Probability, 2

Frank Edwards

Three kinds of probability

Joint probability

The joint probability of two events (A and B) occurring is expressed as

$$P(A \text{ and } B)$$

Marginal probability

The marginal probability of an event B is

$$P(B)$$

Conditional probability

The conditional probability of event A occurring given that event B occurred is the ratio of the joint probability of A and B divided by the marginal probability of B

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Working with some real data

Voter files

```r
data("FLVoters")
voters<-na.omit(FLVoters)
head(voters)
```

```
##      surname county VTD age gender  race
## 1     PIEDRA    115  66  58      f white
## 2      LYNCH    115  13  51      m white
## 4    LATHROP    115  80  54      m white
## 5     HUMMEL    115   8  77      f white
## 6 CHRISTISON    115  55  49      m white
## 7      HOMAN    115  84  77      f white
```

Marginal probability

What is the probability that a randomly sampled voter in the population is Black: $P(Black) = ?$

```
voters %>%
  count(race, name = "voters") %>%
  mutate(p = voters/sum(voters))
```

```
##       race voters           p
## 1    asian    175 0.019203336
## 2    black   1194 0.131021617
## 3 hispanic   1192 0.130802151
## 4   native     29 0.003182267
## 5    other    310 0.034017338
## 6    white   6213 0.681773291
```

Is a woman: $P(Woman) = ?$

```
voters %>%
  count(gender) %>%
  mutate(n = n/sum(n))
```

```
##   gender         n
## 1      f 0.5358279
## 2      m 0.4641721
```

Joint probability

What is the probability that a voter is a Black woman:

$P(\text{Black and woman}) = ?$

```
voters %>%
  count(gender, race) %>%
  mutate(n = n/sum(n)) %>%
  pivot_wider(names_from = gender, values_from = n)
```

```
## # A tibble: 6 x 3
##   race           f       m
##   <chr>      <dbl>   <dbl>
## 1 asian    0.00911 0.0101
## 2 black    0.0744  0.0566
## 3 hispanic 0.0731  0.0577
## 4 native   0.00187 0.00132
## 5 other    0.0173  0.0167
## 6 white    0.360   0.322
```

What is the probability that a voter is a woman?

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Use the law of total probability:

$$P(A) = P(A \text{ and } B) + P(A \text{ and not } B)$$

put differently, for all categories of B i:

$$P(A) = \sum_{i=1}^{n} P(A \text{ and } B_i)$$

Conditional probability

If a voter is a man, what is the probability that he is Asian: $P(\text{Asian}|\text{man}) = ?$

```
voters %>%
  filter(gender=="m") %>%
  count(race) %>%
  mutate(n=n/sum(n))
```

```
##       race           n
## 1    asian 0.021749409
## 2    black 0.121985816
## 3 hispanic 0.124349882
## 4   native 0.002836879
## 5    other 0.035933806
## 6    white 0.693144208
```

Conditional probability

Alternatively, we can use the definition of conditional probability as the ratio of the joint probability to the marginal probability:

$$P(\text{Asian}|\text{man}) = \frac{P(\text{Asian and man})}{P(\text{man})}$$

```
voters %>%
  count(gender, race) %>%
  mutate(n = n/sum(n))%>%
  pivot_wider(names_from = gender, values_from = n)
```

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```

Conditioning on more than one variable

What is the probability that a male voter over age 60 is white?

$$P(\text{white} | \text{male and over 60})$$

```r
voters %>%
  mutate(over60=age>60) %>%
  count(over60, gender, race) %>%
  mutate(n=n/sum(n)) %>%
  pivot_wider(names_from = gender, values_from = n)
```

```
## # A tibble: 12 x 4
##    over60 race            f        m
##    <lgl>  <chr>       <dbl>    <dbl>
##  1 FALSE  asian    0.00691  0.00823
##  2 FALSE  black    0.0555   0.0435
##  3 FALSE  hispanic 0.0549   0.0436
##  4 FALSE  native   0.00121  0.000768
##  5 FALSE  other    0.0124   0.0129
##  6 FALSE  white    0.212    0.198
##  7 TRUE   asian    0.00219  0.00187
##  8 TRUE   black    0.0189   0.0132
##  9 TRUE   hispanic 0.0182   0.0142
## 10 TRUE   native   0.000658 0.000549
## 11 TRUE   other    0.00494  0.00373
## 12 TRUE   white    0.148    0.124
```

Conditioning on more than one variable

In general:

$$P(A \text{ and } B | C) = \frac{P(A \text{ and } B \text{ and } C)}{P(C)}$$

and

$$P(A | B \text{ and } C) = \frac{P(A \text{ and } B \text{ and } C)}{P(B \text{ and } C)}$$

Independence

Two events are independent if knowledge of one event gives us no information about the other event.

$P(A|B) = P(A)$ and $P(B|A) = P(B)$

$$A \perp B$$

if and only if

$$P(A \text{ and } B) = P(A)P(B)$$

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Bayes' rule

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Bayes' rule formalizes how we should update our beliefs based on evidence:

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Prior beliefs and evidence

If we have a *prior* belief that event A has $P(A)$ chance of occurring, then we observe some data, represented as event B, we update our beliefs and obtain a *posterior probability* $P(A|B)$.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Example: Detecting breast cancer

How good is a mammogram at detecting breast cancer?

What we know: One percent of women have breast cancer. 80 percent of people who have cancer and take a mammogram test positive. 9.6 percent of people who take a mammogram get a positive result when they do not have breast cancer.

Example: Detecting breast cancer

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If you take a mammogram and get a positive result, what is the probability that you have breast cancer?

Rewriting as probabilities

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$$P(\text{Test positive}|\text{Cancer}) = 0.8$$

9.6 percent of people who take a mammogram get a positive result when they do not have breast cancer

$$P(\text{Test positive}|\text{No cancer}) = 0.096$$

Using Bayes' rule

The prior probability of having cancer is 0.01. How should we update our belief that someone has cancer based on a positive test?

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Using the law of total probability, we can rewrite the denominator as:

$$P(B) = P(B|A)P(A) + P(B|\text{ not A})P(\text{not } A)$$

Using Bayes' rule

We can apply Bayes' rule for A = Cancer, B = positive test:

$$P(\text{Cancer}|\text{Test positive}) =$$

$$\frac{P(\text{Test positive}|\text{Cancer})P(\text{Cancer})}{P(\text{Test positive})}$$

$$P(\text{Cancer}|\text{Test positive}) = \frac{0.8 \times 0.01}{0.8 \times 0.01 + 0.096 \times 0.99}$$

```r
(0.8 *  0.01)/(0.8 *  0.01 + 0.096 * 0.99)
```

```
## [1] 0.07763975
```

The probability that someone has cancer given a prior probability of one percent and a positive test is about 0.078. What would the probability of a true positive be if the test were more sensitive? Say 0.95?

Random Variables

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- The set of all values is exhaustive (the sample space Ω)
- Discrete random variables take a finite number of values (e.g. TRUE, FALSE)
- Continuous random variables are real numbers, and take on an infinite number of values

The simplest random variable: the binary Bernoulli

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Bernoulli (binary) variables are typically represented as $[0, 1]$ or $[T, F]$. They can also be two-level character variables, like $[\text{pass, fail}]$ or $[\text{plaid, stripes}]$.

A coin flip as a Bernoulli random variable

A coin flip can be defined as a discrete random variable X

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$P(X = 1) = p$

Random variable (probability distribution) notation

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In this notation, we name the variable X, note that it is randomly distributed $\sim$, name the distribution it follows *Bernoulli*, and list the parameters for that distribution p.

Let’s flip some coins

```
set.seed(12345)
sample_of_flips<-rbinom(5, 1, 0.5)
table(sample_of_flips)
```

```
## sample_of_flips
## 0 1
## 1 4
```

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```

```
## sample_of_flips
## 0 1
## 1 4
```

This is the result of taking 5 draws from a Bernoulli random variable with probability 0.5.

Describing a probability distribution: probability mass

We use a probability mass function to show how likely each value is in a random variable

The probability mass function (PMF) of a variable X is defined as the probability that a variable takes on a particular value x.

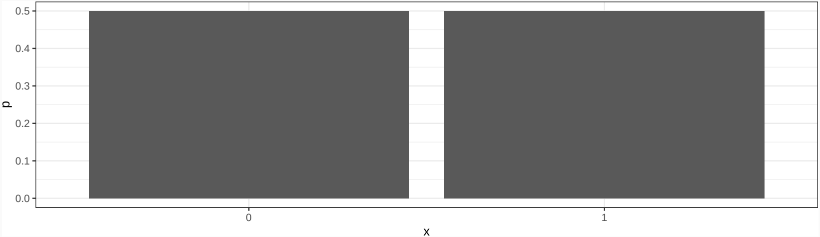
$$PMF(x) = P(X = x)$$

For a Bernoulli variable, $PMF(X = 1) = p$ and $PMF(X = 0) = 1 - p$

The probability mass function for our coin flip

$$PMF(X = 1) = p = 0.5$$

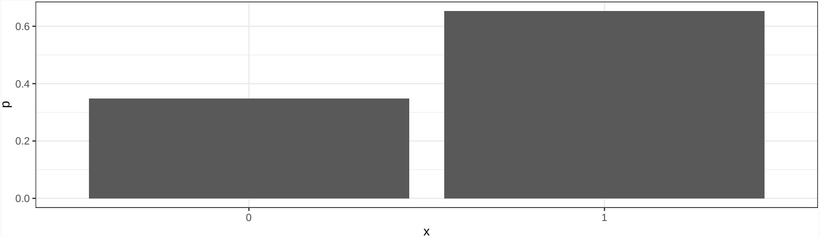
$$PMF(X = 0) = 1 - p = 0.5$$



The probability mass function for passing the bar in NJ (p=0.653)

$$PMF(X=1)=p=0.653$$

$$PMF(X=0)=1-p=0.347$$



Describing a probability distribution: cumulative probability

How likely is a variable to take a value less than or equal to a specified value?

We define the cumulative distribution function as the sum of all probabilities up to a value x

$$CDF(X) = P(X \leq x) = \sum_{k \leq x}^{x} PMF(k)$$

The CDF always ranges from 0 to 1, and never decreases as x increases.

Uniform random variables

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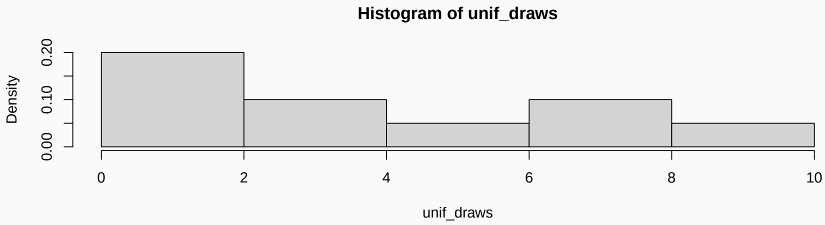
Uniform random variables

Let's simulate it! 10 draws

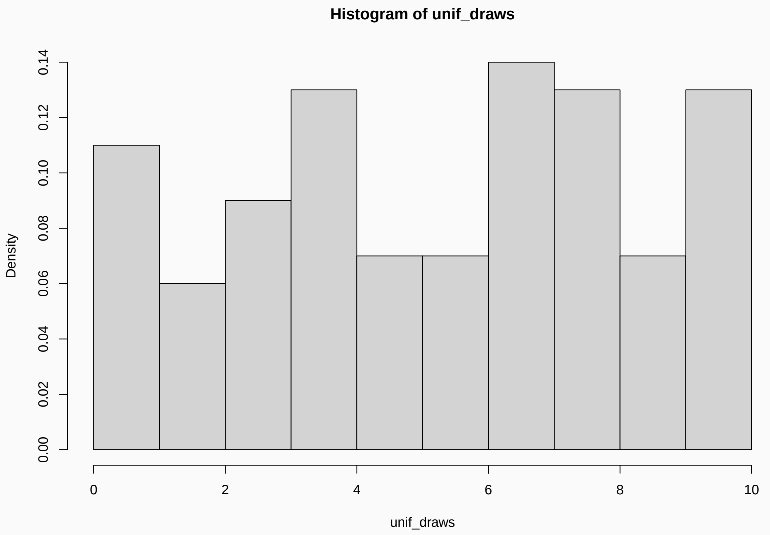
```
unif_draws<-runif(10, min=0, max=10)
unif_draws
```

```
##  [1] 1.66371785 3.25095387 5.09224336 7.27705254 9.89736938 0.34535435
##  [7] 1.52373490 7.35684952 0.01136587 3.91203335
```

```
hist(unif_draws, freq=F)
```



Uniform random variable: 100 draws



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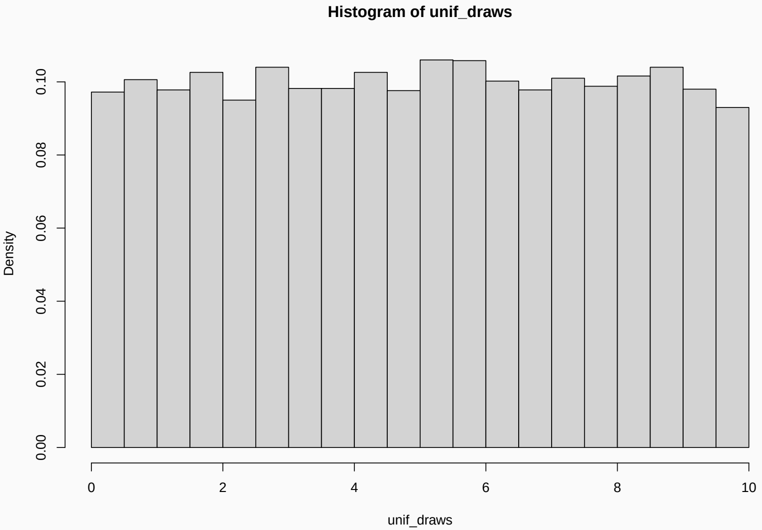


Properties of uniform random variables

For a uniform random variable on the interval $[a, b]$, the probability of drawing any value between a and b is

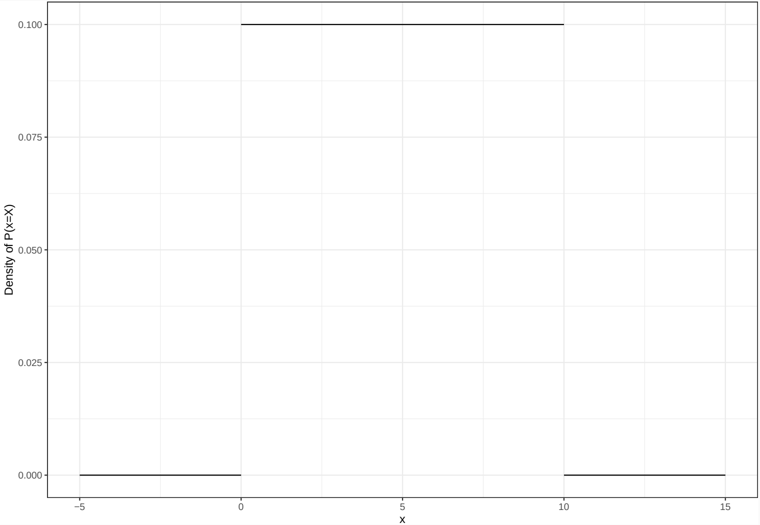
$$\frac{1}{b-a}$$

Formally, the PDF (density, not mass for continuous) and CDF are defined as

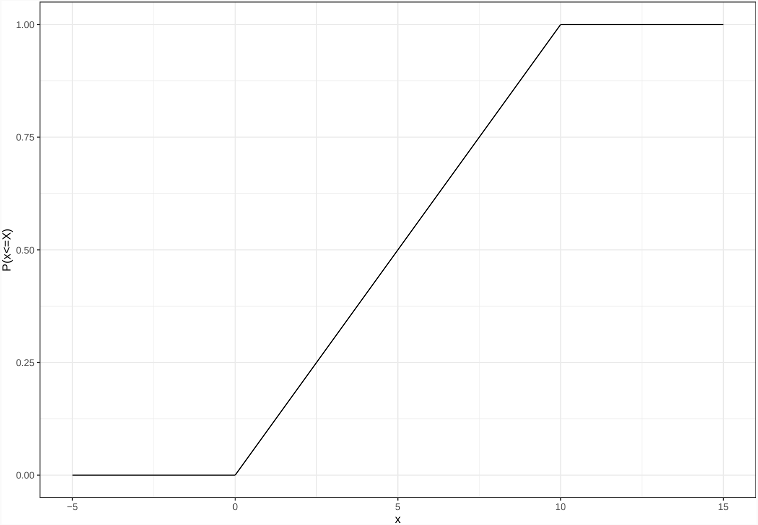
$$\text{PDF:} \begin{cases} \frac{1}{b-a} & \text{for } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

$$\text{CDF:} \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } x \in [a, b) \\ 1 & \text{for } x \geq b \end{cases}$$

Probability Density function for $X \sim Uniform(0, 10)$



Cumulative Distribution Function for $X \sim Uniform(0, 10)$



A note on CDF for continuous variables

Recall that a CDF for a discrete variable is the sum of all probabilities for values $x \leq X$

We can't sum over each value when X is continuous. Instead, we'll take the integral

$$CDF(x) = P(x \leq X) = \int_{-\infty}^{x} PDF(x)dx$$

When we repeat Bernoulli trials many times, we get a binomial random variable.

The binomial distribution

When we repeat Bernoulli trials many times, we get a binomial random variable.

Binomial random variables represent the count of successes in a fixed number of trials of a Bernoulli experiment.

Formally:

A binomial random variable is the sum of n independently and identically distributed (i.i.d) Bernoulli random variables.

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A binomial random variable is the sum of n independently and identically distributed (i.i.d) Bernoulli random variables.

Binomial variables take on integer values between 0 and n

Back to flipping coins

Imagine we flipped a coin 5 times, and then repeated the exercise twice more

```
## [1] 0 1 0 0 0
```

```
## [1] 0 0 0 0 1
```

```
## [1] 1 1 1 1 0
```

Each of these trials is a sample from $X \sim Binomial(n, p)$ where $n = 5$ and $p = 0.5$

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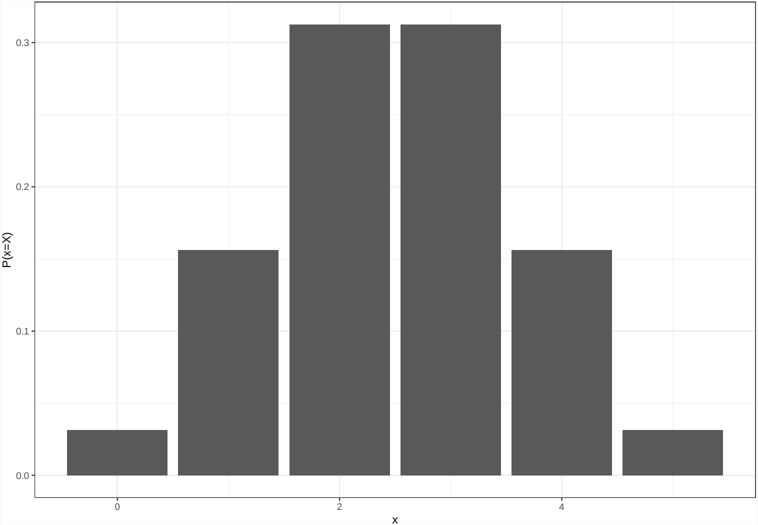
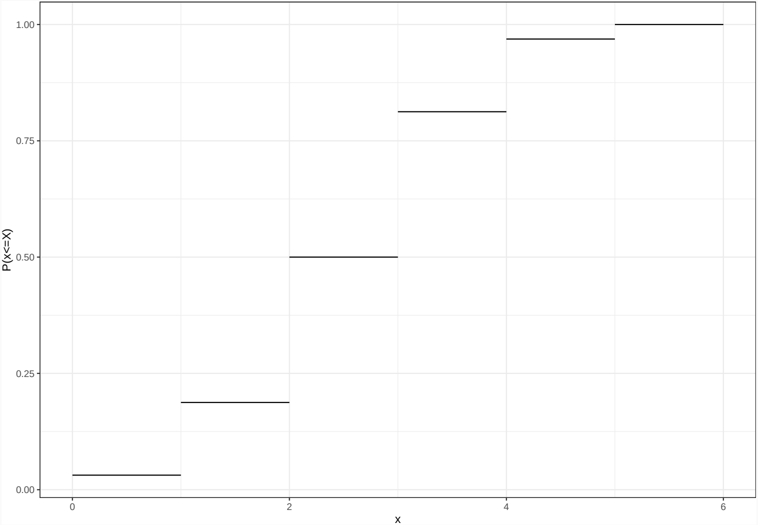
What is x for each trial?

Probability Mass Function for $X \sim Binomial(5, 0.5)$



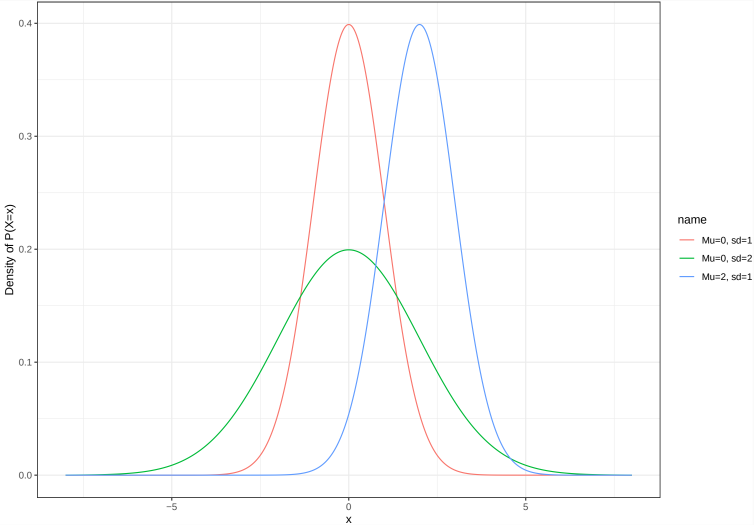
Cumulative Distribution Function for $X \sim Binomial(5, 0.5)$



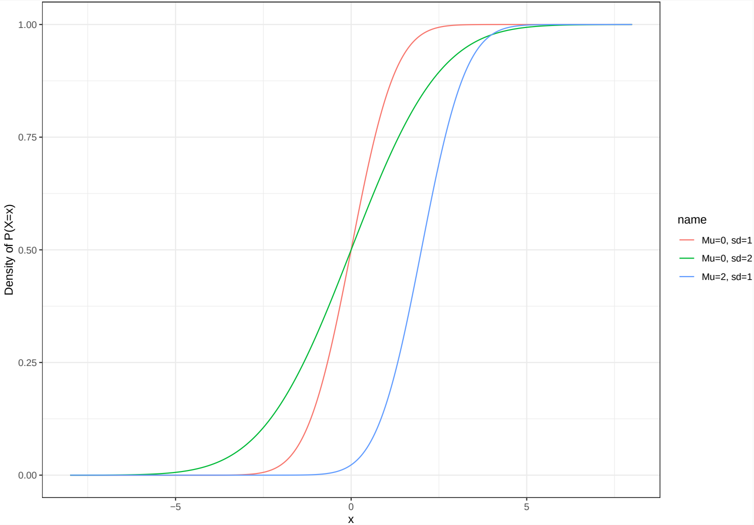
The Normal (Gaussian) distribution is continuous, and takes on values from $[-\infty, \infty]$. It has two parameters, the mean μ and standard deviation σ (or variance σ^2).

- μ determines the location of the distribution
- σ determines the spread of the distribution

The Normal PDF



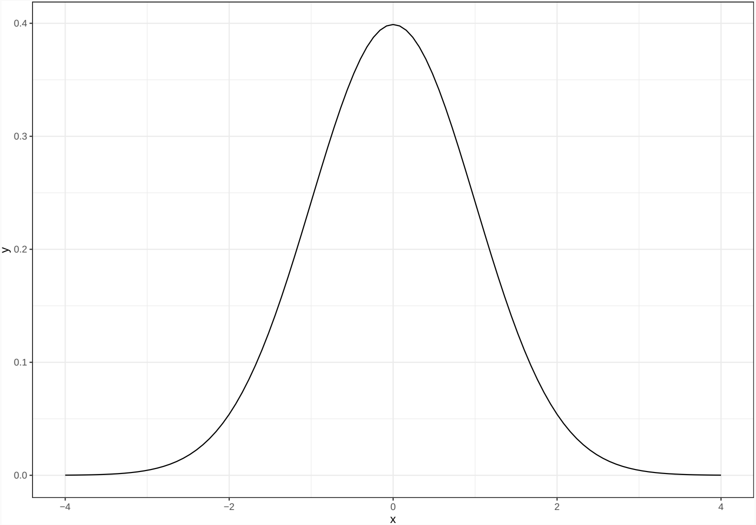
The Normal CDF



Special features of Normal distributions:

- The sum of many random variables from other distributions are often Normal
- For $X \sim N(\mu, \sigma^2)$, $Z = X + c$ is also Normal: $Z \sim (\mu + c, \sigma^2)$
- $Z = cX$ is distributed $Z \sim N(c\mu, (c\sigma)^2)$
- Z-scores of a Normal random variable are $N(0, 1)$

Area under the curve: interpreting the PDF and CDF



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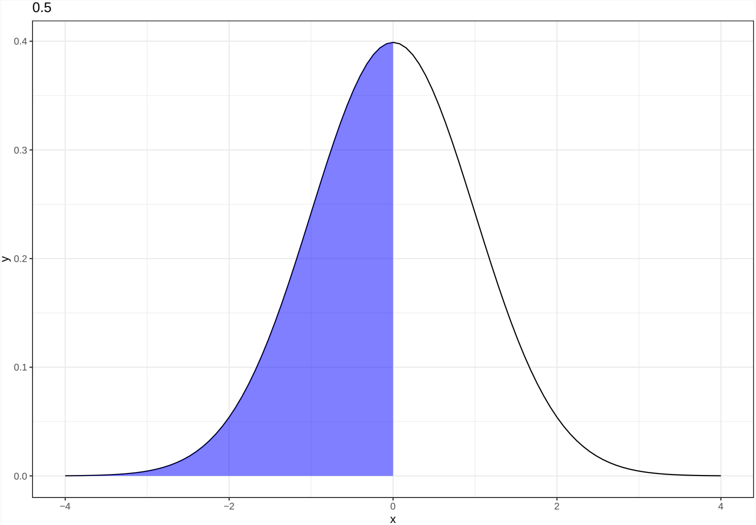

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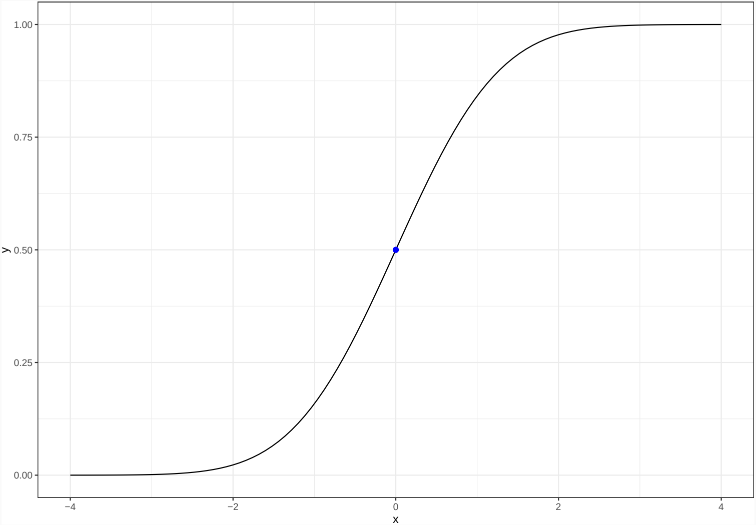

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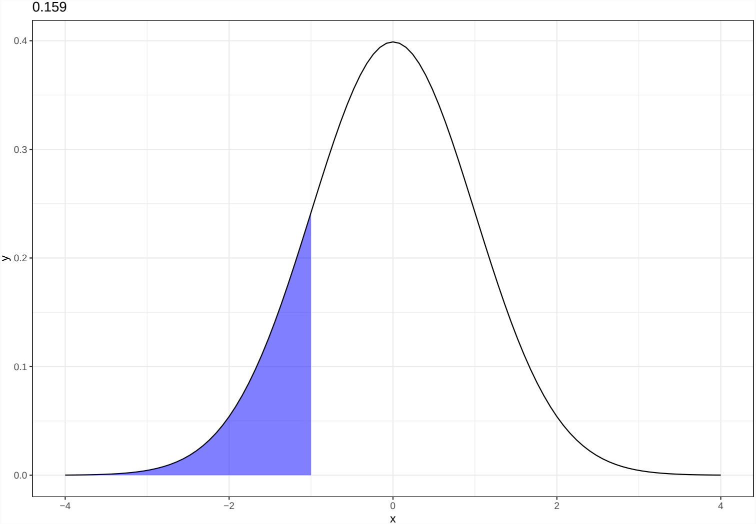

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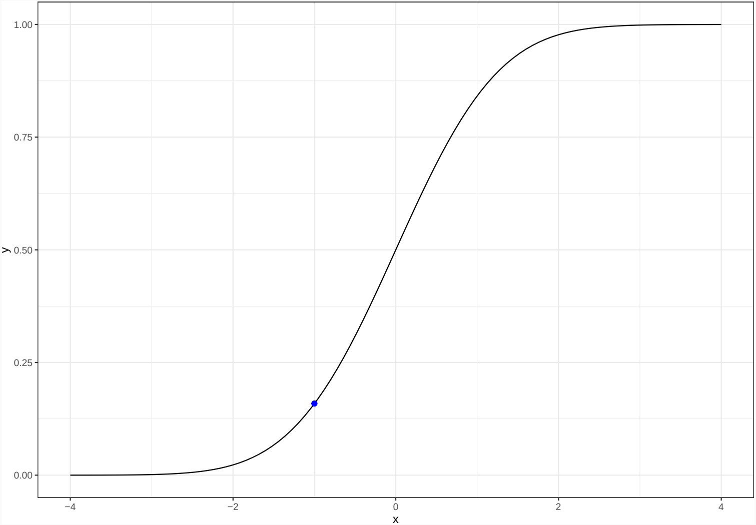

Z-scores and area under the curve

Recall that to obtain a z-score, we subtract the mean and divide by the standard deviation:

$$\text{z-score} = \frac{X - \mu}{\sigma}$$

For a Normal variable, z-scores are distributed $z \sim N(0, 1)$

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What does a z-score of 0 indicate? -1?

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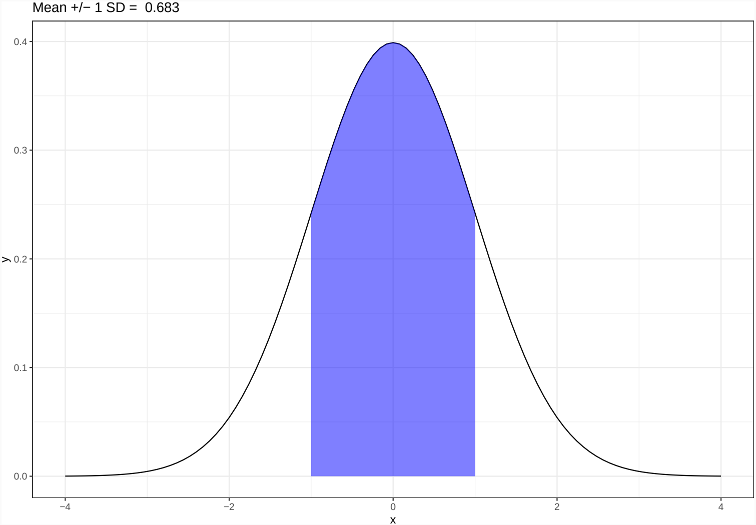
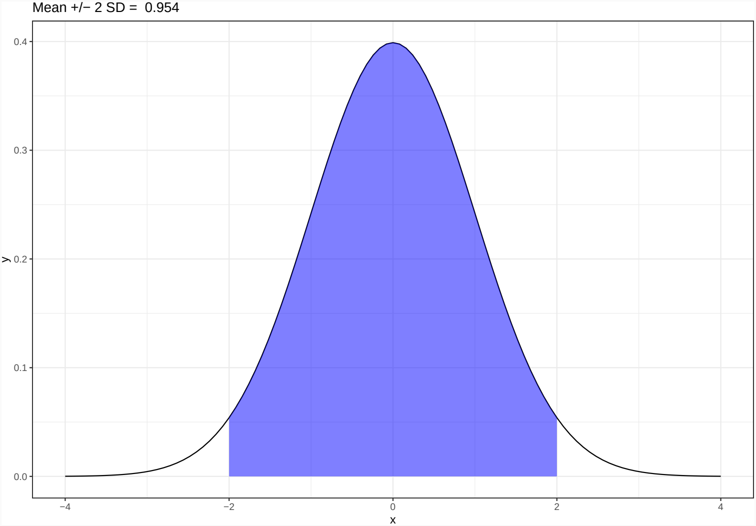
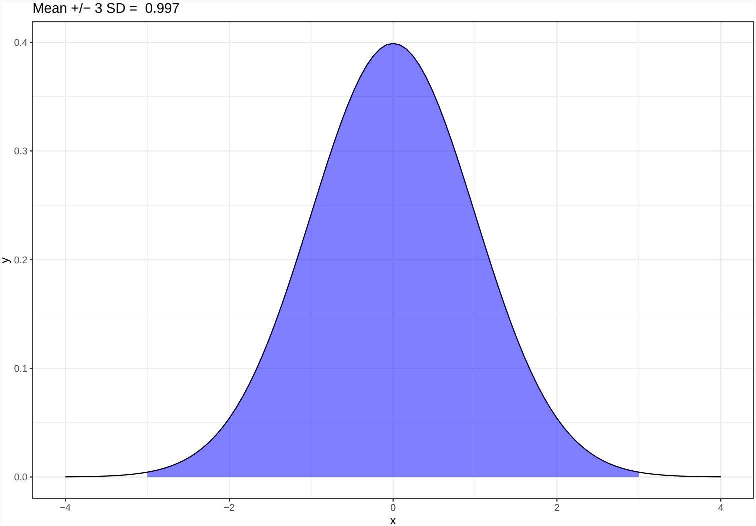
What does a z-score of 0 indicate? -1? 2?


Mean +/− 1 SD = 0.683
0.4
0.3
0.2
0.1
0.0
y
−4
−2
0
2
4
x


Mean +/− 2 SD = 0.954
0.4
0.3
0.2
0.1
0.0
y
−4
−2
0
2
4
x



Mean +/− 3 SD = 0.997
0.4
0.3
0.2
0.1
0.0
y
−4
−2
0
2
4
x

Useful probability distribution functions

```r
### Normal(0,1) probability density function
dnorm(x = 0, mean = 0, sd = 1)
```

```
## [1] 0.3989423
```

```r
### Normal(0,1) cumulative distribution function
pnorm(q = 0, mean = 0, sd = 1)
```

```
## [1] 0.5
```

```r
### Random draw from a normal(0,1) distribution
rnorm(n = 1, mean = 0, sd = 1)
```

```
## [1] 1.231011
```

```r
### CDF position for a given probability (quantile)
qnorm(p = 0.75, mean = 0, sd = 1)
```

```
## [1] 0.6744898
```

```r
### You can also use dbinom(), pbinom(), rbinom(), qbinom()
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The expectation of a random variable

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For a discrete variable, the expectation is the sum of all values of x weighted by their probability, given by the PDF $f(x)$.

$$E(X) = \sum_x x \times f(x)$$

Because continuous variables take on an infinite number of values, we compute the expectation with an integral

$$\int x \times f(x) dx$$

Variance and standard deviation of a random variable

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And the sample variance is sd^2

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Note the similarities in the two equations