Lecture 9: Cardinality

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AY2025/26 Semester 1

Part of the contents here is taken from Dr Wong Tin Lok's lecture notes.

Aaron arrives at Hawaii for a long overdue vacation late one night. Tired, he walks into the famous Hilbert's Hotel looking for a room with a comfortable bed for a good night rest. Unfortunately, the hotel is already full. However, fortunately, the night manager has a clever idea…

Hilbert's Infinite Hotel:
https://www.youtube.com/watch?v=Uj3_KqkI9Zo

9.1 Cardinality

- Pigeonhole principle; dual pigeonhole principle.
- Finite and infinite sets; cardinality; Cantor's definition of same cardinality.
- Proving $|2\mathbb{Z}| = |\mathbb{Z}|$.

9.2 Countably Infinite

- Definition of countably infinite; countable and uncountable sets.
- Proving $\mathbb{Z}$, $\mathbb{Q}^+$, and $\mathbb{Z}^+ \times \mathbb{Z}^+$ are countable.
- Cartesian product; general Cartesian product, unions.

9.3 Countability via Sequences

- Countability and sequences.

9.4 Larger Infinities

- Proving $(0,1)$ is uncountable; Cantor's Diagonalization Argument.
- Cardinality of $\mathbb{R}$.

Reference: Epp's Chapter 7 Section 7.4

9.1 Cardinality

9.1.1 Pigeonhole Principle

Definitions: Injection, surjection, bijection, inverse function

Let X and Y be sets and $f: X \to Y$ be a function.

- f is **injective** iff $\forall x_1, x_2 \in X\ (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$.
- f is **surjective** iff $\forall y \in Y\ \exists x \in X\ \big(y = f(x)\big)$.
- f is **bijective** iff f is injective and surjective, that is,
$$\forall y \in Y\ \exists!\, x \in X\ \big(y = f(x)\big).$$
- $g: Y \to X$ is an inverse of f (also denoted as f^{-1}) iff
$$\forall x \in X\ \forall y \in Y\ \big(y = f(x) \Leftrightarrow x = g(y)\big).$$

Theorem 7.2.3

If $f: X \to Y$ is a bijection, then $f^{-1}: Y \to X$ is also a bijection.
In other words, a function is bijective iff it has an inverse.

What injections and surjections tell us about cardinality.

Pigeonhole Principle

Let A and B be finite sets. If there is an injection $f: A \to B$, then $|A| \leq |B|$.

Contrapositive: Let $m, n \in \mathbb{Z}^+$ with $m > n$. If m pigeons are put into n pigeonholes, then there must be (at least) one pigeonhole with (at least) two pigeons.

Dual Pigeonhole Principle

Let A and B be finite sets. If there is a surjection $f: A \to B$, then $|A| \geq |B|$.

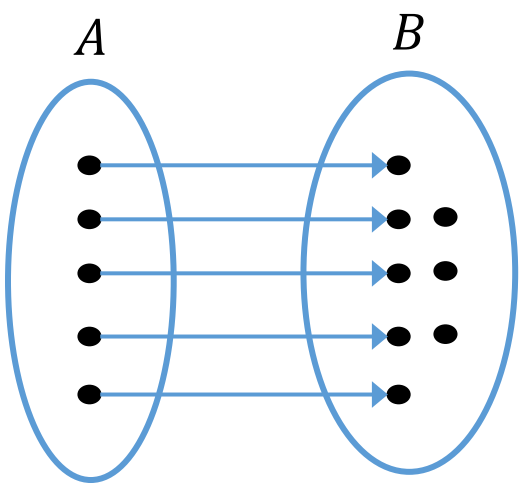
Contrapositive: Let $m, n \in \mathbb{Z}^+$ with $m < n$. If m pigeons are put into n pigeonholes, then there must be (at least) one pigeonhole with no pigeons.

Pigeonhole Principle

Let A and B be finite sets. If there is an injection $f: A \to B$, then $|A| \leq |B|$.

Proof

1. Note that A is finite. Suppose $A = \{a_1, a_2, \cdots, a_m\}$ where $m = |A|$.
2. The injectivity of f tells us that, if $a_i \neq a_j$, then $f(a_i) \neq f(a_j)$.
3. So $f(a_1), f(a_2), \cdots, f(a_m)$ are m different elements of B.
4. This shows $|B| \geq m = |A|$.

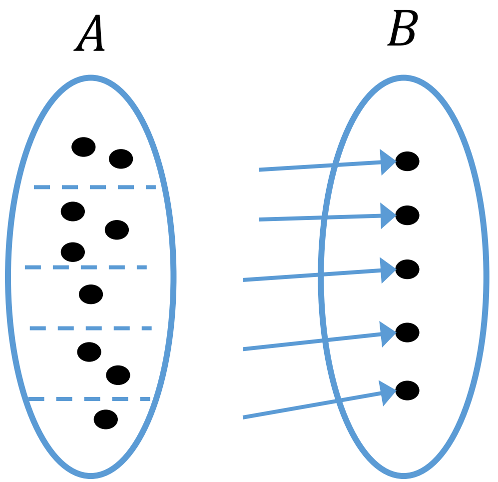


Dual Pigeonhole Principle

Let A and B be finite sets. If there is a surjection $f: A \to B$, then $|A| \geq |B|$.

Proof

1. Note that B is finite. Suppose $B = \{b_1, b_2, \cdots, b_n\}$ where $n = |B|$.
2. For each b_i, use the surjectivity of f to find $a_i \in A$ such that $f(a_i) = b_i$.
3. If $b_i \neq b_j$, then $f(a_i) \neq f(a_j)$ and so $a_i \neq a_j$ as f is a function.
4. So $a_1, a_2, \cdots, a_n$ are n different elements of A.
5. This shows $|A| \geq n = |B|$.



9.1.2 Cardinality

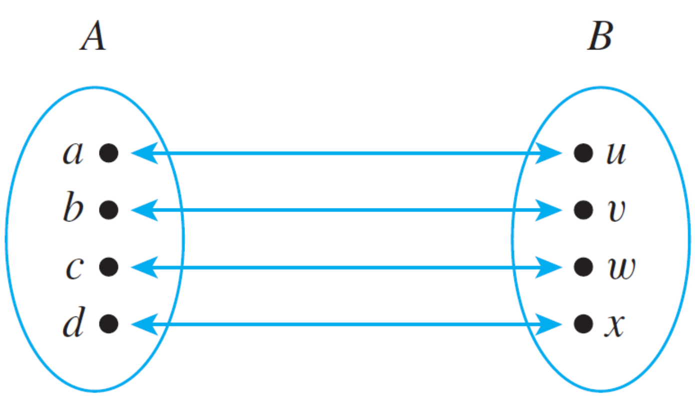
Definitions: Finite set and Infinite set

Let $\mathbb{Z}_n = \{1,2,3,\dots,n\}$, the set of positive integers from 1 to n.

A set S is said to be **finite** iff S is empty, or there exists a bijection from S to $\mathbb{Z}_n$ for some $n \in \mathbb{Z}^+$.

A set S is said to be **infinite** if it is not finite.

We say that two finite sets whose elements can be paired by a bijection have the *same size*.



The elements of set A can be put into a bijection with the elements of B.

Definition: Cardinality

The **cardinality** of a finite set S, denoted $|S|$, is

(i) 0 if $S = \emptyset$, or

(ii) n if $f: S \to \mathbb{Z}_n$ is a bijection.

Theorem: Equality of Cardinality of Finite Sets

Let A and B be any finite sets.
$|A| = |B|$ iff there is a bijection $f: A \to B$.

Proof (sketch)

1. ($\Leftarrow$) This follows from the two Pigeonhole Principles.
2. ($\Rightarrow$) If $A = \{a_1, a_2, \cdots, a_n\}$ and $B = \{b_1, b_2, \cdots, b_n\}$, then the function $f: A \to B$ satisfying $f(a_i) = b_i$ for $i \in \{1,2, \cdots, n\}$ is a bijection.

Finite sets

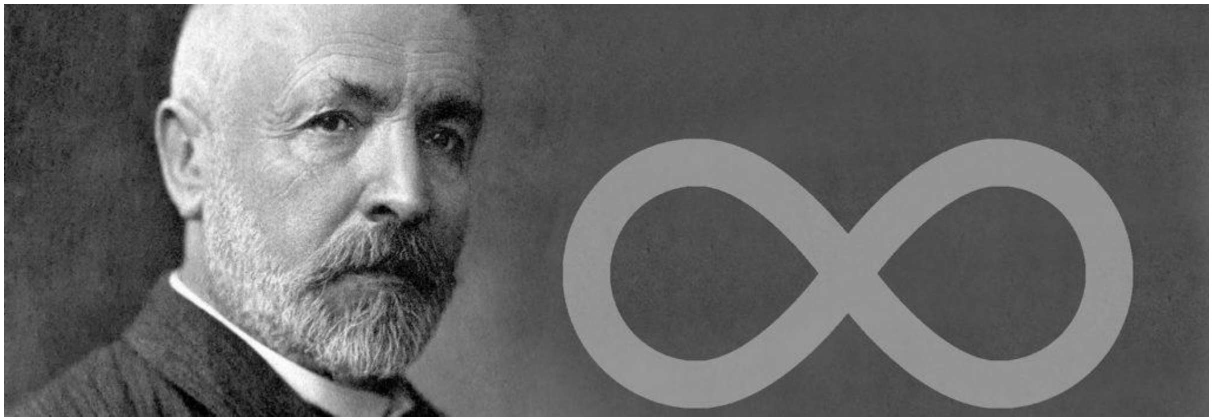
Theorem Cardinality.1: Subset of a Finite Set

Any subset of a finite set is finite.
That is, let $A \subseteq B$. If B is finite, then A is finite.

The contrapositive of this theorem is:
Let $A \subseteq B$. If A is infinite, then B is infinite.

Proof: By mathematical induction (omitted)

What about infinite sets?



Georg Cantor
(1845 – 1918)

https://en.wikipedia.org/wiki/Georg_Cantor

Definition: Same Cardinality (Cantor)

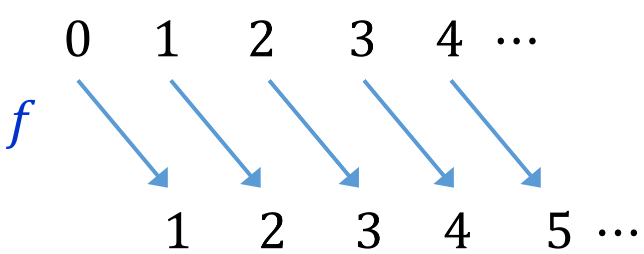
Given any two sets A and B. A is said to have the **same cardinality** as B, written as $|A| = |B|$, iff there is a bijection $f: A \to B$.

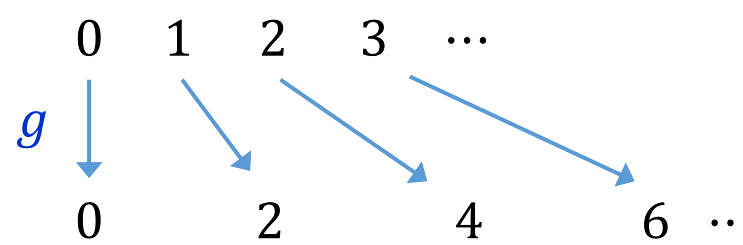
We define what $|A| = |B|$ means
without defining what $|A|$ and $|B|$ mean!

Definition: Same Cardinality (Cantor)

Given any two sets A and B. A is said to have the **same cardinality** as B, written as $|A| = |B|$, iff there is a bijection $f: A \to B$.

Example #1: $|\mathbb{N}| = |\mathbb{N} \setminus \{0\}|$ because the function $f: \mathbb{N} \to \mathbb{N} \setminus \{0\}$ satisfying $f(x) = x + 1$ for all $x \in \mathbb{N}$ is a bijection.





Example #2: $|\mathbb{N}| = |\mathbb{N} \setminus \{1,3,5,\cdots\}|$ because the function $g: \mathbb{N} \to \mathbb{N} \setminus \{1,3,5,\cdots\}$ satisfying $g(x) = 2x$ for all $x \in \mathbb{N}$ is a bijection.

Recall we define $\mathbb{N} = \{0,1,2,3,\cdots\}$.

Theorem 7.4.1 Properties of Cardinality

The same-cardinality relation is an equivalence relation.

For all sets A, B and C:

a. **Reflexive**: $|A| = |A|$.
b. **Symmetric**: $|A| = |B| \rightarrow |B| = |A|$.
c. **Transitive**: $(|A| = |B|) \wedge (|B| = |C|) \rightarrow |A| = |C|$.

Proof (reflexivity): To prove $|A| = |A|$.

It suffices to show that id_A is a bijection $A \rightarrow A$.

1. id_A is injective because if $x_1, x_2 \in A$ such that $id_A(x_1) = id_A(x_2)$, then $x_1 = x_2$.
2. id_A is surjective because given any $x \in A$, we have $id_A(x) = x$.
3. Therefore $|A| = |A|$.

Theorem 7.4.1 Properties of Cardinality

The same-cardinality relation is an equivalence relation.
For all sets A, B and C:

a. **Reflexive**: $|A| = |A|$.
b. **Symmetric**: $|A| = |B| \to |B| = |A|$.
c. **Transitive**: $(|A| = |B|) \wedge (|B| = |C|) \to |A| = |C|$.

Proof (symmetry): To prove $|A| = |B| \to |B| = |A|$.

1. Suppose $|A| = |B|$.
2. Use Cantor's definition of same-cardinality to find a bijection $f: A \to B$.
3. By Theorem 7.2.3, $f^{-1}: B \to A$ is also a bijection.
4. Therefore, $|B| = |A|$.

Theorem 7.2.3

If $f: X \to Y$ is a bijection, then $f^{-1}: Y \to X$ is also a bijection.
In other words, $f: X \to Y$ is bijective iff f has an inverse.

Theorem 7.4.1 Properties of Cardinality

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c. **Transitive**: $(|A| = |B|) \wedge (|B| = |C|) \rightarrow |A| = |C|$.

Proof (transitivity): To prove $(|A| = |B|) \wedge (|B| = |C|) \rightarrow |A| = |C|$.

1. Suppose $|A| = |B|$ and $|B| = |C|$.
2. Use Cantor's definition of same-cardinality to find a bijection $f: A \rightarrow B$ and a bijection $g: B \rightarrow C$.
3. From Tutorial 6 Q2, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
4. In particular, this means $g \circ f$ has an inverse.
5. So by Theorem 7.2.3, $g \circ f$ is a bijection $A \rightarrow C$.
6. Therefore, $|A| = |C|$.

9.1.3 $|2\mathbb{Z}| = |\mathbb{Z}|$

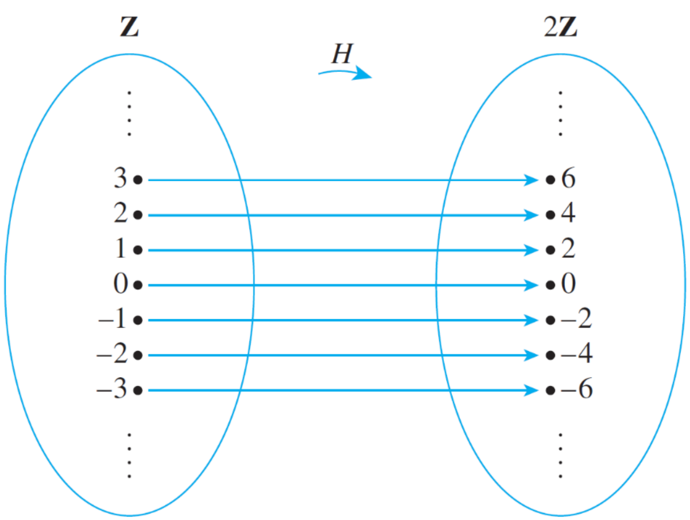
An infinite set can have the same cardinality as a proper subset of itself. (Refer to Example #2.)

Let $2\mathbb{Z}$ be the set of all even integers. Prove that $2\mathbb{Z}$and $\mathbb{Z}$ have the same cardinality.

Consider the function H from $\mathbb{Z}$ to $2\mathbb{Z}$ defined as follows:

$$H(n) = 2n, \forall n \in \mathbb{Z}$$

A partial arrow diagram for H:



Cardinality: $|2\mathbb{Z}| = |\mathbb{Z}|$

1. To show that H is injective:
   1.1 Suppose $H(n_1) = H(n_2)$ for some integers n_1, n_2.
   1.2 Then $2n_1 = 2n_2$ (by the definition of H), and hence $n_1 = n_2$.
   1.3 Therefore H is injective.
2. To show that H is surjective:
   2.1 Suppose $m \in 2\mathbb{Z}$.
   2.2 Then m is an even integer, so $m = 2k$ for some integer k (by the definition of even integer)
   2.3 But $H(k) = 2k = m$.
   2.4 Thus $\exists k \in \mathbb{Z}$ s.t. $H(k) = m$.
   2.5 Therefore H is surjective.
3. Therefore H is a bijection, and so $2\mathbb{Z}$ and $\mathbb{Z}$ have the same cardinality (by Cantor's definition of cardinality).

Cardinality: $|2\mathbb{Z}| = |\mathbb{Z}|$

Note that $2\mathbb{Z}$ is a proper subset of $\mathbb{Z}$, that is, $2\mathbb{Z} \subseteq \mathbb{Z}$ and $2\mathbb{Z} \neq \mathbb{Z}$. And yet $|2\mathbb{Z}| = |\mathbb{Z}|$! How strange!

For a finite set A, any proper subset B of A will have $|B| < |A|$. But this is not true for infinite sets.

Some mathematicians have proposed to use this as the definition of an infinite set. That is, a set A is infinite iff there exists a set B such that $(B \subseteq A) \land (B \neq A) \land (|B| = |A|)$.

9.2 Countably Infinite

9.2.1 Countable Sets

The set $\mathbb{Z}^+$ of counting numbers $\{1,2,3,\dots\}$ is in a sense, the most basic of all infinite sets.

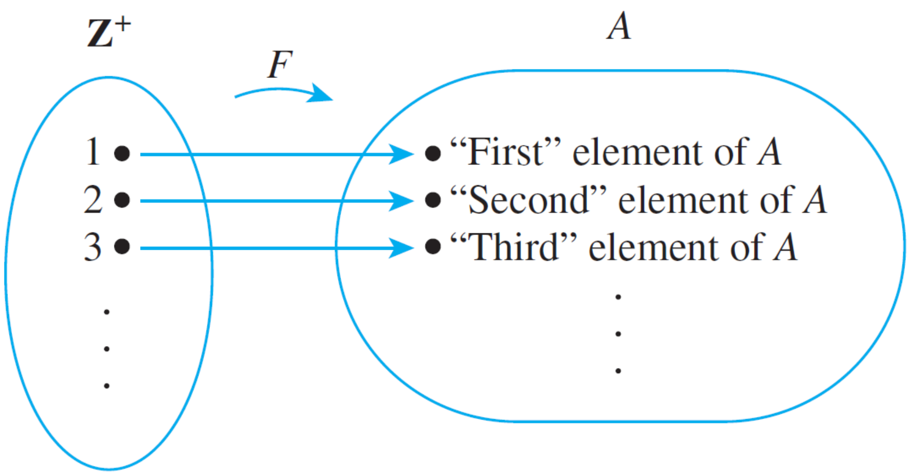
> The set A having the same cardinality as $\mathbb{Z}^+$ is called countably infinite.

The reason is that the bijection between the two sets can be used to "count" the elements of A:

If F is the bijection from $\mathbb{Z}^+$ to A, then "$F(1)$" can be designated as the first element of A, "$F(2)$" as the second element of A, and so forth.

Note: We may use $\mathbb{N}$ or $\mathbb{Z}_{\geq 0}$ instead of $\mathbb{Z}^+$ in the definition.

Countably Infinite: Countable Sets



"Counting" a Countably Infinite Set
Figure 7.4.1

Because F is injective, every element of A is counted at most once; because F is surjective, every element of A is counted at least once.

Countably Infinity: Countable Sets

Definition: Cardinal numbers

Define $\aleph_0 = |\mathbb{Z}^+|$. (Some authors use $\mathbb{N}$ instead of $\mathbb{Z}^+$.)

$\aleph$ is pronounced "aleph", the first letter of the Hebrew alphabet.

This is the first transfinite cardinal number.

Definition: Countably infinite

A set S is said to be **countably infinite** (or, S has the cardinality of natural numbers) iff $|S| = \aleph_0$.

Definitions: Countable set and Uncountable set

A set is said to be **countable** iff it is finite or countably infinite.

A set is said to be **uncountable** if it is not countable

9.2.2 $\mathbb{Z}$ is countable


Example #3: Show that $\mathbb{Z}$ is countable.

(Intuitively, $\mathbb{Z}$ contains twice as many numbers as $\mathbb{Z}^+$, so we expect $|\mathbb{Z}| = 2|\mathbb{Z}^+|$. But this is not the case. They have the same cardinality!)

The set $\mathbb{Z}$ is certainly not finite, so if it is countable, it has to be countably infinite.

To show that $\mathbb{Z}$ is countably infinite, find a bijection from $\mathbb{Z}^+$ to $\mathbb{Z}$.

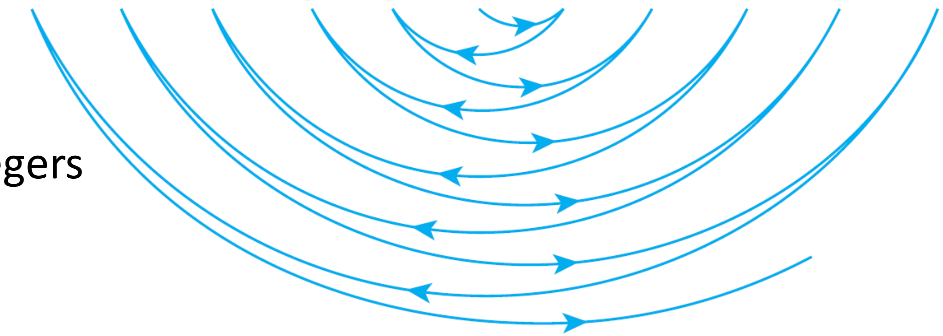
The appears to contradict common sense, as there appear to be more than twice as many integers as there are positive integers:

$$\underbrace{\cdots -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad \overbrace{1 \quad 2 \quad 3 \quad 4 \quad 5 \cdots}^{\text{positive integers}}}_{\text{all integers}}$$

Countably Infinite: $\mathbb{Z}$ is countable

The trick is to <u>start in the middle and work outward</u> systematically. Let the first integer be 0, the second 1, the third -1, the fourth 2, the fifth -2, and so forth:

Integers:	$\cdots$	-5	-4	-3	-2	-1	0	1	2	3	4	5	$\cdots$
The "count" of each integer:		11	9	7	5	3	1	2	4	6	8	10	



"Counting" the Set of All Integers

Figure 7.4.2

Every integer in $\mathbb{Z}$ is counted at most once (so the function is injective) and every integer in $\mathbb{Z}$ is counted at least once (so the function is surjective).

Therefore $\mathbb{Z}$ is countably infinite and hence countable.

Countably Infinite: $\mathbb{Z}$ is countable

The bijection as described earlier can be formulated as follows:

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is an even positive integer} \\ -(n-1)/2, & \text{if } n \text{ is an odd positive integer} \end{cases}$$

$$f(1) = -(1-1)/2 = 0$$
$$f(2) = 2/2 = 1$$
$$f(3) = -(3-1)/2 = -1$$
$$f(4) = 4/2 = 2$$
$$f(5) = -(5-1)/2 = -2$$

etc.

9.2.3 $\mathbb{Q}^+$ is countable

Example #4: Show that $\mathbb{Q}^+$ (the set of all positive rational numbers) is countable.

Display the elements of $\mathbb{Q}^+$ in a grid as shown:

Define a function F from $\mathbb{Z}^+$ to $\mathbb{Q}^+$ by starting to count at $\frac{1}{1}$ and following the arrows as indicated, skipping over any number that has already been counted.

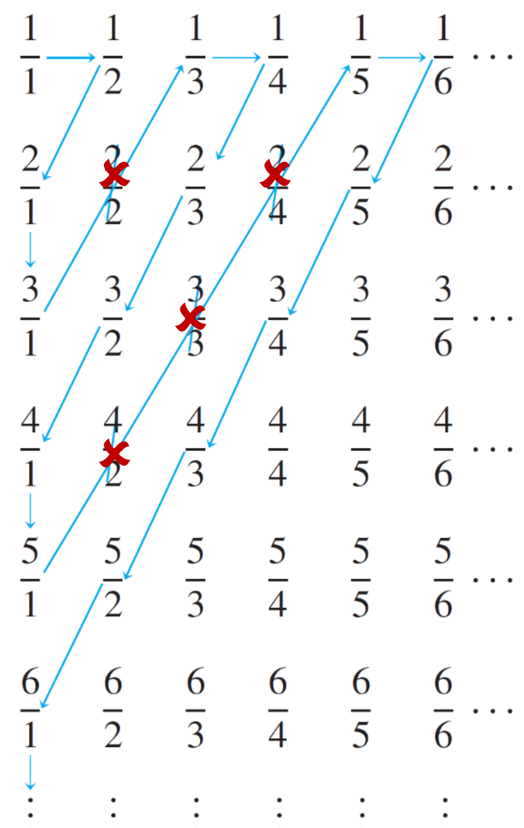


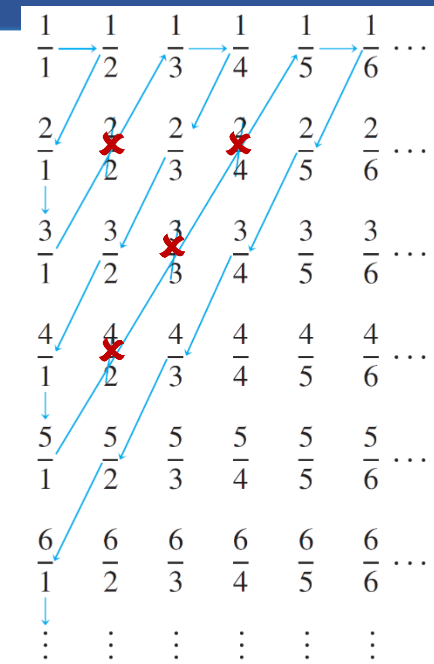
Figure 7.4.3

Larger Infinities

So, set $F(1) = \frac{1}{1}, F(2) = \frac{1}{2}, F(3) = \frac{2}{1}, F(4) = \frac{3}{1}$.

Then skip $\frac{2}{2}$ since $\frac{2}{2} = \frac{1}{1}$ which was counted.

Followed by $F(5) = \frac{1}{3}, F(6) = \frac{1}{4}, F(7) = \frac{2}{3}$, etc.



Note that every positive rational number appears somewhere in the grid, and the counting procedure is set up so that every point in the grid is reached eventually. Thus F is surjective.

Skipping numbers that have already been counted ensures that no number is counted twice. Thus F is injective.

So F is a bijection from $\mathbb{Z}^+$ to $\mathbb{Q}^+$. Therefore $\mathbb{Q}^+$ is countably infinite and hence countable.

9.2.4 $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable

The Infinite Hotel:
https://www.youtube.com/watch?v=Uj3_KqkI9Zo

Theorem: $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

What if an infinite number of buses, each carrying an infinite number of guests, arrive at the Infinite Hotel? Is there room for all of them?

Display the elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$ in a grid as shown:

The ordered pair (x, y) denotes bus x and guest y.

We then count the ordered paired in the following order according to this function $f: \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ by:

$$f(x, y) = \frac{(x+y-2)(x+y-1)}{2} + x$$

Guests

Bus	1	2	3	4	
1	(1,1)	(1,2)	(1,3)	(1,4)	⋯
2	(2,1)	(2,2)	(2,3)	(2,4)	⋯
3	(3,1)	(3,2)	(3,3)	(3,4)	⋯
4	(4,1)	(4,2)	(4,3)	(4,4)	⋯
⋮	⋮	⋮	⋮	⋮	

9.2.5 Theorems

Theorem (Cartesian Product)

If sets A and B are both countably infinite, then so is $A \times B$.

(Proof omitted. Similar to diagonal counting method in example #4.)

Corollary (General Cartesian Product)

Given $n \geq 2$ countably infinite sets $A_1, A_2, \cdots, A_n$, the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ is also countably infinite.

(Proof omitted. Proof by induction on n.)

Theorem (Unions)

The union of countably many countable sets is countable. That is, if $A_1, A_2, \cdots$ are all countable sets, then so is

$$\bigcup_{i=1}^{\infty} A_i$$

(Proof omitted. Similar to diagonal counting method in example #4.)

9.3 Countability via Sequences

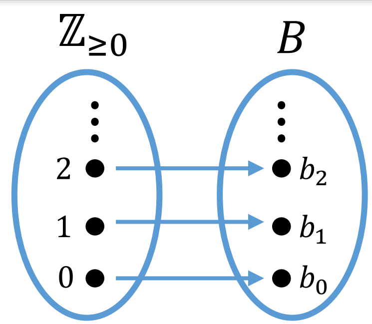
Definition: A set is said to be **countable** iff it is finite or countably infinite, that is, it has the same cardinality as $\mathbb{Z}_{\geq 0}$.

Note: In this section, we use $\mathbb{Z}_{\geq 0}$ instead of $\mathbb{Z}^+$ as we are relating to sequences.

Proposition 9.1

An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \cdots \in B$ in which every element of B appears exactly once.

Justification:



function $f: \mathbb{Z}_{\geq 0} \to B$

$f(0), f(1), f(2), \cdots$

(surjectivity) $\forall b \in B\ \exists i \in \mathbb{Z}_{\geq 0}\ f(i) = b$

(injectivity) $\forall i, j \in \mathbb{Z}_{\geq 0}\ (f(i) = f(j) \Rightarrow i = j)$

sequence of elements of B

$b_0, b_1, b_2, \cdots$

Every $b \in B$ appears at least once.

Every $b \in B$ appears at most once.

Definition: A **sequence** $a_0, a_1, a_2, \cdots$ can be represented by a function a whose domain is $\mathbb{Z}_{\geq 0}$ that satisfies $a(n) = a_n$ for every $n \in \mathbb{Z}_{\geq 0}$.

Proposition 9.1

An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \cdots \in B$ in which every element of B appears exactly once.

Lemma 9.2: Countability via Sequence

An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \cdots$ in which every element of B appears.

Proof:

1. ("only if") This follows directly from Proposition 9.1.
2. ("if")
   - 2.1. Let $b_0, b_1, b_2, \cdots$ be a sequence in which every element of B appears.
   - 2.2. Remove those terms in the sequence that are not in B.
   - 2.3. If an element of B appears more than once, then remove all but the first appearance.
   - 2.4. The result is a sequence in which every element of B appears exactly once.
   - 2.5. So B is countable.

Theorem: $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable. (Revisit)

We will provide a proof sketch using sequence.

Proof sketch:

The figure below describes a sequence: (1,1),(1,2),(2,1),(1,3),(2,2),…

in which every element of $\mathbb{Z}^+ \times \mathbb{Z}^+$ appears.

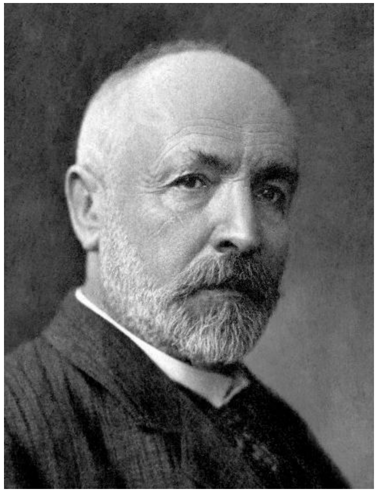
So $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable by Lemma 9.2.

Guests

Bus	1	2	3	4	
1	(1,1)	(1,2)	(1,3)	(1,4)	⋯
2	(2,1)	(2,2)	(2,3)	(2,4)	⋯
3	(3,1)	(3,2)	(3,3)	(3,4)	⋯
4	(4,1)	(4,2)	(4,3)	(4,4)	⋯
⋮	⋮	⋮	⋮	⋮	

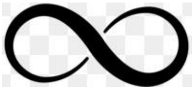
9.4 Larger Infinities

Larger Infinities



Georg Cantor, the man who discovered different infinities.

In 1874 the German mathematician Georg Cantor achieved success in the search for a larger infinity by showing that the set of all real numbers is uncountable. His method of proof was somewhat complicated, however.



The uncountability of the set of all real numbers between 0 and 1 using a simpler technique introduced by Cantor in 1891 is known as the **Cantor's diagonalization process.**

Over the intervening years, this technique and variations on it have been used to establish a number of important results in logic and the theory of computation.

9.4.1 Cantor's Diagonalization Argument

Theorem 7.4.2 (Cantor)

The set of real numbers between 0 and 1,

$$(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$$

is uncountable.

To prove that a set is uncountable means proving that there is no possibility of a bijection from that set to $\mathbb{Z}^+$.

The way to prove is by contradiction. Georg Cantor, who gave us set theory, also gave an ingenious proof for the above theorem, came to be known as the Cantor's Diagonalization Argument.

Sketch of proof (proof by contradiction):

1. Suppose $(0,1)$ is countable.
2. Since it is not finite, it is countably infinite.
3. We list the elements x_i of $(0,1)$ in a sequence as follows:

$$x_1 = 0.\, a_{11}\, a_{12} a_{13} \cdots a_{1n} \cdots$$
$$x_2 = 0.\, a_{21}\, a_{22} a_{23} \cdots a_{2n} \cdots$$
$$x_3 = 0.\, a_{31}\, a_{32} a_{33} \cdots a_{3n} \cdots$$
$$\vdots$$
$$x_n = 0.\, a_{n1}\, a_{n2} a_{n3} \cdots a_{nn} \cdots$$
$$\vdots$$

where each $a_{ij} \in \{0,1,\cdots,9\}$ is a digit.*

* Note that some numbers have two representations, eg: 0.49999… = 0.50000… We agree to use only the latter representation.

Larger Infinities: Cantor's Diagonalization Argument

4. Now, construct a number $d = 0.d_1\, d_2 d_3 \cdots d_n \cdots$ s.t.

$$d_n = \begin{cases} 1, & \text{if } a_{nn} \neq 1; \\ 2, & \text{if } a_{nn} = 1. \end{cases}$$

$$\begin{aligned} x_1 &= 0.\,a_{11}\, a_{12} a_{13} \cdots a_{1n} \cdots \\ x_2 &= 0.\,a_{21}\, a_{22} a_{23} \cdots a_{2n} \cdots \\ x_3 &= 0.\,a_{31}\, a_{32} a_{33} \cdots a_{3n} \cdots \\ &\vdots \\ x_n &= 0.\,a_{n1}\, a_{n2} a_{n3} \cdots a_{nn} \cdots \end{aligned}$$

5. Note that $\forall n \in \mathbb{Z}^+, d_n \neq a_{nn}$. Thus, $d \neq x_n, \forall n \in \mathbb{Z}^+$.
6. But clearly, $d \in (0,1)$, hence a contradiction. Therefore $(0,1)$ is uncountable.

Illustration:

$0.20148802\ldots$	d_1 is 1 because $a_{11} = 2$
$0.11666021\ldots$	d_2 is 2 because $a_{22} = 1$
$0.03853320\ldots$	d_3 is 1 because $a_{33} = 8$
$0.96776809\ldots$	d_4 is 1 because $a_{44} = 7$
$0.00031002\ldots$	d_5 is 2 because $a_{55} = 1$

Hence $d = 0.12112\ldots$, which is not in the list. So, the list is incomplete. This is true regardless of how the elements in (0,1) are listed.

Theorem 7.4.3

Any subset of any countable set is countable.

Proof:

1. Let A be any countable set and B be any subset of A.
2. If A is finite then B must be finite and hence countable – done.
3. Suppose A is countably infinite. If B is finite, then B is countable – done.
4. Suppose B is infinite.
   4.1 Since A is countable, there is a bijection $f: \mathbb{Z}^+ \to A$.
   4.2 Let $M = f^{-1}(B)$ (note that f^{-1} is a bijection), and define a function $g: \mathbb{Z}^+ \to B$ inductively as follows:
   S1. Let $g(1) = f(i_1)$, where i_1 is the minimum element in M.
   S2. If $g(1), g(2), \cdots, g(k-1)$ have been defined, …

Larger Infinities: Any subset of any countable set is countable

4. Suppose B is infinite.

   4.1 Since A is countable, there is a bijection $f\colon \mathbb{Z}^+ \to A$.

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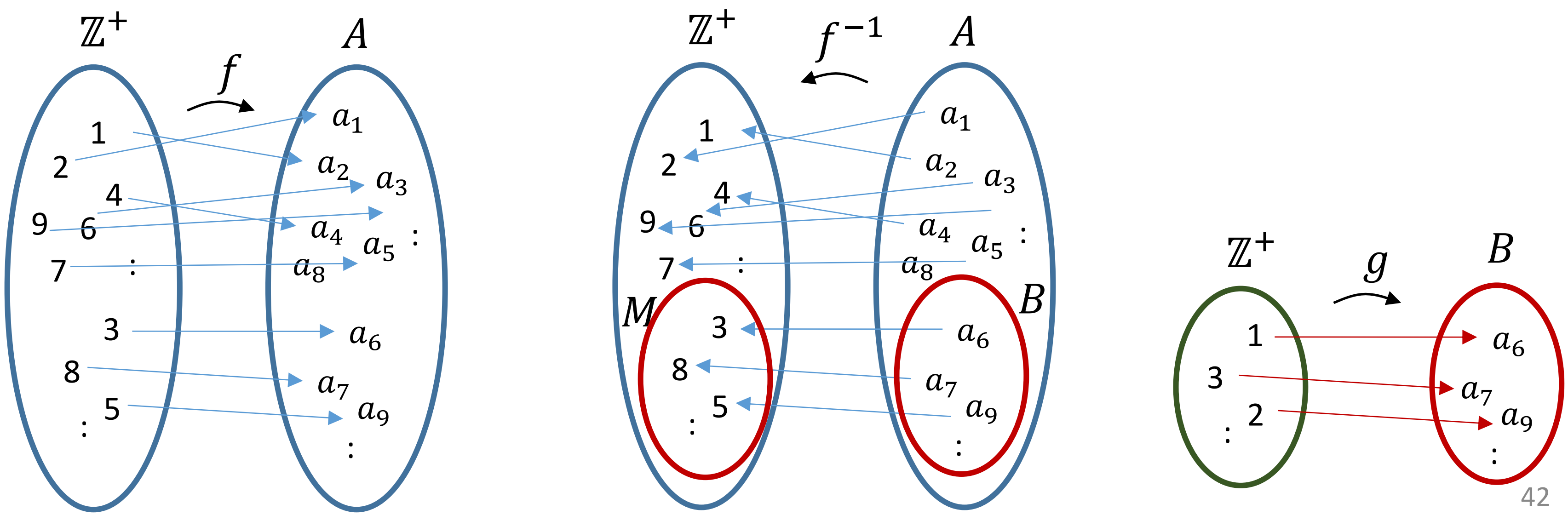
   S1. Let $g(1) = f(i_1)$, where i_1 is the minimum element in M.

   S2. If $g(1), g(2), \cdots, g(k-1)$ have been defined, let

   $$g(k) = f(i_k), \text{ where } i_k = \min\{m\colon m > i_{k-1}, m \in M\}.$$

   4.3 g is a bijection (why?), hence B is countable.



Larger Infinities: Any subset of any countable set is countable

Theorem 7.4.3

Any subset of any countable set is countable.

Corollary 7.4.4 (Contrapositive of Theorem 7.4.3)

Any set with an uncountable subset is uncountable.

Corollary 7.4.4 implies that $\mathbb{R}$ is uncountable since $(0,1) \subseteq \mathbb{R}$ and $(0,1)$ is uncountable.

Larger Infinities: Every infinite set has a countably infinite subset

Proposition 9.3

Every infinite set has a countably infinite subset.

Proof:

1. Let B be an infinite set.
2. Keep choosing elements $b_0, b_1, b_2, \cdots$ from B.
3. When we choose b_n, where $n \in \mathbb{Z}_{\geq 0}$, we can always make sure $b_n \neq b_i$ for any $i < n$, because otherwise B is equal to the finite set $\{b_0, b_1, b_2, \cdots, b_{n-1}\}$ which is a contradiction.
4. The result is a countably infinite set $\{b_0, b_1, b_2, \cdots\} \subseteq B$.

Lemma 9.4: Union of Countably Infinite Sets.

Let A and B be countably infinite sets. Then $A \cup B$ is countable.

Proof:

1. Apply Lemma 9.2 to find a sequence $a_0, a_1, a_2, \cdots$ in which every element of A appears.
2. Apply Lemma 9.2 to find a sequence $b_0, b_1, b_2, \cdots$ in which every element of B appears.
3. Then $a_0, b_0, a_1, b_1, a_2, b_2, \cdots$ is a sequence in which every element of $A \cup B$ appears.
4. So $A \cup B$ is countable by Lemma 9.2.

Lemma 9.2: Countability via Sequence

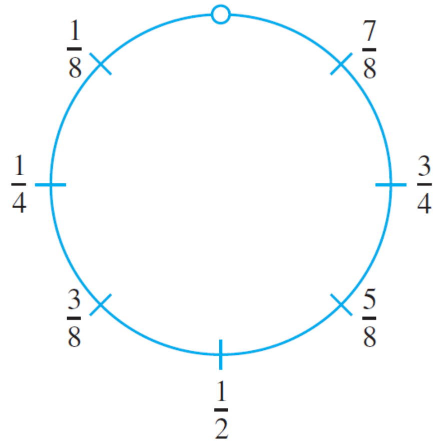
An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \cdots$ in which every element of B appears.

9.4.2 Cardinality of ℝ

Example #5: Show that $|\mathbb{R}| = |(0,1)|$.

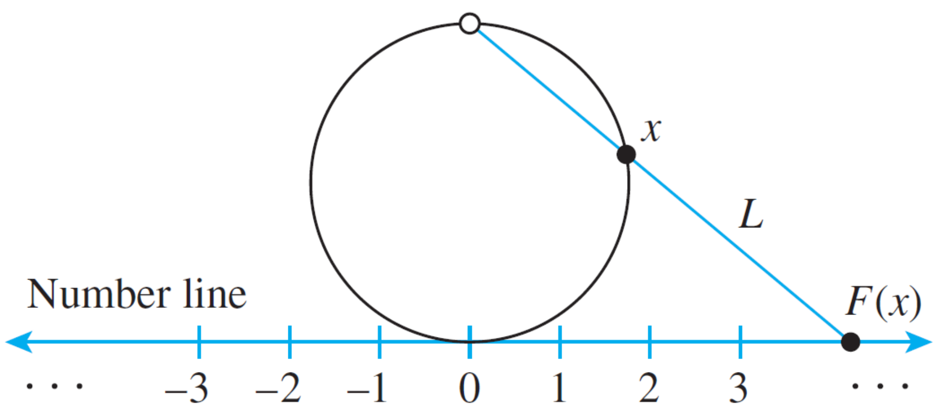
Let $S = (0,1)$, that is, $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

Imagine picking up S and bending it into a circle:



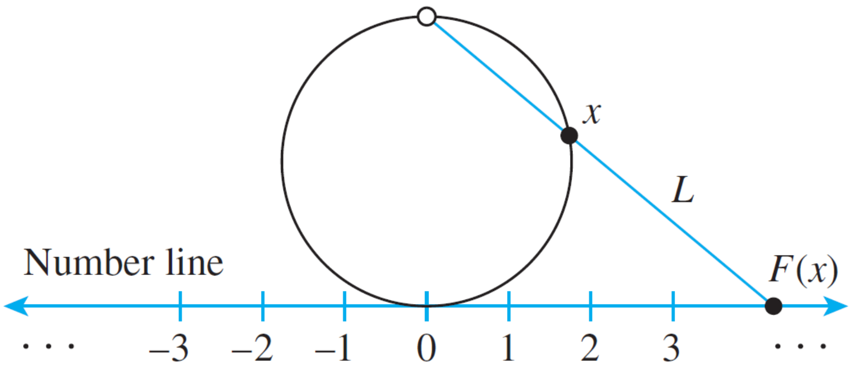
Define a function $F: S \to \mathbb{R}$ as follows:

Draw a number line and place the interval, S, bent into a circle as shown above, tangent to the line above the point 0, as shown below.



Larger Infinities: Cardinality of $\mathbb{R}$

For each point x on the circle representing S, draw a straight line L through the topmost point of the circle and x.



Let $F(x)$ be the point of intersection of L and the number line. ($F(x)$ is called the *projection of* x onto the number line.)

It can be seen that $F(x)$ is injective and surjective.
Hence S and $\mathbb{R}$ have the same cardinality, i.e. $|\mathbb{R}| = |(0,1)|$.

9.4.3 The Continuum Hypothesis

For reading only.

The Continuum Hypothesis

Recall that $\aleph_0 = |\mathbb{Z}^+|$.

Cantor himself wondered if there exists a set A such that:

$$\aleph_0 < |A| < |\mathbb{R}|$$

In 1964, Paul Cohen proved that the Continuum Hypothesis is undecidable.