BEDLAM

Contents

1. Notation

Definition 1.1 (symmetric difference):
Let A, B be sets. The **symmetric difference** is the operation, denoted $S \triangle T$, and defined $(S \setminus T) \cup (T \setminus S)$

Definition 1.2 (half-open rectangle):
Let $a_0, b_0, \ldots, a_n, b_n \in \mathbb{R}$. The set $\bigtimes_{i=0}^{n} [a_i, b_i)$ is called an n-dimensional **half-open rectangle**. The collection of all n-dimensional **half-open rectangles** is denoted with $\mathcal{J}_h^n$.

Definition 1.3 (restriction):
Let $f : X \longrightarrow Y$. Let $X' \subseteq X$. Let Y' such that $f(X') \subseteq Y' \subseteq Y$. A **restriction of f over $X' \times Y'$**, denoted $f|_{\text{X' times Y'}}$ is a function $X' \longrightarrow Y'$ such that $f|_{\text{X times Y}} = \{x \mapsto f(x) \mid x \in X', f(x) \in Y'\}$

Example 1.1 (restriction):
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = x^2$ power operator over the real numbers. Now, consider $g : \mathbb{N} \longrightarrow \mathbb{N}$ such that $g(x) = x^2$ power operator over the natural number only. Then g is a *restriction* of f.
1. $\mathbb{N} \subseteq \mathbb{R}$.
2. $f(\mathbb{N}) \subseteq \mathbb{N} \subseteq \mathbb{R}$.
3. $\{(x, g(x)) |\ x \in \mathbb{N}, y \in \mathbb{N})\} \subseteq \{(x, f(x) | x \in \mathbb{R}, y \in \mathbb{R})\}$

Definition 1.4 (extension):
Let $f : X \longrightarrow Y$. Let $X' \subseteq X$. Let $f|_{\text{X' times Y'}}$ be a *restriction* of f. Then f is said an **extension** of $f|_{\text{X times Y}}$

Definition 1.5 (inverse function):
Let $f : X \longrightarrow Y$ be a function. The **inverse function** $f^{-1} : Y \longrightarrow X$ is a function such that $f^{-1}(y \in Y) = x \in X$ if $f(x) = y$.

Definition 1.6 (preimage):
Let $f : X \longrightarrow Y$ be a function. Let $E \subseteq Y$. The **preimage** is the set $f^{-1}(E) = \{x \in X \mid f(x) \in E\}$.

2. Set Theory

Definition 2.1 (cover):
Let A be a set. A collection of sets $\mathcal{C}$ is a **cover of S** iff. $A \subseteq \bigcup_{C \in \mathcal{C}} C$

Proposition 2.1 (unions as disjoint unions):
Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of sets. Let $\{A'_n\}_{n \in \mathbb{N}}$ be a sequence of set such that $A'_n = A_n \setminus A_1 \setminus \ldots \setminus A_{n-1}$. Then $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} A'_n$ and $\{A'_n\}_{n \in \mathbb{N}}$ are pairwise disjoint.

Proof 2.1 (of *Proposition 2.1*):
1. Let us show that $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} A_n$:
$$\begin{aligned} \bigcup_{n \in \mathbb{N}} A_n &= A_1 \cup A_2 \cup A_3 \cup \ldots \\ &= A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_1 \setminus A_2) \cup \ldots \\ &= A'_1 \cup A'_2 \cup A'_3 \cup \ldots \\ &= \bigcup_{n \in \mathbb{N}} A'_n \end{aligned}$$
2. Let us show that $\{A'_n\}_{n \in \mathbb{N}}$ are pairwise disjoint. Consider A'_a and A'_b where, without loss of genrality, a < b. Then $A'_a \cap A'_b = \emptyset$ since A'_b results from A_b without A_a (among other sets) and $A'_a \subseteq A_a$

■

3. Abstract Algebra

Definition 3.1 (monoid):
$(X, \cdot : X \times X \longrightarrow X)$ is a **monoid** iff.
1. $\forall a, b, c \in X : a \cdot (b \cdot c) = (a \cdot) b \cdot c$. Associativity.
2. $\exists e \in X : \forall a \in X : e \cdot a = a \cdot e = a$. Identity element.

Definition 3.2 (semiring):
$(X, + : X \times X \longrightarrow X, \cdot : X \times X \longrightarrow X)$ is a **semiring** iff.
1. $(X, +)$ is a *monoid* with identity element 0.
2. $(X, \cdot)$ is *monoid* with identity element 1.
3. $+$ is commutative.
4. $a \cdot 0 = 0 \wedge 0 \cdot a = 0$. $\cdot$ is annihilated by the identity element of $+$.
5. $a \cdot (b + c) = a \cdot b + a \cdot c \wedge (b + c) \cdot a = b \cdot a + c \cdot a$. $\cdot$ distributes over $+$.

4. Topology

Definition 4.1 (Topology):
Let X bet a set. A **topology over X** is a subset Σ of 2^X such that:
1. $A \subseteq \Sigma \implies \bigcup_{E \in A} E$. Infinite or finite unions of sets.
2. $A, B \in \Sigma \implies A \cap B \in \Sigma$. Finite intersections of sets.
3. $X \in \Sigma$

Definition 4.2 (Topological Space):
(X, Σ) is a **topological space** iff. Σ is a *topology* of X.

Definition 4.3 (Everywhere dense):
Let (X, Σ) *topological space*, and $H \subseteq X$. H is said **everywhere dense in $\boldsymbol{\Sigma}$** iff. $\forall E \in \Sigma, E \neq \emptyset : H \cap E = \emptyset$. We can find a bit of H in every corner of the *topology* Σ.

Definition 4.4 (Separable):
Let (X, Σ) be a *topological space*. (X, Σ) is said **separable** iff it exists $H \subseteq X$, such that H is countable and H is *everywhere dense* in Σ. There is a set of elements $\{x_n \in X\}_{n=1}^{\infty}$ such that every set in the topology contains at least one them.

Definition 4.5 (Metric Space):
(X, d) is a **metric space** iff.
1. $X \neq \emptyset$
2. $d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ such that (d is a distance):
   1. $\forall x, y \in X : d(x, y) = 0 \implies x = y$. there are no different elements at zero-distance.
   3. $\forall x, y \in X : d(x, y) = d(y, x)$. symmetry.
   2. $\forall x, y, z \in X : d(x, z) \leq d(x, y) + d(y, z)$. triangular inequality.

Definition 4.6 (open ε-ball):
Let (X, d) be a *metric space*, $x \in X$, and $\varepsilon \in \mathbb{R}_{>0}$. We call $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ an **open ε-ball**. A ball of ε radius centered at some point.

Definition 4.7 (Neighborhood):
Let (X, d) be a *metric space*, $S \subseteq X$, $x \in S$, and $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(x) \subseteq S$. Then S is said a **neighborhood of x**. A neighborhood of an element is simply a set that contains an open ball containing the element.

Definition 4.8 (Open Set):
Let (X, d) be a *metric space* and $U \subseteq X$. U is an **open set** iff. $\forall u \in U : \exists \varepsilon \in \mathbb{R}_{>0} : B_\varepsilon(u) \subseteq U$. An open set is simply a set which is also neighborhood for all its points.

Definition 4.9 (Induced Topology):
Let (X, d) be a *metric space*. Σ is said an **induced topology** iff. $\Sigma = \{U \subseteq X \mid U \text{ is an } \textit{open-set} \text{ in } (X, d)\}$

Definition 4.10 (Metrizable):
Let (X, Σ) be a *topological space*. (X, Σ) is said **metrizable** iff. it exists (X, d) *metric space* such that Σ is the *induced topology* by (X, d).

Definition 4.11 (Cauchy Sequence):
Let (X, d) be a *metric space*, $[x_n \in X]$ a sequence. $[x_n]$ is said a **cauchy sequence** iff. $\forall \varepsilon \in \mathbb{R}_{>0} : \exists N \in \mathbb{N} : \forall m, n \in \mathbb{N} : d(x_n, x_m) \leq \varepsilon$. There is a point after which all pairs of elements are close to each other.

Definition 4.12 (Convergent Sequence):
Let (X, d) be a *metric space*, $l \in X$, $[x_n \in X]$ a sequence. $[x_n]$ is said a **convergent sequence to the limit l** iff. $\forall \varepsilon \in \mathbb{R}_{>0} : \exists N \in \mathbb{R}_{>0} : \forall n > N : d(x_n, l) < \varepsilon$. If such a limit exists the sequence is simply said **convergent**.

Definition 4.13 (Complete Metric Space):
Let (X, d) be a *metric space*. (X, d) is said a **complete metric space** iff. every *cauchy sequence* is a *convergent sequence*.

Definition 4.14 (Polish Space):
Let (X, Σ) be a *topological space*. (X, Σ) is said a **Polish Space** iff. (X, Σ) is *separable*, *metrizable*, and a *complete metric space* for some metric.

5. Measure Theory

5.1. Introduction

Definition 5.1.1 (Set algebra):
Let X be a set, and $\mathcal{A} \subseteq 2^X$ such that:
1. $X \in \mathcal{A}$. Unit.
2. $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$. Closed under union.
3. $A \in \mathcal{A} \implies X \setminus A \in \mathcal{A}$. Closed under complement.

Then $(X, \mathcal{A})$ is called a **set algebra**.

Definition 5.1.2 (Set ring):
Let X be a set, and $\mathcal{R} \subseteq 2^X$ such that:
1. $\mathcal{R} \neq \emptyset$. Non-empty.
2. $A, B \in \mathcal{R} \implies A \cap B \in \mathcal{R}$. Closed under intersection.
3. $A, B \in \mathcal{R} \implies A \triangle B \in \mathcal{R}$. Closed under symmetric difference.

Then $(X, \mathcal{R})$ is called a **set ring**

Proposition 5.1.1 (intersection of *set rings* is a *set ring*):
Let $(X, \mathcal{R}_0)$ and $(X, \mathcal{R}_1)$ be two *set rings*. Then $(X, \mathcal{R}_0 \cap \mathcal{R}_1)$ is a *set ring*.

Proof 5.1.1 (of *Proposition 5.1.1*):
Given two *set rings* $(X, \mathcal{R}_0)$ and $(X, \mathcal{R}_1)$. We need to show that $(X, \mathcal{R} = \mathcal{R}_0 \cap \mathcal{R}_1)$ is a *set ring*:
1. Suppose $A_0 \in \mathcal{R}_0$ and $A_1 \in \mathcal{R}_1$ (such A_0 and A_1 exists since $\mathcal{R}_0$ and $\mathcal{R}_1$ are non-empty). Then $\emptyset \in \mathcal{R}_0$ since $\emptyset = A_0 \triangle A_1 \in \mathcal{R}_0$. Similarly, $\emptyset \in \mathcal{R}_1$. Therefore $\emptyset \in \mathcal{R}_0 \cap \mathcal{R}_1$
2. Suppose $A, B \in \mathcal{R}$. Then $A, B \in \mathcal{R}_0$ and $A, B \in \mathcal{R}_1$. Then $A \cap B \in \mathcal{R}_0$ and $\mathcal{R}_1$. Therefore $A \cap B \in \mathcal{R}$.
3. Suppose $A, B \in \mathcal{R}$. Then $A, B \in \mathcal{R}_0$ and $A, B \in \mathcal{R}_1$. Then $A \triangle B \in \mathcal{R}_0$ and $\mathcal{R}_1$. Therefore $A \triangle B \in \mathcal{R}$.

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Definition 5.1.3 (σ-algebra):
Let X be a set. $\Sigma \subseteq 2^X$ is said a **sigma algebra of X** iff.:
1. $X \in \Sigma$
2. $E \in \Sigma \implies X \setminus E \in \Sigma$. close under complement.
3. $\{A_n \in \Sigma\}_{n=1}^{\infty} \implies \bigcup_{i=1}^{\infty} A_i \in \Sigma$. close under infinite unions.

Proposition 5.1.2 (a *σ-algebra* is a *set ring*):
Let (X, Σ) be a *σ-algebra*, then (X, Σ) is a *set ring*

Proof 5.1.2 (of *Proposition 5.1.2*):
We need to show that, given a *σ-algebra* (X, Σ) the axioms of *set rings* hold:
1. $\Sigma \neq \emptyset$. This is true since $X \in \Sigma$.
2. $A, B \in \Sigma \implies A \cap B \in \Sigma$. This is true since $A \cap B = (X \setminus A) \cup (X \setminus B)$ (a *σ-algebra* is closed under $\cup$ and $\setminus$).
3. $A, B \in \Sigma \implies A \triangle B \in \Sigma$. This is true since $A \triangle B = (A \setminus B) \cup (B \setminus A)$ (a *σ-algebra* is closed under $\cup$ and $\setminus$).

∎

Definition 5.1.4 (generate σ-algebra):
Let X be a set and $G \subseteq 2^X$. The **σ-algebra generated by G**, denoted $\sigma_X(G)$, is the smallest σ-*algebra* such that:
1. $G \subseteq \sigma_X(G)$.
2. $\forall \Sigma$ σ-algebra : $G \subseteq \Sigma \Longrightarrow \sigma_X(G) \subseteq \Sigma$. Every other σ-*algebra* that contains G contains also the generated one, $\sigma_X(G)$.

Definition 5.1.5 (borel σ-algebra):
Let (X, G) be a *topological space*. We refer to $\sigma_X(G) = \mathcal{B}(X, G)$ as a **Borel σ-algebra**.

Definition 5.1.6 (σ-algebra product):
Let Σ_1 and Σ_2 be σ-*algebra* on X_1 and X_2 respectively. The **product σ-algebra** denoted $\Sigma_1 \otimes \Sigma_2$ is defined as $\sigma_{X_1 \times X_2}(\{S_1 \times S_2 \mid S_1 \in \Sigma_1, S_2 \in \Sigma_2\})$

Definition 5.1.7 (measurable space):
(X, Σ) is said **measurable** iff. Σ is a σ-*algebra* of X.

Definition 5.1.8 (measure):
Given (X, Σ) *measurable space*. $\mu : \Sigma \longrightarrow \mathbb{R} \cup \{+\infty, -\infty\}$ is said a **measure** iff.
1. $\mu(\emptyset) = 0$ Empty set.
2. $E \in \Sigma \Longrightarrow \mu(E) \geq 0$. Positiveness.
3. $\{E_n \in \Sigma\}_{n \in \mathbb{N}}$ pairwise disjoint $\Longrightarrow \mu(\cup_{n \in \mathbb{N}} E_b) = \sum_{n \in \mathbb{N}} \mu(E_n)$. Countable additivity.

Definition 5.1.9 (measure space):
(X, Σ, μ) is said a **measure space** iff. (X, Σ) is a σ-*algebra* and μ is a *measure* of (X, Σ).

Definition 5.1.10 (measurable function):
Let (X_1, Σ_1) and (X_2, Σ_2) be *measurable spaces*. $f : X_1 \longrightarrow X_2$ is said a **measurable function** iff. $\forall E \in \Sigma_2 : f^{-1}(E) \in \Sigma_1$. The *preimage* of each measurable set is again measurable.

Definition 5.1.11 (pushforward):
Let (X_1, Σ_1, μ) be a *measure space*. Let (X_2, Σ_2) be a *measurable space*. Let $f : X_1 \longrightarrow X_2$ be a *measurable function*. The **pushforwad of μ under f** is the mapping $f_\# \mu : \Sigma_2 \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined as:

$$\forall E \in \Sigma_2 : f_\# \mu = \mu(f^{-1}(E))$$

The pushforward is simply a function that generates a measure for a measurable space starting from a different measure space and a measurable function acting as bridge between the two spaces.

Proposition 5.1.3 (pushforward of a measure is a measure):
Let (X_1, Σ_1, μ) be a *measure space*. Let (X_2, Σ_2) be a *measurable space*. Let $f : X_1 \longrightarrow X_2$ be a *measurable function*. Then $(X_2, \Sigma_2, f_\# \mu)$ is a *measure space*.

Proof 5.1.3 (of *Proposition 5.1.3*):
To prove that statement, we need to prove only the axioms of a *measure*.
1. Let $E \in \Sigma_2$, we need to show that $f_{\#}\mu(E) \geq 0$. This is trivial by definition of *pushforward* and *measure*.
2. Let $[E_n \in \Sigma_2]_{n=1}^{\infty}$ be a sequence of pairwise disjoint sets. We need to show that: $f_{\#}\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} f_{\#}\mu(E_n)$.

$$\begin{aligned} f_{\#}\mu\left(\bigcup_{n=1}^{\infty} E_n\right) &= \mu\left(f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right)\right) \text{ definition of pushforward} \\ &= \mu\left(\bigcup_{n=1}^{\infty} f^{-1}(E_n)\right) \\ &= \sum_{n=1}^{\infty} \mu(f^{-1}(E_n)) \text{ definition of measure} \\ &= \sum_{n=1}^{\infty} f_{\#}\mu(E_n) \text{ definition of pushforward} \end{aligned}$$

3. We need to show that $\exists E \in \Sigma_1$ such that $f_{\#}\mu(E) \geq 0$. Let $E' \in \Sigma_1$ such that $\mu(E') \geq 0$ (such E' exists by defintion of measure). Then, $f(E')$ is a set that meets the requirements, that is

$$f_{\#}\mu(f(E')) = \mu(f^{-1}(f(E'))) = \mu(E') \geq 0$$

■

Example 5.1.1 (pushforward example):
Consider the *measure space* $(\mathbb{N}, 2^{\mathbb{N}}, \mu(E) = |E|)$. Consider the *measurable space* $(\mathbb{R}, \sigma_{\mathbb{R}}(\mathcal{J}_h^n))$. Consider the *measurable function* $f : \mathbb{N} \longrightarrow \mathbb{R}$ such that f(x) = x. Consider *pushforward* $f_{\#}\mu : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$. Then $f_{\#}\mu$ is a *measure* for the *measurable space* $(\mathbb{R}, \sigma_{\mathbb{R}}(\mathcal{J}_h^n))$ since:
1. $f_{\#}\mu(E \in \sigma_{\mathbb{R}}(\mathcal{J}_h^n)) = |\{n \in \mathbb{N} \mid n \in E\}| \geq 0$.
2. Let $\{E_n\}_{n=1}^{\infty}$ pairwise disjoint, then $f_{\#}\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right)\right) = \mu\left(\bigcup_{n=1}^{\infty} f^{-1}(E_n)\right) = \sum_{n=1}^{\infty} \mu(f^{-1}(E_n)) = \sum_{n=1}^{\infty} f_{\#}\mu(E_n)$.
3. $f_{\#}\mu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$

Definition 5.1.12 (pre-measure):
Let (X, Σ) be a *set algebra*. Let $\mu : S \longrightarrow R_{\geq 0} \cup \{+\infty\}$. μ is said a **pre-measure** iff.
1. $\mu(\emptyset) = 0$. Empty set.
2. Given a collection of pairwise disjoint sets $\{A_n \in S\}_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} A_n \in S \Longrightarrow \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$. Countable additivity.
3. $\forall A \in S : \mu(A) \geq 0$. Positiveness.

A *pre-measure* is a precursor of a full-fledge *measure*. The main difference is that a *measure* is defined on σ*-algebras*, meanwhile the *pre-measure* is defined on a simple collection of subsets. Further, given that this collection is not necessarily closed under unions as a σ*-algebra* does, we also need to check that, in the second requirement, the union of A_n is indeed contained in the collection.

Definition 5.1.13 (Outer measure):
Let X be a set. An **outer measure** $\mu : 2^X \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that:
1. `(..) =>` ... empty set.
2. `(..) =>` ... Monotonicity.
3. `(..) =>` ... Countable subadditivity.

An *outer measure*s are weaker wrt. *measure*s as they are only countably subadditive rather than countably additive. However, they are able to measure all subset of X rather than only a σ-*algebra*s.

5.2. Caratheodory Extension Theorem

Proposition 5.2.1 (σ-*algebra* generated by an *outer measure*):
Let X be a set. Let λ be an *outer measure* on X. Let $\Sigma_\lambda = \{A \in 2^X \mid \forall E \in 2^X : \lambda(E) = \lambda(E \cap A) + \lambda(E \cap (X \setminus A))\}$
The set of subsets in which *outer measure* cut X in a "good way" . Then Σ_λ is a σ-*algebra*.

Proof 5.2.1 (of *Proposition 5.2.1*):
We need to show that the axiom a σ-*algebra* hold for Σ_λ:

1. ($X \in \Sigma_\lambda$) Let $E \in 2^x$, we have $\lambda(E \cap X) + \lambda(E \cap (X \setminus X)) = \lambda(E \cap X) = \lambda(X)$.
2. ($A \in \Sigma_\lambda \implies X \setminus A \in \Sigma_\lambda$) Suppose $A \in \Sigma_\lambda \implies \forall E \in 2^X \lambda(E \cap A) + \lambda(E \cap (X \setminus A)) = \lambda(E)$. Now consider $\lambda(E \cap (X \setminus A)) + \lambda(E \cap (X \setminus (X \setminus A))) = \lambda(E \cap (X \setminus A)) + \lambda(E \cap A) = \lambda(E)$.
3. ($\{A_n\}_{n\in\mathbb{N}} \in \Sigma_\lambda \implies \bigcup_{n\in\mathbb{N}} A_n \in \Sigma_\lambda$ (countable unions))
   1. ($A, B \in \Sigma_\lambda \implies A \cup B \in \Sigma_\lambda$ (finite unions)) Suppose $A, B \in \Sigma_\lambda$ then
      - $\forall E \in 2^X : \lambda(E) = \lambda(E \cap A) + \lambda(E \cap (X \setminus A))$ and,
      - $\forall E \in 2^X : \lambda(E) = \lambda(E \cap B) + \lambda(E \cap (X \setminus B))$.

      Then, we have that:

$$\begin{aligned}
&\lambda(E \cap (A \cup B)) + \lambda(E \cap (X \setminus (A \cup B))) \\
&= \lambda(E \cap (A \cup B)) + \lambda(E \cap (X \setminus A) \cap (X \setminus B)) \\
&= \lambda(E \cap (A \cup B) \cap A) + \lambda(E \cap (A \cup B) \cap (X \setminus A)) + \lambda(E \cap (X \setminus A) \cap (X \setminus B)) \\
&= \lambda(E \cap A) + \lambda(E \cap B \cap (X \setminus A)) + \lambda(E \cap (X \setminus A) \cap (X \setminus B)) \\
&= \lambda(E \cap A) + \lambda(E \cap (X \setminus A)) - \lambda(E \cap (X \setminus A) \cap (X \setminus B)) + \lambda(E \cap (X \setminus A) \cap (X \setminus B)) \\
&= \lambda(E \cap A) + \lambda(E \cap (X \setminus A)) \\
&= \lambda(E)
\end{aligned}$$

   2. Now we proceed to the third property. Let $N \in \mathbb{N}$. Let $\{A_n \in \Sigma_\lambda\}_{n\in\mathbb{N}}$. Consider the pairwise disjoint $\{A'_n\}_{n\in\mathbb{N}}$ built as in *Proposition 2.1*. Recall that $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} A'_n = A$.
      - We have that:

$$\begin{aligned}
&\lambda\left(E \cap \left(\bigcup_{n=1}^{N} A'_n\right)\right) + \lambda\left(E \cap \left(X \setminus \bigcup_{n=1}^{N} A'_n\right)\right) && \Sigma_\lambda \text{ is closed under finite unions.} \\
&\leq \lambda\left(E \cap \left(X \setminus \bigcup_{n=1}^{N} A'_n\right)\right) + \sum_{n=1}^{N} \lambda(E \cap A'_n) && \text{definition of outer measure} \\
&= \lambda(E \cap (X \setminus A'_1) \cap \ldots \cap (X \setminus A'_N)) + \lambda(E \cap A'_1) + \ldots + \lambda(E \cap A'_N) && \text{expansion} \\
&= \lambda(E \cap (X \setminus A'_2) \cap \ldots \cap (X \setminus A'_N)) + \lambda(E \cap A'_2) + \ldots + \lambda(E \cap A'_N) && \text{definition of } \Sigma_\lambda \\
&\ldots \\
&= \lambda(E)
\end{aligned}$$

        This relation holds for any N, even for $N = \infty$. Therefore:

$$\lambda(E \cap A) + \lambda(E \cap (X \setminus A)) \leq \lambda(E)$$

      - On the other hand:

$$\lambda(E) = \lambda((E \cap A) \cup (E \cap (X \setminus A))) \leq \lambda(E \cap A) + \lambda(E \cap (X \setminus A))$$

      Therefore, we have:

$$\lambda(E) \leq \lambda(E \cap A) + \lambda(E \cap (X \setminus A)) \leq \lambda(E)$$

      Or, equivalently:

$$\lambda(E) = \lambda(E \cap A) + \lambda(E \cap (X \setminus A))$$

Thus: $A \in \Sigma_\lambda$

Therefore, Σ_λ is a σ*-algebra*. ■

Proposition 5.2.2:
The restriction of the *outer measure* λ to the σ*-algebra* Σ_λ as in *Proposition 5.2.1* is a *measure* for Σ_λ.

Proof 5.2.2 (of *Proposition 5.2.2*):
We need to show that the axioms of a measure hold:

1. $(\lambda(\emptyset) = 0)$.
   This holds since λ is an *outer measure* and $\emptyset \in \Sigma_\lambda$.
2. $(E \in \Sigma \Longrightarrow \lambda(E) \geq 0)$.
   This holds since λ is an *outer measure* and $\forall A \in \Sigma_\lambda : \emptyset \subseteq A$
3. $(\{E_n \in \Sigma\}_{n \in \mathbb{N}}$ pairwise disjoint $\Longrightarrow \lambda(\cup_{n \in \mathbb{N}} E_b) = \sum_{n \in \mathbb{N}} \lambda(E_n))$.
   1. $\lambda(\cup_{n \in \mathbb{N}} E_n) \leq \sum_{n \in \mathbb{N}} \lambda(E_n)$.
      This hold since λ is an *outer measure*.
   2. $\lambda(\cup_{n \in \mathbb{N}} E_n) \geq \sum_{n \in \mathbb{N}} \lambda(E_n)$.
      From *Proof 5.2.1*, we know that:
      $$\lambda(A \cap (X \setminus (\cup_{n \in \mathbb{N}} E_n))) + \sum_{n \in \mathbb{N}} (A \cap E_n) \leq \lambda(A)$$
      Since this inequality must hold for any A in Σ_λ, it must hold also for $\cup_{n \in \mathbb{N}} E_n$:
      $$\begin{aligned} \lambda((\cup_{n \in \mathbb{N}} E_n) \cap (X \setminus (\cup_{n \in \mathbb{N}} E_n))) + \sum_{n \in \mathbb{N}} \lambda((\cup_{n \in \mathbb{N}} E_n) \cap E_n) &\leq \lambda(\cup_{n \in \mathbb{N}} E_n) \\ \lambda(\emptyset) + \sum_{n \in \mathbb{N}} \lambda(E_n) &\leq \lambda(\cup_{n \in \mathbb{N}} E_n) \\ \sum_{n \in \mathbb{N}} \lambda(E_n) &\leq \lambda(\cup_{n \in \mathbb{N}} E_n) \end{aligned}$$

   Since $\lambda(\cup_{n \in \mathbb{N}} E_n) \leq \sum_{n \in \mathbb{N}} \lambda(E_n)$ and $\lambda(\cup_{n \in \mathbb{N}} E_n) \geq \sum_{n \in \mathbb{N}} \lambda(E_n)$, we have that $\lambda(\cup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \lambda(E_n)$

■

Proposition 5.2.3 (measuring outside the measure domain):
Let $(\mathcal{R}, X)$ be s *set ring*. Let $\mu : \mathcal{R} \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a *measure* $\mathcal{R}$. Let $E \subset X$. Let $\mathcal{A} = \left\{\{A_n\}_{n \in \mathbb{N}} \mid A_n \in \mathcal{R}, E \subseteq \bigcup_{n \in \mathbb{N}} A_n\right)\}$ (Set of all *covers* of E). Let $\lambda_\mu = \inf_{\{A_n\}_{n \in \mathbb{N}} \in \mathcal{A}} \left\{\sum_{n \in \mathbb{N}} \mu(A_n)\right\}$. Then λ_μ is an *outer measure* on X.

Proof 5.2.3 (of *Proposition 5.2.3*):
We need to proove the three axiom for an *outer measure*.

1. $(\lambda_\mu(\emptyset) = 0)$.

   Since nothing in $\mathcal{R}$, we have that $\{\emptyset\} \in \mathcal{A}$. Therefore $\lambda_\mu(\emptyset) = 0$.
2. $(\forall A, B : A \subseteq B \Longrightarrow \lambda_\mu(A) \leq \lambda_\mu(B))$.

   A *cover* for B is also a *cover* of A since $A \subseteq B$. Therefore $\lambda_\mu(A) \leq \lambda_\mu(B)$
3. $(\forall \{A_n\}_{n \in \mathbb{N}} : \lambda_\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \lambda_\mu(A_n))$.

Let $\{A_{n,m}\}_{m\in\mathbb{N}}$ be the respective *cover* for A_n. Without loss of generality given by *Proposition 2.1*, let each *cover* be pairwise disjoint. Since the union of *cover* of sets *cover*s the unions of sets ($\bigcup A_n \subseteq \bigcup A_{m,n}$), we have

$$\begin{aligned}
\lambda_\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) &\leq \lambda_\mu\left(\bigcup_{m\in\mathbb{N}}\bigcup_{n\in\mathbb{N}} A_{n,m}\right) && \text{previous axiom} \\
&\leq \sum_{n\in\mathbb{N}} \mu\left(\bigcup_{m\in\mathbb{N}} A_{n,m}\right) && \text{None of the covers} A_{n,m} \text{ achieves the infinum} \\
&= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} \mu(A_{n,m}) && A_{n,m_0} \cap A_{n,m_1} = \emptyset
\end{aligned}$$

■

Proposition 5.2.4 (restriction of λ_μ):
The restriction of λ_μ as in *Proposition 5.2.3* on $\mathcal{R}$ is μ.

Proof 5.2.4 (of *Proposition 5.2.4*):
1. $\lambda_\mu(A) \leq \mu(A)$.
   For any $A \in \mathcal{R}$, A itself is a *cover* for A, hence $\lambda_\mu(A) \leq \mu(A)$ (λ_μ is the inf. of the measure of all *cover*s).
2. $\lambda_\mu(A) \geq \mu(A)$. Suppose there is a *cover* $\mathcal{B} = \{B_n \in \mathcal{R}\}_{n\in\mathbb{N}}$ of A. Let $\mathcal{B}' = \{B'_n\}_{n\in\mathbb{N}}$ be the respective pairwise disjoint cover according to *Proposition 2.1*. Let $\mathcal{B}'' = \{B'_n \cap A\}_{n\in\mathbb{N}}$. We have that $\mathcal{B}''$ is pairwise disjoint and $A = \cup_{B\in\mathcal{B}''} B \subseteq \cup_{B\in\mathcal{B}} B$ We have that $\cup_{B\in\mathcal{B}''} B = A$ since $\mathcal{B}''$ is a *cover* for A intersected with A. Then, $\mu(A) = \mu(\cup_{B\in\mathcal{B}''} B) = \sum_{B\in\mathcal{B}''} \mu(B) \leq \sum_{B\in\mathcal{B}'} \mu(B) = \sum_{B\in\mathcal{B}} \mu(B)$ since μ is a *measure*. Therefore, for any *cover* $\mathcal{B}$, we have shown that $\mu(A) \leq \sum_{B\in\mathcal{B}} \mu(B)$. Then, $\mu(A) \leq \lambda_\mu(A)$, since λ_μ is the inf. of all *cover*s.

Putting all togheter:

$$\mu(A) \leq \lambda_\mu(A) \leq \mu(A)$$

Therefore, $\mu(A) = \lambda_\mu(A)$.

■

Theorem 5.2.1 (Carathéodory's Extension Theorem):
Let (X, Σ_0) be a *set ring*. Let $\mu : \Sigma_0 \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a *pre-measure* (X, Σ_0). Then exists unique $\mu : \sigma(\Sigma_0) \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that (X, Σ, μ) is a *measure space* and $\mu|_{\Sigma_0} = \mu_0$ (μ is a *measure* and an *extension* of μ_0).

Proof 5.2.5 (of *Theorem 5.2.1*):
TODO ■

5.3. Lebesgue Measure

Definition 5.3.1 (Lebesgue pre-measure):
The **Lebesgue pre-measure** is a mapping $\lambda^n : \mathcal{I}_h^n \longrightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$ ($\mathcal{I}_h^n$ denotes the set half open rectangle) such that $\lambda^n\left(\bigtimes_{i=1}^n [a_i, b_i)\right) = \prod_{i=1}^n (b_i - a_i)$ for $a_i, b_i \in \mathbb{R}$ and $a_i \leq b_i$.

Proposition 5.3.1:
The Lebesgue pre-measure is a *pre-measure*.

Proof 5.3.1 (of Proposition *Proposition 5.3.1*):

1. $\lambda^n(\emptyset) = \lambda^n\left(\bigtimes_{i=1}^n [a_i, a_i)\right) = \prod_{i=1}^n (a_i - a_i) = 0$
2. Let $I = \bigtimes_{i=1}^n [a_i, b_i)$ and $I' = \bigtimes_{i=1}^n [a'_i, b'_i)$ be disjoint *half-open rectangles*. The $I \cup I'$ belongs to $\mathcal{I}_h^n$ if we can stitch one to the other. This can only happen if there is an i such that:
   1. $j = i \Longrightarrow b_j = a'_j$.
   2. $j \neq i \Longrightarrow b_j = b'_j$.
   3. $j \neq i \Longrightarrow a_j = a'_j$.

   This can be intuitively visualized in Figure 1 where two 2-dimensional half open rectangles met at one side. The only difference between the rectangles is that one is shifted along a single dimension, in such a way that they met at the open and close edges.

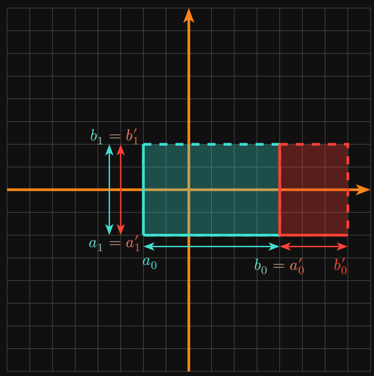

Figure 1: Two half open rectangles that can be stitched together.

   In this situation we have that:

$$\begin{aligned}
\lambda^n(I) + \lambda^n(I') &= \prod_{j=1}^n (b_j - a_j) + \prod_{j=1}^n (b'_j, a'_j) && \text{Lebesgue pre-measure definition} \\
&= ((b_i - a_i) + (b'_i - a'_i)) \prod_{\substack{j=1 \\ j \neq i}}^n b_j - a_j && \text{factoring out } \prod_{\substack{j=1 \\ j \neq i}}^n b_j - a_j \\
&= ((b_i - a'_i)) \prod_{\substack{j=1 \\ j \neq i}}^n b_j - a_j && \text{stitching half-open rectangles together} \\
&= \lambda^n(I \cup I')
\end{aligned}$$

   Thus it is verified that λ^n is finitely additive.
3. The $\forall E \in \mathcal{I}_h^n : \lambda^n(E) \geq 0$ since the product of positive terms is positive.

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Given a *pre-measure* on a *set algebra* is always possible to extend this *pre-measure* to a full-fledge *measure* over a *σ-algebra* generated by the *set algebra*. Further, such a *measure* is unique. This is the subject of the following theorem.

Definition 5.3.2 (Lebesgue Measure):
TODO

Theorem 5.3.1 (Lebesgue Measure Existence and Uniqueness):
TODO

Proof 5.3.2 (of *Theorem 5.3.1*):
TODO ∎

6. Probability Theory

Definition 6.1 (Probability Space):
(Ω, Σ, p) is said a **probability space** iff.
1. (Ω, Σ, p) is a *measure space*.
2. $p(\Omega) = 1$.

Intuitively, Ω represents the set of all possible outcomes, it is also known as **sample space**. Σ represents the set of all possible events. These are nothing more than set of outcomes. It is also known as **event space**. p is a *measure* on the event space, it is also known as **probability function**. It maps events to their likelihood.

Example 6.1 (Fair Die):
Consider the *probability space* (Ω, Σ, p), where:
1. $\Omega = \{1, 2, 3, 4, 5, 6\}$ is the sample space, representing the possible outcomes of rolling a standard six-sided die.
2. $\Sigma = 2^\Omega$ is the event space.
3. $p : \Sigma \longrightarrow [0, 1]$ is the probability *measure* function, defined as $P(E) = \frac{|E|}{6}$ for any event $E \in \Sigma$.

For example, consider the event $A = \{1, 2, 3\}$, which represents rolling a 1, 2, or 3. This event is an element of Σ. The probability of event A occurring is $p(A) = \frac{|A|}{6} = \frac{3}{6} = \frac{1}{2}$.

Definition 6.2 (Coupling):
Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be *probability spaces*. A **coupling** is a *probability space* $(\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2, \gamma)$ such that:
1. $\forall E \in \Sigma_1 : \gamma(E \times \Omega_2) = \mu_1(E)$. The left marginal of γ is μ_1.
2. $\forall E \in \Sigma_2 : \gamma(\Omega_1 \times E) = \mu_2(E)$. The right marginal of γ is μ_2.

Example 6.2 (Coupling a Dice and a Coin):
Consider a *probability space* $\mathcal{F}_1 = \left(\Omega_1 = \{1, 2, 3, 4\}, \Sigma_1 = 2^{\Omega_1}, p_1 = A \mapsto \frac{|A|}{4}\right)$ (The *probability space* corresponding to a 4 sided die). Further, consider a *probability space* $\mathcal{F}_2 = \left(\Omega_2 = \{1, 2\}, \Sigma_2 = 2^{\Omega_2}, p_2 = A \mapsto \frac{|A|}{2}\right)$ (The *probability space* corresponding to a coin). We can define a *probability space* $\mathcal{F} = (\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2, p)$ by *coupling* $\mathcal{F}_1$ and $\mathcal{F}_2$. Here, sample space and event space are already decided, we need to provide only a proper *measure* p. Such a *measure* can be built by providing a *coupling* table:

$$\begin{pmatrix} p & \{1\} & \{2\} & \{3\} & \{4\} & p_1 \\ \{1\} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} \\ \{2\} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ p_2 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1 \end{pmatrix}$$

On the top row, we have the possible singleton events from $\mathcal{F}_1$. On the left column, we have the possible singleton event from $\mathcal{F}_2$. The last row and column corresponds to marginal distributions. These marginals match p_2 and p_1 as required by the definition of *coupling*. The central body of this matrix represents join probabilities of the die and coin. For example, $p(\{1\} \times \{3\}) = \frac{1}{4}$.

Note that we could fill this matrix in such a way that we have a *probability space* but not a *coupling* by breaking the marginal axioms.

Retrieving event probabilities from singleton events is only matter of applying traditional probability rules.

7. Optimal Transport

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