Measure Theory

Definition 0.1 (inverse function):
Let $f : X \longrightarrow Y$ be a function. The **inverse function** $f^{-1} : Y \longrightarrow X$ is a function such that $f^{-1}(y \in Y) = x \in X$ if $f(x) = y$.

Definition 1: inverse function

Definition 0.2 (preimage):
Let $f : X \longrightarrow Y$ be a function. Let $E \subseteq Y$. The **preimage** is the set $f^{-1}(E) = \{x \in X \mid f(x) \in E\}$.

Definition 2: preimage

Definition 0.3 (σ-algebra):
Let X be a set. $\Sigma \subseteq 2^X$ is said a **sigma algebra of X** iff.:

1. $X \in \Sigma$
2. $E \in \Sigma \Longrightarrow X \setminus E \in \Sigma$. close under complement.
3. $\{A_n \in \Sigma\}_{n=1}^{\infty} \Longrightarrow \bigcup_{i=1}^{\infty} A_i \in \Sigma$. close under infinite unions.

Definition 3: σ-algebra

Definition 0.4 (generate σ-algebra):
Let X be a set and $G \subseteq 2^X$. The σ**-algebra generated by** G, denoted $\sigma_X(G)$, is the smallest σ-algebra such that:

1. $G \subseteq \sigma_X(G)$.
2. $\forall \Sigma$ σ-algebra $: G \subseteq \Sigma \Longrightarrow \sigma_X(G) \subseteq \Sigma$. Every other σ-algebra that contains G contains also the generated one, $\sigma_X(G)$.

Definition 4: generate σ-algebra

Definition 0.5 (σ-algebra product):
Let Σ_1 and Σ_2 be σ-algebras on X_1 and X_2 respectively. The **product** σ**-algebra** denoted $\Sigma_1 \otimes \Sigma_2$ is defined as $\sigma_{X_1 \times X_2}(\{S_1 \times S_2 \mid S_1 \in \Sigma_1, S_2 \in \Sigma_2\})$

Definition 5: σ-algebra product

Definition 0.6 (measurable space):
(X, Σ) is said **measurable** iff. Σ is a sigma-algebra of X.

Definition 6: measurable space

Definition 0.7 (measure):
Given (X, Σ) measurable space. $\mu : \Sigma \longrightarrow \mathbb{R} \cup \{+\infty, -\infty\}$ is said a **measure** iff.

1. $E \in \Sigma \Longrightarrow \mu(E) \geq 0$. positive.
2. $\{E_n \in \Sigma\}_{n=1}^{\infty}$ such that $E_i \cap E_j$ for $i \neq j \Longrightarrow \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$. The measure of disjoint sets is is the sum of the measures of each set.
3. $\mu(\emptyset) = 0$.

Definition 7: measure

Definition 0.8 (measure space):
(X, Σ, μ) is said a **measure space** iff. (X, Σ) is a sigma algebra and μ is a measure of (X, Σ).

Definition 8: measure space

Definition 0.9 (measurable function):
Let (X_1, Σ_1) and (X_2, Σ_2) be a measurable spaces. $f : X_1 \longrightarrow X_2$ is said a **measurable function** iff. $\forall E \in \Sigma_2 : f^{-1}(E) \in \Sigma_1$. The preimage of each measurable set is again measurable.

Definition 9: measurable function

Definition 0.10 (pushforward):
Let (X_1, Σ_1, μ) be a measure space. Let (X_2, Σ_2) be a measurable space. Let $f : X_1 \longrightarrow X_2$ be a measurable function. The **pushforwad of μ under** f is the mapping $f_{\#}\mu : \Sigma_2 \longrightarrow \mathbb{R}_{\geq 0}$ defined as:

$$\forall E \in \Sigma_2 : f_{\#}\mu(E) = \mu\left(f^{-1}(E)\right)$$

Definition 10: pushforward

Proposition 0.1 (pushforward of a measure is a measure):
Let (X_1, Σ_1, μ) be a measure space. Let (X_2, Σ_2) be a measurable space. Let $f : X_1 \longrightarrow X_2$ be a measurable function. Then $\left(X_2, \Sigma_2, f_{\#}\mu\right)$ is a measure space.

Proposition 11: pushforward of a measure is a measure

Proof 0.1 (of Proposition 11):

To prove that statement, we need to prove only the axioms of a measure.

1. Let $E \in \Sigma_2$, we need to show that $f_{\#}\mu(E) \geq 0$. This is trivial by definition of pushforward and measure.
2. Let $[E_n \in \Sigma_2]_{n=1}^{\infty}$ be a sequence of pairwise disjoint sets. We need to show that: $f_{\#}\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} f_{\#}\mu(E_n)$.

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\begin{aligned}
f_{\#}\mu\left(\bigcup_{n=1}^{\infty} E_n\right) &= \mu\left(f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right)\right) \text{ definition of pushforward} \\
&= \mu\left(\bigcup_{n=1}^{\infty} f^{-1}(E_n)\right) \\
&= \sum_{n=1}^{\infty} \mu(f^{-1}(E_n)) \text{ definition of measure} \\
&= \sum_{n=1}^{\infty} f_{\#}\mu(E_n) \text{ definition of pushforward}
\end{aligned}
$$

3. We need to show that $\exists E \in \Sigma_1$ such that $f_{\#}(E) \geq 0$. Let $E' \in \Sigma_1$ such that $\mu(E') \geq 0$ (such E' exists by defintion of measure). Then, $f(E')$ is a set that meets the requirements, that is

$$f_{\#}(f(E')) = \mu(f^{-1}(f(E'))) = \mu(E') \geq 0$$

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Proof 12: of Proposition 11

Example 0.1 (pushforward example):

Consider the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu(E) = |E|)$. Consider the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Consider the measurable function $f : \mathbb{N} \longrightarrow \mathbb{R}$ such that f(x) = x. Consider pushforward $f_{\#}\mu : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$. Then $f_{\#}\mu$ is a measure for the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ since:

1. $f_{\#}\mu(E \in \mathcal{B}(\mathbb{R})) = |\{n \in \mathbb{N} \mid n \in E\}| \geq 0$.
2. Let $\{E_n\}_{n=1}^{\infty}$ pairwise disjoint, then $f_{\#}\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right)\right) = \mu\left(\bigcup_{n=1}^{\infty} f^{-1}(E_n)\right) = \sum_{n=1}^{\infty} \mu(f^{-1}(E_n)) = \sum_{n=1}^{\infty} f_{\#}\mu(E_n)$.
3. $f_{\#}\mu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$

Example 13: pushforward example

TODO: add borel sigma algebra