2: Foundational math for statistics

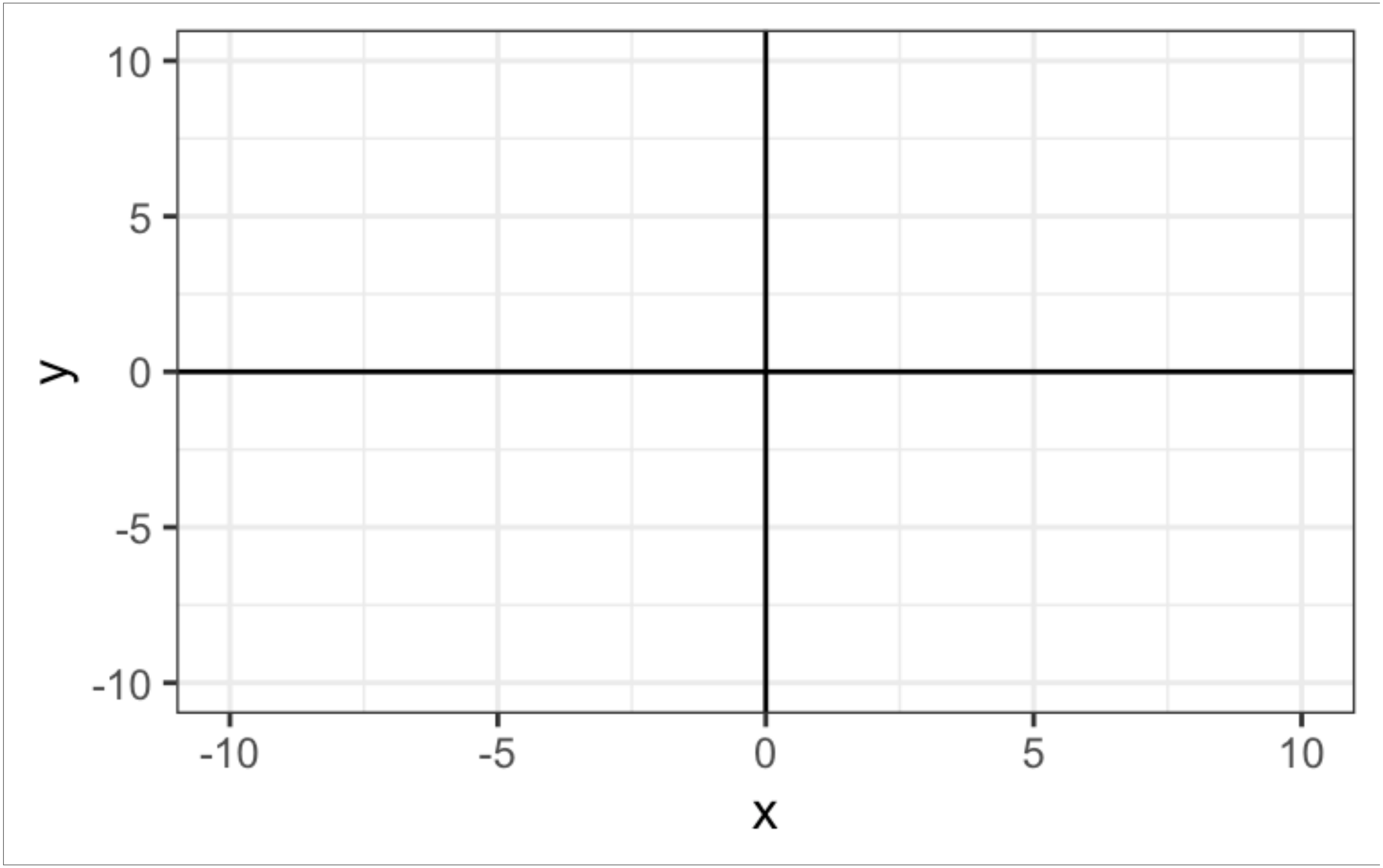
Welcome back: math time!

Agenda today

1. Plotting, functions
2. The summation operator
3. Matrices and vectors

Coordinates and lines

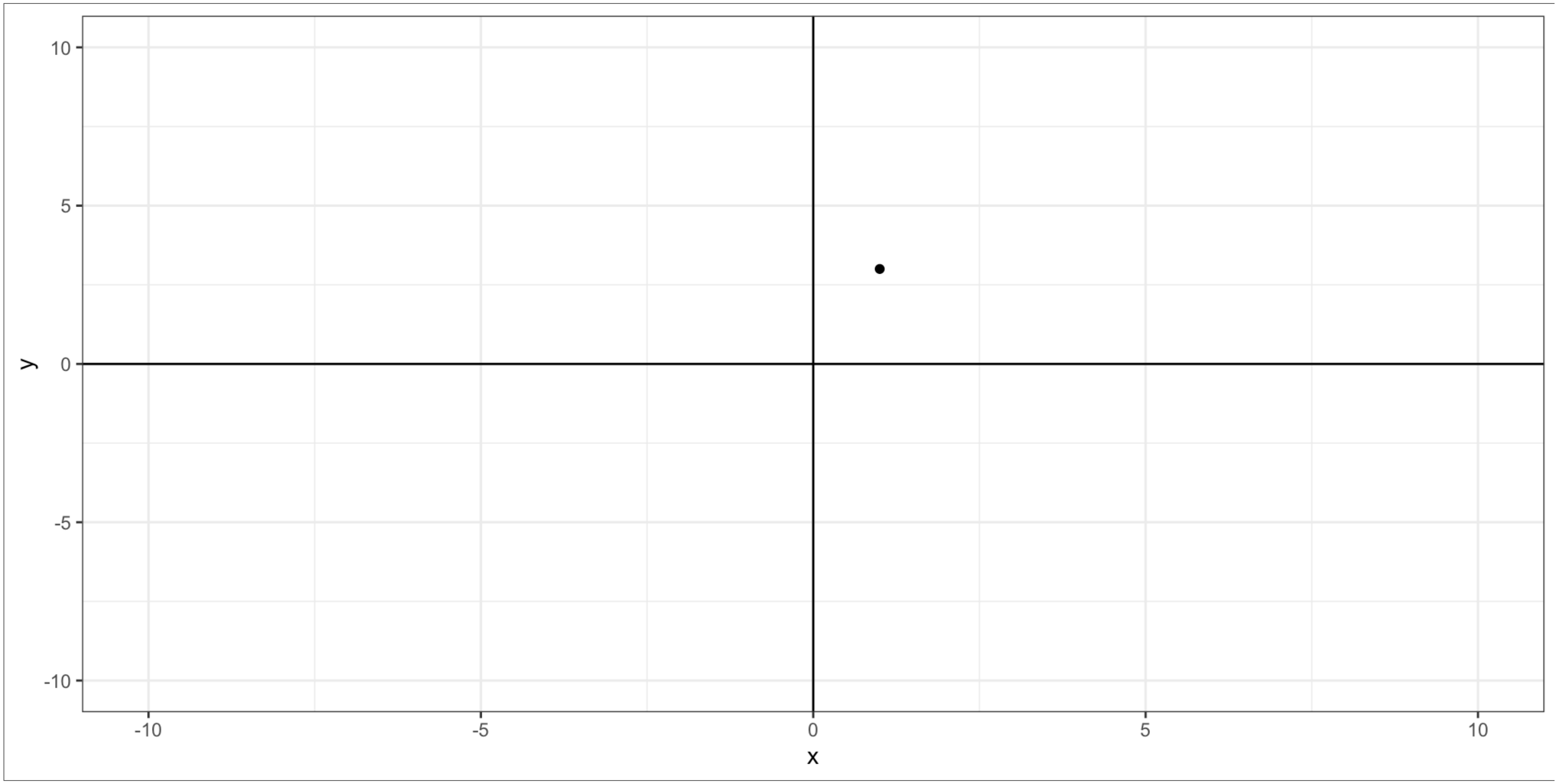
The coordinate plane



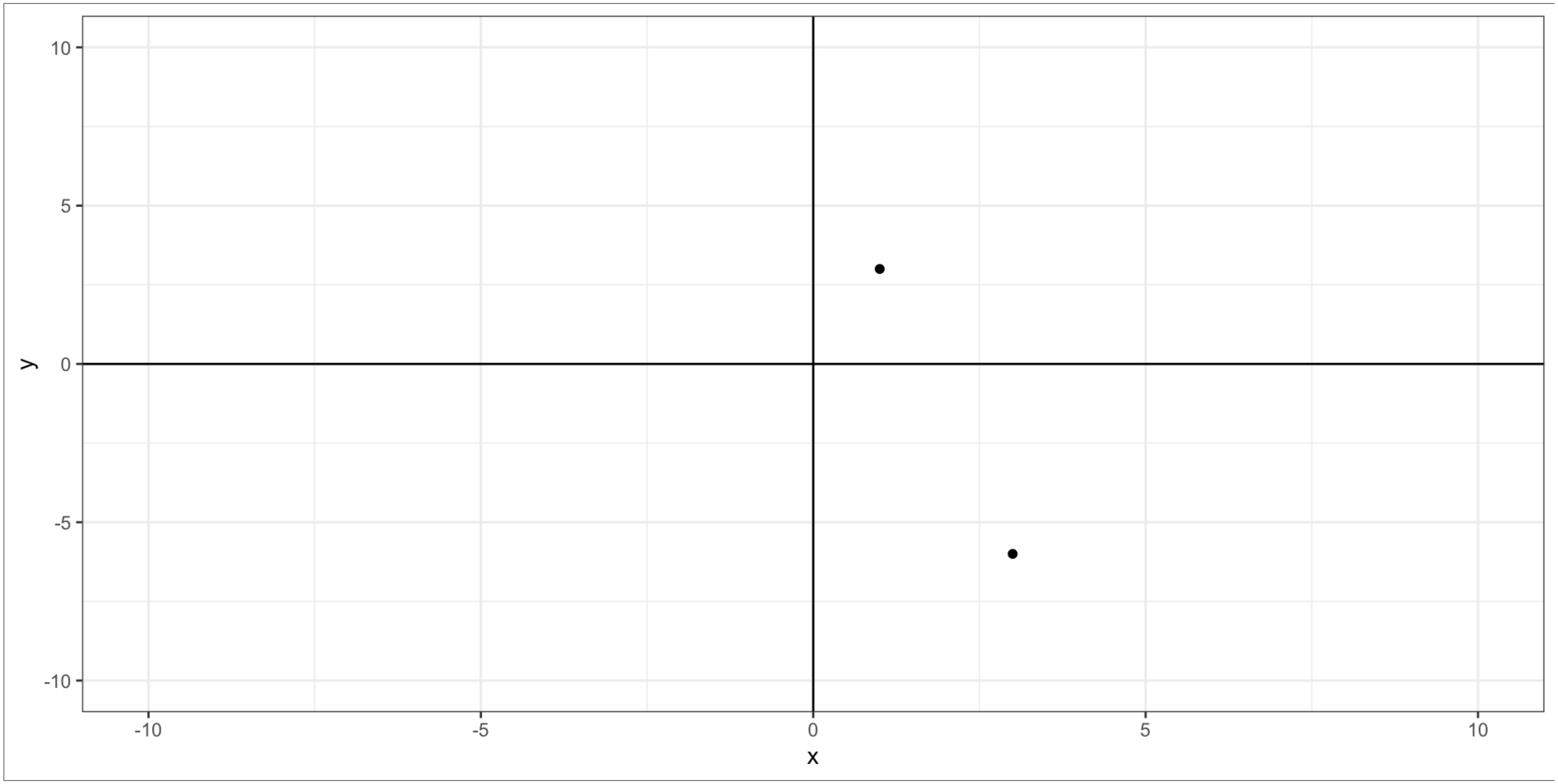
Plotting points

For coordinate pair (x_1, y_1), we can plot along an x and y axis.

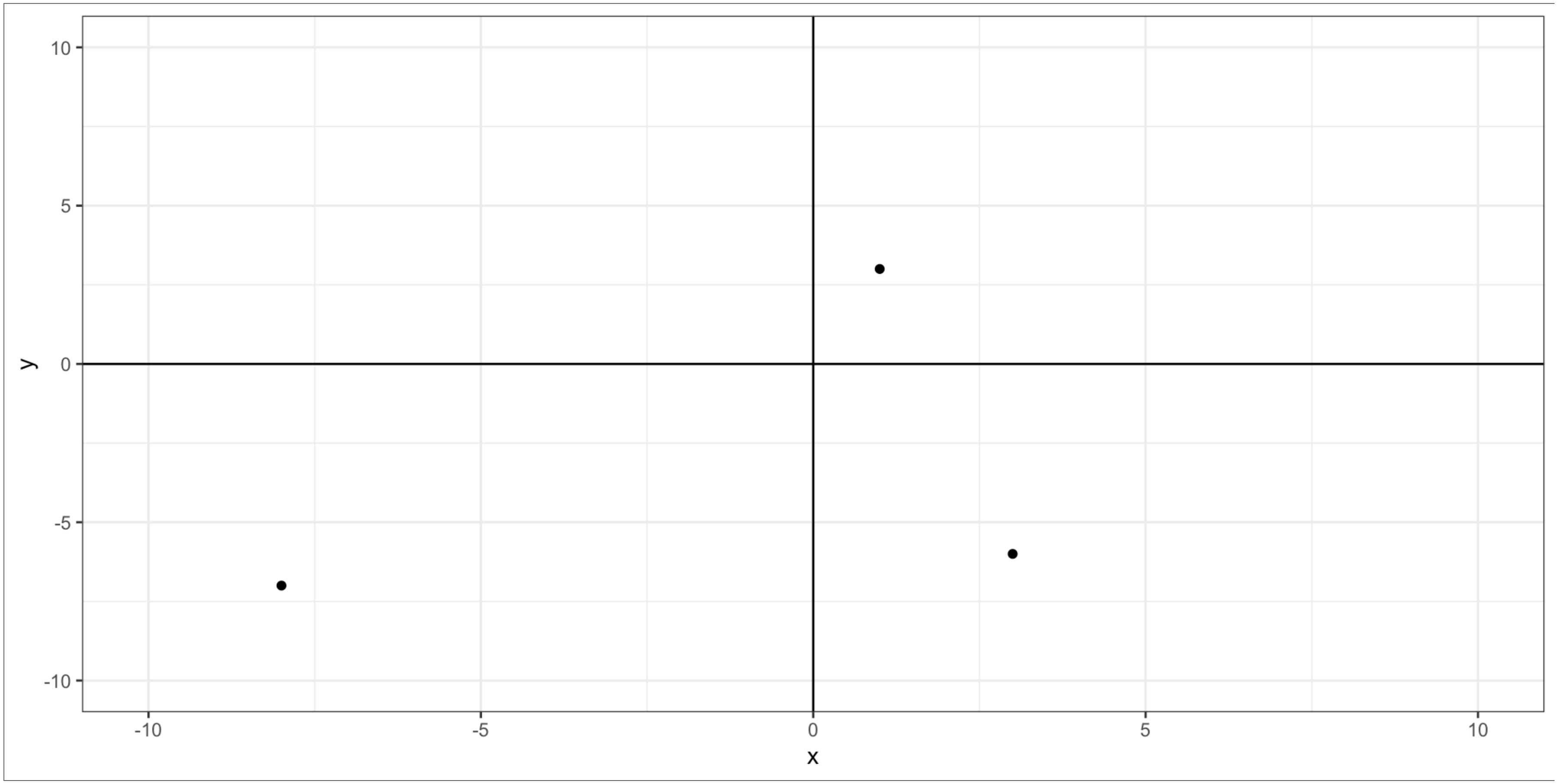
Example: (1,3)



Example: (1,3), (3,-6)



Example: (1,3), (3,-6), (-8, -7)



Practice

Sketch out a coordinate plane on paper. Plot the following points.
Remember we give points as (x, y), where x represents the horizontal location and y represents the vertical location

- (0, 1)
- (-3, 2)
- (5, -6)
- (5, 5)

Lines

The typical equation for a line is $y = mx + b$ where m is the slope and by is the y-intercept.
You will often see a line expressed as a regression equation:
$y = \beta_0 + \beta_1 x$
where β_0 is the y-intercept and β_1 is the slope.

The slope

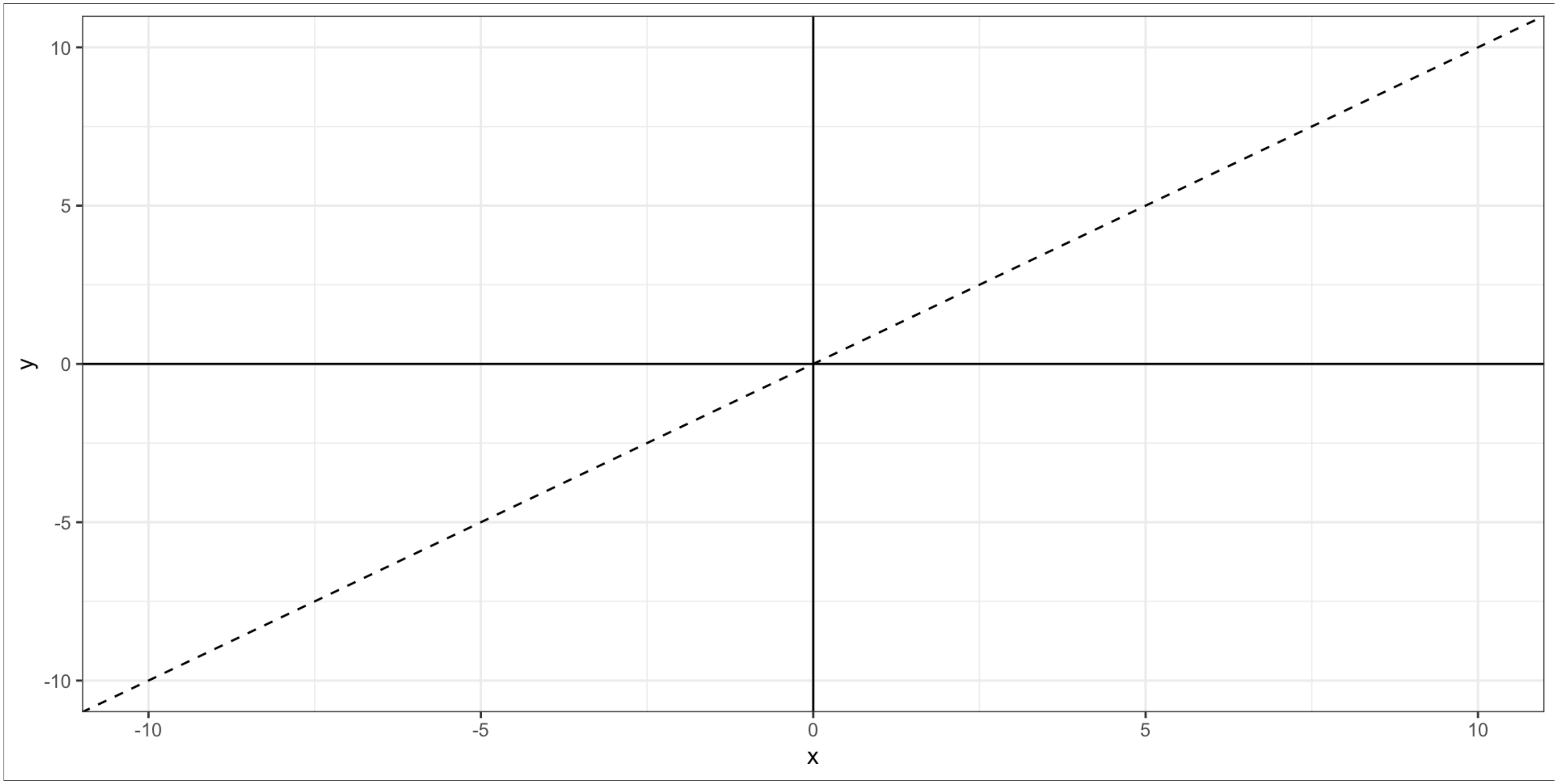
Slope measures the steepness of a line. A line with a positive slope has increasing values of y as x increases. A line with a negative slope has decreasing values of y for increasing values of x.
We can calculate the slope with two coordinates on the line $(x_1, y_1), (x_2, y_2)$
The slope is the ratio of the difference in y values to the difference in x values.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

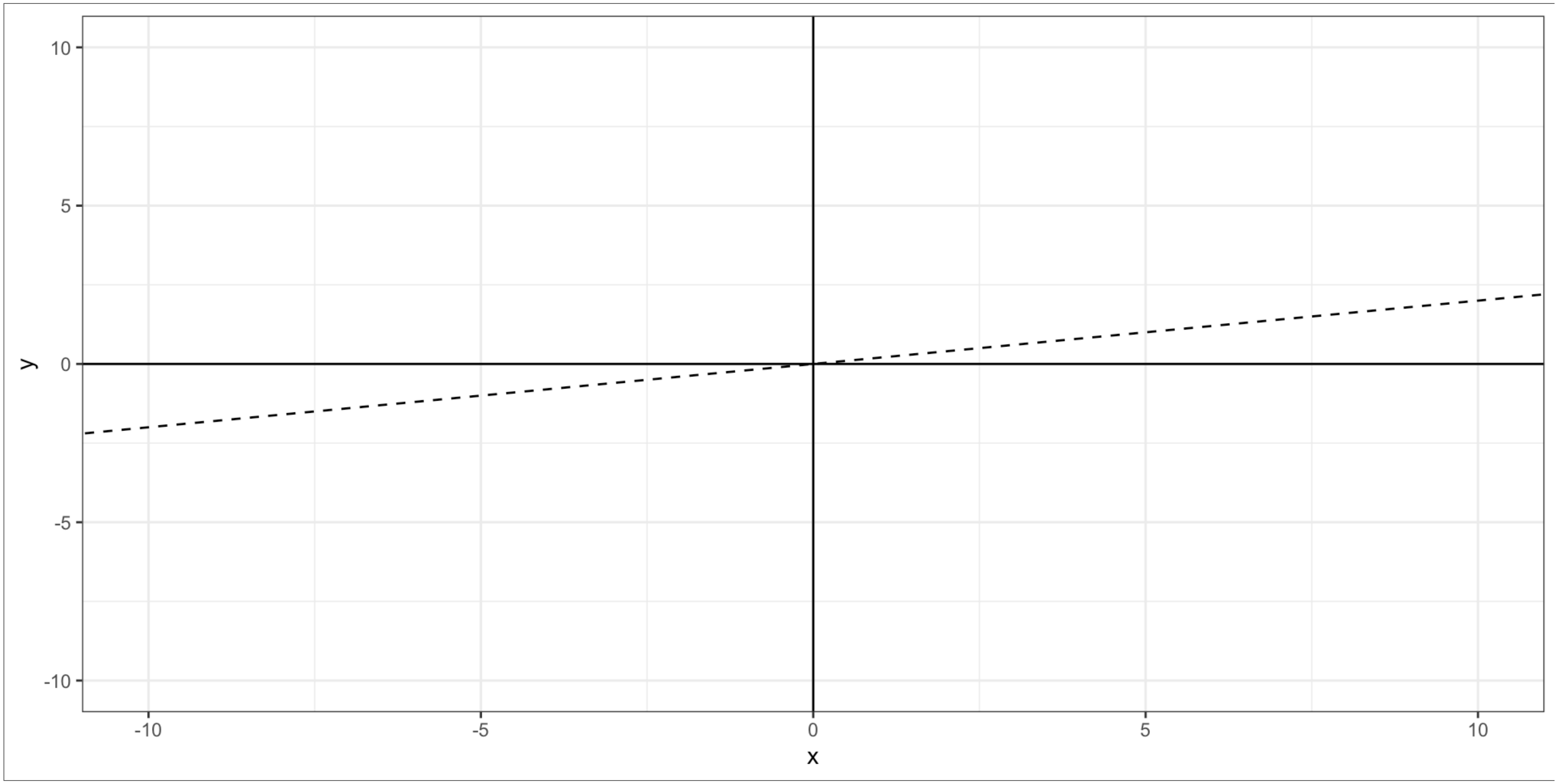
The y-intercept

The y-intercept is the value of y when x = 0. If we have the value of one point on the line, and the slope, we can obtain the y-intercept
$b = y_1 - m \cdot x_1$

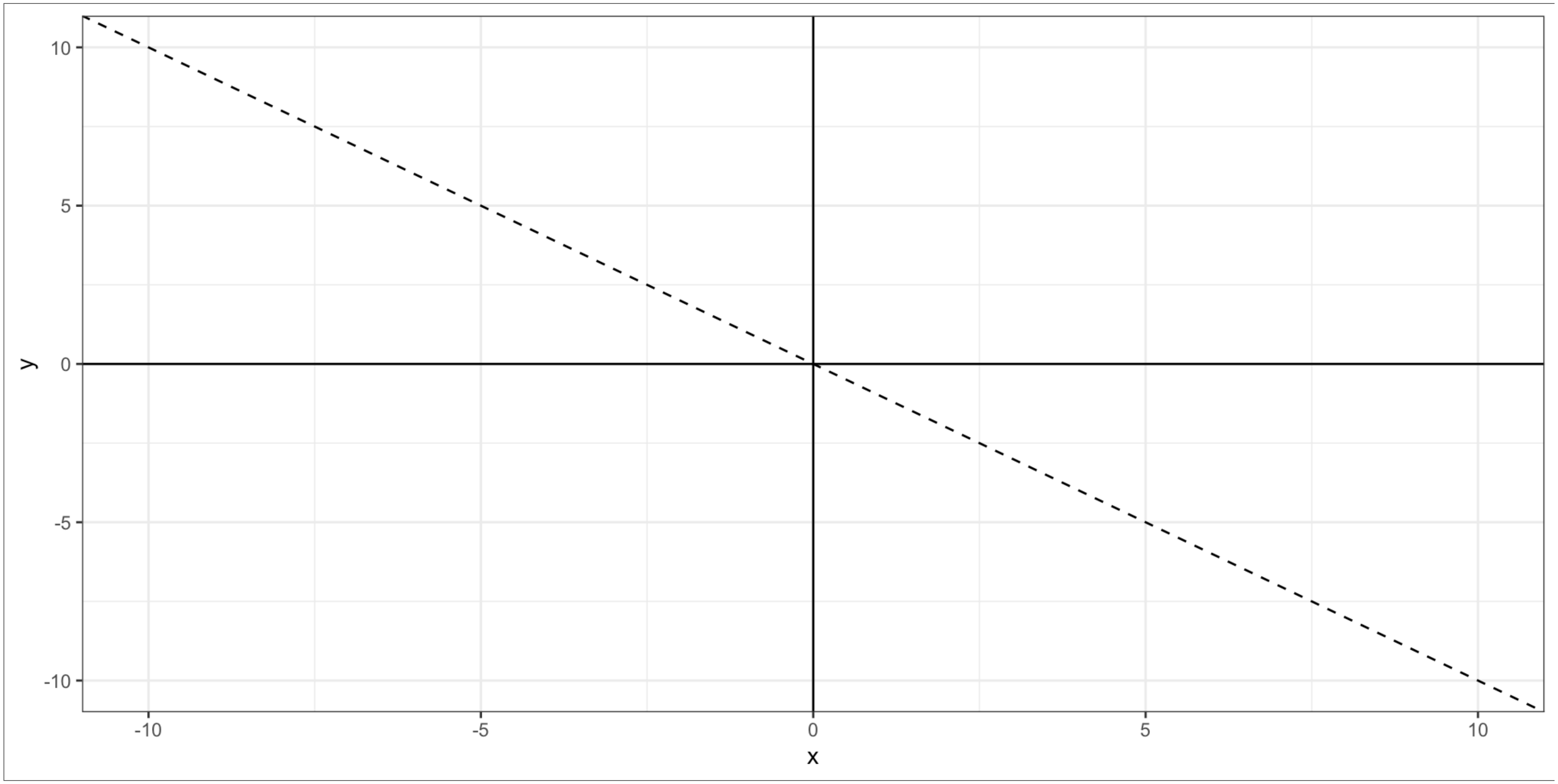
Example: intercept = 0, slope = 1



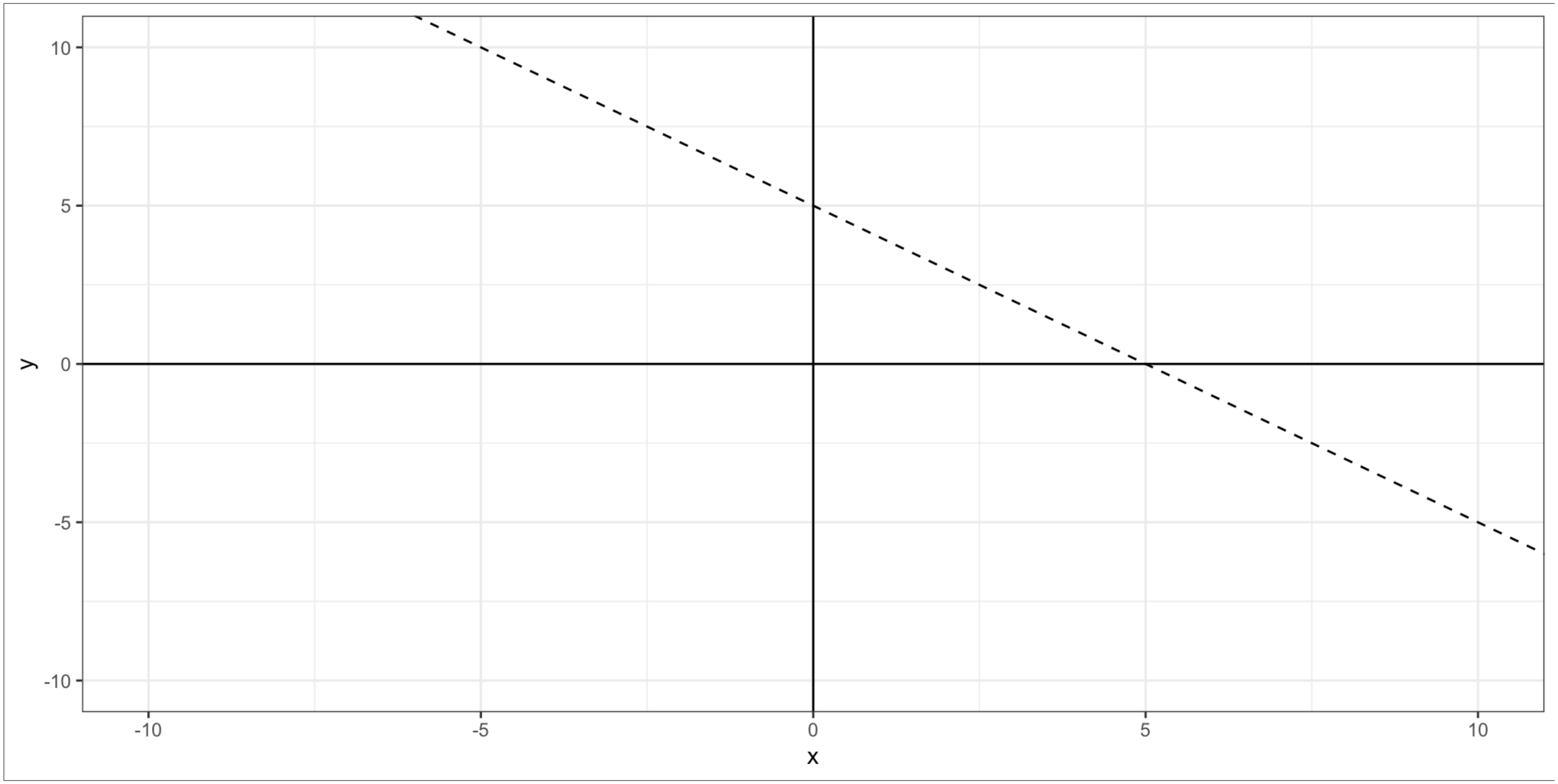
Example: intercept = 0, slope = 0.2



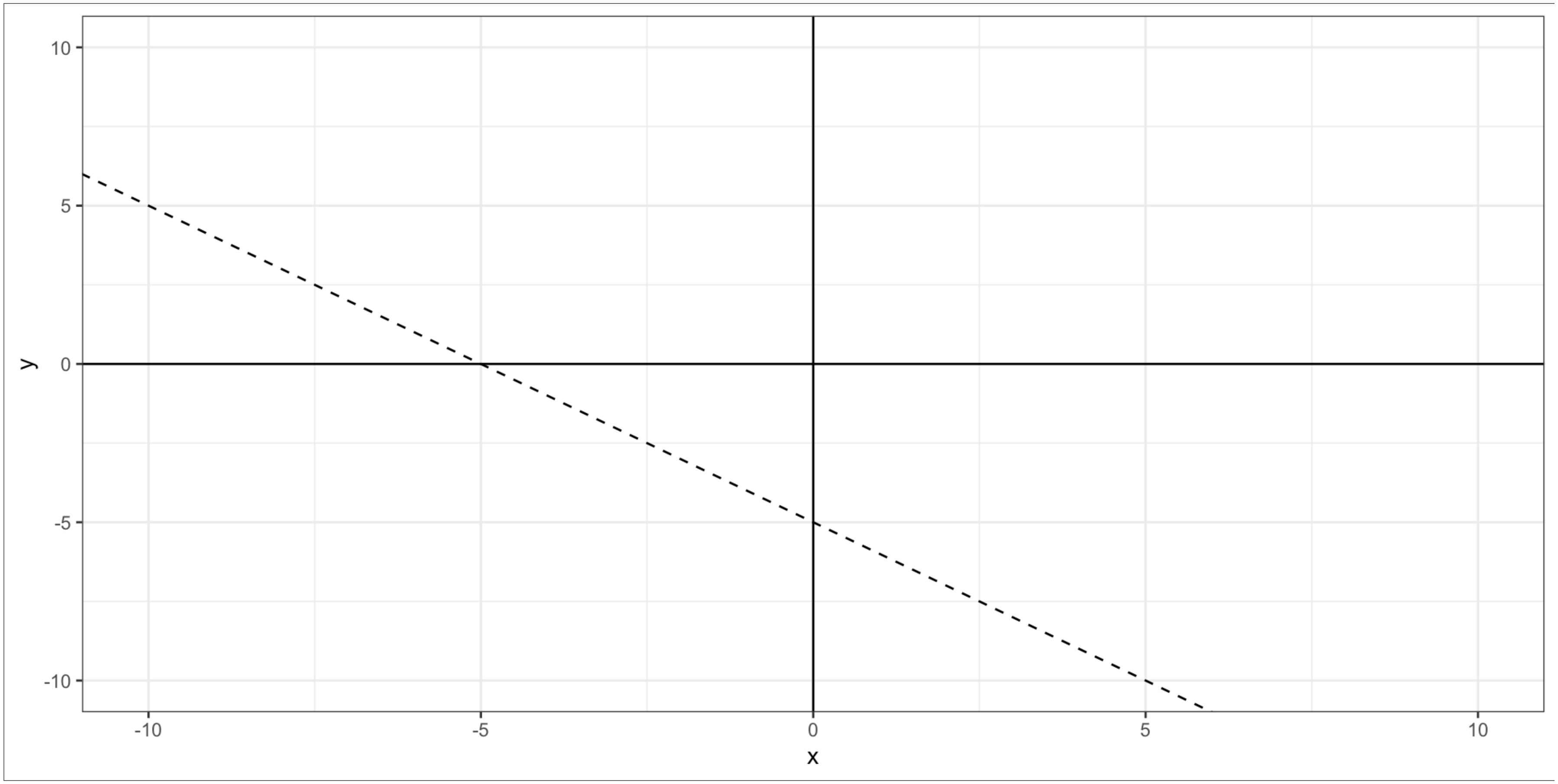
Example: intercept = 0, slope = -1



Example: intercept = 5, slope = -1

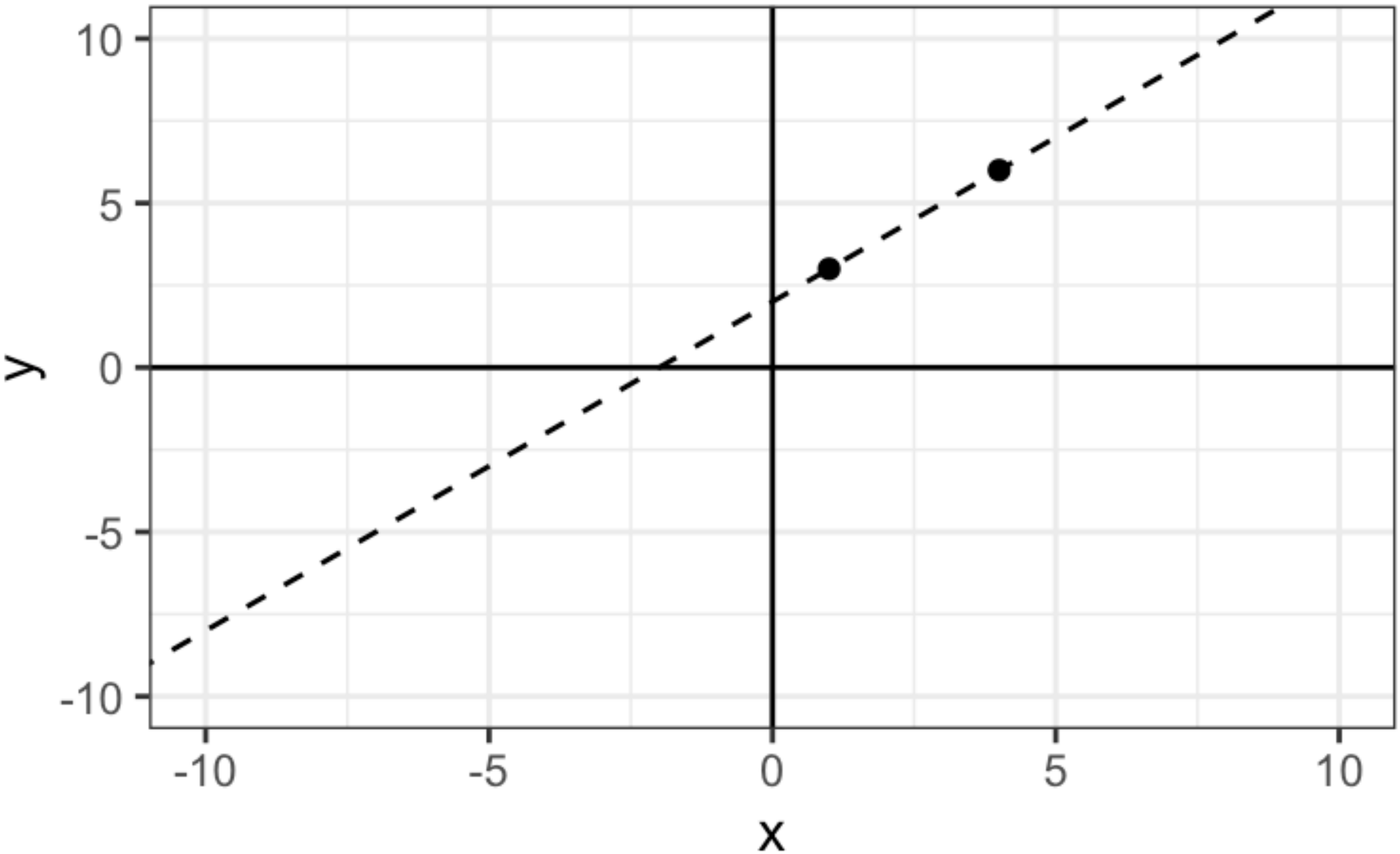


Example: intercept = -5, slope = -1



Example

Given the points $(1, 3)$ and $(4, 6)$, the slope is $m = \frac{6-3}{4-1} = 1$ and the y-intercept is $b = 3 - 1 \cdot 1 = 2$. The equation of the line is $y = 1x + 5$



Exercises: Compute slope, y-intercept, and sketch a line for each pair of coordinates

The equation for a line is $y = mx + b$
The slope is computed as $m = \frac{y2-y1}{x2-x1}$
The y intercept is $ b = y_1 - m x_1$

1. (2,3), (5,6)
2. (-2,4), (0, 2)
3. (10, 12), (-5, -2)

Functions

Functions, in general

A function maps each element in a set X to an element in set Y

- Linear function: $f(x) = x + 5$
- Quadratic function: $f(x) = x^2 + 2x + 3$
- Exponential function: $f(x) = e^{2x} + 6$

Graphical forms of functions: linear

$f(x) = 2x + 5$

Graphical forms of functions: quadratic

$f(x) = x^2 + 2x + 3$

Graphical forms of functions: exponential

$f(x) = e^{2x} + 6$

Graphical forms of functions: logarithmic

$f(x) = log(x)$

Exercises

Evaluate these functions for $x = 1$, $x = 2.5$, and $x = -6$

1. $f(x) = 2x$
2. $f(x) = \frac{x}{2}$
3. $f(x) = 2(x + 1)^3$

Summation

Represented as $\sum$, with integer begin and end points

$$\sum_{x=1}^{3} x$$

Summation

$$\sum_{x=1}^{3} x = 1 + 2 + 3 = 6$$

Summation in R

In R, we can calculate a sum using the sum() function

```
# make an integer vector from 1 to 3
x<-1:3
# x<-c(1,2,3) is equivalent
sum(x)
```

```
[1] 6
```

Summation exercises

Compute the following by hand, and then in R

$$\sum_{x=3}^{8}(x+1)$$

$$\sum_{x=1}^{4} 2x$$

Exercises (solutions)

Compute the following in R

$$\sum_{x=3}^{8}(x+1)$$

```r
x<-3:8 # or c(3, 4, 5, 6, 7, 8)
sum(x+1)
```

$$\sum_{x=1}^{4}2x$$

```r
x<-c(1, 2, 3, 4) # or 1:4
sum(2*x)
```

Vectors

Vectors are one-dimensional arrays of values. They have dimension n, where n is the number of elements in the vector

$x = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$

Indexing

We can define each element with a position index i.
$x = \begin{bmatrix} 3 & 1 & 5 & 2 \end{bmatrix}$

- What is x_2?
- What is x_4

Vector operations

We can compute operations on vectors using single values, called scalars. Simply perform the operation on each element of the vector

$x = \begin{bmatrix} 3 & 1 & 5 & 2 \end{bmatrix}$

$2 + x = \begin{bmatrix} 5 & 3 & 7 & 4 \end{bmatrix}$

$2x = \begin{bmatrix} 6 & 2 & 10 & 4 \end{bmatrix}$

Vector operations

We can compute math operations on two vectors of equal dimension by performing the operation on each element pair by index

$x = \begin{bmatrix} 3 & 1 & 5 & 2 \end{bmatrix}$
$y = \begin{bmatrix} 4 & -2 & 3 & 0 \end{bmatrix}$
$x + y = \begin{bmatrix} 7 & -1 & 8 & 2 \end{bmatrix}$

Exercises: compute the following

$$x = \begin{bmatrix} 2 & 1 & 4 & -2 \end{bmatrix}$$
$$y = \begin{bmatrix} 0 & 5 & 3 & 8 \end{bmatrix}$$

1. x + 2
2. 2y
3. x + y
4. 2x - y

Matrices

A matrix is a rectangular array of numbers, with dimensions expressed as rows $\times$ columns

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

A is a 3 x 3 matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

B is a 2 x 3 matrix

$$\mathbf{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

C is a 3 x 2 matrix

Matrix notation

We can identify each element of a matrix with its column and row position, where x_{ij} refers to the value in the ith row and jth column of matrix X. Note that we use uppercase letters for a matrix, and lowercase letters for elements of a matrix.

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$$

Matrix notation

$$\mathbf{X} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

What is

- $X_{1,2}$
- $X_{2,1}$
- $X_{1,3}$

Diagonal Matrix

A diagonal matrix has zero values except on the diagonal:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Identity Matrix

An identity matrix is a special case of a diagonal matrix, where all values on the diagonal are equal to 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagnoal and Identity matrices are also symmetric, where all $x_{ij} = x_{ji}$. Symmetric matrices are square.

Matrix operations

Matrix transpose

A transpose interchanges the rows and columns of a matrix, or rotates it. The dimensions are switched, so an $n \times k$ matrix becomes a $k \times n$ matrix. We denote a transpose with a T

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \mathbf{B}^{\mathbf{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Addition and subtraction

Two matrices (or vectors) can be added or subtracted only if they have identical dimensions. Then add or substract the corresponding elements of the matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix}$$

Multiplication by scalar

Matrices and vectors can be multiplied by constant values (called scalars).

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} 4\mathbf{B} = \begin{bmatrix} 4 & 8 & 12 \\ 16 & 20 & 24 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} 3\mathbf{C} = \begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \end{bmatrix}$$

Matrices and vectors in R

The c() function makes a vector

We can define a vector in R using the `c()` function

```
x <- c(3, 5, 8)
x
```

```
[1] 3 5 8
```

Vector indexing in R

We can call a vector index i using brackets

```
x
```

```
[1] 3 5 8
```

```
x[1]
```

```
[1] 3
```

```
x[3]
```

```
[1] 8
```

Operations on vectors

We can also easily perform scalar operations on vectors.
Try the following

- $x + 2$
- $2x$
- $(x + 2)^2$

Further vector operations

In R, define $x = [1, 4, 6, 9]$ and $y = [0, -2, 5, 7]$
Compute:

- $x + y$
- $2x - y$
- $(2x + 2) \times (y - 3)$

Making a matrix

```
Z<-matrix(data = c(1, 2, 4, 5, 6, 7, 8, 9),
          nrow = 4, ncol = 2)

Z
```

```
     [,1] [,2]
[1,]    1    6
[2,]    2    7
[3,]    4    8
[4,]    5    9
```

- What is $Z_{2,2}$?
- What is $Z_{4,1}$?

Matrices are made up of vectors

Bracket notation in R gets us thinking in vectors.

- Try this code `Z[,1]`. What does that return? What kind of object is it?
- What do you think you will get if you run `3 + Z[,2]`

Exercises

```r
Z
```

```
     [,1] [,2]
[1,]    1    6
[2,]    2    7
[3,]    4    8
[4,]    5    9
```

- How could you retrieve the 3rd row of Z in R?
- Multiply the 1st column of R by 2

data.frames in R

R has a special kind of object called a data.frame(). It is a matrix-like object that can have names for columns and rows.
This is a data.frame

```
           Murder Assault UrbanPop Rape
Alabama      13.2     236       58 21.2
Alaska       10.0     263       48 44.5
Arizona       8.1     294       80 31.0
Arkansas      8.8     190       50 19.5
California    9.0     276       91 40.6
Colorado      7.9     204       78 38.7
```

Indexing with brackets

We can use bracket indexing on data.frames

```
USArrests[1, 1]
```

```
[1] 13.2
```

```
USArrests[2,3]
```

```
[1] 48
```

Or we can pull whole rows or columns

```
USArrests[1,]
```

```
        Murder Assault UrbanPop Rape
Alabama   13.2     236       58 21.2
```

```
USArrests[,2]
```

```
 [1] 236 263 294 190 276 204 110 238 335 211  46 120 249 113  56 115
109 249  83
[20] 300 149 255  72 259 178 109 102 252  57 159 285 254 337  45 120
151 159 106
[39] 174 279  86 188 201 120  48 156 145  81  53 161
```

Name indexing

We can use the name of columns in data.frames for indexing using the `$` operator

These functions can be helpful to remember the dimensions and structure of an object: `str()`, `names()`