LIMITS & CONTINUITY

Definition: $\lim_{x\to c} f(x) = L$ if $\forall \epsilon > 0, \exists \delta > 0 : |x-c| < \delta \Rightarrow |f(x) - L| < \epsilon$

One-sided: $\lim_{x\to c^-} f(x)$ (left), $\lim_{x\to c^+} f(x)$ (right)

$\lim_{x\to c} f(x)$ exists $\Leftrightarrow$ both one-sided limits exist and are equal

Limit Laws:
- $\lim[f \pm g] = \lim f \pm \lim g$
- $\lim[cf] = c \lim f$
- $\lim[fg] = (\lim f)(\lim g)$
- $\lim[\frac{f}{g}] = \frac{\lim f}{\lim g}$ if $\lim g \neq 0$
- If g continuous at b, $\lim_{x\to c} f(x) = b$, then $\lim_{x\to c} g(f(x)) = g(b)$

Indeterminate Forms: $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty, 0^0, 1^\infty, \infty^0$

Key Limits:
- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to 0} \frac{\tan x}{x} = 1$
- $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$
- $\lim_{x\to 0} (1+x)^{1/x} = e$
- $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$
- $\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$ for $n > 0$

For $\frac{P(x)}{Q(x)}$ as $x \to \pm\infty$: Compare leading terms

Squeeze Theorem: If $g(x) \leq f(x) \leq h(x)$ near c and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x) = L$

L'Hôpital's Rule: If $\lim \frac{f}{g}$ is $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim \frac{f}{g} = \lim \frac{f'}{g'}$ (if RHS exists)

For $0\cdot\infty$: rewrite as $\frac{0}{0}$ or $\frac{\infty}{\infty}$

For $\infty - \infty$: combine fractions

For $0^0, 1^\infty, \infty^0$: take ln, use L'H, then exponentiate

Continuity: f continuous at c if:
- $f(c)$ is defined
- $\lim_{x\to c} f(x)$ exists
- $\lim_{x\to c} f(x) = f(c)$

At endpoints: use one-sided limits

Continuous Functions: polynomials, rational (where defined), trig, exp, log, combinations

IVT: If f continuous on $[a,b]$ and k between $f(a)$ and $f(b)$, then $\exists c \in [a,b] : f(c) = k$

Infinite Limits: $\lim_{x\to c} \frac{1}{(x-c)^2} = \infty$

$\lim_{x\to c^+} \frac{1}{x-c} = \infty$, $\lim_{x\to c^-} \frac{1}{x-c} = -\infty$

Horizontal Asymptote: $y = L$ if $\lim_{x\to\pm\infty} f(x) = L$

Vertical Asymptote: $x = c$ if $\lim_{x\to c^\pm} f(x) = \pm\infty$

DERIVATIVES

Definition: $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$

Alternative: $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$

Notations: $f'(x) = \frac{df}{dx} = \frac{dy}{dx} = Df(x) = y'$

Differentiability $\Rightarrow$ **Continuity** (converse false)

Basic Rules:
- Constant: $(c)' = 0$
- Power: $(x^n)' = nx^{n-1}$
- Constant Multiple: $(cf)' = cf'$
- Sum/Difference: $(f \pm g)' = f' \pm g'$

Product Rule: $(uv)' = u'v + uv'$

Quotient Rule: $(\frac{u}{v})' = \frac{u'v - uv'}{v^2}$

Chain Rule: $(f(g(x)))' = f'(g(x))g'(x)$

Notation: $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

Implicit Differentiation: Differentiate both sides w.r.t. x, treat y as function of x, solve for $\frac{dy}{dx}$

Example: $x^2 + y^2 = 25 \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

Inverse Function: $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$

Or: $\frac{dx}{dy} = \frac{1}{dy/dx}$

Logarithmic Differentiation: For $y = f(x)^{g(x)}$ or products/quotients:
- Take ln of both sides
- Differentiate implicitly
- Solve for $\frac{dy}{dx}$
- Substitute y back

Parametric Equations: $x = f(t), y = g(t)$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$

$\frac{d^2y}{dx^2} = \frac{d}{dt}(\frac{dy}{dx})\cdot\frac{1}{dx/dt} = \frac{g''f' - g'f''}{(f')^3}$

Higher Order: $f''(x) = \frac{d^2f}{dx^2}$, $f'''(x) = \frac{d^3f}{dx^3}$, etc.

$\frac{d^n}{dx^n}(x^n) = n!$

APPLICATIONS OF DERIVATIVES

Tangent Line at $(a, f(a))$: $y - f(a) = f'(a)(x-a)$

Normal Line at $(a, f(a))$: $y - f(a) = -\frac{1}{f'(a)}(x-a)$ (if $f'(a) \neq 0$)

Linear Approximation: $f(x) \approx f(a) + f'(a)(x-a)$ near $x = a$

Increasing/Decreasing:
- $f' > 0$ on $(a,b) \Rightarrow f$ increasing on $[a,b]$
- $f' < 0$ on $(a,b) \Rightarrow f$ decreasing on $[a,b]$

Concavity:
- $f'' > 0 \Rightarrow$ concave up
- $f'' < 0 \Rightarrow$ concave down

Inflection Point: Concavity changes; $f'' = 0$ or f'' DNE

Critical Point: c is critical if:
- c is interior point AND
- $f'(c) = 0$ or $f'(c)$ DNE

Local Extrema: Can only occur at critical points or endpoints

First Derivative Test: At critical point c:
- f' changes $+$ to $-$: local max
- f' changes $-$ to $+$: local min
- f' doesn't change sign: neither

Second Derivative Test: If $f'(c) = 0$:
- $f''(c) > 0$: local min
- $f''(c) < 0$: local max
- $f''(c) = 0$: inconclusive

Absolute Extrema on $[a,b]$**:**
1. Find critical points in (a,b)
2. Evaluate f at critical points and endpoints
3. Largest value is absolute max, smallest is absolute min

Extreme Value Theorem: If f continuous on $[a,b]$, then f has absolute max and min on $[a,b]$

Rolle's Theorem: If f continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$, then $\exists c \in (a,b) : f'(c) = 0$

Mean Value Theorem: If f continuous on $[a,b]$, differentiable on (a,b), then $\exists c \in (a,b) : f'(c) = \frac{f(b)-f(a)}{b-a}$

Related Rates:
1. Draw diagram, label variables
2. Write equation relating variables
3. Differentiate w.r.t. time t
4. Substitute known values
5. Solve for unknown rate

Optimization:
1. Identify quantity to optimize
2. Write as function of one variable
3. Find domain
4. Find critical points
5. Use First Deriv Test or evaluate at critical pts and endpoints

INTEGRATION

Antiderivative: $F'(x) = f(x) \Rightarrow F$ is antiderivative of f

Indefinite Integral: $\int f(x)dx = F(x) + C$ where $F' = f$

Properties:
- $\int[f \pm g]dx = \int f dx \pm \int g dx$
- $\int cf dx = c\int f dx$

Riemann Sum: $\sum_{i=1}^n f(x_i^*)\Delta x$ where $\Delta x = \frac{b-a}{n}$

Definite Integral: $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x$

FTC Part 1: $\int_a^b f(x)dx = F(b) - F(a)$ where $F' = f$

Notation: $[F(x)]_a^b = F(b) - F(a)$

FTC Part 2: $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

More generally: $\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))g'(x)$

Properties of Definite Integrals:
- $\int_a^a f = 0$
- $\int_a^b f = -\int_b^a f$
- $\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$
- $\int_a^b cf = c\int_a^b f$
- $\int_a^b f = \int_a^c f + \int_c^b f$
- If $f \geq 0$ on $[a,b]$: $\int_a^b f \geq 0$
- If $f \geq g$ on $[a,b]$: $\int_a^b f \geq \int_a^b g$
- If $m \leq f \leq M$ on $[a,b]$: $m(b-a) \leq \int_a^b f \leq M(b-a)$
- Even function: $\int_{-a}^a f = 2\int_0^a f$
- Odd function: $\int_{-a}^a f = 0$

Substitution (Indefinite): $\int f(g(x))g'(x)dx = \int f(u)du$ where $u = g(x)$, $du = g'(x)dx$

Substitution (Definite): $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Choose u by LIATE: Logarithmic, Inverse trig, Algebraic, Trig, Exponential

Partial Fractions: Decompose $\frac{P(x)}{Q(x)}$:
- If $\deg(P) \geq \deg(Q)$: long division first
- $(ax+b)$ factor: $\frac{A}{ax+b}$
- $(ax+b)^2$ factor: $\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
- (ax^2+bx+c) irreducible: $\frac{Ax+B}{ax^2+bx+c}$

Trigonometric Substitution:
- $\sqrt{a^2-x^2}$: use $x = a\sin\theta$, $dx = a\cos\theta d\theta$
- $\sqrt{a^2+x^2}$: use $x = a\tan\theta$, $dx = a\sec^2\theta d\theta$
- $\sqrt{x^2-a^2}$: use $x = a\sec\theta$, $dx = a\sec\theta\tan\theta d\theta$

Trig Integrals:
- $\int \sin^m x\cos^n x dx$: if m or n odd, use substitution
- If both even: use power-reducing formulas
- $\int \tan^m x\sec^n x dx$: if n even, factor $\sec^2 x$; if m odd, factor $\sec x\tan x$

Improper Integrals:

Type I: $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$

Type II (discontinuity at c): $\int_a^b f = \lim_{\epsilon\to 0^+}[\int_a^{c-\epsilon} f + \int_{c+\epsilon}^b f]$

Converges if limit is finite; diverges otherwise

Comparison Test: If $0 \leq f \leq g$:
- $\int g$ converges $\Rightarrow \int f$ converges
- $\int f$ diverges $\Rightarrow \int g$ diverges

APPLICATIONS OF INTEGRATION

Area Between Curves:

Vertical strips: $A = \int_a^b [f(x) - g(x)]dx$ where $f(x) \geq g(x)$

Horizontal strips: $A = \int_c^d [h(y) - k(y)]dy$ where $h(y) \geq k(y)$

General: $A = \int |f - g|$ (split where functions cross)

Volume by Slicing: $V = \int_a^b A(x)dx$ where $A(x)$ is cross-sectional area

Disk Method: Revolve around x-axis: $V = \pi\int_a^b [f(x)]^2 dx$

Revolve around y-axis: $V = \pi\int_c^d [g(y)]^2 dy$

Washer Method: $V = \pi\int_a^b ([R(x)]^2 - [r(x)]^2)dx$ where $R(x) =$ outer radius, $r(x) =$ inner radius

Shell Method: Revolve around y-axis: $V = 2\pi\int_a^b x\cdot f(x)dx$

Revolve around x-axis: $V = 2\pi\int_c^d y\cdot g(y)dy$

General: $V = 2\pi\int$(radius)(height)

Arc Length:

For $y = f(x)$, $a \leq x \leq b$: $L = \int_a^b \sqrt{1+[f'(x)]^2}dx$

For $x = g(y)$, $c \leq y \leq d$: $L = \int_c^d \sqrt{1+[g'(y)]^2}dy$

Parametric $(x(t), y(t))$, $\alpha \leq t \leq \beta$: $L = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2}dt$

Surface Area of Revolution:

Around x-axis: $S = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}dx$

Around y-axis: $S = 2\pi\int_a^b x\sqrt{1+[f'(x)]^2}dx$

SEQUENCES & SERIES

Sequence: $\{a_n\}_{n=1}^\infty = a_1, a_2, a_3, \ldots$

Limit: $\lim_{n\to\infty} a_n = L$ (converges) or DNE (diverges)

Monotonic: Increasing: $a_n \leq a_{n+1}$; Decreasing: $a_n \geq a_{n+1}$

Bounded: $\exists M : |a_n| \leq M$ for all n

Monotone Convergence: If $\{a_n\}$ monotonic and bounded, then converges

Squeeze Theorem: If $a_n \leq b_n \leq c_n$ and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$

Important Limits:
- $\lim_{n\to\infty} \frac{1}{n^p} = 0$ if $p > 0$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$
- $\lim_{n\to\infty} \sqrt[n]{a} = 1$ if $a > 0$
- $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$
- $\lim_{n\to\infty} (1+\frac{x}{n})^n = e^x$
- $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any x

Series: $\sum_{n=1}^\infty a_n = a_1 + a_2 + a_3 + \cdots$

Partial Sum: $S_n = \sum_{k=1}^n a_k$

Series converges if $\lim_{n\to\infty} S_n$ exists (finite); otherwise diverges

Geometric Series: $\sum_{n=0}^\infty ar^n = \frac{a}{1-r}$ if $|r| < 1$; diverges if $|r| \geq 1$

Telescoping Series: $\sum(b_n - b_{n+1}) = b_1 - \lim b_n$

Properties: If $\sum a_n, \sum b_n$ converge:
- $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$
- $\sum ca_n = c\sum a_n$
- Changing first few terms doesn't affect convergence

n-th Term (Divergence) Test: If $\lim_{n\to\infty} a_n \neq 0$ or DNE, then $\sum a_n$ diverges

Caution: $\lim a_n = 0$ does NOT imply $\sum a_n$ converges

Integral Test: If f positive, continuous, decreasing on $[1,\infty)$ with $f(n) = a_n$, then $\sum_{n=1}^\infty a_n$ and $\int_1^\infty f(x)dx$ both converge or both diverge

p-Series: $\sum_{n=1}^\infty \frac{1}{n^p}$ converges iff $p > 1$

Comparison Test: Suppose $0 \leq a_n \leq b_n$ for all n:
- If $\sum b_n$ converges, then $\sum a_n$ converges
- If $\sum a_n$ diverges, then $\sum b_n$ diverges

Limit Comparison Test: If $a_n, b_n > 0$ and $\lim_{n\to\infty}\frac{a_n}{b_n} = c$ where $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge

Ratio Test: Let $L = \lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|$
- $L < 1$: series converges absolutely
- $L > 1$: series diverges
- $L = 1$: inconclusive

Root Test: Let $L = \lim_{n\to\infty}\sqrt[n]{|a_n|}$
- $L < 1$: series converges absolutely
- $L > 1$: series diverges
- $L = 1$: inconclusive

Alternating Series: $\sum_{n=1}^\infty (-1)^{n-1}b_n$ or $\sum_{n=1}^\infty (-1)^n b_n$ where $b_n > 0$

Alternating Series Test: If b_n decreasing and $\lim_{n\to\infty} b_n = 0$, then series converges

Alternating Series Error: $|S - S_n| \leq b_{n+1}$ (error $\leq$ first omitted term)

Absolute Convergence: $\sum a_n$ converges absolutely if $\sum |a_n|$ converges

Conditional Convergence: $\sum a_n$ converges but $\sum |a_n|$ diverges

Theorem: Absolute convergence $\Rightarrow$ convergence (converse false)

POWER SERIES

Form: $\sum_{n=0}^\infty c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$

Center: a; Coefficients: c_n

Convergence: Always converges at $x = a$

Radius of Convergence R**:**
- $R = 0$: converges only at $x = a$
- $R = \infty$: converges for all x
- $0 < R < \infty$: converges for $|x-a| < R$, diverges for $|x-a| > R$

Finding R**:** Use Ratio or Root Test on $\sum c_n(x-a)^n$

If $L = \lim|\frac{c_{n+1}}{c_n}|$ or $L = \lim\sqrt[n]{|c_n|}$ exists, then $R = \frac{1}{L}$

Interval of Convergence: $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, or $[a-R, a+R]$

Must test endpoints $x = a \pm R$ separately

Operations: Within interval of convergence:
- Addition: $\sum c_n(x-a)^n + \sum d_n(x-a)^n = \sum(c_n+d_n)(x-a)^n$
- Multiplication by $x-a$: $(x-a)\sum c_n(x-a)^n = \sum c_n(x-a)^{n+1}$
- Differentiation: $[\sum c_n(x-a)^n]' = \sum nc_n(x-a)^{n-1}$
- Integration: $\int\sum c_n(x-a)^n dx = C + \sum\frac{c_n(x-a)^{n+1}}{n+1}$

Differentiation and integration preserve radius of convergence

Taylor Series: $f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$

Maclaurin Series: Taylor series at $a = 0$

$f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!}x^n$

Important Series: (all Maclaurin)
- $\frac{1}{1-x} = \sum_{n=0}^\infty x^n$, $|x| < 1$
- $\frac{1}{1+x} = \sum_{n=0}^\infty (-1)^n x^n$, $|x| < 1$
- $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$, all x
- $\sin x = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, all x
- $\cos x = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!}$, all x
- $\ln(1+x) = \sum_{n=1}^\infty \frac{(-1)^{n-1}x^n}{n}$, $-1 < x \leq 1$
- $\tan^{-1} x = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{2n+1}$, $|x| \leq 1$
- $(1+x)^k = \sum_{n=0}^\infty \binom{k}{n}x^n$, $|x| < 1$ (binomial series)

Taylor Polynomial: $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$
Taylor's Remainder: $R_n(x) = f(x) - T_n(x)$
If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then $|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$

VECTORS IN SPACE

Vector: $\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$
Magnitude: $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
Unit Vector: $\hat{v} = \frac{\vec{v}}{|\vec{v}|}$ (has magnitude 1)
Standard Basis: $\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$, $\vec{k} = \langle 0,0,1 \rangle$
Vector from P_1 to P_2: $\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$
Operations:

- Addition: $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
- Scalar mult: $c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$
- $\vec{a} = \vec{b}$ iff all components equal

Parallel: $\vec{a} \parallel \vec{b}$ iff $\vec{a} = c\vec{b}$ for some scalar c
Dot Product: $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ where θ is angle between vectors
Properties of Dot Product:

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$
- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

Angle: $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$
Orthogonal (Perpendicular): $\vec{a} \perp \vec{b}$ iff $\vec{a} \cdot \vec{b} = 0$
Projection: $\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\vec{a}$
Scalar Component: $\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$
Cross Product: $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
$= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$
Properties of Cross Product:

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (anti-commutative)
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
- $\vec{a} \times \vec{a} = \vec{0}$
- $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$

Magnitude: $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$
Parallel: $\vec{a} \parallel \vec{b}$ iff $\vec{a} \times \vec{b} = \vec{0}$
Area of Parallelogram: $|\vec{a} \times \vec{b}|$
Area of Triangle: $\frac{1}{2}|\vec{a} \times \vec{b}|$
Direction: Right-hand rule
Dist. Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
Midpoint: $M = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2})$
Sphere: $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$
Center: (h,k,l), Radius: r

LINES & PLANES

Line (Vector Eq): $\vec{r}(t) = \vec{r}_0 + t\vec{v}$
where $\vec{r}_0$ is position vector of point on line, $\vec{v}$ is direction vector
Line (Parametric): $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$
Direction numbers: a, b, c
Line through P_1, P_2: Use $\vec{v} = \overrightarrow{P_1P_2}$ as direction vector
Parallel Lines: Direction vectors are parallel
Skew Lines: Neither parallel nor intersecting
Angle between Lines: Angle between direction vectors
Distance from Point to Line: $d = \frac{|\overrightarrow{PQ} \times \vec{v}|}{|\vec{v}|}$
where Q is point on line, $\vec{v}$ is direction vector
Plane (Vector Eq): $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$ or $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$
where $\vec{n}$ is normal vector, $\vec{r}_0$ is position vector of point on plane
Plane (Scalar Eq): $ax + by + cz = d$ or $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
Normal vector: $\vec{n} = \langle a, b, c \rangle$
Plane through 3 Points: Use $\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$
Parallel Planes: Normal vectors are parallel
Angle between Planes: Angle between normal vectors: $\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$
Distance (Point to Plane): $d = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$
Distance (Parallel Planes): Use distance from point on one plane to other plane
Line of Intersection: Solve plane equations simultaneously

VECTOR FUNCTIONS

Vector Function: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$
Limit: $\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$
Continuity: $\vec{r}$ continuous at a if $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$
Derivative: $\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$
$\vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$
Tangent Vector: $\vec{r}'(t)$ is tangent to curve at $\vec{r}(t)$
Tangent Line: $\vec{L}(s) = \vec{r}(t_0) + s\vec{r}'(t_0)$
Differentiation Rules:

- $[\vec{u} + \vec{v}]' = \vec{u}' + \vec{v}'$
- $[c\vec{u}]' = c\vec{u}'$
- $[f(t)\vec{u}(t)]' = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
- $[\vec{u} \cdot \vec{v}]' = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$
- $[\vec{u} \times \vec{v}]' = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$

Integration: $\int \vec{r}(t)dt = \langle \int f(t)dt, \int g(t)dt, \int h(t)dt \rangle$
Definite Integral: $\int_a^b \vec{r}(t)dt = \langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \rangle$
Arc Length: $L = \int_a^b |\vec{r}'(t)|dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt$

MULTIVARIABLE FUNCTIONS

Function of 2 Variables: $z = f(x,y)$
Domain $D \subseteq \mathbb{R}^2$, Range $\subseteq \mathbb{R}$
Graph: Surface $z = f(x,y)$ in $\mathbb{R}^3$
Level Curve: $f(x,y) = k$ (in xy-plane)
Contour Plot: Multiple level curves for various k
Function of 3 Variables: $w = f(x,y,z)$
Level Surface: $f(x,y,z) = k$ (in $\mathbb{R}^3$)
Partial Derivatives:
$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$
$f_y(x,y) = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$
Hold other variable constant, differentiate
Higher Order: $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$, $f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$
Clairaut's Theorem: If f_{xy} and f_{yx} continuous, then $f_{xy} = f_{yx}$
Chain Rule (1 param): $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$
Chain Rule (2 param):
$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$
$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$
Implicit Differentiation: For $F(x,y,z) = 0$:
$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$ (if $F_z \neq 0$)
Tangent Plane to $z = f(x,y)$ at $(a, b, f(a,b))$:
$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
Normal vector: $\langle f_x(a,b), f_y(a,b), -1 \rangle$
Tangent Plane to Level Surface $F(x,y,z) = k$ at (x_0, y_0, z_0):
$F_x(x_0,y_0,z_0)(x - x_0) + F_y(x_0,y_0,z_0)(y - y_0) + F_z(x_0,y_0,z_0)(z - z_0) = 0$
Normal vector: $\nabla F(x_0, y_0, z_0)$
Linear Approximation: $f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
Differentials: $dz = f_x(x,y)dx + f_y(x,y)dy$
$\Delta z \approx dz$ for small Δx, Δy
Gradient: $\nabla f = \langle f_x, f_y \rangle$ (2D) or $\nabla f = \langle f_x, f_y, f_z \rangle$ (3D)
Directional Derivative: $D_{\vec{u}}f = \nabla f \cdot \vec{u}$ where $|\vec{u}| = 1$
Rate of change of f in direction $\vec{u}$
Maximum Rate of Increase: $|\nabla f|$ in direction of ∇f
Maximum Rate of Decrease: $-|\nabla f|$ in direction of $-\nabla f$
Gradient Perpendicular to Level Curves/Surfaces: $\nabla f \perp$ level curve/surface
Critical Point: (a,b) where $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or one doesn't exist
Second Derivative Test: At critical point (a,b):
$D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$

- $D > 0$ and $f_{xx}(a,b) > 0$: local minimum
- $D > 0$ and $f_{xx}(a,b) < 0$: local maximum
- $D < 0$: saddle point
- $D = 0$: inconclusive

Absolute Extrema on Closed Bounded Domain:

1. Find critical points in interior
2. Find extreme values on boundary
3. Compare all values

DOUBLE INTEGRALS

Riemann Sum: $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$ where $\Delta A = \Delta x \Delta y$
Double Integral: $\iint_R f(x,y)dA = \lim_{m,n \to \infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$
Volume: If $f(x,y) \geq 0$, then $V = \iint_R f(x,y)dA$
Fubini's Theorem: For rectangle $R = [a,b] \times [c,d]$:
$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dy\,dx = \int_c^d \int_a^b f(x,y)dx\,dy$
Type I R.: $D = \{(x,y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$
$\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx$
Type II R.: $D = \{(x,y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$
$\iint_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\,dy$
Properties:

- $\iint_D [f \pm g]dA = \iint_D f dA \pm \iint_D g dA$
- $\iint_D cf dA = c\iint_D f dA$
- If $f \geq g$: $\iint_D f dA \geq \iint_D g dA$
- If $D = D_1 \cup D_2$: $\iint_D f = \iint_{D_1} f + \iint_{D_2} f$

Area: $A(D) = \iint_D 1\,dA$
Average Value: $f_{avg} = \frac{1}{A(D)}\iint_D f(x,y)dA$
Polar Coordinates: $x = r\cos\theta$, $y = r\sin\theta$
$r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$
$dA = r\,dr\,d\theta$ (don't forget the r!)
Polar Rectangle: $R = \{(r,\theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta\}$
$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$
When to Use Polar:

- Region involves circles, sectors
- Integrand contains $x^2 + y^2$
- Simplifies the integral

Surface Area: For $z = f(x,y)$ over region D:
$S = \iint_D \sqrt{1 + [f_x]^2 + [f_y]^2}\,dA$

DIFFERENTIAL EQUATIONS

Separable: $\frac{dy}{dx} = f(x)g(y)$
Separate: $\frac{dy}{g(y)} = f(x)dx$, then integrate both sides
First Order Linear: $\frac{dy}{dx} + P(x)y = Q(x)$
Integrating factor: $I(x) = e^{\int P(x)dx}$
Multiply by $I(x)$: $\frac{d}{dx}[y \cdot I(x)] = Q(x) \cdot I(x)$
Solution: $y \cdot I(x) = \int Q(x)I(x)dx$
Bernoulli Equation: $y' + p(x)y = q(x)y^n$ where $n \neq 0, 1$
Substitute $u = y^{1-n}$: $u' + (1-n)p(x)u = (1-n)q(x)$ (linear in u)
If $n > 0$: $y = 0$ is also a solution
Homogeneous: $y' = g(\frac{y}{x})$
Substitute $v = \frac{y}{x}$, so $y = xv$, $y' = v + xv'$
Get: $v + xv' = g(v)$ or $xv' = g(v) - v$ (separable in v)
Exact Form $y' = f(ax + by)$: Substitute $u = ax + by$
Applications:

- Exponential growth/decay: $\frac{dP}{dt} = kP$, solution $P = P_0e^{kt}$
- Newton's cooling: $\frac{dT}{dt} = k(T - T_{env})$
- Mixing problems: Track amount in/out
- Half-life: $t_{1/2} = \frac{\ln 2}{k}$

QUADRIC SURFACES

Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
All cross-sections are ellipses
Elliptic Paraboloid: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$
Opens upward if $c > 0$, downward if $c < 0$
Hyperbolic Paraboloid (Saddle): $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$
Cylinder: Missing one variable
$x^2 + y^2 = r^2$: circular cylinder (axis: z-axis)
$y = x^2$: parabolic cylinder

TRICKS & TIPS

Completing the Square: $x^2 + bx = (x + \frac{b}{2})^2 - \frac{b^2}{4}$
Partial Fractions Setup:

- $(x-a)$: $\frac{A}{x-a}$
- $(x-a)^2$: $\frac{A}{x-a} + \frac{B}{(x-a)^2}$
- $(x^2 + bx + c)$ irreducible: $\frac{Ax+B}{x^2+bx+c}$

Trig Identities Quick Reference:

- $\sin^2 + \cos^2 = 1$
- $\sec^2 = 1 + \tan^2$
- $\csc^2 = 1 + \cot^2$
- Double angle: $\sin 2x = 2\sin x\cos x$
- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- Power reducing: $\sin^2 x = \frac{1 - \cos 2x}{2}$, $\cos^2 x = \frac{1 + \cos 2x}{2}$

Common Mistakes to Avoid:

- Forgetting r in polar: $dA = r\,dr\,d\theta$
- Chain rule in partial derivatives
- Testing endpoints in power series
- Normal vector for tangent plane
- Direction vector must be unit vector for directional derivative

Quick Tests for Series:

1. Always try n-th term test first
2. p-series or geometric? Use formulas
3. Factorials or exponentials? Try ratio test
4. Powers of n? Try root test
5. Can't determine? Try comparison
6. Alternating? Check alternating series test