

Lecture #12: Graphs Summary

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10. Graphs and Trees

10.1 Graphs: Definitions and Basic Properties

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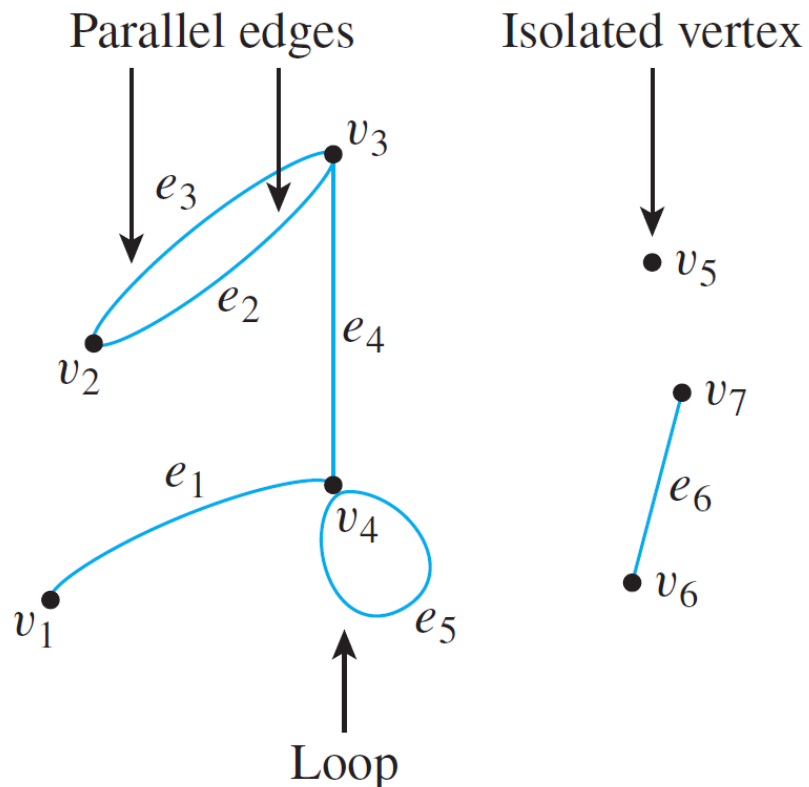
Reference: Epp's Chapter 10 Graphs and Trees

Summary

10.1 Definitions and Basic Properties

An **undirected graph** $G = (V, E)$ consists of

- a set of vertices $V = \{v_1, v_2, \dots, v_n\}$, and
- a set of (undirected) edges $E = \{e_1, e_2, \dots, e_k\}$.
- An (undirected) edge e connecting v_i and v_j is denoted as $e = \{v_i, v_j\}$.



$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$
$$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$e_1 = \{v_1, v_4\}$$
$$e_2 = e_3 = \{v_2, v_3\}$$
$$e_4 = \{v_3, v_4\}$$
$$e_5 = \{v_4, v_4\}$$
$$e_6 = \{v_6, v_7\}$$

Edges **incident** on v_4 : e_1, e_4 and e_5 .
Vertices **adjacent** to v_4 : v_1, v_3 and v_4 .
Edges **adjacent** to e_2 : e_3 and e_4 .

Definition: Undirected Graph

An undirected **graph** G consists of 2 finite sets: a nonempty set V of **vertices** and a set E of **edges**, where each (undirected) edge is associated with a set consisting of either one or two vertices called its **endpoints**.

An edge is said to **connect** its endpoints; two vertices that are connected by an edge are called **adjacent vertices**; and a vertex that is an endpoint of a loop is said to be **adjacent to itself**.

An edge is said to be **incident on** each of its endpoints, and two edges incident on the same endpoint are called **adjacent edges**.

We write $e = \{v, w\}$ for an undirected edge e incident on vertices v and w .

Definition: Directed Graph

A **directed graph**, or **digraph**, G , consists of 2 finite sets: a nonempty set V of **vertices** and a set E of **directed edges**, where each (directed) edge is associated with an **ordered pair** of vertices called its **endpoints**.

We write $e = (v, w)$ for a directed edge e from vertex v to vertex w .

Summary

10.1 Definitions and Basic Properties

Definition: Simple Graph

A **simple graph** is an undirected graph that does not have any loops or parallel edges. (That is, there is at most one edge between each pair of distinct vertices.)

Definition: Complete Graph

A **complete graph** on n vertices, $n > 0$, denoted K_n , is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices.

Definition: Bipartite Graph

A **bipartite graph** (or bigraph) is a simple graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V .

Definition: Complete Bipartite Graph

A **complete bipartite graph** is a bipartite graph on two disjoint sets U and V such that every vertex in U connects to every vertex in V .

If $|U| = m$ and $|V| = n$, the complete bipartite graph is denoted as $K_{m,n}$.

Summary

10.1 Definitions and Basic Properties

Definition: Subgraph of a Graph

A graph H is said to be a **subgraph** of graph G iff every vertex in H is also a vertex in G , every edge in H is also an edge in G , and every edge in H has the same endpoints as it has in G .

Definition: Degree of a Vertex and Total Degree of a Graph

Let G be a graph and v a vertex of G . The **degree** of v , denoted $\deg(v)$, equals the number of edges that are incident on v , with an edge that is a loop counted twice.

The **total degree of G** is the sum of the degrees of all the vertices of G .

Theorem 10.1.1 The Handshake Theorem



If the vertices of G are v_1, v_2, \dots, v_n , where $n \geq 0$, then the **total degree of G**
 $= \deg(v_1) + \deg(v_2) + \dots + \deg(v_n) = 2 \times (\text{the number of edges of } G).$

Corollary 10.1.2

The total degree of a graph is even.

Proposition 10.1.3

In any graph there are an even number of vertices of odd degree.

Definition: Indegree and outdegree of a Vertex of a Directed Graph

Let $G=(V,E)$ be a directed graph and v a vertex of G . The **indegree** of v , denoted $\mathbf{deg}^-(v)$, is the number of directed edges that end at v . The **outdegree** of v , denoted $\mathbf{deg}^+(v)$, is the number of directed edges that originate from v .

Note that
$$\sum_{v \in V} \mathbf{deg}^-(v) = \sum_{v \in V} \mathbf{deg}^+(v) = |E|$$

Definitions

Let G be a graph, and let v and w be vertices of G .

A **walk from v to w** is a finite alternating sequence of adjacent vertices and edges of G . Thus a walk has the form $v_0 e_1 v_1 e_2 \dots v_{n-1} e_n v_n$, where the v 's represent vertices, the e 's represent edges, $v_0=v$, $v_n=w$, and for all $i \in \{1, 2, \dots, n\}$, v_{i-1} and v_i are the endpoints of e_i . The number of edges, n , is the **length** of the walk.

The **trivial walk** from v to v consists of the single vertex v .

A **trail from v to w** is a walk from v to w that does not contain a repeated edge.

A **path from v to w** is a trail that does not contain a repeated vertex.

A **closed walk** is a walk that starts and ends at the same vertex.

A **circuit** (or **cycle**) is a closed walk of length at least 3 that does not contain a repeated edge.

A **simple circuit** (or **simple cycle**) is a circuit that does not have any other repeated vertex except the first and last.

An undirected graph is **cyclic** if it contains a loop or a cycle; otherwise, it is **acyclic**.

Summary

10.2 Trails, Paths, and Circuits

Definition: Connectedness

Two vertices v and w of a graph G are **connected** iff there is a walk from v to w .

The graph G is connected iff given *any* two vertices v and w in G , there is a walk from v to w . Symbolically, G is connected iff \forall vertices $v, w \in V(G)$, \exists a walk from v to w .

Lemma 10.2.1

Let G be a graph.

- If G is connected, then any two distinct vertices of G can be connected by a path.
- If vertices v and w are part of a circuit in G and one edge is removed from the circuit, then there still exists a trail from v to w in G .
- If G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnecting G .

Definition: Connected Component

A graph H is a **connected component** of a graph G iff

- The graph H is a subgraph of G ;
- The graph H is connected; and
- No connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H .

Summary

10.2 Trails, Paths, and Circuits

Definitions: Euler Circuit and Eulerian Graph

Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G .

An **Eulerian graph** is a graph that contains an Euler circuit.

Theorem 10.2.2

If a graph has an Euler circuit, then every vertex of the graph has positive even degree.

Contrapositive Version of Theorem 10.2.2

If some vertex of a graph has odd degree, then the graph doesn't have an Euler circuit.

Theorem 10.2.3

If a graph G is connected and the degree of every vertex of G is a positive even integer, then G has an Euler circuit.

Theorem 10.2.4

A graph G has an Euler circuit iff G is connected and every vertex of G has positive even degree.

Summary

10.2 Trails, Paths, and Circuits

Definition: Euler Trail

Let G be a graph, and let v and w be two distinct vertices of G . An **Euler trail/path from v to w** is a sequence of adjacent edges and vertices that starts at v , ends at w , passes through every vertex of G at least once, and traverses every edge of G exactly once.

Corollary 10.2.5

Let G be a graph, and let v and w be two distinct vertices of G . There is an Euler trail from v to w iff G is connected, v and w have odd degree, and all other vertices of G have positive even degree.

Summary

10.2 Trails, Paths, and Circuits

Definitions: Hamiltonian Circuit and Hamiltonian Graph

Given a graph G , a **Hamiltonian circuit** for G is a simple circuit that includes every vertex of G . (That is, every vertex appears exactly once, except for the first and the last, which are the same.)

A **Hamiltonian graph** (also called **Hamilton graph**) is a graph that contains a Hamiltonian circuit.

Proposition 10.2.6

If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties:

1. H contains every vertex of G .
2. H is connected.
3. H has the same number of edges as vertices.
4. Every vertex of H has degree 2.

Definition: Adjacency Matrix of a Directed Graph

Let G be a directed graph with ordered vertices v_1, v_2, \dots, v_n . The **adjacency matrix of G** is the $n \times n$ matrix $\mathbf{A} = (a_{ij})$ over the set of non-negative integers such that

$$a_{ij} = \text{the number of arrows from } v_i \text{ to } v_j \text{ for all } i, j = 1, 2, \dots, n.$$

Definition: Adjacency Matrix of an Undirected Graph

Let G be an undirected graph with ordered vertices v_1, v_2, \dots, v_n . The **adjacency matrix of G** is the $n \times n$ matrix $\mathbf{A} = (a_{ij})$ over the set of non-negative integers such that

$$a_{ij} = \text{the number of edges connecting } v_i \text{ and } v_j \text{ for all } i, j = 1, 2, \dots, n.$$

Definition: Symmetric Matrix

An $n \times n$ square matrix $\mathbf{A} = (a_{ij})$ is called **symmetric** iff for all $i, j = 1, 2, \dots, n$,

$$a_{ij} = a_{ji}.$$

Summary

10.3 Matrix Representations of Graphs

Definition: n^{th} Power of a Matrix

For any $n \times n$ matrix \mathbf{A} , the **powers of \mathbf{A}** are defined as follows:

$\mathbf{A}^0 = \mathbf{I}$ where \mathbf{I} is the $n \times n$ identity matrix

$\mathbf{A}^n = \mathbf{A} \mathbf{A}^{n-1}$ for all integers $n \geq 1$

Theorem 10.3.2

If G is a graph with vertices v_1, v_2, \dots, v_m and \mathbf{A} is the adjacency matrix of G , then for each positive integer n and for all integers $i, j = 1, 2, \dots, m$,

the ij -th entry of \mathbf{A}^n = the number of walks of length n from v_i to v_j .

Definition: Isomorphic Graph

Let $G = (V_G, E_G)$ and $G' = (V_{G'}, E_{G'})$ be two graphs.

G is isomorphic to G' , denoted $G \cong G'$, if and only if there exist bijections $g: V_G \rightarrow V_{G'}$ and $h: E_G \rightarrow E_{G'}$ that preserve the edge-endpoint functions of G and G' in the sense that for all $v \in V_G$ and $e \in E_G$,

$$v \text{ is an endpoint of } e \Leftrightarrow g(v) \text{ is an endpoint of } h(e).$$

Alternative definition (for simple graphs)

Let $G = (V_G, E_G)$ and $G' = (V_{G'}, E_{G'})$ be two simple graphs.

G is isomorphic to G' if and only if there exists a permutation $\pi: V_G \rightarrow V_{G'}$ such that $\{u, v\} \in E_G \Leftrightarrow \{\pi(u), \pi(v)\} \in E_{G'}$.

Theorem 10.4.1 Graph Isomorphism is an Equivalence Relation

Let S be a set of graphs and let \cong be the relation of graph isomorphism on S . Then \cong is an equivalence relation on S .

Definition: Planar Graph

A **planar graph** is a graph that can be drawn on a (two-dimensional) plane without edges crossing.

Euler's Formula

For a connected planar simple graph $G = (V, E)$ with $e = |E|$ and $v = |V|$, if we let f be the number of faces, then

$$f = e - v + 2$$