Supplementary Material for "Thermal Transport Signatures of Broken-Symmetry Phases in Graphene"

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I. THERMAL CONDUCTANCE AND THERMAL WAVELENGTH IN TWO DIMENSIONS

Assuming black-body radiation, the thermal conductance can be obtained from [1]

$$G_{\rm th} = \frac{d}{dT}\frac{W}{\pi}\int d\omega u(\omega)\partial_k\omega. \tag{1}$$

with the spectral density $u(\omega) = n_B(\omega)\nu(\omega)\omega$. Using $\nu(\omega) = k/2\pi\partial_k\omega$ one obtains the expression given in the main text, $G_{\rm th} = \frac{d}{dT}(W/2\pi^2)\int dk n_B[\omega(k)]\omega(k)k\partial_k\omega$. For a gapped linear spectrum, $\omega = \sqrt{\Delta^2 + v^2k^2}$, the thermal conductance is given by

$$G_{\rm th} = \frac{3WT^2}{2\pi^2 v} f(\Delta/T), \tag{2}$$

where we have defined $f(z) = \int_z^\infty dx x\sqrt{x^2 - z^2}/(e^x - 1)$ and we have set $k_B = 1$. The function f decays monotonically and can be approximated for large arguments by $f(z) \simeq e^{-z}\sqrt{z^3\pi/2}$. At temperatures $T = \Delta/4$ the conductance is reduced by an order of magnitude. For a gapless dispersion, $\Delta = 0$, we recover the Stefan–Boltzmann relation in two dimensions

$$G_{\rm th}(\Delta = 0) = \frac{3\zeta(3)WT^2}{\pi^2 v}. \tag{3}$$

A quadratic dispersion, $\omega(k) = k^2/2m$, has a weaker temperature dependence, instead,

$$G_{\rm th} = \frac{9\zeta(5/2)}{4\sqrt{2\pi^3}} W\sqrt{m}T^{3/2}. \tag{4}$$

The thermal wavelength corresponds to the wavelength λ where the spectral density $u(\lambda)$ becomes maximal. For a linear dispersion $\omega = vk = 2\pi v/\lambda$ in two dimensions we have

$$u(\omega)d\omega \propto \frac{d\lambda}{\lambda^4}\frac{1}{e^{2\pi v/\lambda T} - 1} \tag{5}$$

which has a maximum at $\lambda_{\rm max} \simeq 1.6 \times v/T$.

II. PHYSICAL PARAMETERS

We estimate the physical parameters of our model mainly from the experiment in Ref. [2] for a perpendicular magnetic field of $B_\perp = 5\,\mathrm{T}$. We emphasize, however, that physical parameters may vary between different experiments in particular as screening by nearby gate electrodes can influence the effective interactions. The parameter $u_\perp$ can be obtained from the critical Zeeman field at the CAF-FM transition, when magnetic field is rotated. For a perpendicular field $B_\perp = 5\,\mathrm{T}$ the critical total field has been determined as $B^c_{\rm tot} = 21\,\mathrm{T}$ [2]. With a Zeeman energy $\epsilon_Z/B_{\rm tot} = 0.7\,\mathrm{KT}^{-1}$ this yields an anisotropy energy $u_\perp(B_\perp = 5\,\mathrm{T}) = \epsilon_Z/2 = 7.3\,\mathrm{K}$. In a total field of $B_{\rm tot} = 5\,\mathrm{T}$ we have a Zeeman energy of $\epsilon_Z = 3.5\,\mathrm{K}$ which yields a canting angle $\theta \sim 0.4\pi$. The anisotropy energy u_z can be estimated from the phase transition to the layer polarized phase $u_\perp + u_z = \epsilon_V^2/(u_z - u_\perp) + \epsilon_Z^2/2u_\perp$, where $\epsilon_V = e\mathcal{E}a_z/2$ with $\mathcal{E}$ the electric field and $a_z = 3.5\,\text{Å}$ the interlayer distance. From [2] we obtain $u_z/B_\perp \simeq 14 - 20\,\mathrm{KT}^{-1}$. For the purpose of presentation it is useful to choose a somewhat lower value here, $u_z = 20\,\mathrm{K}$, although our results remain qualitatively valid for higher values of u_z.

The bare Coulomb energy $e^2/l_B = 650\,\mathrm{K}\sqrt{B_\perp}$, where $B_\perp$ is measured in Tesla, can be screened by an electronic polarization in graphene or by a substrate or gate electrodes [3]. We therefore assume a renormalized Coulomb energy $U_c = e^2/\epsilon l_B$ with an effective dielectric constant ϵ that takes all these effects into account. Its value can be estimated from the single-particle gap in the system and may vary from sample to sample. For the $\nu = 0$ state in suspended bilayer graphene Ref. [4] reports a gap of 100 K at $B_\perp = 5\,\mathrm{T}$ and we therefore set $U_c = e^2/\epsilon l_B = 100\,\mathrm{K}$, which corresponds to an effective dielectric constant of $\epsilon \simeq 15$.

With these parameters we obtain a spin wave velocity in the CAF phase

$$v = l_B \sin\theta \sqrt{|u_\perp| U_c'} = 2.4 \times 10^4\,\mathrm{m/s}. \tag{6}$$

Assuming a width $W = 5\,\mu\mathrm{m}$ and $T = 1\,\mathrm{K}$ the thermal conductance of the Goldstone mode in the CAF phase can be estimated from Eq. (3) as $G_{\mathrm{th}} \simeq 17\,\mathrm{pW/K}$. The thermal wavelength at 1 K is $\lambda \simeq 300\,\mathrm{nm}$ and thus considerably shorter than typical sample sizes in the range of $1 - 10\,\mu\mathrm{m}$ (see, e.g., Refs. 5 and 6).

III. CALCULATION OF THE PARTICLE-HOLE EXCITATION SPECTRUM

We shall perform the calculation in several steps. We first evaluate the matrix elements of the Hamiltonian in the single-particle eigenbasis of the various broken-symmetry phases of bilayer graphene. This is followed by an evaluation of the single-particle self energy in Sec. III C. Finally, we evaluate the response function in Sec. III D and find the excitation spectrum from its poles.

A. Matrix elements in SU(4) spin space

As introduced in the main text we use a notation, where the eight states in the zero-energy Landau level are labeled by index $a = \{n_a, \alpha\}$ that combines orbital index $n_a = 0, 1$ and SU(4) index α. The Landau level wavefunctions are $\psi_{a,k}(\mathbf{r}) = \chi_\alpha \langle \mathbf{r}|n_a, k\rangle$, where χ_α is an SU(4) spinor and $|n, k\rangle$ is the orbital wavefunction.

1. *Eigenbases of broken-symmetry states*

The single particle Hamiltonian $\mathcal{H}_s = -\epsilon_Z\sigma_z - \epsilon_V\tau_z$ is given in the spinor basis $\chi^0 = (\chi_{K,\uparrow}, \chi_{K,\downarrow}, \chi_{K',\uparrow}, \chi_{K',\downarrow})^T$. The SU(4)-spin symmetric part of the Hamiltonian is minimized when the four electrons occupy the single-particle states $\chi_\alpha|0\rangle$, $\chi_\alpha|1\rangle$, $\chi_\beta|0\rangle$, $\chi_\beta|1\rangle$ [7–9] where α and β are SU(4) indices. Taking into account the short-range anisotropic interactions, the system can be in four different phases: CAF, FM, PLP, or FLP. These four phases can be described by two classes of single-particle eigenstates [3, 9].

(a) *CAF and FM.* In the canted antiferromagnet phase, we have $\alpha = \{K, \mathbf{s}_\alpha\}$ and $\beta = \{K', \mathbf{s}_\beta\}$ with the spin polarization vectors $\mathbf{s}_{\alpha/\beta} = (\pm\sin\theta_s\cos\varphi_s, \pm\sin\theta_s\sin\varphi_s, \cos\theta_s)$. The spinor $\chi^{\mathrm{CAF}} = (\chi_{K,\mathbf{s}_\alpha}, \chi_{K,-\mathbf{s}_\alpha}, \chi_{K',\mathbf{s}_\beta}, \chi_{K',-\mathbf{s}_\beta})^T$ is related to χ^0 by the transformation $\chi^{\mathrm{CAF}} = U_{\mathrm{CAF}}^\dagger \chi^0$ with

$$U_{\mathrm{CAF}} = e^{i\sigma_y\tau_z\theta_s/2} e^{i\sigma_z\varphi_s/2}. \tag{7}$$

The ferromagnetic phase is realized for $\theta_s = 0$.

(b) *PLP and FLP.* In the partially layer polarized state we have $\alpha = \{\mathbf{m}, \uparrow\}$ and $\beta = \{\mathbf{m}, \downarrow\}$ with the isospin polarization vector $\mathbf{m} = (\sin\theta_m\cos\varphi_m, \sin\theta_m\sin\varphi_m, \cos\theta_m)$. The spinor $\chi^{\mathrm{PLP}} = (\chi_{\mathbf{m},\uparrow}, \chi_{\mathbf{m},\downarrow}, \chi_{-\mathbf{m},\uparrow}, \chi_{-\mathbf{m},\downarrow})^T$ is related to χ^0 by the transformation $\chi^{\mathrm{PLP}} = U_{\mathrm{PLP}}^\dagger \chi^0$ with

$$U_{\mathrm{PLP}} = e^{i\tau_y\theta_m/2} e^{i\tau_z\varphi_m/2}. \tag{8}$$

The fully layer polarized phase is realized for $\theta_m = 0$.

2. *Useful expressions*

In the CAF and PLP eigenbases the single-particle Hamiltonian $\mathcal{H}_s = -\epsilon_V\tau_z - \epsilon_Z\sigma_z$ is given by

$$\mathcal{H}_s^{\mathrm{CAF}} = U_{\mathrm{CAF}}^\dagger \mathcal{H}_s U_{\mathrm{CAF}} = -\epsilon_V\tau_z - \epsilon_Z(\sigma_z\cos\theta_s + \tau_z\sigma_x\sin\theta_s\cos\varphi_s + \tau_z\sigma_y\sin\theta_s\sin\varphi_s), \tag{9}$$

$$\mathcal{H}_s^{\mathrm{PLP}} = -\epsilon_V(\tau_z\cos\theta_m + \tau_x\sin\theta_m\cos\varphi_m + \tau_y\sin\theta_m\sin\varphi_m) - \epsilon_Z\sigma_z. \tag{10}$$

It is useful to evaluate certain expressions involving the anisotropic interaction $W_{\alpha\beta\gamma\delta}$ in the eigenbases that will be needed later. For the evaluation of the self energy in Sec. III C we will need the 4×4 matrix

$$N_{\alpha\beta} = \frac{1}{\pi l_B^2}\sum_\lambda f_\lambda (W_{\alpha\lambda\beta\lambda} - W_{\alpha\lambda\lambda\beta}), \tag{11}$$

where f_λ is the occupation number. In the eigenbases we find

$$N^{\rm CAF}_{\alpha\beta} = u_z\delta_{\alpha\beta}f_\alpha + 2u_\perp\left[\frac{1}{2}(1-\sigma_z) + \cos\theta_s(\sigma_z\cos\theta_s + \tau_z\sigma_x\sin\theta_s\cos\varphi_s + \tau_z\sigma_y\sin\theta_s\sin\varphi_s)\right]_{\alpha\beta}, \tag{12}$$

$$\begin{aligned} N^{\rm PLP}_{\alpha\beta} =& \delta_{\alpha\beta}(1-f_\alpha)(2u_\perp+u_z) - (u_z\cos^2\theta_m + u_\perp\sin^2\theta_m)(\tau_z)_{\alpha\beta} \\ &+ (u_\perp - u_z)\cos\theta_m\sin\theta_m(\cos\varphi_m\tau_x + \sin\varphi_m\tau_y)_{\alpha\beta}. \end{aligned} \tag{13}$$

During the evaluation of the response function in Sec. III D, W appears in the combination

$$M_{\alpha\beta;\lambda\mu} = \frac{W_{\alpha\mu\lambda\beta} - W_{\alpha\mu\beta\lambda}}{\pi l_B^2} \tag{14}$$

for all pairs (α,β) and (λ,μ) such that each pair includes one occupied and one empty state. For each pair there are eight possible combinations and thus we can write M as an 8×8 matrix. For simplicity we set $\phi_m = \phi_s = 0$ as the spectrum does not depend on the azimuthal angle.

In the CAF and FM phase we chose the basis $|\lambda\mu\rangle = \chi^{\rm CAF}_\lambda \otimes \chi^{\rm CAF}_\mu$. Thus $|\lambda\rangle$ is occupied for $\lambda = 1,3$ and empty for $\lambda = 2,4$. Writing the matrix M in the basis $(|12\rangle, |21\rangle, |34\rangle, |43\rangle, |14\rangle, |23\rangle, |32\rangle, |41\rangle)$ we find after some calculation

$$M_{\rm CAF} = -u_z + \begin{pmatrix} 0 & A_1 & 0 & 0 \\ A_1 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{pmatrix}, \tag{15}$$

where we have defined the 2×2 matrices $A_1 = 2u_\perp(-\cos^2\theta_s + \rho_x\sin^2\theta_s)$ and $A_2 = 2u_z + u_\perp - u_\perp\cos 2\theta_s - 2\rho_x u_\perp\sin^2\theta_s$ and ρ_x is a Pauli matrix.

In the PLP and FLP phase we chose the basis $|\lambda\mu\rangle = \chi^{\rm PLP}_\lambda \otimes \chi^{\rm PLP}_\mu$. In this case $|\lambda\rangle$ is occupied for $\lambda = 1,2$ and empty for $\lambda = 3,4$. We obtain M in the basis $(|13\rangle, |31\rangle, |24\rangle, |42\rangle, |14\rangle, |32\rangle, |23\rangle, |41\rangle)$ as

$$M_{\rm PLP} = 2u_\perp + u_z + \begin{pmatrix} 0 & B_1 & 0 & 0 \\ B_1 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & B_2 \end{pmatrix} \tag{16}$$

with 2×2 matrices $B_1 = (3u_\perp + u_z + (u_\perp - u_z)\cos 2\theta_m)/2 + (u_z - u_\perp)\sin^2\theta_m\rho_x$ and $B_2 = u_z\cos^2\theta_m + u_\perp\sin^2\theta_m - 2u_\perp - u_z - (u_z - u_\perp)\sin^2\theta_m\rho_x$.

B. Interaction matrix elements in orbital space

The Coulomb interaction can be decomposed into a long-range isotropic and a short-range anisotropic contribution

$$V_{abcd}(q_1,q_2,q_3,q_4) = \int d^2r_1 d^2r_2 \psi^*_{a,q_1}(\mathbf{r}_1)\psi^*_{b,q_2}(\mathbf{r}_2)\psi_{c,q_3}(\mathbf{r}_2)\psi_{d,q_4}(\mathbf{r}_1)V_{\alpha\beta\gamma\delta}(\mathbf{r}_1-\mathbf{r}_2), \tag{17}$$

where $V_{\alpha\beta\gamma\delta}(\mathbf{r}) = V_0(\mathbf{r})\delta_{\alpha\delta}\delta_{\beta\gamma} + \delta(\mathbf{r})W_{\alpha\beta\gamma\delta}$. The long-range Coulomb interaction $V_0(\mathbf{r}) = e^2/\epsilon r$ includes an effective dielectric constant, which takes into account screening by electrons, the substrate and nearby gates. The short-range anisotropic interaction is

$$W_{\alpha\beta\gamma\delta} = \pi l_B^2 \sum_{j=x,y,z} u_j (\tau_j)_{\alpha\delta}(\tau_j)_{\beta\gamma}. \tag{18}$$

After introducing relative and center of mass coordinates, $x_{1/2} = X \pm x/2$ and $y_{1/2} = Y \pm y/2$, and shifting $X \to X + (q_1 + q_2)l_B^2/2$ we find

$$V_{abcd}(q_1, q_2, q_3, q_4) = \delta_{q_1+q_2, q_3+q_4} \tilde{V}_{abcd}(q_1 - q_3, q_2 - q_3) \tag{19}$$

with

$$\begin{aligned}\tilde{V}_{abcd}(q, q') =& \frac{e^{-q'^2 l_B^2/2}}{\pi l_B^2 L_y \left(2^{n_a+n_b+n_c+n_d}\right)^{1/2}} \int dx dy\, e^{-(x+ql_B^2)^2/2l_B^2} e^{-iyq'} V_{\alpha\beta\gamma\delta}(\mathbf{r}) \int dX e^{-2X^2/l_B^2} \\ & \times H_{n_a}(X + \frac{x}{2} + (q - q')l_B^2/2) H_{n_b}(X - \frac{x}{2} - (q - q')l_B^2/2) H_{n_c}(X - \frac{x}{2} - (q + q')l_B^2/2) \\ & \times H_{n_d}(X + \frac{x}{2} + (q + q')l_B^2/2).\end{aligned} \tag{20}$$

where we have used the Landau level wavefunction

$$\langle \mathbf{r}|n, k\rangle = e^{iky} \frac{1}{\sqrt{L_y l_B 2^n \sqrt{\pi}}} H_n[(x + kl_B^2)/l_B] e^{-(x+kl_B^2)^2/2l_B^2} \tag{21}$$

with $H_0(x) = 1$ and $H_1(x) = x$. We will need the partial Fourier transforms

$$\tilde{V}^{(1)}_{abcd}(\mathbf{k}) = \sum_{q'} e^{iq'k_x l_B^2} \tilde{V}_{abcd}(k_y, q'), \tag{22}$$

$$\tilde{V}^{(2)}_{abcd}(\mathbf{k}) = \sum_{q} e^{iqk_x l_B^2} \tilde{V}_{abcd}(q, k_y). \tag{23}$$

The isotropic parts read [10]

$$\begin{aligned}\tilde{V}^{(1,i)}_{abcd}(\mathbf{k}) =& \delta_{\alpha\delta}\delta_{\beta\gamma} \sqrt{\frac{2^{n_b+n_c}}{2^{n_a+n_d}}} \int \frac{d^2r}{2\pi l_B^2} e^{-r^2/2l_B^2} V_0(\mathbf{r} - \mathbf{k} \times \hat{z} l_B^2) \left(\frac{x + iy}{l_B}\right)^{n_a - n_c} \left(\frac{x - iy}{l_B}\right)^{n_d - n_b} \\ & \times L_{n_c}^{n_a - n_c}(r^2/2l_B^2) L_{n_b}^{n_d - n_b}(r^2/2l_B^2)\end{aligned} \tag{24}$$

and

$$\begin{aligned}\tilde{V}^{(2,i)}_{abcd}(\mathbf{k}) =& \delta_{\alpha\delta}\delta_{\beta\gamma} \sqrt{\frac{2^{n_b+n_d}}{2^{n_a+n_c}}} \int \frac{d^2r}{2\pi l_B^2} e^{-i\mathbf{kr}} V_0(\mathbf{r}) l_B^{n_a - n_d + n_c - n_b} (ik_x - k_y)^{n_a - n_d} (-ik_x - k_y)^{n_c - n_b} \\ & \times L_{n_d}^{n_a - n_d}(k^2 l_B^2/2) L_{n_b}^{n_c - n_b}(k^2 l_B^2/2) e^{-k^2 l_B^2/2}.\end{aligned} \tag{25}$$

The anisotropic parts are

$$\begin{aligned}\tilde{V}^{(1,a)}_{abcd}(\mathbf{k}) =& \sqrt{\frac{2^{n_b+n_c}}{2^{n_a+n_d}}} \frac{W_{\alpha\beta\gamma\delta}}{2\pi l_B^2} l_B^{n_a - n_c + n_d - n_b} (k_y - ik_x)^{n_a - n_c} (k_y + ik_x)^{n_d - n_b} \\ & \times L_{n_c}^{n_a - n_c}(k^2 l_B^2/2) L_{n_b}^{n_d - n_b}(k^2 l_B^2/2) e^{-k^2 l_B^2/2}\end{aligned} \tag{26}$$

and

$$\begin{aligned}\tilde{V}^{(2,a)}_{abcd}(\mathbf{k}) =& \sqrt{\frac{2^{n_b+n_d}}{2^{n_a+n_c}}} \frac{W_{\alpha\beta\gamma\delta}}{2\pi l_B^2} l_B^{n_a - n_d + n_c - n_b} (ik_x - k_y)^{n_a - n_d} (-ik_x - k_y)^{n_c - n_b} \\ & \times L_{n_d}^{n_a - n_d}(k^2 l_B^2/2) L_{n_b}^{n_c - n_b}(k^2 l_B^2/2) e^{-k^2 l_B^2/2}.\end{aligned} \tag{27}$$

Here the L_n^m are the associated Laguerre polynomials, $L_0^0(x) = L_0^1(x) = 1$ and $L_1^0(x) = 1 - x$. Note that the order m of L_n^m should always be non-negative in the above expressions and indices should be interchanged if necessary. We can rewrite these results by defining the form factors

$$F_{00}(\mathbf{k}) = 1, \qquad F_{11}(\mathbf{k}) = 1 - \frac{k^2 l_B^2}{2}, \tag{28}$$

$$F_{01}(\mathbf{k}) = \frac{1}{\sqrt{2}}(k_y + ik_x) l_B, \qquad F_{10}(\mathbf{k}) = [F_{01}(\mathbf{k})]^*. \tag{29}$$

The Hartree contribution of the isotropic interaction will not be needed as we only consider intra Landau level excitations. The Fock contribution reads

$$V^{(1,i)}_{ambl}(\mathbf{r}) = \delta_{\alpha\lambda}\delta_{\beta\mu}e^2 \int \frac{d^2r}{2\pi l_B^2} \frac{\exp(-r^2/2l_B^2)}{|\mathbf{r} - l_B^2\mathbf{k}\times\hat{z}|} F_{ab}(\hat{z}\times\mathbf{r}/l_B^2)[F_{lm}(\hat{z}\times\mathbf{r}/l_B^2)]^*. \tag{30}$$

The integrals are elementary and the solutions can expressed in terms of Bessel functions. The anisotropic interaction will appear as a difference of Fock and Hartree terms

$$\tilde{V}^{(1,a)}_{ambl}(\mathbf{k}) - \tilde{V}^{(2,a)}_{amlb}(-\mathbf{k}) = \frac{1}{2\pi l_B^2}(W_{\alpha\mu\lambda\beta} - W_{\alpha\mu\beta\lambda})e^{-k^2 l_B^2/2}F_{ab}(\mathbf{k})[F_{lm}(\mathbf{k})]^*. \tag{31}$$

C. Self energy

The single-particle Green function is given by

$$G_a(\omega) = \frac{1}{\omega - (\Sigma + \mathcal{H}_s)_a + i\epsilon_a} \tag{32}$$

with $\epsilon_a < 0$ ($\epsilon_a > 0$) if the state a is occupied (empty). Here $\mathcal{H}_s$ is the single particle Hamiltonian and Σ the self energy. We work in an eigenbasis such that the Green function is diagonal. Thus we require the sum $\Sigma + \mathcal{H}_s$ to be diagonal, although the self energy Σ may have offdiagonal elements.

In order to calculate the self energy we need to sum the Hartree and Fock contributions from anisotropic short-range interactions and isotropic long-range interactions. The long-range Coulomb interaction has only a Fock contribution, since the Hartree part is canceled by the ionic background. We find

$$\begin{aligned}\Sigma_{ab}(\omega, q) =& i\sum_c \int \frac{d\omega'}{2\pi} \sum_{q'} G_a(\omega+\omega')e^{i(\omega+\omega')\delta}V^i_{acbc}(q,q',q,q') \\ &+ i\sum_c \int \frac{d\omega'}{2\pi} \sum_{q'} G_c(\omega+\omega')e^{i(\omega+\omega')\delta}V^a_{acbc}(q,q',q,q') \\ &- i\sum_c \int \frac{d\omega'}{2\pi} \sum_{q'} G_c(\omega+\omega')e^{i(\omega+\omega')\delta}V^a_{accb}(q,q',q',q).\end{aligned} \tag{33}$$

It is easy to see that the self energy does not depend on energy or momentum and we obtain

$$\Sigma_{ab} = -\sum_c f_c \left[\tilde{V}^{(i,1)}_{acbc}(0) + \tilde{V}^{(a,1)}_{acbc}(0) - \tilde{V}^{(a,2)}_{accb}(0)\right], \tag{34}$$

where f_c is the occupation number. At this point we must proceed with care. The sum over intermediate states c should run over all occupied Landau levels with $n_c \leq -2$ in addition to the partially filled zero-energy Landau level $n = 0, 1$. The contribution of the fully occupied Landau levels induces a "Lamb shift" between the $n = 0$ and $n = 1$ state [8]. For the anisotropic contribution $\sum_{n_c\leq -2,\gamma} \tilde{V}^{(a,1)}_{acbc}(0) - \tilde{V}^{(a,2)}_{accb}(0)$ yields just a constant energy shift that we can safely ignore. The isotropic part, however, depends on the orbital index n_a. Following Ref. [8] we can rewrite the sum over completely filled Landau levels as

$$\sum_{n_c\leq 2,\gamma} f_c \tilde{V}^{(i,1)}_{acbc}(0) = -\frac{1}{2}\delta_{\alpha\beta} \sum_{n_c=0,1} \tilde{V}^{(i,1)}_{acbc}(0). \tag{35}$$

Using Eqs. (12), (13), (31), and (30) and introducing $U_c = \sqrt{\pi/2}e^2/\epsilon l_B$ we find the self energy in the eigenbases introduced in Sec. III A 1

$$\begin{aligned}\Sigma^{\mathrm{CAF}}_{ab} =& \delta_{ab}(1/2 - f_\alpha)U_c\left(\frac{3}{2} - n_a\frac{1}{4}\right) - \delta_{ab}f_\alpha u_z \\ &- \delta_{n_a n_b}\left[u_\perp(1-\sigma_z) + 2u_\perp\cos\theta_s(\sigma_z\cos\theta_s + \tau_z\sigma_x\sin\theta_s\cos\varphi_s + \tau_z\sigma_y\sin\theta_s\sin\varphi_s)\right]_{\alpha\beta}\end{aligned} \tag{36}$$

and

$$\Sigma^{\rm PLP}_{ab} = \delta_{ab}(1/2-f_\alpha)U_c\left(\frac{3}{2}-n_a\frac{1}{4}\right)-\delta_{ab}(1-f_\alpha)(2u_\perp+u_z) + \delta_{n_a n_b}[(u_z\cos^2\theta_m+u_\perp\sin^2\theta_m)\tau_z+(u_z-u_\perp)\cos\theta_m\sin\theta_m(\cos\varphi_m\tau_x+\sin\varphi_m\tau_y)]_{\alpha\beta}. \quad (37)$$

Note that the self energies are diagonal in orbital space. The single-particle Hamiltonian is given by Eqs. (9) and (10). In the CAF basis, we find that $\Sigma+\mathcal{H}_s$ is diagonal if $\cos\theta_s=-\epsilon_Z/2u_\perp$ (CAF) or $\theta_s=0$ (FM) whereas in the PLP basis we find $\cos\theta_m=\epsilon_V/(u_z-u_\perp)$ (PLP) or $\theta_m=0$ (FLP) in agreement with Ref. [9]

$$(\Sigma+\mathcal{H}_s)^{\rm CAF} = \delta_{ab}(1/2-f_\alpha)U_c\left(\frac{3}{2}-n_a\frac{1}{4}\right)-\delta_{ab}(f_\alpha u_z+u_\perp) + [-\epsilon_V\tau_z+u_\perp\sigma_z-\sigma_z\cos\theta_s(2u_\perp\cos\theta_s+\epsilon_Z)]_{\alpha\beta}\,, \quad (38)$$

$$(\Sigma+\mathcal{H}_s)^{\rm PLP} = \delta_{ab}(1/2-f_\alpha)U_c\left(\frac{3}{2}-n_a\frac{1}{4}\right)+\delta_{ab}[f_\alpha(2u_\perp+u_z)-(u_\perp+u_z)] + [-\epsilon_Z\sigma_z-\tau_z\cos\theta_m[\epsilon_V-(u_z-u_\perp)\cos\theta_m]]_{\alpha\beta}\,. \quad (39)$$

D. Response function

We now calculate the response function of operator θ_A

$$\chi_A(\mathbf{k},\omega)=\sum_{ab}\sum_{q_1q_2}\left\langle q_1a\right|e^{i\mathbf{kr}}\theta_A^\dagger\left|q_2b\right\rangle\int\frac{d\omega'}{2\pi}G_a(\omega+\omega')G_b(\omega')\Gamma_{A,ab}(q_1,q_2;\mathbf{k},\omega), \quad (40)$$

closely following [10]. The single-particle Green function is given by Eq. (32) and the vertex part can be determined from the diagrams shown in Fig. 1(b) of the main text as

$$\Gamma_{A,ab}(q_1,q_2;\mathbf{k},\omega)=\left\langle q_2b\right|e^{-i\mathbf{kr}}\theta_A\left|q_1a\right\rangle + i\sum_{lm}\sum_{q_3q_4}V_{ambl}(q_1,q_4,q_2,q_3)\int\frac{d\omega'}{2\pi}G_l(\omega+\omega')G_m(\omega')\Gamma_{A,lm}(q_3,q_4,\mathbf{k},\omega) - i\sum_{lm}\sum_{q_3q_4}V_{amlb}(q_1,q_4,q_3,q_2)\int\frac{d\omega'}{2\pi}G_l(\omega+\omega')G_m(\omega')\Gamma_{A,lm}(q_3,q_4,\mathbf{k},\omega), \quad (41)$$

where $V_{abcd}(q_1,q_2,q_3,q_4)$ is given by Eq. (17). We change variables to total and relative momentum $Q=(q_1+q_2)/2$ and $q=q_1-q_2$ and using Eq. (19) we write

$$\Gamma_{A,ab}(Q,q;\mathbf{k},\omega)=\delta_{q,k_y}e^{(q_1+q_2+ik_x)^2l_B^2/4-q_1^2l_B^2/2-q_2^2l_B^2/2}(\theta_A)_{ba} + i\sum_{lm}\sum_{q_4}\tilde{V}_{ambl}(q,q_4-Q)\int\frac{d\omega'}{2\pi}G_l(\omega+\omega')G_m(\omega')\Gamma_{A,lm}(q_4+q/2,q_4-q/2,\mathbf{k},\omega) - i\sum_{lm}\sum_{q_4}\tilde{V}_{amlb}(Q-q_4,-q)\int\frac{d\omega'}{2\pi}G_l(\omega+\omega')G_m(\omega')\Gamma_{A,lm}(q_4+q/2,q_4-q/2,\mathbf{k},\omega). \quad (42)$$

Introducing the partial Fourier transform

$$\tilde{\Gamma}_{A,ab}(p,q;\mathbf{k},\omega)=\sum_Q e^{ipQl_B^2}\Gamma_{A,ab}(Q,q;\mathbf{k},\omega) \quad (43)$$

we obtain

$$\tilde{\Gamma}_{A,ab}(p,q;\mathbf{k},\omega)=\delta_{q,k_y}\delta_{p,-k_x}e^{-(k_x^2+k_y^2)l_B^2/4}(\theta_A)_{ba}+i\sum_{lm}\tilde{V}^{(1)}_{ambl}(q,-p)\int\frac{d\omega'}{2\pi}G_l(\omega+\omega')G_m(\omega')\tilde{\Gamma}_{A,lm}(p,q,\mathbf{k},\omega) \quad (44)$$

$$-i\sum_{lm}\tilde{V}^{(2)}_{amlb}(p,-q)\int\frac{d\omega'}{2\pi}G_l(\omega+\omega')G_m(\omega')\tilde{\Gamma}_{A,lm}(p,q,\mathbf{k},\omega), \quad (45)$$

where $\tilde{V}^{(1/2)}$ have been defined in Eqs. (22) and (23). When introducing

$$\Gamma'_{A,ab}(\mathbf{k},\omega) = \delta_{q,k_y}\delta_{p,-k_x}\tilde{\Gamma}_{A,ab}(p,q,\mathbf{k},\omega), \tag{46}$$

we can rewrite the matrix equation for the vertex part as

$$\Gamma'_{A,ab}(\mathbf{k},\omega) = e^{-(k_x^2+k_y^2)l_B^2/4}(\theta_A)_{ba} + \sum_{lm}\left[\tilde{V}^{(1)}_{ambl}(\mathbf{k}) - \tilde{V}^{(2)}_{amlb}(-\mathbf{k})\right] D_{lm}(\omega)\Gamma'_{A,lm}(\mathbf{k},\omega), \tag{47}$$

where D_{lm} is the two-particle propagator

$$D_{lm}(\omega) = i\int \frac{d\omega'}{2\pi} G_l(\omega+\omega')G_m(\omega') \tag{48}$$

$$= \frac{f_l(1-f_m)}{\omega - [(\Sigma+\mathcal{H}_s)_l - (\Sigma+\mathcal{H}_s)_m] - i\epsilon} - \frac{f_m(1-f_l)}{\omega - [(\Sigma+\mathcal{H}_s)_l - (\Sigma+\mathcal{H}_s)_m] + i\epsilon}. \tag{49}$$

Using Eqs. (38) and (39) we obtain the exchange energy $E_{\mathrm{xc},lm} = (\Sigma+\mathcal{H}_s)_l - (\Sigma+\mathcal{H}_s)_m$ for l occupied and m empty as

$$E^{\mathrm{CAF}}_{\mathrm{xc},lm} = -U_c\left[\frac{3}{2} - \frac{1}{8}(n_l+n_m)\right] + 2u_\perp - u_z - \epsilon_V[(\tau_z)_{ll} - (\tau_z)_{mm}] - 2\cos\theta_s(2u_\perp\cos\theta_s + \epsilon_Z), \tag{50}$$

$$E^{\mathrm{PLP}}_{\mathrm{xc},lm} = -U_c\left[\frac{3}{2} - \frac{1}{8}(n_l+n_m)\right] + 2u_\perp + u_z - \epsilon_Z[(\sigma_z)_{ll} - (\sigma_z)_{mm}] - 2\cos\theta_m[\epsilon_V - (u_z-u_\perp)\cos\theta_m] \tag{51}$$

and thus

$$D_{lm}(\omega) = \frac{f_l(1-f_m)}{\omega - E_{\mathrm{xc},lm} - i\epsilon} - \frac{f_m(1-f_l)}{\omega + E_{\mathrm{xc},ml} + i\epsilon}. \tag{52}$$

The equation for the vertex part can be rewritten as

$$\sum_{lm}\left[\delta_{al}\delta_{bm}(D_{lm})^{-1} - \tilde{V}^{(1)}_{ambl}(\mathbf{k}) + \tilde{V}^{(2)}_{amlb}(-\mathbf{k})\right]\Pi_{Alm}(\mathbf{k},\omega) = e^{-(k_x^2+k_y^2)l_B^2/4}(\theta_A)_{ba}, \tag{53}$$

where we have introduced

$$\Pi_{Alm}(\mathbf{k},\omega) = D_{lm}(\omega)\Gamma'_{A,lm}(\mathbf{k},\omega). \tag{54}$$

This equation must be satisfied for all pairs (a,b) such that one state is occupied and one is empty. The excitation spectrum is given by poles $\omega(k)$ of the response function which are solutions of the equation

$$\sum_{lm}\left[\delta_{al}\delta_{bm}(D_{lm})^{-1} - \tilde{V}^{(1)}_{ambl}(\mathbf{k}) + \tilde{V}^{(2)}_{amlb}(-\mathbf{k})\right]B_{lm} = 0, \tag{55}$$

where the summation runs over particle-hole pairs (l,m) such that one state is occupied and one is empty, since Π_{lm} vanishes otherwise. Defining $\tilde{V}_{ablm} = \tilde{V}^{(1)}_{ambl} + \tilde{V}^{(2)}_{amlb}$ we arrive at Eq. (2) of the main text

$$\sum_{lm}\left[\delta_{al}\delta_{bm}[(f_l-f_m)\omega - E_{\mathrm{xc},lm}] - \tilde{V}^{(1)}_{ambl}(\mathbf{k}) + \tilde{V}^{(2)}_{amlb}(-\mathbf{k})\right]B_{lm} = 0. \tag{56}$$

Note that the Hartree part of the long-range interaction $\tilde{V}^{(2,i)}_{amlb}(\mathbf{k}) \propto \delta_{\alpha\beta}\delta_{\lambda\mu}$ vanishes, as we only consider intra-Landau level excitations for which $\alpha \neq \beta$ and $\lambda \neq \mu$.

E. Low-energy excitation spectrum

We now derive the spectrum at low-energies, $\omega \ll U_c$, and long-wavelengths, $kl_B \ll 1$, in the case that long-range Coulomb interaction dominates all other scales $U_c \gg u_z, u_\perp, \epsilon_V, \epsilon_Z$. Equation (56) can be written as a matrix in a 32-dimensional space with four components in orbital space and eight components in SU(4) spin space of particle hole pairs (a,b).

We start by diagonalizing the eigenvalue problem in four dimensional orbital space within the basis $|n_a n_b\rangle = [|00\rangle, |01\rangle, |10\rangle, |11\rangle]_{ab}$. The Hartree and Fock contributions of the anisotropic interaction are given by Eq. (31) as

$$\tilde{V}^{(1,a)}_{ambl}(\mathbf{k}) - \tilde{V}^{(2,a)}_{amlb}(-\mathbf{k}) = \frac{1}{2\pi l_B^2}(W_{\alpha\mu\lambda\beta} - W_{\alpha\mu\beta\lambda})e^{-k^2 l_B^2/2} F_{n_a,n_b}(\mathbf{k}) F_{n_l,n_m}(\mathbf{k}). \tag{57}$$

In orbital space this expression has only one nonzero eigenvalue with the corresponding eigenvector $\nu(\mathbf{k}) = [F_{00}(\mathbf{k}), F_{01}(\mathbf{k}), F_{10}(\mathbf{k}), F_{11}(\mathbf{k})]$. Thus the anisotropic contribution vanishes outside the subspace spanned by the vector $\nu(\mathbf{k})$.

We now turn to the remaining part $\delta_{ac}\delta_{bd}E_{\mathrm{xc},cd} + \tilde{V}^{(1,i)}_{ambl}(\mathbf{k})$. At $k = 0$ and only keeping terms of order U_c, this matrix is an identity matrix in SU(4) spin space and has eigenvalues U_c, $7U_c/8$, $7U_c/8$, and 0 in orbital space. Incidentally the zero-eigenvalue subspace is spanned by the vector $\nu(\mathbf{k} = 0) = (1, 0, 0, 1)$. Hence, the low-energy excitations at $k = 0$ comprise superpositions of electrons and holes in the same orbital state $|00\rangle + |11\rangle$.

The vector $\nu(\mathbf{k})$ therefore defines the low-energy subspace near $k = 0$ with four modes and their particle-hole symmetric counterparts. We project onto this subspace using a Schrieffer-Wolff transformation and obtain an effective 8×8 eigenvalue problem in SU(4) spin space expanded to second order in kl_B

$$\sum_{lm}\left[\delta_{al}\delta_{bm}[(f_l - f_m)\omega + U_c' k^2 l_B^2 - \Delta_{lm} - \frac{1 - k^2 l_B^2/2}{2\pi l_B^2}(W_{\alpha\mu\lambda\beta} - W_{\alpha\mu\beta\lambda})\right] B_{lm} = 0 \tag{58}$$

with $U_c' = N_0 U_c$, $N_0 = 89/224$, and

$$\Delta^{\mathrm{CAF}}_{lm} = 2u_\perp - u_z - (f_l - f_m)\epsilon_V[(\tau_z)_{ll} - (\tau_z)_{mm}] - 2\cos\theta_s(2u_\perp \cos\theta_s + \epsilon_Z), \tag{59}$$

$$\Delta^{\mathrm{PLP}}_{lm} = 2u_\perp + u_z - (f_l - f_m)\epsilon_Z[(\sigma_z)_{ll} - (\sigma_z)_{mm}] - 2\cos\theta_m[\epsilon_V - (u_z - u_\perp)\cos\theta_m]. \tag{60}$$

The anisotropic contribution $W_{\alpha\mu\lambda\beta} - W_{\alpha\mu\beta\lambda}$ is given by Eqs. (14)–(16). Equation (58) contains the solutions in all four phases depending on the choice of basis and the definitions of $\theta_{s/m}$. The 8×8 matrix written in the basis used in Eq. (15) and (16) decomposes into a 4×4 block and two 2×2 blocks, which can be readily diagonalized. The solutions in the CAF and PLP phase are given as Eqs. (3)–(7) of the main text.

In the FM phase we can set $\theta_s = 0$ and find

$$\omega_1 = 2\epsilon_Z + k^2 l_B^2 U_c', \tag{61}$$

$$\omega_2 = 2(\epsilon_Z + 2u_\perp) + k^2 l_B^2 U_c', \tag{62}$$

$$\omega_{3,4} = 2(\epsilon_Z + u_z + u_\perp \pm \epsilon_V) + k^2 l_B^2 U_c'. \tag{63}$$

We see that the gap of all four low-energy modes increases as the Zeeman field increases. Gap closings occur for $\epsilon_V = -2u \perp$ at the CAF-FM transitions and for $\epsilon_V - \epsilon_Z = u_z + u_\perp$ at the FM-FLP transition [9].

In the FLP phase we set $\theta_p = 0$ and obtain

$$\omega_{1/2} = 2(\epsilon_V - u_z \pm u_\perp) + k^2 l_B^2 U_c', \tag{64}$$

$$\omega_{3/4} = 2(\epsilon_V - u_z - u_\perp \pm \epsilon_Z) + k^2 l_B^2 U_c'. \tag{65}$$

As in the FM phase all modes have a gap, which increases with ϵ_V. A gap closing occurs at the PLP-FLP transition, where $\epsilon_V = u_z - u_\perp$, and the FM-FLP transition, where $\epsilon_V - \epsilon_Z = u_z + u_\perp$ [9].

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