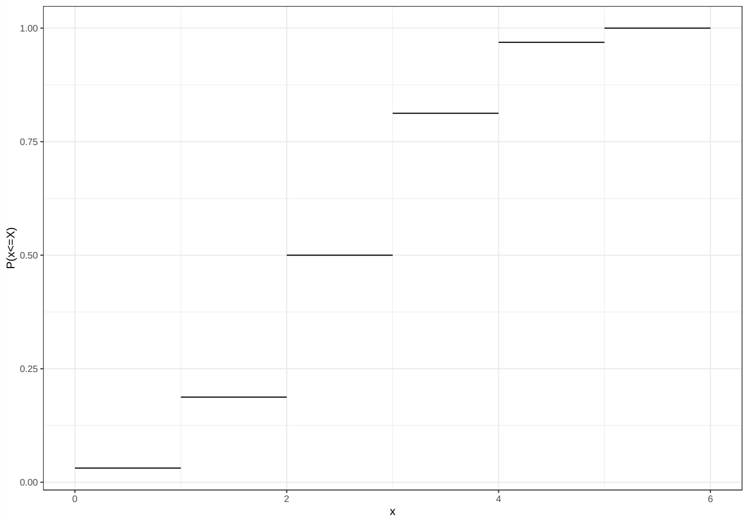
What is x for each trial?

Probability Mass Function for $X \sim Binomial(5, 0.5)$



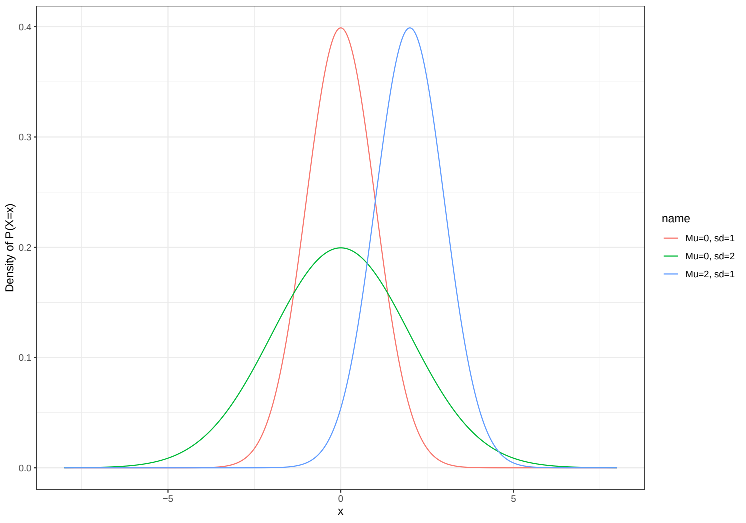
Cumulative Distribution Function for $X \sim Binomial(5, 0.5)$



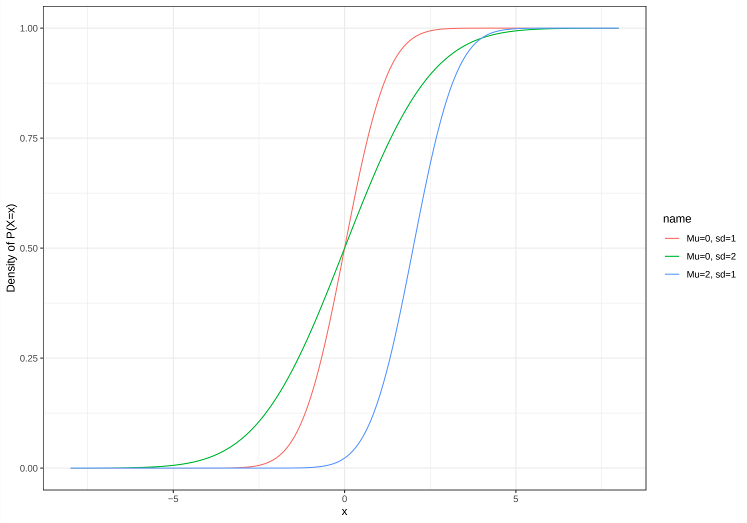
The Normal (Gaussian) distribution is continuous, and takes on values from $[-\infty, \infty]$. It has two parameters, the mean μ and standard deviation σ (or variance σ^2).

- μ determines the location of the distribution
- σ determines the spread of the distribution

The Normal PDF



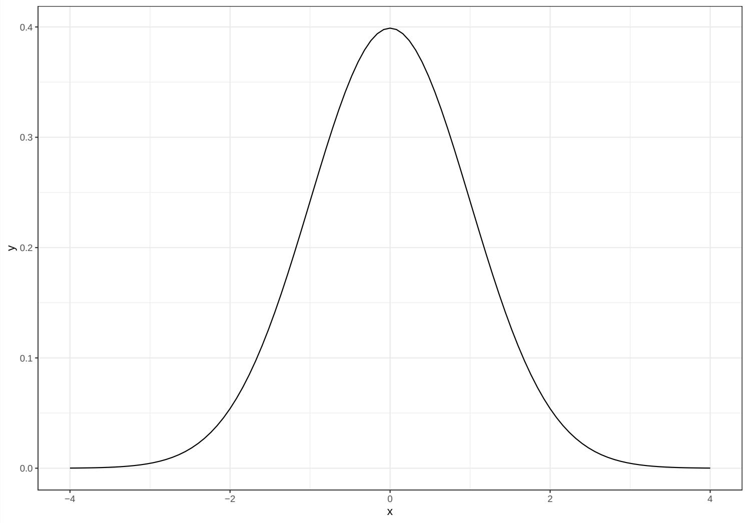
The Normal CDF



Special features of Normal distributions:

- The sum of many random variables from other distributions are often Normal
- For $X \sim N(\mu, \sigma^2)$, $Z = X + c$ is also Normal: $Z \sim (\mu + c, \sigma^2)$
- $Z = cX$ is distributed $Z \sim N(c\mu, (c\sigma)^2)$
- Z-scores of a Normal random variable are $N(0, 1)$

Area under the curve: interpreting the PDF and CDF



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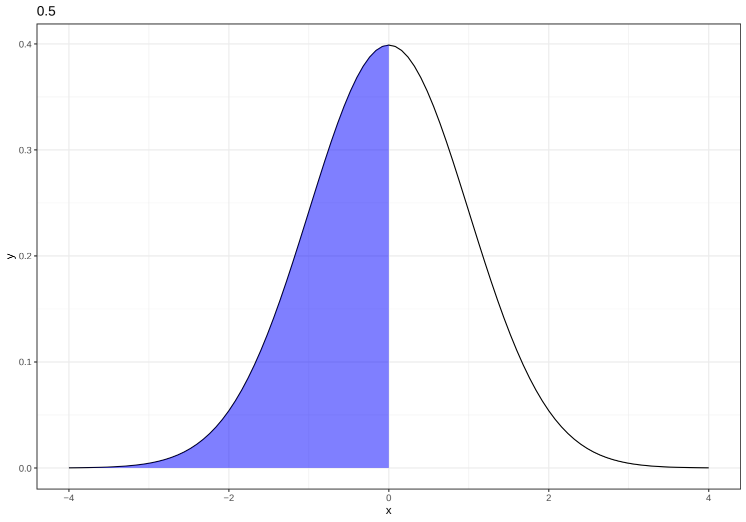
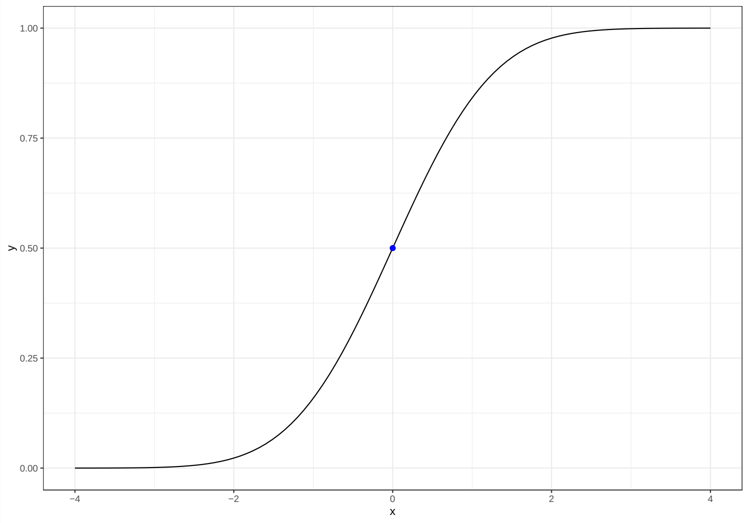


Area under the curve: interpreting the PDF and CDF



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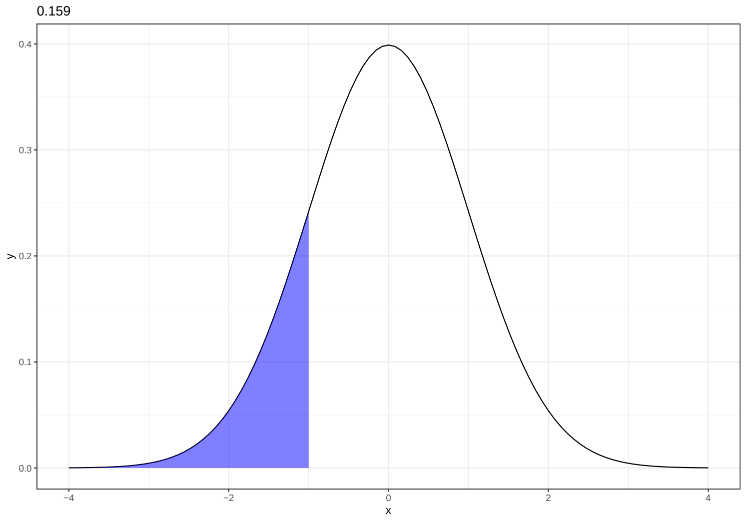

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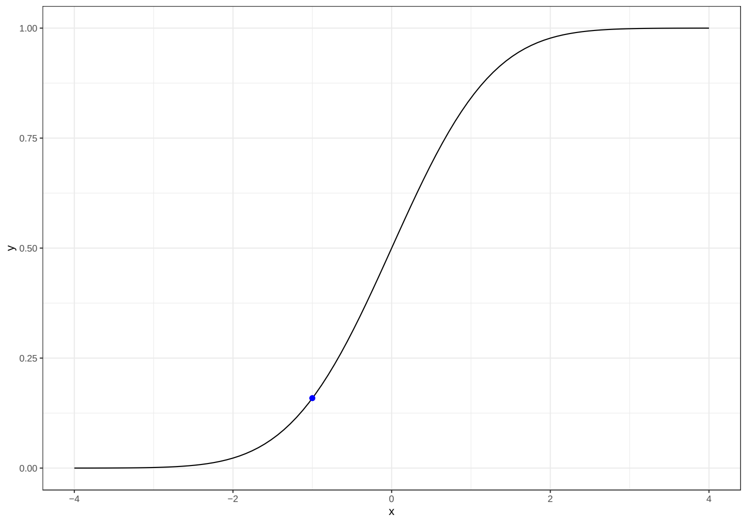

Z-scores and area under the curve

Recall that to obtain a z-score, we subtract the mean and divide by the standard deviation:

$$\text{z-score} = \frac{X - \mu}{\sigma}$$

For a Normal variable, z-scores are distributed $z \sim N(0, 1)$

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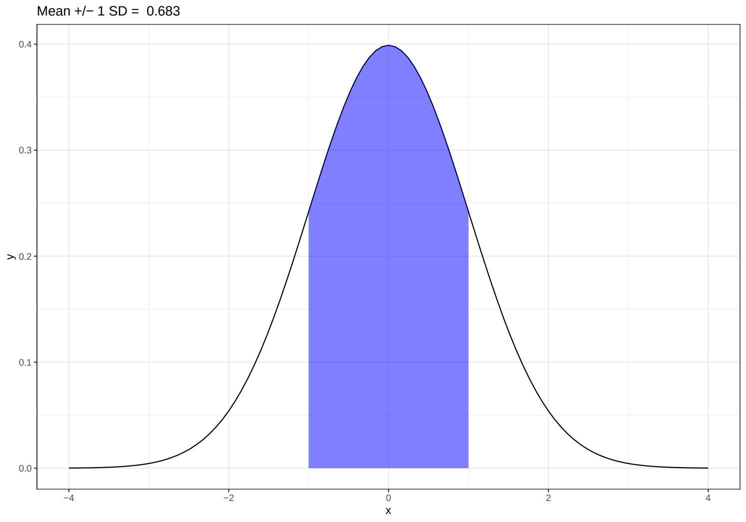
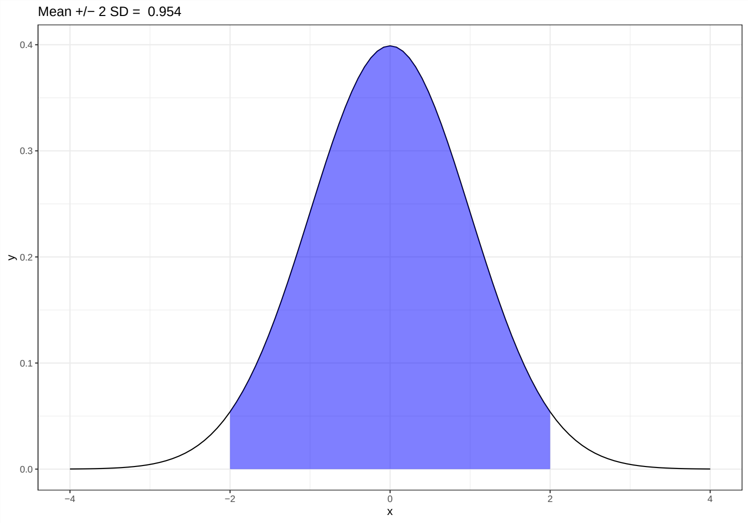
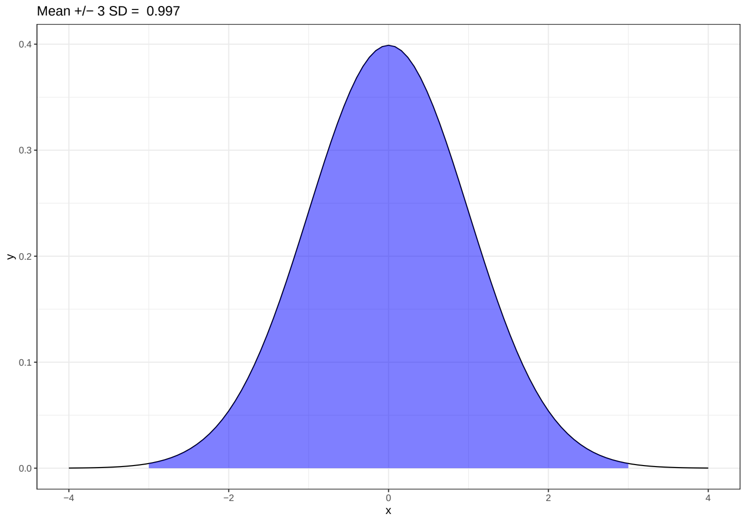
What does a z-score of 0 indicate? -1? 2?



Mean +/− 1 SD = 0.683
0.4
0.3
0.2
0.1
0.0
y
−4
−2
0
2
4
x


Mean +/− 2 SD = 0.954
0.4
0.3
0.2
0.1
0.0
y
−4
−2
0
2
4
x



Mean +/− 3 SD = 0.997
0.4
0.3
0.2
0.1
0.0
y
−4
−2
0
2
4
x

Useful probability distribution functions

```r
### Normal(0,1) probability density function
dnorm(x = 0, mean = 0, sd = 1)
```

```
## [1] 0.3989423
```

```r
### Normal(0,1) cumulative distribution function
pnorm(q = 0, mean = 0, sd = 1)
```

```
## [1] 0.5
```

```r
### Random draw from a normal(0,1) distribution
rnorm(n = 1, mean = 0, sd = 1)
```

```
## [1] 1.231011
```

```r
### CDF position for a given probability (quantile)
qnorm(p = 0.75, mean = 0, sd = 1)
```

```
## [1] 0.6744898
```

```r
### You can also use dbinom(), pbinom(), rbinom(), qbinom()
```

The expectation of a random variable

The expectation of a random variable $E(X)$ is the mean of a random variable.

Be careful not to confuse $E(X)$ and $\bar{x}$.

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For a discrete variable, the expectation is the sum of all values of x weighted by their probability, given by the PDF $f(x)$.

$$E(X) = \sum_x x \times f(x)$$

Because continuous variables take on an infinite number of values, we compute the expectation with an integral

$$\int x \times f(x) dx$$

Variance and standard deviation of a random variable

Recall that for a sample, the standard deviation *sd* is

$$sd = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

And the sample variance is sd^2

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$$V(X) = E[\{X - E(X)\}^2]$$

Note the similarities in the two equations

Large sample (asymptotic) theorems

The law of large numbers

As a sample of draws from a random variable increases, the sample mean converges to the population mean $E(X)$

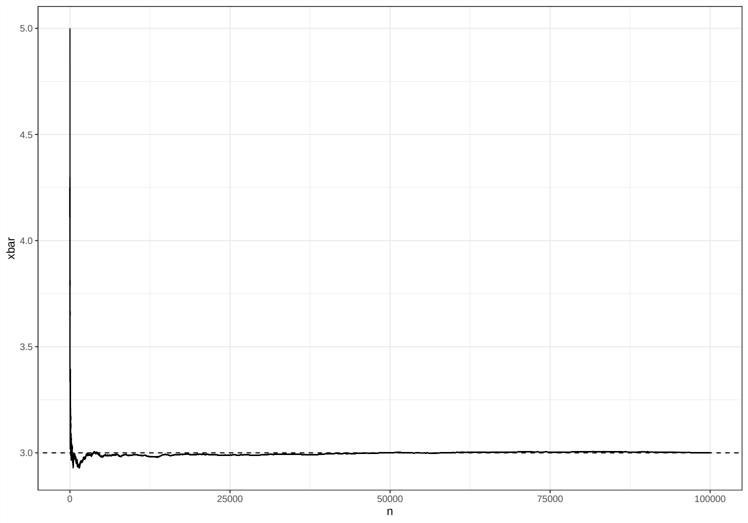
$$\bar{x}_n \rightarrow E(X)$$

Monte Carlo simulation for the mean of a binomial variable

To test the law of large numbers, let's draw from a binomial variable with varying sample sizes.

We expect that $\bar{x}$ will converge to $E(X)$ as the sample size n increases

```
## MC simulation, 1000 reps
sims<-100000
## Take 1000 draws from binomial(0.3, 10)
x<-rbinom(sims, p = 0.3, size = 10)
### output df
out<-data.frame(n=1:sims, xbar = NA)
for(i in 1:sims){
  out$xbar[i]<-sum(x[1:i])/i
}
## or use xbar<-cumsum(x)/1:sims
```


5.0
4.5
4.0
3.5
3.0
xbar
0
25000
50000
75000
100000
n

- As n increases, the distribution of the sample mean $\bar{x}$ approaches a Normal distribution.

The Central Limit Theorem

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- This relationship holds for many distributions (Bernoulli, Binomial, Normal, others we'll discuss later)

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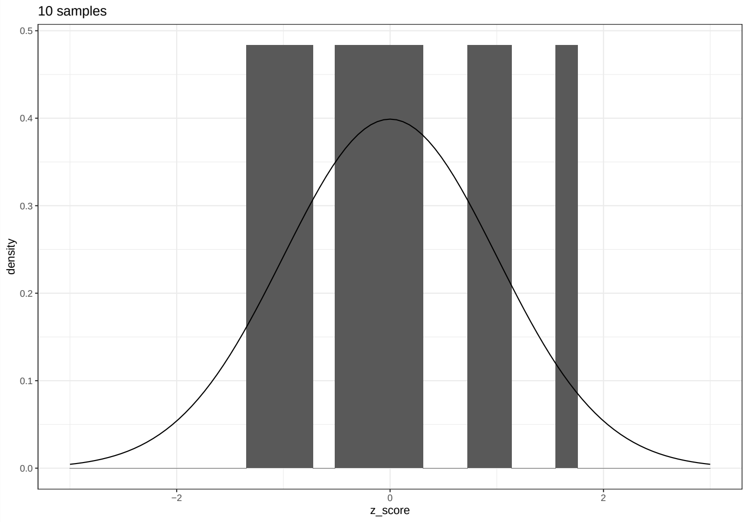
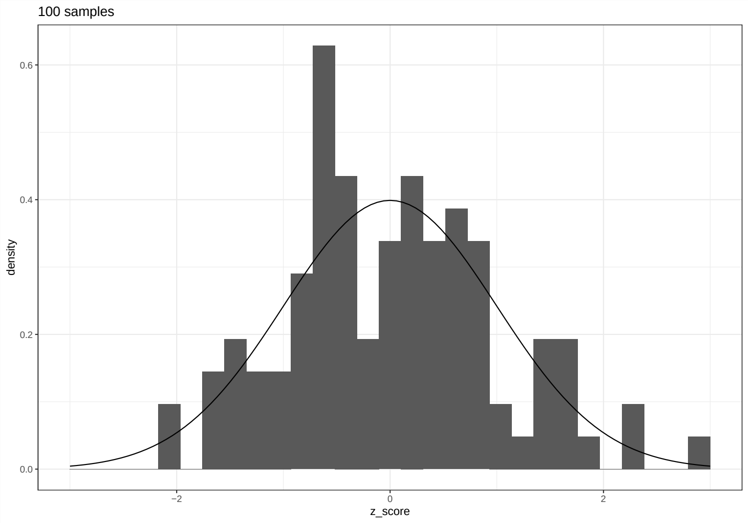
- If our samples are independent, and each observation within the sample is iid, the distribution of z-scores of sample means converges to a $Normal(0, 1)$ distribution

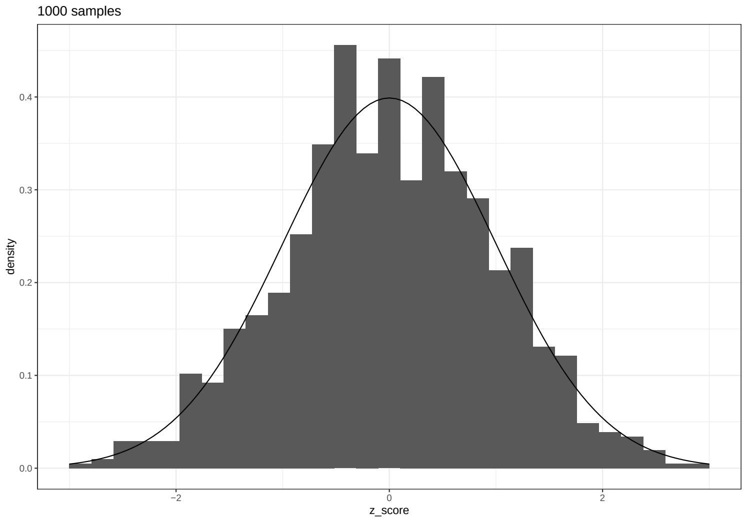
- As n increases, the distribution of the sample mean $\bar{x}$ approaches a Normal distribution.
- This relationship holds for many distributions (Bernoulli, Binomial, Normal, others we'll discuss later)
- If our samples are independent, and each observation within the sample is iid, the distribution of z-scores of sample means converges to a $Normal(0, 1)$ distribution
- The Central Limit Theorem allows us to make statements about uncertainty when we haven't observed the population mean or variance

Monte Carlo simulations of a binomial variable p=0.7, n=10

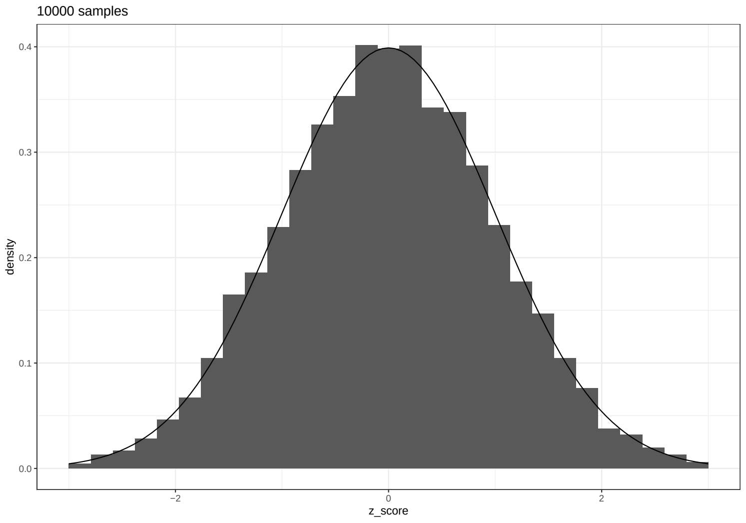
```
### Binomial random variable, 10 observations, probability of success = 0.7
## simulate each sample size for 1000 replications
n<-10 # 10 students in a class
p<-0.7 # 70% chance of a 1
xbar_10<-rep(NA, 10)
xbar_100<-rep(NA, 100)
xbar_1000<-rep(NA, 1000)
xbar_10000<-rep(NA, 10000)

for(i in 1:10){
  x_10<-rbinom(10, p=p, size=n)
  xbar_10[i]<-(mean(x_10))
}

for(i in 1:100){
  x_100<-rbinom(100, p=p, size=n)
  xbar_100[i]<-(mean(x_100))
}

for(i in 1:1000){
  x_1000<-rbinom(1000, p=p, size=n)
  xbar_1000[i]<-(mean(x_1000))
}

for(i in 1:10000){
  x_10000<-rbinom(10000, p=p, size=n)
  xbar_10000[i]<-(mean(x_10000))
}
```


10 samples
density
0.0
0.1
0.2
0.3
0.4
0.5
−2
0
2
z_score


100 samples
density
0.6
0.4
0.2
0.0
−2
0
2
z_score


1000 samples
density
0.4
0.3
0.2
0.1
0.0
−2
0
2
z_score

```
ggplot(data.frame(z_score=scale(xbar_10000)),
       aes(x=z_score))+
  geom_histogram(aes(y = ..density..)) +
  stat_function(fun=dnorm) +
  ggtitle("10000 samples")+
  xlim(c(-3,3))
```



Lab, HW

- Optional Homework: Khan Academy probability and distributions.
- Lab: simulating variables and working with probability distributions

Lab

- Conduct an experiment where you flip a fair coin 10 times. How many heads do you observe? (hint, `?rbinom`)
- Now, use a loop to repeat the experiment 10 times. Visualize your results with a histogram
- Repeat the experiment 1000 times. Visualize your results with a histogram

- How likely are you to observe 3 heads when you flip 10 fair coins? (hint, `dbinom`, the probability density)
- How likely are you to observe 3 or fewer heads when you flip 10 fair coins (hint, `pbinom`, the cumulative density)
- How likely are you to observe 7 heads when you flip 10 fair coins?
- How likely are you to observe 7 or fewer heads when you flip 10 fair coins?

Generate 1000 draws each from the following variables (`rnorm`)

- $y_1 \sim N(0, 1)$
- $y_2 \sim N(0, 2)$
- $y_3 \sim N(0, 4)$

Describe these variables using descriptive statistics and or visualizations

The Normal distribution

Assume that a tree's height is a function of it's age a, such that $height_i \sim N(2 + a_i, 2)$

$$E(height) = 2 + a$$

$$height_i \sim N(\mu = 2 + a_i, \sigma = 2)$$

- Predict a single tree height for trees ranging in age from 1 to 50
- Visualize this distribution
- Now predict 100 tree heights for each age in the range from 1 to 50
- Visualize this distribution