Regression and uncertainty part 2: stochastic error

Frank Edwards

The linear regression model

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

β_0: The value of y when x is equal to zero

The linear regression model

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

β_0: The value of y when x is equal to zero

β_1: The average increase in y when x increases by one unit

The linear regression model

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

β_0: The value of y when x is equal to zero

β_1: The average increase in y when x increases by one unit

ε: The distance between the line $y = \beta_0 + \beta_1 X$ and the actual observed values of y. Allows us to estimate the line, even when x and y do not fall exactly on a line.

The linear regression model

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

β_0: The value of y when x is equal to zero

β_1: The average increase in y when x increases by one unit

ε: The distance between the line $y = \beta_0 + \beta_1 X$ and the actual observed values of y. Allows us to estimate the line, even when x and y do not fall exactly on a line.

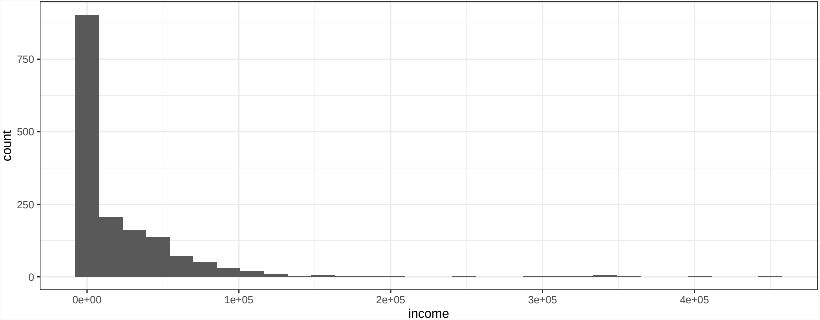
Data for today

```r
dat <- read_csv("https://www.openintro.org/data/csv/acs12.csv")
glimpse(dat)
```

```
## Rows: 2,000
## Columns: 13
## $ income       <dbl> 60000, 0, NA, 0, 0, 1700, NA, NA, NA, 45000, NA, 8600, 0,~
## $ employment   <chr> "not in labor force", "not in labor force", NA, "not in l~
## $ hrs_work     <dbl> 40, NA, NA, NA, NA, 40, NA, NA, NA, 84, NA, 23, NA, NA, N~
## $ race         <chr> "white", "white", "white", "white", "white", "other", "wh~
## $ age          <dbl> 68, 88, 12, 17, 77, 35, 11, 7, 6, 27, 8, 69, 69, 17, 10, ~
## $ gender       <chr> "female", "male", "female", "male", "female", "female", "~
## $ citizen      <chr> "yes", "yes", "yes", "yes", "yes", "yes", "yes", "yes", "~
## $ time_to_work <dbl> NA, NA, NA, NA, NA, 15, NA, NA, NA, 40, NA, 5, NA, NA, NA~
## $ lang         <chr> "english", "english", "english", "other", "other", "other~
## $ married      <chr> "no", "no", "no", "no", "no", "yes", "no", "no", "no", "y~
## $ edu          <chr> "college", "hs or lower", "hs or lower", "hs or lower", "~
## $ disability   <chr> "no", "yes", "no", "no", "yes", "yes", "no", "yes", "no",~
## $ birth_qrtr   <chr> "jul thru sep", "jan thru mar", "oct thru dec", "oct thru~
```

Let's look at income for this ACS 2012 sample

```
ggplot(dat, aes(x = income)) + geom_histogram()
```



OK, what could cause variation in income?

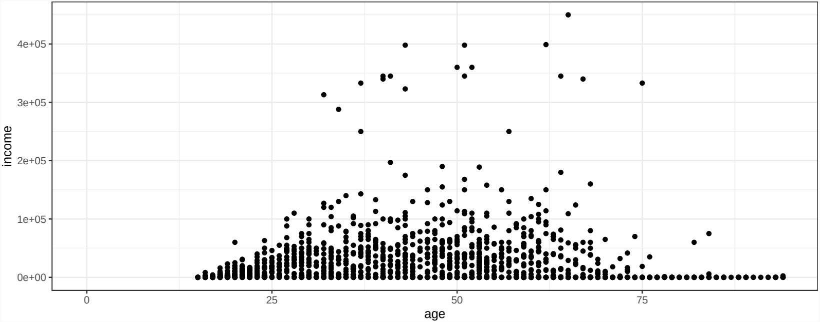
```r
glimpse(dat)
```

```
## Rows: 2,000
## Columns: 13
## $ income       <dbl> 60000, 0, NA, 0, 0, 1700, NA, NA, NA, 45000, NA, 8600, 0,~
## $ employment   <chr> "not in labor force", "not in labor force", NA, "not in l~
## $ hrs_work     <dbl> 40, NA, NA, NA, NA, 40, NA, NA, NA, 84, NA, 23, NA, NA, N~
## $ race         <chr> "white", "white", "white", "white", "white", "other", "wh~
## $ age          <dbl> 68, 88, 12, 17, 77, 35, 11, 7, 6, 27, 8, 69, 69, 17, 10, ~
## $ gender       <chr> "female", "male", "female", "male", "female", "female", "~
## $ citizen      <chr> "yes", "yes", "yes", "yes", "yes", "yes", "yes", "yes", "~
## $ time_to_work <dbl> NA, NA, NA, NA, NA, 15, NA, NA, NA, 40, NA, 5, NA, NA, NA~
## $ lang         <chr> "english", "english", "english", "other", "other", "other~
## $ married      <chr> "no", "no", "no", "no", "no", "yes", "no", "no", "no", "y~
## $ edu          <chr> "college", "hs or lower", "hs or lower", "hs or lower", "~
## $ disability   <chr> "no", "yes", "no", "no", "yes", "yes", "no", "yes", "no",~
## $ birth_qrtr   <chr> "jul thru sep", "jan thru mar", "oct thru dec", "oct thru~
```

Visual checks

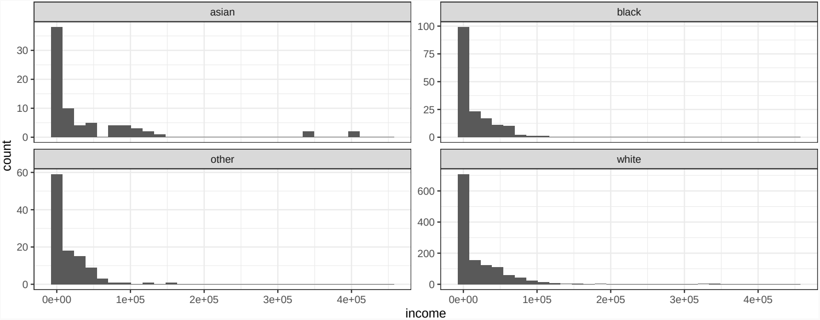
Causation requires association (though it's not always unconditional!). So let's evaluate

```
ggplot(dat, aes(y = income, x = age)) + geom_point()
```



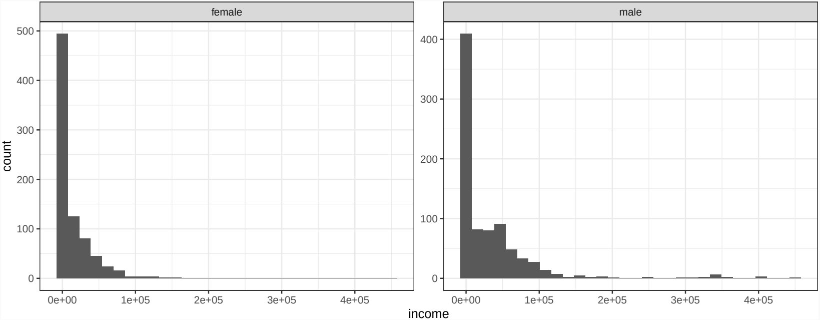
Visual checks

```
ggplot(dat, aes(x = income)) + geom_histogram() + facet_wrap(~race, scales = "free_y")
```



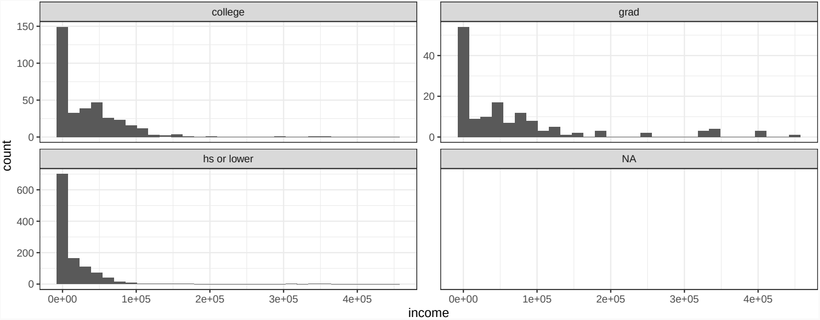
Visual checks

```
ggplot(dat, aes(x = income)) + geom_histogram() + facet_wrap(~gender, scales = "free_y")
```



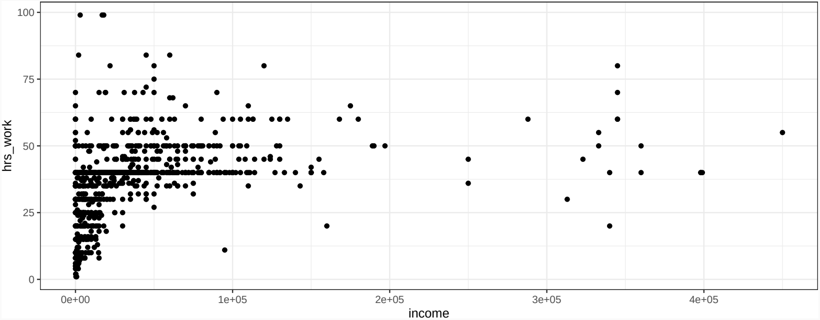
Visual checks

```
ggplot(dat, aes(x = income)) + geom_histogram() + facet_wrap(~edu, scales = "free_y")
```



Visual checks

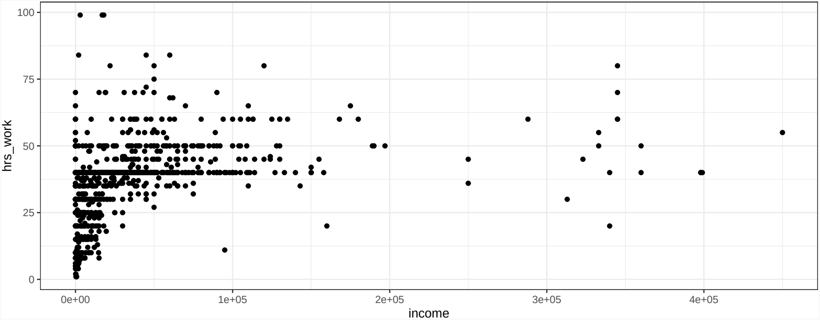
```
ggplot(dat, aes(x = income, y = hrs_work)) + geom_point()
```



Let’s build a **causal model** to formalize what we think causes variation in income across the population.

Visual checks

```
ggplot(dat, aes(x = income, y = hrs_work)) + geom_point()
```



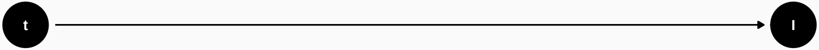
Let's build a **causal model** to formalize what we think causes variation in income across the population.

To do this, we'll use *Directed Acyclic Graphs*, or *DAGs* for short.

Let's start with a simple model

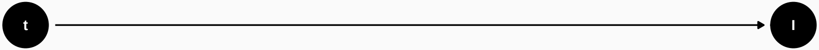
Based on our deep scientific knowledge we suspect that *hours worked t* has direct effects on *income I*

```
library(dagitty)
library(ggdag)
d1 <- dagitty("dag {
            t->I
            t [exposure]
            I [outcome]
            }")
coordinates(d1) <- list(x = c(t = 1, I = 2), y = c(t = 1, I = 1))
ggdag(d1) + theme_dag()
```



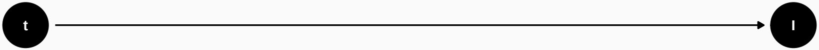
Basic features of a DAG

DAGs contain *nodes* that represent variables, and *edges* that represent causal relationships between variables. In this case, we have two nodes, hours worked and income, and one edge, representing the effect of time spent working on income.



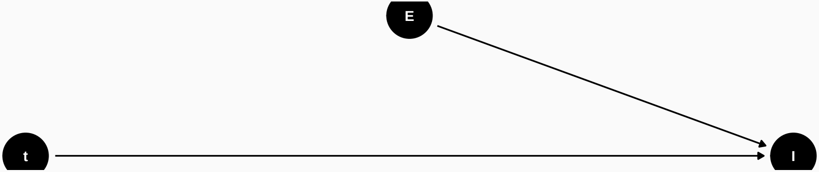
Basic features of a DAG

DAGs compactly represent our theoretical models. What is the theory presented here and do we believe it is adequate?

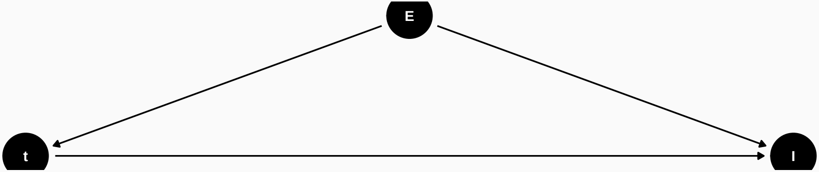


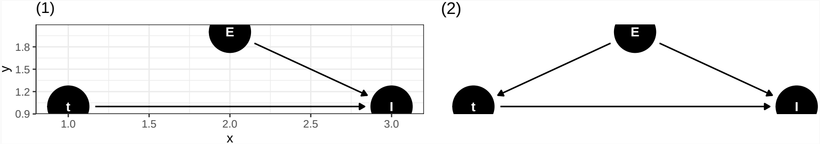
Adding complexity

Let's add level of education to our model. What theoretical relationships does this model suggest?



Which model is more plausible?

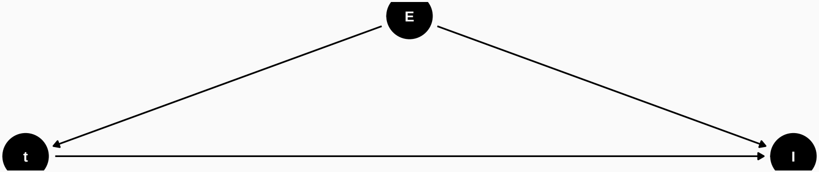


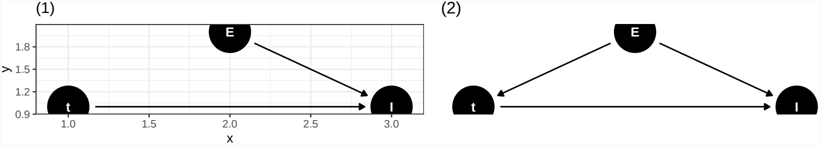


Confounding

We cannot obtain a valid estimate of the effect of *t* on *I* if model (2) is correct, unless we adjust for *E*.

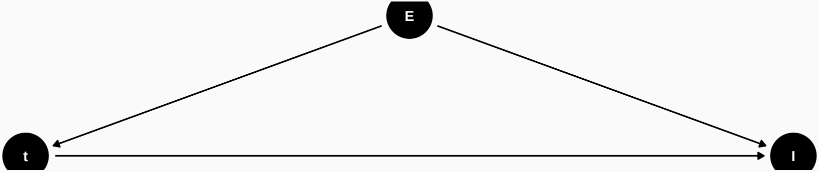
This is a case of *confounding*. A relationship between two variables *X* and *Y* is confounded when a third variable *Z* also causes *X* and *Y*.





Confounding

We cannot provide an unbiased estimate of the effect of t and I if we don't adjust for E



Let’s try it: unconditional linear relationship

```r
library(broom)
m0 <- lm(income ~ hrs_work, data = dat)
tidy(m0)
```

```
## # A tibble: 2 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)  -12905.     4983.     -2.59 9.75e- 3
## 2 hrs_work       1392.      124.     11.3  1.11e-27
```

Let’s try it: additive linear relationship

```r
library(broom)
m1 <- lm(income ~ hrs_work + edu, data = dat)
tidy(m1)
```

```
## # A tibble: 4 x 5
##   term           estimate std.error statistic  p.value
##   <chr>             <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)       1315.     5438.     0.242 8.09e- 1
## 2 hrs_work          1198.      116.    10.3   1.19e-23
## 3 edugrad          42472.     5589.     7.60  7.08e-14
## 4 eduhs or lower  -18598.     3579.    -5.20  2.48e- 7
```

Multiple regression (regression with more than 1 predictor)

We can generalize the linear regression

$$Y = \beta_0 + \beta_1 X + \varepsilon$$
$$varepsilon \sim N(0, \sigma^2)$$

Multiple regression (regression with more than 1 predictor)

We can generalize the linear regression

$$Y = \beta_0 + \beta_1 X + \varepsilon$$
$$varepsilon \sim N(0, \sigma^2)$$

as

$$Y = \beta_0 + \beta_1 x_1 \cdots \beta_k x_k + \varepsilon$$
$$\varepsilon \sim N(0, \sigma^2)$$

Where k is the number of predictor variables we include in the model. Our only constraint is that k must be smaller than the number of observations n in our data.

Our theoretical model tells us that if we want to learn about $t \longrightarrow I$, we must adjust for the effects that E has on both t and I.

We tried this with the model:

(Keep in mind that edu is going to be treated as the number of categories in the variable - 1 extra parameters).

Interpreting this model

```r
table(dat$edu)
```

```
##
##      college         grad hs or lower
##          359          144        1439
```

```r
tidy(m1)
```

```
## # A tibble: 4 x 5
##   term           estimate std.error statistic  p.value
##   <chr>             <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)       1315.     5438.     0.242 8.09e- 1
## 2 hrs_work          1198.      116.    10.3   1.19e-23
## 3 edugrad          42472.     5589.     7.60  7.08e-14
## 4 eduhs or lower  -18598.     3579.    -5.20  2.48e- 7
```

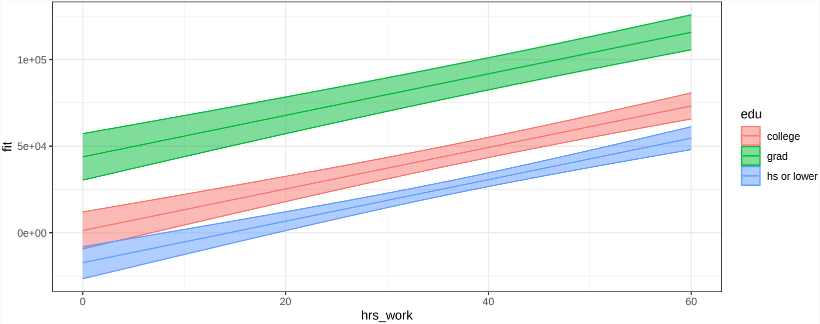
Visualizing model expectations: setup

```r
# set up prediction data with values of interest
hrs_work <- 0:60
edu <- c("college", "hs or lower", "grad")
pred_dat <- expand_grid(hrs_work, edu)
# generate expected values and CI, join pred_dat
e_y <- predict(m1, newdata = pred_dat, interval = "confidence") %>%
    bind_cols(pred_dat)
# inspect
head(e_y)
```

```
## # A tibble: 6 x 5
##       fit     lwr    upr hrs_work edu
##     <dbl>   <dbl>  <dbl>    <int> <chr>
## 1   1315.  -9356. 11986.        0 college
## 2 -17283. -26515. -8051.        0 hs or lower
## 3  43787.  30390. 57184.        0 grad
## 4   2513.  -7968. 12994.        1 college
## 5 -16085. -25110. -7060.        1 hs or lower
## 6  44985.  31752. 58218.        1 grad
```

Visualizing model expectations

```
ggplot(e_y, aes(y = fit, ymin = lwr, ymax = upr, x = hrs_work, fill = edu, color = edu)) +
    geom_ribbon(alpha = 0.5) + geom_line()
```



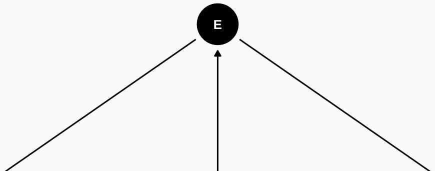
Multiple regression basics

1. Categorical predictors act as intercepts, or differences in level
2. Continuous predictors act as slopes

Maybe we think age also plays a role. Let's assume this causal model, where A is age. Now, we have to condition on A and E to close all *back door* paths between t and I and adjust for confounding

```
d4 <- dagitty("dag {
            E->I
            t->I
            E->t
            A->I
            A->t
            A->E
            t [exposure]
            I [outcome]
            }")

coordinates(d4) <- list(x = c(t = 1, I = 3, E = 2, A = 2), y = c(t = 1, I = 1, E = 2,
    A = 0))
ggdag(d4) + theme_dag()
```



Estimating the model

```r
m2 <- lm(income ~ hrs_work + edu + age, data = dat)
tidy(m2)
```

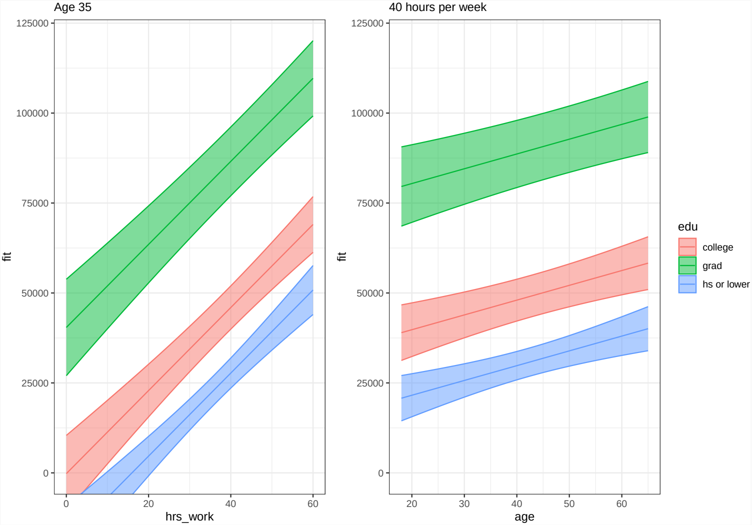
```
## # A tibble: 5 x 5
##   term           estimate std.error statistic  p.value
##   <chr>             <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)     -14596.     6739.     -2.17 3.06e- 2
## 2 hrs_work          1154.      116.      9.95 2.99e-22
## 3 edugrad          40626.     5566.      7.30 6.13e-13
## 4 eduhs or lower  -18215.     3553.     -5.13 3.58e- 7
## 5 age               411.      104.      3.94 8.67e- 5
```

Visualizing model expectations: setup

```r
# set up prediction data with values of interest
hrs_work <- c(0:60)
age <- c(18:65)
edu <- c("college", "hs or lower", "grad")
pred_dat <- expand_grid(hrs_work, edu, age)
# generate expected values and CI, join pred_dat
e_y <- predict(m2, newdata = pred_dat, interval = "confidence") %>%
    bind_cols(pred_dat)
# inspect
head(e_y)
```

```
## # A tibble: 6 x 6
##      fit     lwr   upr hrs_work edu       age
##    <dbl>   <dbl> <dbl>    <int> <chr>   <int>
## 1 -7197. -18604. 4211.        0 college    18
## 2 -6785. -18118. 4547.        0 college    19
## 3 -6374. -17636. 4887.        0 college    20
## 4 -5963. -17157. 5230.        0 college    21
## 5 -5552. -16681. 5577.        0 college    22
## 6 -5141. -16209. 5927.        0 college    23
```

Visualizing model expectations

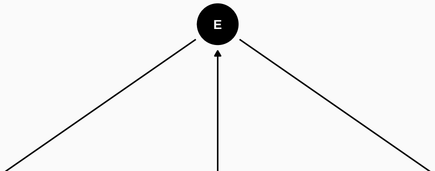


The difference between a DAG and a specification

This DAG can help us theorize how to adjust our models, but it does not tell us the correct regression specification. Is the relationship between A and I linear?

```
d4 <- dagitty("dag {
            E->I
            t->I
            E->t
            A->I
            A->t
            A->E
            t [exposure]
            I [outcome]
            }")

coordinates(d4) <- list(x = c(t = 1, I = 3, E = 2, A = 2), y = c(t = 1, I = 1, E = 2,
    A = 0))
ggdag(d4) + theme_dag()
```

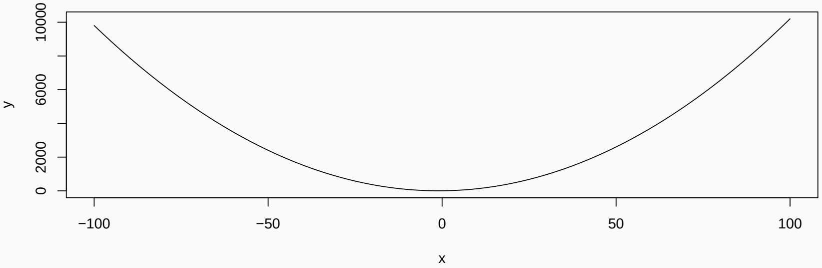


Adding complexity: quadratic terms

We know that earnings for people less than age 18 and greater than age 70 tend to be very low (or zero). We can try to use a parabola (a quadratic equation) to model this process.

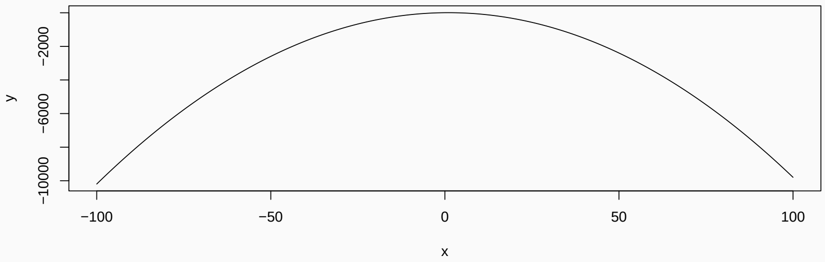
Quadratics take a form that looks like this

```
x <- -100:100
y <- 5 + 2 * x + x^2
plot(x, y, type = "l")
```



Adding complexity: negative sign

```
x <- -100:100
y <- 5 + 2 * x - x^2
plot(x, y, type = "l")
```



Fitting a quadratic term

We use the I() function to require R to evaluate math statements inside formula objects

```
m3 <- lm(income ~ hrs_work * edu + (age + I(age^2)), data = dat)
tidy(m3)
```

```
## # A tibble: 8 x 5
##   term                   estimate std.error statistic  p.value
##   <chr>                     <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)            -56158.   14084.      -3.99  7.20e- 5
## 2 hrs_work                 1530.     214.       7.14  1.90e-12
## 3 edugrad                 19318.   20210.       0.956 3.39e- 1
## 4 eduhs or lower          12569.   10480.       1.20  2.31e- 1
## 5 age                      1718.     582.       2.95  3.25e- 3
## 6 I(age^2)                  -14.4      6.49    -2.21  2.70e- 2
## 7 hrs_work:edugrad          471.     465.       1.01  3.11e- 1
## 8 hrs_work:eduhs or lower  -792.     257.      -3.08  2.10e- 3
```

Visualizing model expectations

```
## # A tibble: 6 x 6
##       fit     lwr     upr hrs_work edu       age
##     <dbl>   <dbl>   <dbl>    <int> <chr>   <int>
## 1 -29889. -48189. -11589.        0 college    18
## 2 -28703. -46828. -10577.        0 college    19
## 3 -27545. -45523.  -9567.        0 college    20
## 4 -26416. -44273.  -8560.        0 college    21
## 5 -25316. -43075.  -7558.        0 college    22
## 6 -24245. -41927.  -6563.        0 college    23
```