NOTES

1.1. **Problem Statement.** The problem we want to solve is as follows

Problem 1. Let $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$, and $T \in \mathbb{R}_{>0}$ be given. Determine P and Q symmetric positive definite matrices and a polynomial $p\colon [0,T] \to \mathbb{R}_{>0}$ such that

$$\langle \nabla V(x), f(x) \rangle < 0 \qquad \forall x \in \mathbb{R}^{n+m} \times [0,T]$$

where $x := (x_p, \tau) = (x_1, x_2, \tau)$ and for all $x \in \mathbb{R}^{n+m} \times [0,T]$

$$V(x) := x_1^\top P x_1 + p(\tau) x_2^\top Q x_2$$

$$f(x) := \begin{pmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \\ -1 \end{pmatrix}$$

Alternatively

$$V(x) := x_1^\top P x_1 + x_2^\top Q(\tau) x_2$$

where $Q(\tau)$ is a polynomial matrix with entries of degree at most d.

1.2. **Positive polynomials over intervals.** This is a direct application of classical Positivstellensatz for positive polynomials over intervals. We include a proof for the sake of self-containedness.

Lemma 1.1. *Let $T > 0$, and let $p \in \mathbb{R}[\tau]$ be of degree d and nonnegative on $[0,T]$. Then there exists sums of squares $\sigma_0, \sigma_1 \in \mathbb{R}[\tau]$, of degree $\deg(\sigma_0) \le 2d$ and $\deg(\sigma_1) \le 2d-2$ such that*

$$p(\tau) = \sigma_0(\tau) + \big(\tau T - \tau^2\big)\sigma_1(\tau)$$

for all $\tau \in \mathbb{R}$.

Proof. We define the linear transformation $\varphi(\tau) = \frac{T\tau+T}{2}$ and its inverse $\varphi^{-1}(t) = \frac{2t-T}{T}$. Remark that φ and φ^{-1} send $[-1,1]$ and $[0,T]$ to each other, and deduce that the polynomial

$$f(t) = p(\varphi(t)) = p\left(\frac{Tt+T}{2}\right)$$

is nonnegative for all $t \in [-1,1]$. By [1, Th.2.4(a)], there exists sums of squares $\overline{\sigma_0}, \overline{\sigma_1} \in \mathbb{R}[t]$ of degree $\deg(\overline{\sigma_0}) \le 2d$ and $\deg(\overline{\sigma_1}) \le 2d-2$ such that

$$f = \overline{\sigma_0} + (1-t^2)\overline{\sigma_1}$$

for all $t \in \mathbb{R}$. Hence one can apply the inverse transformation $t = \varphi^{-1}(\tau)$ to get

$$p(\tau) = f(\varphi^{-1}(\tau)) = \overline{\sigma_0}(\varphi^{-1}(\tau)) + (1 - (\varphi^{-1}(\tau))^2)\overline{\sigma_1}(\varphi^{-1}(\tau))$$

and we conclude by observing that $\sigma_i := \frac{4}{T^2}\overline{\sigma_i}(\varphi^{-1}(\tau))$ are sums of squares in $\mathbb{R}[\tau]$, with degree bounded by that of the $\overline{\sigma_i}$. $\square$

Hence one can express positivity of p via a semidefinite program with two blocks, exactly as follows:

$$p(\tau) = v_d(\tau)^T X_0 v_d(\tau) + \left(\tau T - \tau^2\right) v_{d-1}(\tau)^T X_1 v_{d-1}(\tau)$$
$$\begin{pmatrix} X_0 & \\ & X_1 \end{pmatrix} \succeq 0.$$

where $v_i(\tau)$ is the column vector with entries $1, \tau, \tau^2, \ldots, \tau^i$. The first line is a linear system in the $\binom{d+1}{2} + \binom{d}{2}$ variables of X_0 and X_1, and with $d+1$ constraints. Hence the LMI to be solved will have

$$\begin{aligned} \text{size } &= d + d - 1 = 2d - 1 \\ \text{number of variables } &= \binom{d+1}{2} + \binom{d}{2} - (d+1) = d^2 - d - 1. \end{aligned}$$

1.3. **Positive matrix polynomials over intervals.** We consider the second model for the Lyapunov function V, that is

$$V(x) := x_1^\top P x_1 + x_2^\top Q(\tau) x_2.$$

Since P has to be positive definite, we impose that $Q(\tau)$ is positive semidefinite for $\tau \in [0,T]$, which gives a sufficient condition for $V(x) > 0$.

We recall that $Q(\tau) = \sum_{i=0}^d Q_i \tau^i$ has degree at most d as polynomial matrix. The following Lemma gives a certificate of positivity over $[0,T]$.

Lemma 1.2. *Let $Q = Q(\tau)$ be a symmetric polynomial matrix of degree at most d and positive definite over $[0,T]$. Then UNDER ASSUMPTIONS TO BE CHECKED there exist an integer N and positive definite symmetric matrices X_α, for $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 + \alpha_2 = N + d$ such that*

$$Q = \sum_\alpha X_\alpha \tau^{\alpha_1} (T - \tau)^{\alpha_2}$$

Proof. Remark that $[0,T]$ is convex, compact and defined by the linear inequalities $\tau \geq 0, T - \tau \geq 0$. Then the thesis follows from the Matrix Handelman's Positivstellensatz [2, Th.3]. □

The same theorem [2, Th.3] gives upper bounds for N that are quadratic on d. We deduce from Lemma 1.2 that the condition $Q(\tau) \succeq 0$ can be tested by solving the linear matrix inequality

$$Q(\tau) = \sum_{\alpha_1+\alpha_2=N+d} X_\alpha \tau^{\alpha_1} (T-\tau)^{\alpha_2}$$
$$\bigoplus_{\alpha_1+\alpha_2=N+d} X_\alpha \succeq 0.$$

If Q is $m \times m$, then the previous LMI has

$$\begin{aligned} \text{size } &= m(N+d+1) \\ \text{number of variables } &= (N+d+1)\binom{m+1}{2} - \binom{m+1}{2} = (N+d)\binom{m+1}{2}. \end{aligned}$$

References

[1] Lasserre, J-B. An introduction to polynomial and semi-algebraic optimization. Vol. 52. Cambridge University Press, 2015.

[2] Lê, C-T. and Du, T-H-B. Handelman's Positivstellensatz for polynomial matrices positive definite on polyhedra. Positivity 22.2 (2018): 449-460.