Table 1. Quantiles, Tilt Factors, and Estimated Powers for y and $y^{1/3}$, y Exponentially Distributed

q	y_q	y_{1-q}	$\tau(y, q)$	$p(y)$	$\tau(y^{1/3}, q)$	$p(y^{1/3})$
$\frac{1}{4}$	.415	2.0	1.710	.38	1.023	.91
$\frac{1}{8}$	.193	3.0	2.477	.49	1.047	.89
$\frac{1}{16}$	.093	4.0	3.308	.57	1.074	.87
$\frac{1}{32}$	.046	5.0	4.192	.64	1.105	.85
$\frac{1}{64}$	.023	6.0	5.116	.69	1.140	.83

Emerson and Stoto may expose problems, such plots are often inconclusive in small samples. If the method is to be used, then diagnostic statistics of the transformed data should be calculated and the process iterated; the method should work better for p near 1.

In practice, when transformation parameters are estimated, they are usually rounded to the nearest "simple fraction" before further analyses are performed. A simple procedure is (a) choose a grid of possible powers p, (b) calculate for each p the likelihood and/or the tilt factor for a few quantiles, and (c) choose a transformation on the basis of the results. The Emerson-Stoto procedure must iterate and is thus computationally no easier than this. Further, it has the disadvantage that, unless the iterations are repeated until convergence (when the Emerson-Stoto estimator coincides with that of Hinkley), there is no indication of how far the estimate is from the root of (1). If iterations are repeated until convergence, then an efficient numerical root-finder may as well be used to solve (1) directly or to calculate the Box-Cox estimate.

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Rejoinder

JOHN D. EMERSON and MICHAEL A. STOTO*

Our transformation plot for symmetry (Emerson and Stoto 1982) emerges from the first three terms of the expansion of $y_q{}^p$ and $y_{1-q}{}^p$ in powers of $(y_q - M)$ and $(y_{1-q} - M)$, respectively. Cameron argues convincingly that the method "cannot be used automatically," and we fully agree. The second-order approximation is not a good one for all data sets, especially when a suitable index p is far from 1 and either y_q or y_{1-q} is very far from M.

Coincidentally, on the day Cameron's manuscript reached us, we also learned of a data set that exhibits behavior much like that of his exponential example. Thus, while Cameron's caveats may be mathematically oriented, data analysts need to bear them in mind and proceed with their eyes open. We illustrate below some ways to do this, and we further indicate how the transformation plot can be used effectively in exploring y-versus-x data.

We first applied the transformation plot to Cameron's exponential example. The plot showed strong curvature, but the various estimates of p suggested a square-root transformation as a first step. After taking square roots and looking at a second transformation plot, we obtained an unambiguous indication of a need to further transform the y data; the estimates of p were .61, .60, .59, .58, and .57. We chose .60 and reexpressed the raw data with power $p = .6 \times .5 = .3$. Thus the second application of the transformation plot, carried out in a region where the second-order series approximation is much more suitable, brought us to the "right" reexpression.

Analyses of Cameron's lognormal example and of our empirical data were similar, but they required one further iteration of the process. Once in the right neighborhood, the plot worked as it was supposed to and no longer displayed the tell-tale pronounced curvature.

Cameron's analysis and our own further investigations lead us to the following recommendations for exploratory analyses of highly skewed data sets:

1. The points in the transformation plot for the raw data corresponding to the first quantiles ($q = \frac{1}{4}$ and perhaps $q = \frac{1}{8}$) are more reliable for indicating an initial reexpression.
2. As with many exploratory methods (for example, median polish or the resistant line), iteration of the transformation plot may be helpful. In practice, one can often

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March 1984, Volume 79, Number 385
Theory and Methods Section

make an initial guess about p, and then use the transformation plot to help find a more precise estimate.

3. Repeated use of the transformation plot, coupled with inspection of the midsummaries of the reexpressed data, can quickly lead to a very effective power transformation to symmetry when one is available. Two steps often suffice when a first step falls short. The calculations for the transformation plot can be performed easily with a few lines of code, even on a programmable calculator.

4. When inferences are to follow transformation, we recommend that the Box-Cox (Box and Cox 1964) techniques be used in combination with Hinkley's more robust approach (1975). The former method permits confidence interval construction for the estimate of a power and, at least in some instances, it can be followed by inferential procedures in the transformed scale; see the recent articles by Bickel and Doksum (1981), Box and Cox (1982), and Doksum and Wong (1983). However, the Box-Cox techniques assume normal error distributions, and may not be robust to departures from this assumption. The method of Hinkley is based on sample quantiles and thus gains robustness; its implementation relies on a numerical root-finder.

The issues raised by Cameron can of course also be raised with regard to our transformation plot for straightening y-versus-x data. We carried out a small empirical investigation along the lines of that outlined above, and our findings were similar. Our proposed method can produce the right power for data sets with moderate curvature, but further iteration of the process may be needed for strongly curved data. For example, we used 30 equally spaced x-values and explored $y = x^2$, x^3, and x^4. For $y = x^2$, the method worked well in one step; for $y = x^3$ it also gave the correct power ($\frac{1}{3}$) in one step, but the plot showed departure from linearity; for $y = x^4$, we needed two iterations to produce the desired power transformation.

As Cameron's article suggests, there is no substitute for an alert and intelligent examination of the results of any step of an exploratory analysis of data. Our experience suggests that transformation plots can be convenient and useful tools for guiding the exploration.

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