

## Lab 5: Scan Matching

Instructor: Rahul Mangharam

Name: STUDENT NAME, StudentID: ID



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**Course Policy:** Read all the instructions below carefully before you start working on the assignment, and before you make a submission. All sources of material must be cited. The University Academic Code of Conduct will be strictly enforced.

**THIS IS A GROUP ASSIGNMENT.** Submit one from each team.

## 1 Learning outcomes

The following fundamentals should be understood by the students upon completion of this lab:

- Localization
- Odometry Estimation
- Convex optimization
- C++ ROS2
- Quadratic Programming

## 2 Overview

This section of the class deals with the problem of localization in Robotics and provides an introduction to localization and why it is important in the autonomy stack. Through the lab, one of the most fundamental algorithms of localization, scan matching, is implemented. It uses the Iterative Closest Point algorithm which has been introduced in class. You can take reference from the [Andre Censi PLICP paper](#) which is referenced in the pinned piazza post. By the end of this lab you will have a certain level of knowledge and expertise in localization of a robot given a mapped environment and how it is important in path planning and trajectory tracking.

### 3 Theoretical Questions

$$1. M_i = \begin{pmatrix} 1 & 0 & p_{i0} & -p_{i1} \\ 0 & 1 & p_{i1} & p_{i0} \end{pmatrix}$$

- (a) Show that  $B_i := M_i^T M_i$  is symmetric.
- (b) Demonstrate that  $B_i$  is positive semi-definite

2. The following is the optimization problem:

$$x^* = \operatorname{argmin}_{x \in \mathbb{R}^4} \sum_{i=1}^n \|M_i x - \pi_i\|_2^2 \quad \text{s.t.} \quad x_3^2 + x_4^2 = 1$$

- (a) Find the matrices M, W and g which give you the formulation

$$x^* = \operatorname{argmin}_{x \in \mathbb{R}^4} x^T M x + g^T x \quad \text{s.t.} \quad x^T W x = 1$$

- (b) Show that M and W are positive semi definite.

### 4 Skeleton Code Explanation

The provided skeleton consists of three main program files and their respective header files which are needed to run the `scan_matching` package. You have to work on only two of these program files. `Correspond.cpp` and `transform.cpp`. You will not necessarily have to change anything anywhere unless explicitly required by your code. In which case, you should specify the change made in your `README`.

#### 4.1 Folder Contents

1. **scan\_match.cpp** Contains the main function and the driver code for running the scan matching algorithm. The node built is through this file and is called `scanmatch_node`.
2. **correspond.cpp and correspond.h** Contains the function definitions(in the `correspond.h` header file), the various structs needed for calculations (in the `correspond.h` header file) and the implementation of the functions needed for performing fast correspondence search and naive correspondence search.

In `correspond.cpp` you have to fill in the function: `getCorrespondence()`

#### (a) Input:

- i. `old_points`: vector of struct points containing the old points of the scan
- ii. `trans_points`: vector of struct points containing the new points transformed in the previous scans frame
- iii. `jump_Table`: jump table created from the helper function
- iv. `c` : vector of struct correspondences as a references. Being a reference this will also be the output.

#### (b) Output:

- i. `c`: vector of struct correspondences as a reference will be changed in place.

(c) **Helper Functions:**

- i. `computeJump`: computes the jump table based on the previous table and current scan points

3. **transform.cpp and transform.h** Contains the function definitions, function implementations and structs needed to perform transformation of points and updating the transforms by closed form ICP optimization technique taught in class.

You have to fill in the function: `updateTransform()`

(a) **Input:**

- i. `corresponds`: vector of struct correspondence containing corresponding points
- ii. `curr_trans`: contains the transform found by the previous optimization step. Being a reference, you will update this with the new transform.

(b) **Output**

- i. `curr_trans`: to be updated in place as the new transform using the previous transform.

(c) **Helper Functions**

- i. `transformPoints()`: you can use this function apply any transform to the set of points. Input the required transform and the points to be transformed.
- ii. `get_cubic_root`: use this function to find the cubic roots given the 4 input coefficients of a cubic polynomial.
- iii. `greatest_real_root()`: use this function to find the greatest real root of a quartic polynomial using the 5 coefficients of the quartic polynomial as the input.

4. **visualization.cpp and visualization.h**

You will not necessarily have to make any changes to these files. If you correctly fill in the previous files you will be able to visualize the odometry computed through the ICP scan matching algorithm.

5. Comments:

- We understand that the performance of the scan matching will not be ideal, the report should reflect an analysis of the performance and explanation of the limitations.
- The initial implementation uses the naive correspondence search. This can be compared to the performance after implementing the improved correspondence search.

## 4.2 Data Structures

The skeleton uses a few data structures which are defined in the different program files. You are free to use the data structures provided or make new ones of your own. The functions however should be implemented as defined.

1. **Point**

The struct the radial distance and angular distance of a point from the car frame. In this struct there are few functions implemented which can be used to derive other information from the points:

- `distToPoint(point P)`: find the distance to another point

- `distToPoint2(point P)`: find the square of the distance to another point
  - `adialGap(point P)`: find the radial gap to another point
  - `getX()`: get the x coordinate of the point
  - `getY()`: get the y coordinate of the point
  - `wrapTheta()`: wrap theta around 2pi during rotation
  - `rotate(phi)`: rotate the point by angle phi
  - `translate(x,y)`: translate a point by distance x and y
  - `getVector()`: get the vector (in x and y)
2. **Correspondence** This struct stores the correspondence values which you find through the fast search algorithm. It contains the transformed points:  $P$ , original points :  $P_0$  , first best point:  $P_{j1}$ , second best point  $P_{j2}$ .
- `getNormal()`: get normal of the correspondence
  - `getPiGeo()`: get correspondence point as a geometry message
  - `getPiVec()`: get correspondence point as a vector message
3. **Transform** The struct stores the transform which you calculate through optimization at every step. it contains the x translation, y translation and the theta rotation.
- `apply(point P)`: apply the transform to a provided point
  - `getMatrix()`: get the transformation matrix from the found transform

## 5 Algorithm Implementation

In order to implement the scan matching algorithm you have to solve the following optimization problem:

$$\min_x \mathbf{x}^T \mathbf{M} \mathbf{x} + \mathbf{g}^T \mathbf{x} \quad (5.1)$$

$$\mathbf{x} \left( \underbrace{\sum_i \mathbf{M}_i^T \mathbf{C}_i \mathbf{M}_i}_M \right) \mathbf{x} + \left( \underbrace{\sum_i -2\pi_i^T \mathbf{C}_i \mathbf{M}_i}_g \right) \mathbf{x} \quad \text{s.t.} \quad \mathbf{x}^T \mathbf{W} \mathbf{x} = 1 \quad (5.2)$$

Follow the steps below to solve the optimization problem above.

1. First, we define the following variables:

- (a) The optimization variable which contains the position and orientation transforms to be optimized.

$$\mathbf{x} = [x_1, x_2, x_3, x_4] = [t_x, t_y, \cos\theta, \sin\theta]$$

- (b)  $\mathbf{W}$  is the weight matrix which is needed for applying the constraint  $x_3^2 + x_4^2 = 1$

$$\mathbf{W} = \begin{bmatrix} \mathbf{0}_{2 \times 2} & \mathbf{0}_{2 \times 2} \\ \mathbf{0}_{2 \times 2} & \mathbf{I}_{2 \times 2} \end{bmatrix}$$

- (c) Matrix  $\mathbf{M}_i$

$$\mathbf{M}_i = \begin{bmatrix} 1 & 0 & p_{i0} & -p_{i1} \\ 0 & 1 & p_{i1} & p_{i0} \end{bmatrix}$$

(d) Points  $\mathbf{p}_i$  in the scans

$$\mathbf{p}_i = (p_{i0}, p_{i1})$$

(e) Matrix  $\mathbf{C}_i$

$$\mathbf{C}_i = w_i \mathbf{n}_i \mathbf{n}_i^\top$$

where  $\mathbf{n}_i \in \mathbb{R}^2$  is the vector normal to the line.

2. Now, we solve the optimization problem as follows:

(a) Using Lagrange multipliers so the solution to this problem can be found and the solution takes the following form:

$$\mathbf{x} = -(2\mathbf{M} + 2\lambda\mathbf{W}^{-T})\mathbf{g} \quad (5.3)$$

where  $\mathbf{W}$  is defined in the previous step and  $\mathbf{M}$  and  $\mathbf{g}$  are known to us. See equation (5.2). We just need to find  $\lambda$  to solve this equation.

(b) We can find  $\lambda$  using some algebra tricks.

i. Write the expression of the previous equation in the form:

$$2\mathbf{M} + 2\lambda\mathbf{W} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{D} + 2\lambda\mathbf{I} \end{bmatrix} \quad (5.4)$$

and then find the values for  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{D}$ .

ii. Then, calculate  $\mathbf{S}$  in the following expression:

$$(\mathbf{D} - \mathbf{B}^\top \mathbf{A}^{-1} \mathbf{B} + 2\lambda\mathbf{I}) \triangleq (\mathbf{S} + 2\lambda\mathbf{I}) \quad (5.5)$$

iii. Calculate  $S^A$

$$S^A = \det(\mathbf{S}) \cdot \mathbf{S}^{-1} \quad (5.6)$$

iv. Calculate  $p(\lambda)$ :

$$p(\lambda) = \det(\mathbf{S} + 2\lambda\mathbf{I}) \quad (5.7)$$

v. Use the parameters found about to solve the following quartic equation to find  $\lambda$ .  $p$  is a function of  $\lambda$  To use helper functions which have been provided in the code, you only need to write out the equation and find the coefficients of quartic equation that follows:

$$\begin{aligned} [p(\lambda)^2] = & 4\lambda^2 \mathbf{g}^\top \begin{bmatrix} \mathbf{A}^{-1} \mathbf{B} \mathbf{B}^\top \mathbf{A}^{-\top} & -\mathbf{A}^{-1} \mathbf{B} \\ \text{(symm)} & \mathbf{I} \end{bmatrix} \mathbf{g} \\ & + 4\mathbf{g}^\top \begin{bmatrix} \mathbf{A}^{-1} \mathbf{B} \mathbf{S}^A \mathbf{B}^\top \mathbf{A}^{-\top} & -\mathbf{A}^{-1} \mathbf{B} \mathbf{S}^A \\ \text{(symm)} & \mathbf{S}^A \end{bmatrix} \mathbf{g} \\ & + \mathbf{g}^\top \begin{bmatrix} \mathbf{A}^{-1} \mathbf{B} \mathbf{S}^A \mathbf{S}^A \mathbf{B}^\top \mathbf{A}^{-\top} & -\mathbf{A}^{-1} \mathbf{B} \mathbf{S}^A \mathbf{S}^A \\ \text{(symm)} & \mathbf{S}^A \mathbf{S}^A \end{bmatrix} \mathbf{g} \end{aligned} \quad (5.8)$$

vi. Now that you have  $\lambda$ , plug  $\lambda$  back into Equation (5.3) to solve for  $\mathbf{x}$ .

vii. Repeat until convergence. This can be a fixed number of iterations or until the difference in the metric between iterations is below some threshold.

## 6 Fast Correspondence Search

The pseudo code given in the censi paper [Andre Censi PLICP](#) in the appendix is the best resource to get started on the fast correspondence search algorithm. It is intuitive and the slides provide sufficient understandings for the concepts.

Here are a few pointers to take care of while coding the algorithm.

- Jump table is computed for one scan set and not for each scan point. The scan table implementation is already present in the skeleton.
- Start the correspondence search from the previous best point correspondence you found for each point.
- It is safe to make the assumption that the second best point is adjacent to the best point you have found. Decreasing or increasing the index ( while taking care of the edge cases) should work. You are free to implement better ways.
- Make use of early stopping criterion cleverly. It helps in making the search faster.
- For finding the distances between the points, the functions implemented in the Point struct will be useful. Be careful, the point to line metric is not the metric for finding the closest point. It is only the metric for optimization. For correspondence you only search for the closest point by euclidean distance.

## 7 Grading

You will be evaluated on the implementation of the scan matching algorithm.

### 7.1 Rubric

Topics	Points
Compilation	10
Correctly Calculate Transform	40
Correctly Calculate Correspondences	40
Video	10
<b>Total</b>	<b>100</b>