UCB EE120: Assignment #2

Prof. Murat Arcak , Fall 2019

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Problem 1

We can divide $\hat{x}(t)$ into two parts, which are $\hat{x}_1(t) = x\left(t - \frac{1}{2}\right)$ and $\hat{x}_2 = -x\left(t + \frac{1}{2}\right)$.
Acording to the linearity and time shift properties of the Fourier Series, we can derive that the coefficients of $\hat{x}_1$ are as follows:

$$\hat{a}_{k_1} = \begin{cases} \frac{1}{2} \text{ if } k = 0, \\ \frac{1-e^{-jk\pi}}{2jk\pi} = \frac{1-(-1)^k}{2jk\pi} \text{ otherwise.} \end{cases}$$

And of $\hat{x}_2$:

$$\hat{a}_{k_2} = \begin{cases} -\frac{1}{2} \text{ if } k = 0, \\ \frac{1-e^{jk\pi}}{2jk\pi} = \frac{1-(-1)^k}{2jk\pi} \text{ otherwise.} \end{cases}$$

Add these two together, we get the coefficients of $\hat{x}$:

$$\hat{a}_k = \begin{cases} 0 \text{ if } k = 0, \\ \frac{1-(-1)^k}{jk\pi} \text{ otherwise.} \end{cases}$$

Finally, simplify the expression, we get:

$$\hat{a}_k = \begin{cases} 0 \ k : \text{even} \\ \frac{2}{jk\pi} \ k : \text{odd} \end{cases}$$

Problem 2

a) $x(t) = x(-t) \Rightarrow a_k = a_{-k} = a_k^* \Rightarrow a_k$ is purely real
b) $x(t) = -x(-t) \Rightarrow a_k = -a_{-k} = -a_k^* \Rightarrow a_k$ is purely imaginary
c) $x(t) = -x\left(t + \frac{T}{2}\right) \Rightarrow a_k = -a_k e^{-jk\pi} = a_k(-1)^{k+1} \Rightarrow a_k = 0$ for every even k
d) Omitted.

Problem 3

a)
$$\boldsymbol{W} = \begin{bmatrix} \Phi_0[0] & \Phi_1[0] & \dots & \Phi_{N-1}[0] \\ \Phi_0[1] & \Phi_1[1] & \dots & \Phi_{N-1}[1] \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_0[N-1] & \Phi_1[N-1] & \dots & \Phi_{N-1}[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & e^{j\frac{2\pi}{N}} & \dots & e^{j(N-1)\frac{2\pi}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j(N-1)\frac{2\pi}{N}} & \dots & e^{j(N-1)^2\frac{2\pi}{N}} \end{bmatrix}$$

b)
$$T = \frac{2\pi}{N} = \frac{2}{3}\pi \begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{W} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{3}} & e^{j\frac{4\pi}{3}} \\ 1 & e^{j\frac{4\pi}{3}} & e^{j\frac{8\pi}{3}} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Thus, $\begin{bmatrix} a_0 \\ a_1 \\ a_1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

c)
$$\boldsymbol{W}^{-1} = \frac{1}{N} \begin{bmatrix} \Phi_0[0] & \Phi_0[1] & \dots & \Phi_0[N-1] \\ \Phi_1[0] & \Phi_1[1] & \dots & \Phi_1[N-1] \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{N-1}[0] & \Phi_{N-1}[1] & \dots & \Phi_{N-1}[N-1] \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & e^{-j\frac{2\pi}{N}} & \dots & e^{-j(N-1)\frac{2\pi}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j(N-1)\frac{2\pi}{N}} & \dots & e^{-j(N-1)^2\frac{2\pi}{N}} \end{bmatrix}$$

d)
$$\boldsymbol{W}\boldsymbol{W}^{-1} = \begin{bmatrix} \Phi_0[0] & \Phi_1[0] & \Phi_2[0] \\ \Phi_0[1] & \Phi_1[1] & \Phi_2[1] \\ \Phi_0[2] & \Phi_1[2] & \Phi_2[2] \end{bmatrix} \begin{bmatrix} \Phi_0[0] & \Phi_0[1] & \Phi_0[2] \\ \Phi_1[0] & \Phi_1[1] & \Phi_1[2] \\ \Phi_2[0] & \Phi_2[1] & \Phi_2[2] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

Problem 4

a)
$$\begin{aligned} \frac{\mathrm{d}X(\omega)}{\mathrm{d}\omega} &= \frac{j}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{d}e^{-t^2}}{\mathrm{d}t} e^{-j\omega t}\,\mathrm{d}t \\ &= -\frac{j}{2}\int_{-\infty}^{+\infty} e^{-t^2}\,\mathrm{d}e^{-j\omega t} \\ &= -\frac{\omega}{2}\int_{-\infty}^{+\infty} e^{-t^2} e^{-j\omega t}\,\mathrm{d}t \\ &= -\frac{\omega}{2}X(\omega) \end{aligned}$$

b) Obviously, $\frac{\mathrm{d}X(\omega)}{\mathrm{d}\omega} = -\frac{2\omega}{\beta} \cdot \alpha e^{-\frac{\omega^2}{\beta}} = -\frac{2\omega}{\beta}X(\omega)$
To make the equation hold, we need to have $\beta = 4$
And we have $X(0) = \alpha = \sqrt{\pi}$, thus the solution is $\alpha = \sqrt{\pi}, \beta = 4$

Problem 5

$r(t)\cos(\omega_0 t) = s(t)\cos(\omega_0 t + \varphi)\cos(\omega_0) = s(t) \cdot \frac{\cos\varphi + \cos(2\omega_0 + \varphi)}{2}$
Since the ideal lowpass filter has a cutoff frequency ω_0, $\cos(2\omega_0 + \varphi)$ will be dropped.
For the passband gain, we multiply the remaining term by 2 to get $y(t) = s(t)\cos(\varphi)$.
When $\varphi = 90^\circ$, $y(t) \equiv 0$.