2: Foundational math for statistics

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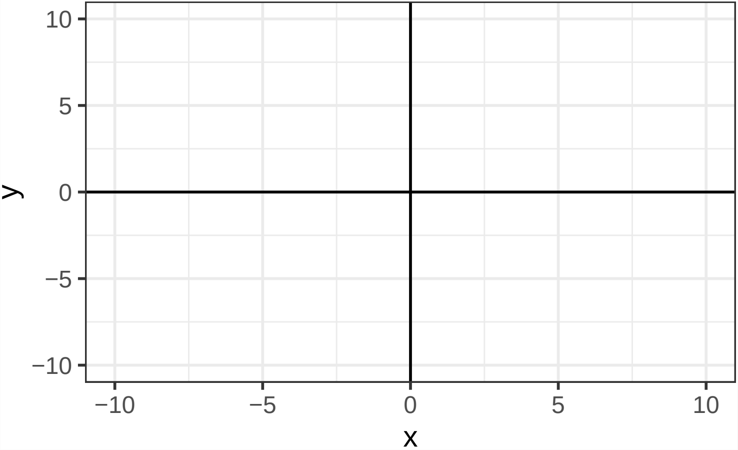
Welcome back: math time!

Agenda today

1. Plotting, functions
2. The summation operator
3. Matrices and vectors

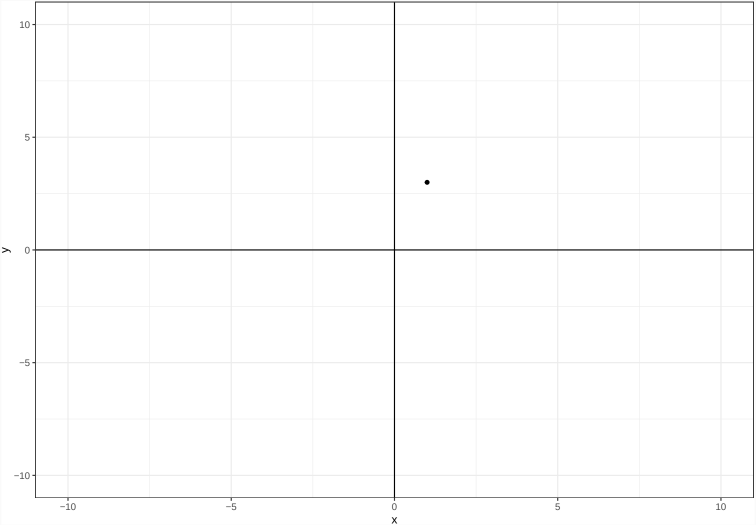
Coordinates and lines


10
5
0
−5
−10
y
−10
−5
0
5
10
x

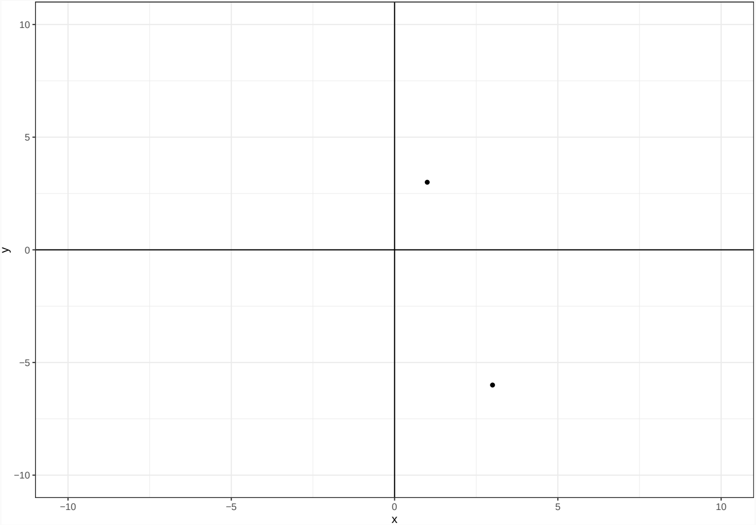
Plotting points

For coordinate pair (x_1, y_1), we can plot along an x and y axis.

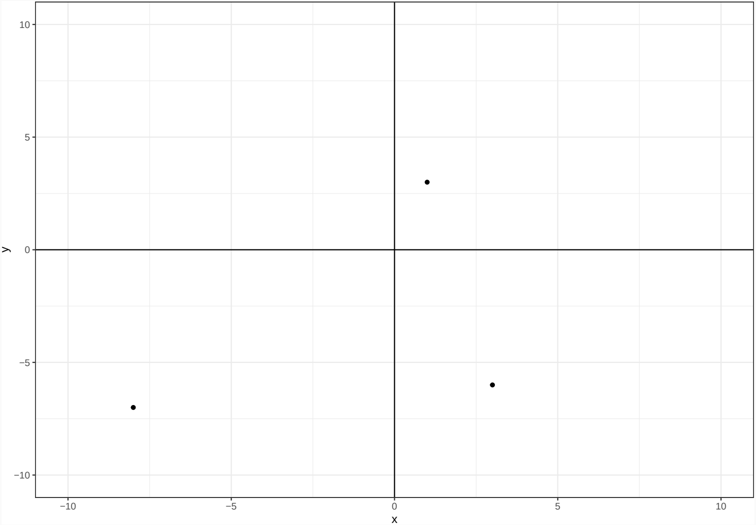
Example: (1,3)



Example: (1,3), (3,-6)



Example: (1,3), (3,-6), (-8, -7)



Practice

Sketch out a coordinate plane on paper. Plot the following points.

Remember we give points as (x, y), where x represents the horizontal location and y represents the vertical location

- (0, 1)
- (-3, 2)
- (5, -6)
- (5, 5)

Lines

The typical equation for a line is $y = mx + b$ where m is the slope and by is the y-intercept.

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You will often see a line expressed as a regression equation:

$$y = \beta_0 + \beta_1 x$$

where β_0 is the y-intercept and β_1 is the slope.

The slope

Slope measures the steepness of a line. A line with a positive slope has increasing values of y as x increases. A line with a negative slope has decreasing values of y for increasing values of x.

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We can calculate the slope with two coordinates on the line $(x_1, y_1), (x_2, y_2)$

The slope is the ratio of the difference in y values to the difference in x values.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The y-intercept

The y-intercept is the value of y when $x = 0$. If we have the value of one point on the line, and the slope, we can obtain the y-intercept

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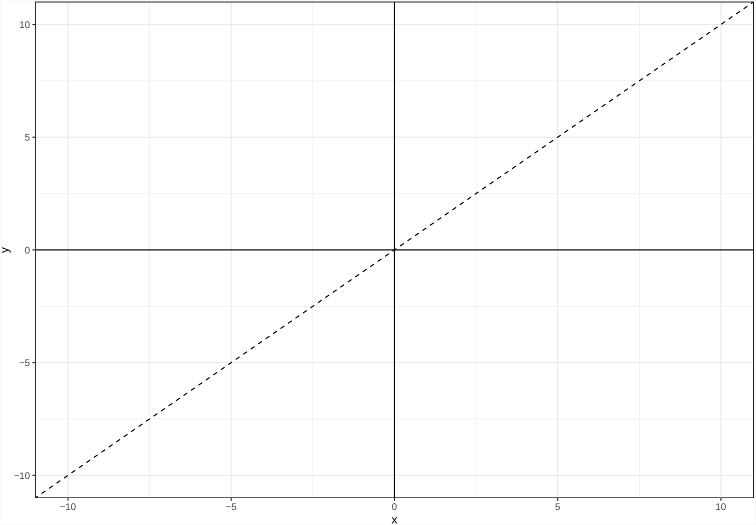
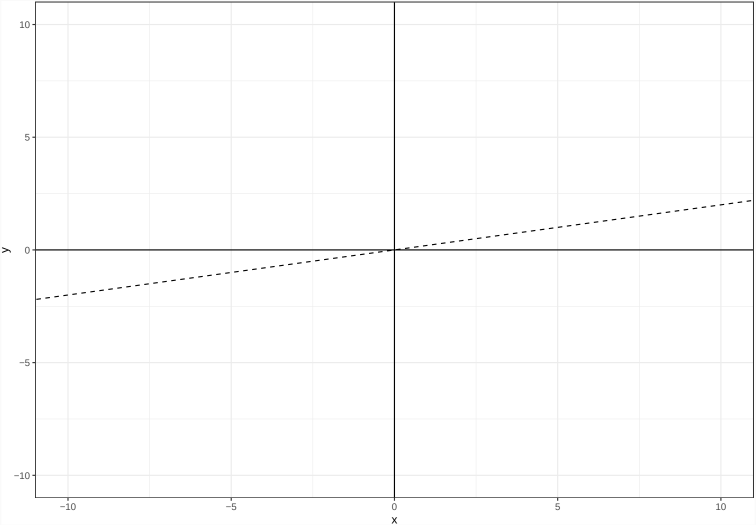
$$b = y_1 - m \cdot x_1$$

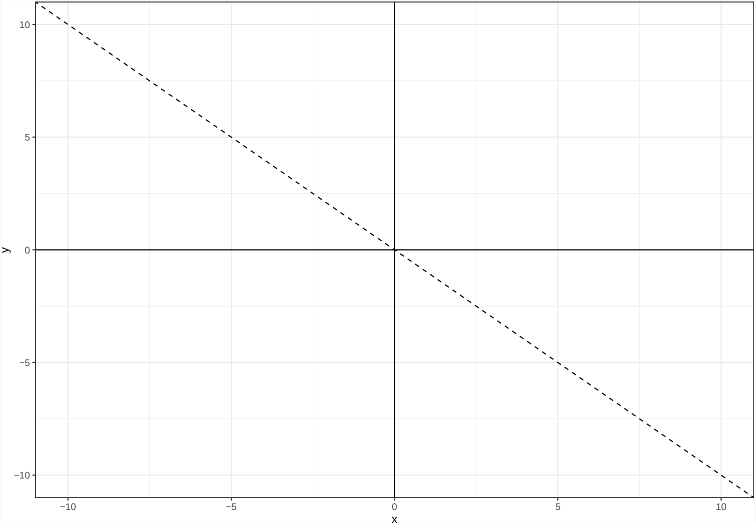
Example: intercept = 0, slope = 1



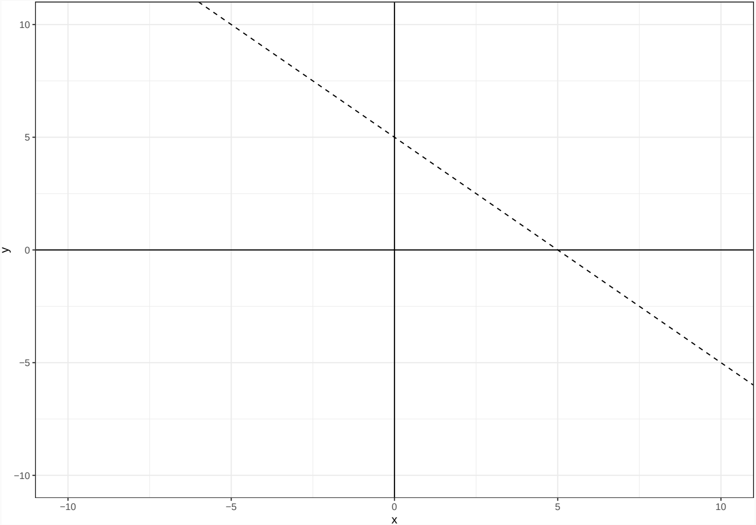
Example: intercept = 0, slope = 0.2



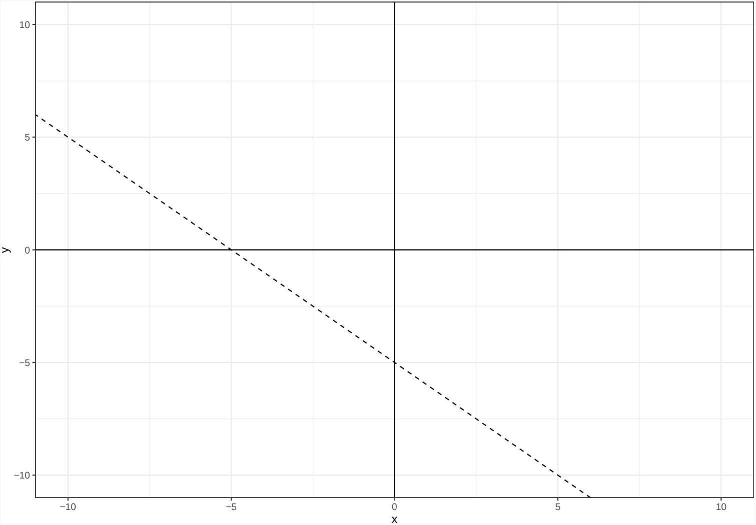
Example: intercept = 0, slope = -1



Example: intercept = 5, slope = -1

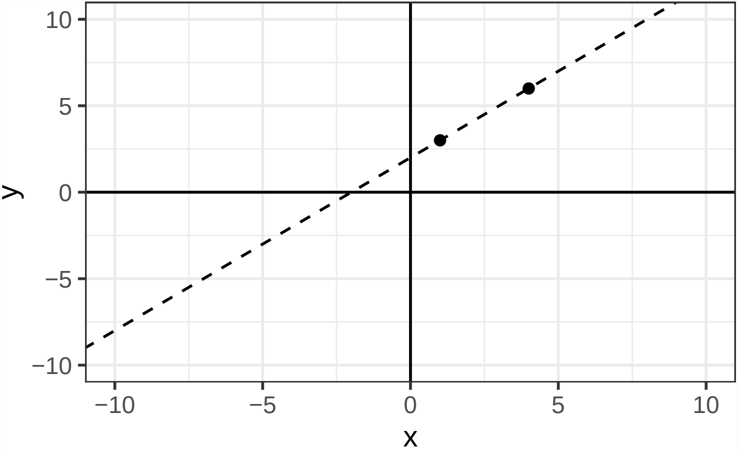


Example: intercept = -5, slope = -1



Example

Given the points $(1, 3)$ and $(4, 6)$, the slope is $m = \frac{6-3}{4-1} = 1$ and the y-intercept is $b = 3 - 1 \cdot 1 = 2$. The equation of the line is $y = 1x + 5$



Exercises: Compute slope, y-intercept, and sketch a line for each pair of coordinates

The equation for a line is $y = mx + b$

The slope is computed as $m = \frac{y2-y1}{x2-x1}$

The y intercept is $b = y_1 - m \cdot x_1$

1. $(2, 3), (5, 6)$
2. $(-2, 4), (0, 2)$
3. $(10, 12), (-5, -2)$

Functions

Functions, in general

A function maps each element in a set X to an element in set Y

- Linear function: $f(x) = x + 5$

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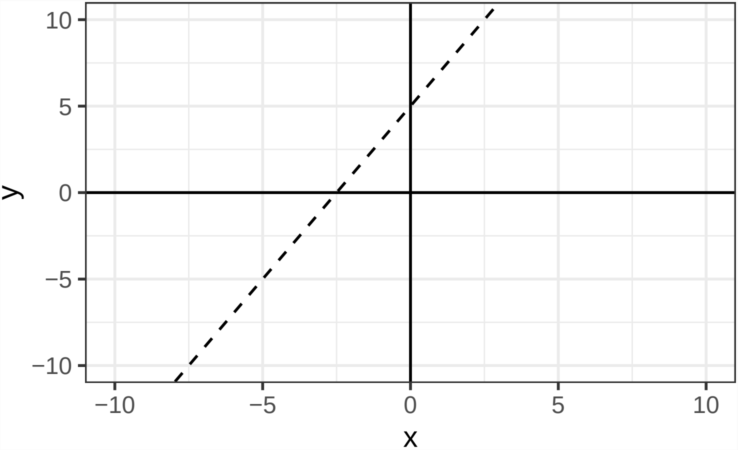
- Linear function: $f(x) = x + 5$
- Quadratic function: $f(x) = x^2 + 2x + 3$
- Exponential function: $f(x) = e^{2x} + 6$

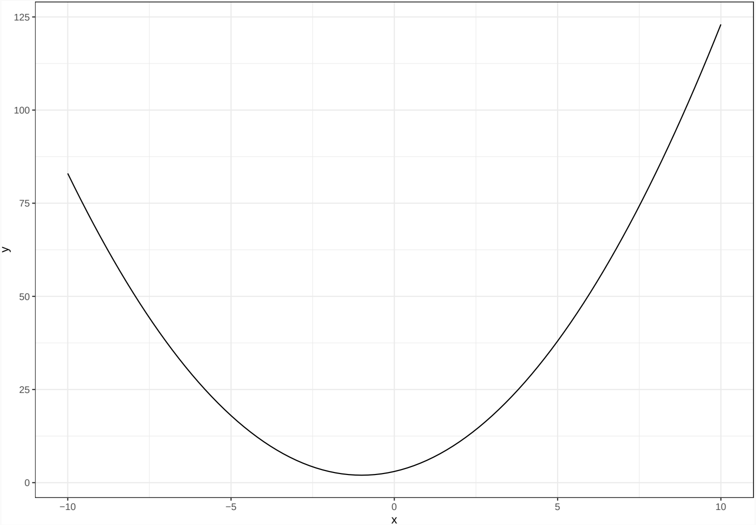
Graphical forms of functions: linear

$f(x) = 2x + 5$



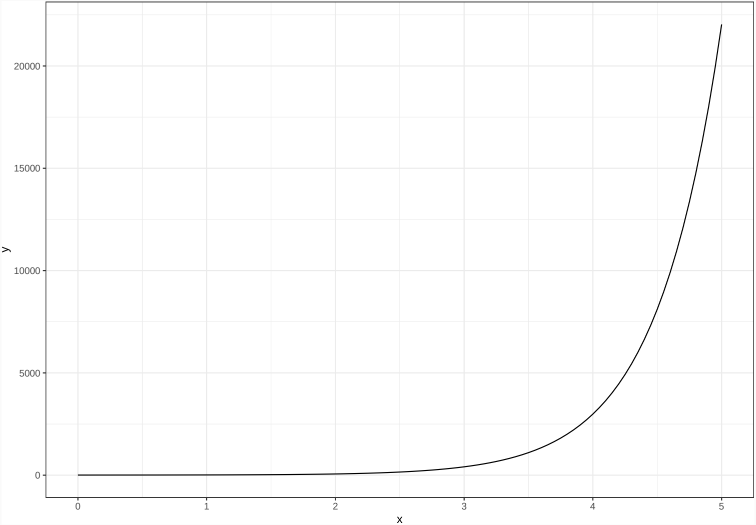
Graphical forms of functions: quadratic

$f(x) = x^2 + 2x + 3$



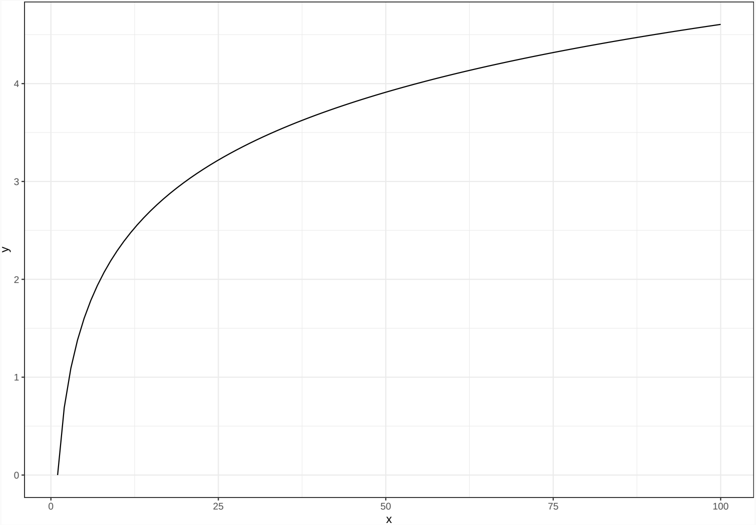
Graphical forms of functions: exponential

$f(x) = e^{2x} + 6$



Graphical forms of functions: logarithmic

$f(x) = log(x)$



Evaluate these functions for $x = 1$, $x = 2.5$, and $x = -6$

1. $f(x) = 2x$
2. $f(x) = \frac{x}{2}$
3. $f(x) = 2(x + 1)^3$

Summation

Represented as $\sum$, with integer begin and end points

$$\sum_{x=1}^{3} x$$

Summation

$$\sum_{x=1}^{3} x = 1 + 2 + 3 = 6$$

Summation in R

In R, we can calculate a sum using the sum() function

```r
# make an integer vector from 1 to 3
x<-1:3
# x<-c(1,2,3) is equivalent
sum(x)
```

```
## [1] 6
```

Compute the following by hand, and then in R

$$\sum_{x=3}^{8}(x+1)$$

$$\sum_{x=1}^{4}2x$$

Exercises (solutions)

Compute the following in R

$$\sum_{x=3}^{8}(x+1)$$

```
x<-3:8 # or c(3, 4, 5, 6, 7, 8)
sum(x+1)
```

$$\sum_{x=1}^{4}2x$$

```
x<-c(1, 2, 3, 4) # or 1:4
sum(2*x)
```

Vectors

Vectors are one-dimensional arrays of values. They have dimension n, where n is the number of elements in the vector

$$x = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

Indexing

We can define each element with a position index i.

$$x = \begin{bmatrix} 3 & 1 & 5 & 2 \end{bmatrix}$$

- What is x_2?
- What is x_4

Vector operations

We can compute operations on vectors using single values, called scalars.

Simply perform the operation on each element of the vector

$$x = \begin{bmatrix} 3 & 1 & 5 & 2 \end{bmatrix}$$

$$2 + x = \begin{bmatrix} 5 & 3 & 7 & 4 \end{bmatrix}$$

$$2x = \begin{bmatrix} 6 & 2 & 10 & 4 \end{bmatrix}$$

Vector operations

We can compute math operations on two vectors of equal dimension by performing the operation on each element pair by index

$$x = \begin{bmatrix} 3 & 1 & 5 & 2 \end{bmatrix}$$

$$y = \begin{bmatrix} 4 & -2 & 3 & 0 \end{bmatrix}$$

$$x + y = \begin{bmatrix} 7 & -1 & 8 & 2 \end{bmatrix}$$

Exercises: compute the following

$$x = \begin{bmatrix} 2 & 1 & 4 & -2 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 5 & 3 & 8 \end{bmatrix}$$

1. x + 2
2. 2y
3. x + y
4. 2x - y

Matrices

A matrix is a rectangular array of numbers, with dimensions expressed as rows $\times$ columns

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

A is a 3 x 3 matrix

$$\mathsf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

B is a 2 x 3 matrix

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

C is a 3 x 2 matrix

Matrix notation

We can identify each element of a matrix with its column and row position, where x_{ij} refers to the value in the ith row and jth column of matrix X. Note that we use uppercase letters for a matrix, and lowercase letters for elements of a matrix.

$$\mathsf{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$$

Matrix notation

$$\mathsf{X} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

What is

- $X_{1,2}$
- $X_{2,1}$
- $X_{1,3}$

Diagonal Matrix

A diagonal matrix has zero values except on the diagonal:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Identity Matrix

An identity matrix is a special case of a diagonal matrix, where all values on the diagonal are equal to 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagnoal and Identity matrices are also symmetric, where all $x_{ij} = x_{ji}$. Symmetric matrices are square.

Matrix operations

Matrix transpose

A transpose interchanges the rows and columns of a matrix, or rotates it.

The dimensions are switched, so an $n \times k$ matrix becomes a $k \times n$ matrix.

We denote a transpose with a T

$$\mathsf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \mathsf{B}^{\mathsf{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Addition and subtraction

Two matrices (or vectors) can be added or subtracted only if they have identical dimensions. Then add or substract the corresponding elements of the matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix}$$

Multiplication by scalar

Matrices and vectors can be multiplied by constant values (called scalars).

$$\mathsf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} 4\mathsf{B} = \begin{bmatrix} 4 & 8 & 12 \\ 16 & 20 & 24 \end{bmatrix}$$

$$\mathsf{C} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} 3\mathsf{C} = \begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \end{bmatrix}$$

Matrices and vectors in R

The c() function makes a vector

We can define a vector in R using the `c()` function

```r
x <- c(3, 5, 8)
x
```

```
## [1] 3 5 8
```

Vector indexing in R

We can call a vector index *i* using brackets

```r
x
```

```
## [1] 3 5 8
```

```r
x[1]
```

```
## [1] 3
```

```r
x[3]
```

```
## [1] 8
```

We can also easily perform scalar operations on vectors.

Try the following

- $x + 2$
- $2x$
- $(x + 2)^2$

Further vector operations

In R, define $x = [1, 4, 6, 9]$ and $y = [0, -2, 5, 7]$

Compute:

- $x + y$
- $2x - y$
- $(2x + 2) \times (y - 3)$

Making a matrix

```r
Z<-matrix(data = c(1, 2, 4, 5, 6, 7, 8, 9),
          nrow = 4, ncol = 2)

Z
```

```
##      [,1] [,2]
## [1,]    1    6
## [2,]    2    7
## [3,]    4    8
## [4,]    5    9
```

- What is $Z_{2,2}$?
- What is $Z_{4,1}$?

Matrices are made up of vectors

Bracket notation in R gets us thinking in vectors.

- Try this code `Z[,1]`. What does that return? What kind of object is it?
- What do you think you will get if you run `3 + Z[,2]`

Exercises

```r
Z
```

```
##      [,1] [,2]
## [1,]    1    6
## [2,]    2    7
## [3,]    4    8
## [4,]    5    9
```

- How could you retrieve the 3rd row of Z in R?
- Multiply the 1st column of R by 2

R has a special kind of object called a data.frame(). It is a matrix-like object that can have names for columns and rows.

This is a data.frame

```
##            Murder Assault UrbanPop Rape
## Alabama      13.2     236       58 21.2
## Alaska       10.0     263       48 44.5
## Arizona       8.1     294       80 31.0
## Arkansas      8.8     190       50 19.5
## California    9.0     276       91 40.6
## Colorado      7.9     204       78 38.7
```

Indexing with brackets

We can use bracket indexing on data.frames

```
USArrests[1, 1]
```

```
## [1] 13.2
```

```
USArrests[2,3]
```

```
## [1] 48
```

Or we can pull whole rows or columns

```r
USArrests[1,]
```

```
##         Murder Assault UrbanPop Rape
## Alabama   13.2     236       58 21.2
```

```r
USArrests[,2]
```

```
##  [1] 236 263 294 190 276 204 110 238 335 211  46 120 249 1
## [20] 300 149 255  72 259 178 109 102 252  57 159 285 254 3
## [39] 174 279  86 188 201 120  48 156 145  81  53 161
```

Name indexing

We can use the name of columns in data.frames for indexing using the `$` operator

These functions can be helpful to remember the dimensions and structure of an object: `str()`, `names()`