Measure Theory

Definition 0.1 (half open rectangle):
Let $a_0, b_0, \dots, a_n, b_n \in \mathbb{R}$. The set $\bigtimes_{i=0}^{n} [a_i, b_i)$ is called an n-dimensional **half open rectangle**. The collection of all n-dimensional **half-open-rectangles** is denoted with $\mathcal{J}_h^n$.

Definition 1: half open rectangle

Definition 0.2 (restriction):
Let $f : X \longrightarrow Y$. Let $X' \subseteq X$. Let Y' such that $f(X') \subseteq Y' \subseteq Y$. A **restriction of f over** $X' \times Y'$, denoted $f|_{X' \times Y'}$ is a function $X' \longrightarrow Y'$ such that $f|_{X \times Y} = \{(x, f(x)) \mid x \in X, f(x) \in Y\}$

Definition 2: restriction

Example 0.1 (restriction):
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = x^2$ power operator over the real numbers. Now, consider $g : \mathbb{N} \longrightarrow \mathbb{N}$ such that $g(x) = x^2$ power operator over the natural number only. Then g is a restriction of f.

1. $\mathbb{N} \subseteq \mathbb{R}$.
2. $f(\mathbb{N}) \subseteq \mathbb{N} \subseteq \mathbb{R}$.
3. $\{(x, g(x))|\ x \in \mathbb{N}, y \in \mathbb{N})\} \subseteq \{(x, f(x)| x \in \mathbb{R}, y \in \mathbb{R})\}$

Example 3: restriction

Definition 0.3 (inverse function):
Let $f : X \longrightarrow Y$ be a function. The **inverse function** $f^{-1} : Y \longrightarrow X$ is a function such that $f^{-1}(y \in Y) = x \in X$ if $f(x) = y$.

Definition 4: inverse function

Definition 0.4 (preimage):
Let $f : X \longrightarrow Y$ be a function. Let $E \subseteq Y$. The **preimage** is the set $f^{-1}(E) = \{x \in X \mid f(x) \in E\}$.

Definition 5: preimage

Definition 0.5 (σ-algebra):
Let X be a set. $\Sigma \subseteq 2^X$ is said a **sigma algebra of X** iff.:

1. $X \in \Sigma$
2. $E \in \Sigma \Longrightarrow X \setminus E \in \Sigma$. close under complement.
3. $\{A_n \in \Sigma\}_{n=1}^{\infty} \Longrightarrow \bigcup_{i=1}^{\infty} A_i \in \Sigma$. close under infinite unions.

Definition 6: σ-algebra

Definition 0.6 (generate σ-algebra):

Let X be a set and $G \subseteq 2^X$. The **σ-algebra generated by** G, denoted $\sigma_X(G)$, is the smallest σ-algebra such that:

1. $G \subseteq \sigma_X(G)$.
2. $\forall \Sigma$ σ-algebra : $G \subseteq \Sigma \Longrightarrow \sigma_X(G) \subseteq \Sigma$. Every other σ-algebra that contains G contains also the generated one, $\sigma_X(G)$.

Definition 7: generate σ-algebra

Definition 0.7 (borel σ-algebra):

Let (X, G) be a topological space. We refer to $\sigma_X(G) = \mathcal{B}(X, G)$ as a **Borel σ-algebra**.

Definition 8: borel σ-algebra

Definition 0.8 (σ-algebra product):

Let Σ_1 and Σ_2 be σ-algebras on X_1 and X_2 respectively. The **product σ-algebra** denoted $\Sigma_1 \otimes \Sigma_2$ is defined as $\sigma_{X_1 \times X_2}(\{S_1 \times S_2 \mid S_1 \in \Sigma_1, S_2 \in \Sigma_2\})$

Definition 9: σ-algebra product

Definition 0.9 (measurable space):

(X, Σ) is said **measurable** iff. Σ is a sigma-algebra of X.

Definition 10: measurable space

Definition 0.10 (measure):

Given (X, Σ) measurable space. $\mu : \Sigma \longrightarrow \mathbb{R} \cup \{+\infty, -\infty\}$ is said a **measure** iff.

1. $E \in \Sigma \Longrightarrow \mu(E) \geq 0$. positive.
2. $\{E_n \in \Sigma\}_{n=1}^{\infty}$ such that $E_i \cap E_j$ for $i \neq j \Longrightarrow \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$. The measure of disjoint sets is is the sum of the measures of each set.
3. $\mu(\emptyset) = 0$.

Definition 11: measure

Definition 0.11 (measure space):

(X, Σ, μ) is said a **measure space** iff. (X, Σ) is a sigma algebra and µ is a measure of (X, Σ).

Definition 12: measure space

Definition 0.12 (measurable function):

Let (X_1, Σ_1) and (X_2, Σ_2) be a measurable spaces. $f : X_1 \longrightarrow X_2$ is said a **measurable function** iff. $\forall E \in \Sigma_2 : f^{-1}(E) \in \Sigma_1$. The preimage of each measurable set is again measurable.

Definition 13: measurable function

Definition 0.13 (pushforward):

Let (X_1, Σ_1, μ) be a measure space. Let (X_2, Σ_2) be a measurable space. Let $f : X_1 \longrightarrow X_2$ be a measurable function. The **pushforwad of μ under** f is the mapping $f_{\#}\mu : \Sigma_2 \longrightarrow \mathbb{R}_{\geq 0}$ defined as:

$$\forall E \in \Sigma_2 : f_{\#}\mu(E) = \mu(f^{-1}(E))$$

Definition 14: pushforward

The pushforward is simply a function that generates a measure for a measurable space starting from a different measure space and a measurable function acting as bridge between the two spaces.

Proposition 0.1 (pushforward of a measure is a measure):

Let (X_1, Σ_1, μ) be a measure space. Let (X_2, Σ_2) be a measurable space. Let $f : X_1 \longrightarrow X_2$ be a measurable function. Then $\left(X_2, \Sigma_2, f_{\#}\mu\right)$ is a measure space.

Proposition 15: pushforward of a measure is a measure

Proof 0.1 (of Proposition 15):

To prove that statement, we need to prove only the axioms of a measure.

1. Let $E \in \Sigma_2$, we need to show that $f_{\#}\mu(E) \geq 0$. This is trivial by definition of pushforward and measure.
2. Let $[E_n \in \Sigma_2]_{n=1}^{\infty}$ be a sequence of pairwise disjoint sets. We need to show that: $f_{\#}\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} f_{\#}\mu(E_n)$.

$$\begin{aligned} f_{\#}\mu\left(\bigcup_{n=1}^{\infty} E_n\right) &= \mu\left(f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right)\right) \text{ definition of pushforward} \\ &= \mu\left(\bigcup_{n=1}^{\infty} f^{-1}(E_n)\right) \\ &= \sum_{n=1}^{\infty} \mu(f^{-1}(E_n)) \text{ definition of measure} \\ &= \sum_{n=1}^{\infty} f_{\#}\mu(E_n) \text{ definition of pushforward} \end{aligned}$$

3. We need to show that $\exists E \in \Sigma_1$ such that $f_{\#}(E) \geq 0$. Let $E' \in \Sigma_1$ such that $\mu(E') \geq 0$ (such E' exists by defintion of measure). Then, $f(E')$ is a set that meets the requirements, that is

$$f_{\#}(f(E')) = \mu(f^{-1}(f(E'))) = \mu(E') \geq 0$$

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Proof 16: of Proposition 15

Example 0.2 (pushforward example):

Consider the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu(E) = |E|)$. Consider the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Consider the measurable function $f : \mathbb{N} \longrightarrow \mathbb{R}$ such that f(x) = x. Consider pushforward $f_{\#}\mu : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$. Then $f_{\#}\mu$ is a measure for the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ since:

1. $f_{\#}\mu(E \in \mathcal{B}(\mathbb{R})) = |\{n \in \mathbb{N} \mid n \in E\}| \geq 0.$
2. Let $\{E_n\}_{n=1}^{\infty}$ pairwise disjoint, then $f_{\#}\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right)\right) = \mu\left(\bigcup_{n=1}^{\infty} f^{-1}(E_n)\right) = \sum_{n=1}^{\infty} \mu(f^{-1}(E_n)) = \sum_{n=1}^{\infty} f_{\#}\mu(E_n).$
3. $f_{\#}\mu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$

Example 17: pushforward example

Lebesgue Measure

Definition 0.14 (pre-measure):

Let (X, Σ) such that $\emptyset \in S$. Let $\mu : S \longrightarrow R_{\geq 0} + \{+\infty\}$. μ is said a **pre-measure** iff.

1. $\mu(\emptyset) = 0.$
2. Given a collection of pairwise disjoint sets $\{A_n \in S\}_{n \in \mathbb{N}}$ such that $\bigcup_{n \in \mathbb{N}} A_n \in S \Longrightarrow \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n).$
3. $\forall A \in S : \mu(A) \geq 0.$

Definition 18: pre-measure

A pre-measure is a precursor of a full-fledge measure. The main difference is that a measure is defined on sigma algebras, meanwhile the pre-measure is defined on a simple collection of subsets. Further, given that this collection is not necessarily closed under unions as a sigma algebra does, we also need to check that, in the second requirement, the union of A_n is indeed contained in the collection.

Definition 0.15 (Lebesgue pre-measure):

The **Lebesgue pre-measure** is a mapping $\lambda^n : \mathcal{J}_h^n \longrightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$ ($\mathcal{J}_h^n$ denotes the set half open rectangle) such that $\lambda^n\left(\bigtimes_{i=1}^{n} [a_i, b_i)\right) = \prod_{i=1}^{n} (b_i - a_i)$ for $a_i, b_i \in \mathbb{R}$ and $a_i \leq b_i$.

Definition 19: Lebesgue pre-measure

Proposition 0.2:

The Lebesgue pre-measure is a pre-measure.

Proof 0.2 (of Proposition Proposition 20):

1. $$\lambda^n(\emptyset) = \lambda^n\left(\bigtimes_{i=1}^n [a_i, a_i)\right) = \prod_{i=1}^n (a_i - a_i) = 0$$
2. Let $I = \bigtimes_{i=1}^n [a_i, b_i)$ and $I' = \bigtimes_{i=1}^n [a_i', b_i')$ be disjoint half open rectangles.

The $I \cup I'$ belongs to $\mathcal{I}_h^n$ if we can stitch one to the other (see Fig.)

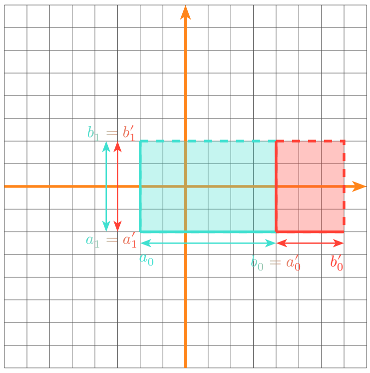

Figure 1: aaa

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Proof 21: of Proposition Proposition 20