Lebesgue Measure

Definition 0.1 (pre-measure):

Let (X, Σ) such that $\emptyset \in S$. Let $\mu : S \longrightarrow R_{\geq 0} + \{+\infty\}$. μ is said a **pre-measure** iff.

1. $$\mu(\emptyset) = 0.$$
2. Given a collection of pairwise disjoint sets $\{A_n \in S\}_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} A_n \in S \Longrightarrow$
$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n).$$
3. $$\forall A \in S : \mu(A) \geq 0.$$

Definition 1: pre-measure

A pre-measure is a precursor of a full-fledge measure. The main difference is that a measure is defined on sigma algebras, meanwhile the pre-measure is defined on a simple collection of subsets. Further, given that this collection is not necessarily closed under unions as a sigma algebra does, we also need to check that, in the second requirement, the union of A_n is indeed contained in the collection.

Definition 0.2 (Lebesgue pre-measure):

The **Lebesgue pre-measure** is a mapping $\lambda^n : \mathcal{J}_h^n \longrightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$ ($\mathcal{J}_h^n$ denotes the set half open rectangle) such that $\lambda^n\left(\bigtimes_{i=1}^n [a_i, b_i)\right) = \prod_{i=1}^n (b_i - a_i)$ for $a_i, b_i \in \mathbb{R}$ and $a_i \leq b_i$.

Definition 2: Lebesgue pre-measure

Proposition 0.1:

The Lebesgue pre-measure is a pre-measure.

Proof 0.1 (of Proposition Proposition 3):

1. $$\lambda^n(\emptyset) = \lambda^n\left(\bigtimes_{i=1}^n [a_i, a_i)\right) = \prod_{i=1}^n (a_i - a_i) = 0$$

2. Let $I = \bigtimes_{i=1}^n [a_i, b_i)$ and $I' = \bigtimes_{i=1}^n [a_i', b_i')$ be disjoint half open rectangles. The $I \cup I'$ belongs to $\mathcal{I}_h^n$ if we can stitch one to the other. This can only happen if there is an i such that:
   1. $j = i \implies b_j = a_j'$.
   2. $j \neq i \implies b_j = b_j'$.
   3. $j \neq i \implies a_j = a_j'$.

   This can be intuitively visualized in Figure 1 where two 2-dimensional half open rectangles met at one side. The only difference between the rectangles is that one is shifted along a single dimension, in such a way that they met at the open and close edges.

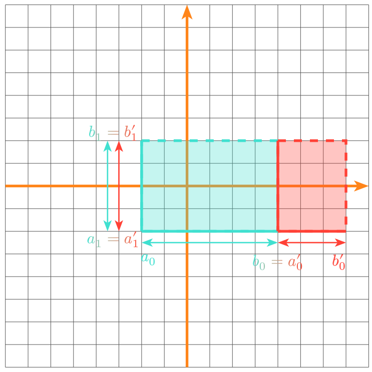

Figure 1: Two half open rectangles that can be stitched together.

In this situation we have that:

$$\begin{aligned}
\lambda^n(I) + \lambda^n(I') &= \prod_{j=1}^n (b_j - a_j) + \prod (j=1)^n (b_j', a_j') && \text{Lebesgue pre-measure definition} \\
&= ((b_i - a_i) + (b_i' - a_i')) \prod_{\substack{j=1 \\ j \neq i}}^n b_j - a_j && \text{factoring out } \prod_{\substack{j=1 \\ j \neq i}}^n b_j - a_j \\
&= ((b_i - a_i')) \prod_{\substack{j=1 \\ j \neq i}}^n b_j - a_j && \text{stitching half open rectangles together} \\
&= \lambda^n(I \cup I')
\end{aligned}$$

Thus it is verified that λ^n is finitely additive.

3. The $\forall E \in \mathcal{I}_h^n : \lambda^n(E) \geq 0$ since the product of positive terms is positive.

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Proof 4: of Proposition Proposition 3