UNIVERSITÀ DEGLI STUDI DI TRENTO

DEPARTMENT OF PHYSICS

BACHELOR DEGREE IN PHYSICS

An investigation of
HURST EXPONENT

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1 Introduction

The main purpose of the present work is to show how to estimate the Hurst exponent for a mono-dimensional time series, using the Detrended Fluctuation Analysis and how to interpret it.
This technique is developed as a C program, which can be found online[1], and applied to a time series about sunspots.

1.1 Hurst Exponent

The Hurst exponent is a dimensionless estimator used to evaluate self-similarity and long-range dependence properties of time series.
It was introduced by Harold Edwin Hurst to describe the regularities of Nile water level.
He defined it as

$$\mathbb{E}\left[\frac{R(N)}{S(N)}\right] \propto N^H \quad \text{for } N \to \infty \tag{1.1}$$

where $H \in [0, 1]$ is the Hurst exponent, $S(N)$ is the standard deviation and $R(N)$ is the range, the difference between maximum and minimum values of a given time series with N points.

A modern definition is discussed in section 2.

1.2 Detrended Fluctuation Analysis

It is often not advisable to estimate the Hurst exponent of measured data by Equation 1.1, because often they carry not interesting and explicit trends, or, more often, do not exhibit a clear scaling.

So it is useful to introduce a particular way of detrending, called Detrended Fluctuation Analysis (DFA), which was developed by Peng *et al.*[2] for the analysis of DNA.

The advantages of DFA over other methods are that of permitting the detection of long-range correlations in time series with non-stationary and of avoiding the spurious detection of apparent long-range correlations that are an artefact of non-stationary.

2 Long memory of time series

Long memory is a relative new topic. There are several definitions, which, unless further conditions are imposed, are not necessarily equivalent[3].

For the purpose of this article, the discussion about theory of long memory is short, limited to only stationary processes, considered from a time domain perspective and focused on the interpretation of results.

The following section is a summary, mainly inspired by Pipiras *et al.*[3] and Beran[4]. The theory about long memory time series usually concerns specific examples as fractional Brownian motion and fractional Gaussian noise, which I, for brevity, do not report.

2.1 Notation

Let be $X(t)$ and $Y(t)$ random variables.

$$
\begin{aligned}
\text{Mean} \quad & \mu_X = \mathbb{E}[X(t)] \\
\text{Covariance} \quad & \operatorname{Cov}(X(t), Y(t)) = \mathbb{E}[(X(t) - \mu_X)(Y(t) - \mu_Y)] \\
\text{Variance} \quad & \sigma_X^2 = \operatorname{Var}(X(t)) = \operatorname{Cov}(X(t), X(t)) \\
\text{Correlation} \quad & \operatorname{Corr}(X(t), Y(t)) = \frac{\operatorname{Cov}(X(t), Y(t))}{\sigma_X \sigma_Y} \\
\text{Autocovariance} \quad & \gamma_X(t, \tau) = \operatorname{Cov}(X(t+\tau), X(t)) \\
\text{Autocorrelation} \quad & \rho_X(t, \tau) = \frac{\gamma_X(t, \tau)}{\sigma_X^2}
\end{aligned}
$$

2.2 Stochastic process

Definition 2.1. A **stochastic processes** $\{X(t)\}_{t \in T}$ is a collection of random variables $X(t)$ on some probability space, indexed by the time parameter $t \in T$.

A stochastic process can be distinguished in continuous or discrete if T is a real or a integer set.

Definition 2.2. A stochastic process $\{X(t)\}_{t \in T}$ **finite-dimensional distribution** is the probability distribution

$$
\mathcal{P}(X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n) \qquad t_i \in T, x_i \in \mathbb{R}, n \geq 1
$$

of a random vertical vector $(X(t_1), \ldots, X(t_n))'$ with $t_i \in T, x_i \in \mathbb{R}, n \geq 1$.

The law of $\{X(t)\}_{t \in T}$ is characterized by its finite-dimensional distribution.

Two stochastic processes $\{X(t)\}_{t \in T}$, $\{Y(t)\}_{t \in T}$ have the same law if their finite-dimensional distributions are identical. Equality in distribution is denoted by $\stackrel{d}{=}$. Thus $\{X(t)\}_{t \in T} \stackrel{d}{=} \{Y(t)\}_{t \in T}$ means

$$
\mathcal{P}(X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n) = \mathcal{P}(Y(t_1) \leq x_1, \ldots, Y(t_n) \leq x_n) \qquad t_i \in T, x_i \in \mathbb{R}, n \geq 1
$$

2.3 Stationary

Heuristically, the idea of stationary is equivalent to that of statistical invariance under time shifts.

Definition 2.3. A stochastic process $\{X(t)\}_{t \in T}$ is **strictly stationary** if $T = \mathbb{R}$ or $\mathbb{Z}$ or $\mathbb{R}_+$ or $\mathbb{Z}_+$, and for any $h \in T$

$$
\{X(t)\}_{t \in T} \stackrel{d}{=} \{X(t+h)\}_{t \in T}
$$

However, this is sometimes a too strong requirement. Instead, one often requires that only second-order properties do not change with time.

Definition 2.4. A stochastic process $\{X(t)\}_{t \in T}$ is **weakly or second-order stationary** if $T = \mathbb{R}$ or $\mathbb{Z}$, and for any $t, s \in T$

$$
\mathbb{E}[X(t)] = \mathbb{E}[X(0)] \quad \operatorname{Cov}(X(t), X(s)) = \operatorname{Cov}(X(t-s), X(0))
$$

The time difference $h = t - s$ above is called the *time lag*.

For a stationary process autocovariance and autocorrelation become particularly interesting, because they do not depend on the choice of t.

$$\gamma_X(h) = \mathrm{Cov}(X(t+h), X(t)) = \mathrm{Cov}(X(h), X(0)) \qquad \rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)}$$

2.4 Time series

Time series is a common term which indicates a discrete set of data sampled at certain times. However, in this section, a more strict and precise definition will be used.

Definition 2.5. A discrete weakly stationary stochastic process $\{X_n\}_{n\in\mathbb{Z}}$ is called **time series**.

Notice that in a experimental view, this definition is not usable. Indeed, data points are a finite set and a regular interval between their sampling times is not guaranteed.

2.5 Long Range Dependent

Long-range dependence, also called long memory or strong dependence or long range correlation, is usually defined for time series. However can be generalized for continuous stationary processes easily[3] and for non-stationary ones[5].

Definition 2.6. A function L is **slowly varying at infinity** if it is positive on $[c, \infty)$ with $c \leq 0$ and, for any $a > 0$,

$$\lim_{u\to\infty} \frac{L(au)}{L(a)} = 1$$

In many instances, for example in statistical estimation, the slowly varying functions are such that $L(u) \sim \mathrm{const} > 0$.

As said before, there are several definition of long memory. Only the most interesting and practical ones are here reported.

Definition 2.7. A time series $X = \{X_n\}_{n\in\mathbb{Z}}$ is called **long-range dependent** if one of the non-equivalent conditions, Condition 2.7.1, Condition 2.7.2, Condition 2.7.3 holds. The parameter $d \in (0, 1/2)$ is called a long-range dependence parameter.

Condition 2.7.1. The autocovariance function of the time series $X = \{X_n\}_{n\in\mathbb{Z}}$ satisfies

$$\gamma_X(k) = L_1(k)k^{2d-1} \qquad k = 0, 1, \ldots$$

where L_1 is a slowly varying function at infinity.

Can be shown that Condition 2.7.1 implies the following ones.

Condition 2.7.2. The autocovariances of the time series $X = \{X_n\}_{n\in\mathbb{Z}}$ are not absolutely summable:

$$\sum_{k=-\infty}^{+\infty} |\gamma_X(k)| = \infty$$

Condition 2.7.3. The time series $X = \{X_n\}_{n\in\mathbb{Z}}$ satisfies

$$\mathrm{Var}(X_1 + \cdots + X_N) = L_2(N)N^{2d+1} \qquad N = 1, 2, \ldots$$

where L_2 is a slowly varying function at infinity.

Definition 2.8. A time series $X = \{X_n\}_{n\in\mathbb{Z}}$ is called **short-range dependent** if its autocovariances are absolutely summable:

$$\sum_{k=-\infty}^{+\infty} |\gamma_X(k)| < \infty$$

There is an interesting special type of short-range dependent series, whose autocovariances are absolutely summable as in Definition 2.8, but they sum up to zero.

Definition 2.9. A short-range dependent time series $X = \{X_n\}_{n\in\mathbb{Z}}$ is called **antipersistent** if

$$\sum_{k=-\infty}^{+\infty} |\gamma_X(k)| = 0$$

For an antipersistent time series, the long-range dependent parameter in Condition 2.7.1 and Condition 2.7.3 is negative, $d < 0$.

2.6 Self-similarity

Definition 2.10. A stochastic process $\{X(t)\}_{t\in\mathbb{R}}$ is called **self-similar** if there is $H > 0$ such that, for all $c > 0$,

$$\{X(ct)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{c^H X(t)\}_{t\in\mathbb{R}}$$

The parameter H is called the self-similarity parameter (also Hurst index or Hurst parameter or **Hurst exponent**).

Actually self-similarity is equivalent to $H = 1$. The correct term should be "self-affinity". However, the Definition 2.10 is acceptable, because it is wildly used.
Intuitively, self-similarity means that a stochastic process scaled in time (that is plotted with a different time scale) looks statistically the same as the original process when properly rescaled in space.

A self-similar process cannot be strictly stationary, however can have stationary increment.

Definition 2.11. A stochastic process $\{X(t)\}_{t\in T}$ has **strictly stationary increments** if $T = \mathbb{R}$ or $\mathbb{Z}$ or $\mathbb{R}_+$ or $\mathbb{Z}_+$, and for any $h \in T$

$$\{X(t+h) - X(h)\}_{t\in T} \stackrel{d}{=} \{X(t) - X(0)\}_{t\in T}$$

The Definition 2.11 implies that, for a process $\{X(t)\}_{t\in T}$ with stationary increment and for any $h \in \mathbb{R}$, the process

$$Y(t) = X(t+h) - X(t) \qquad t \in \mathbb{R}$$

is stationary.

2.7 Relationship between Long-Range Dependent and Self Similarity

Consider a self-similar process $Y = \{Y(t)\}_{t\in\mathbb{R}}$ with strictly stationary increment and $H \in (0,1)$ and a strictly stationary series $X_n = Y(n) - Y(n-1)$ with $n \in \mathbb{Z}$.

The series X has mean $\mu_X = 0$, $\mathbb{E}X_n^2 = \mathbb{E}Y^2(1)$ and autocovariance

$$\gamma_X(k) \sim \mathbb{E}Y^2(1)H(2H-1)k^{2H-2} \qquad \text{as } k \to \infty \tag{2.1}$$

Comparing Condition 2.7.1 and Equation 2.1, the series X is long-range dependent if $H \in (1/2, 1)$, with

$$H = d + \frac{1}{2}$$

2.8 Summary

Consider a time series $X = \{X_n\}_{n\in\mathbb{Z}}$, characterized by the Hurst exponent $H \in (0,1]$. X is defined strictly self-similar if $H = 1$, else self-affinal.

Moreover, X is

- short range dependent and all autocovariances at non zero time lags are zero, if $H = 1/2$[4]
- long range dependent if $H \in (1/2, 1)$
- antipersistent short range dependent if $H \in (0, 1/2)$

3 Detrended Fluctuation Analysis

The purpose of DFA is to estimate the variance of partial sums of the series $X = \{X_n\}_{n\in\mathbb{Z}}$. In this way, the Condition 2.7.3 allows to estimate the long-range dependence parameter.

The Multifractal Detrended Fluctuation Analysis (MF-DFA) is a generalization of the standard DFA to all moment q.

3.1 Multifractal Detrended Fluctuation Analysis

The procedure is described for a continuous stochastic process $\{X(t)\}_{t\in[0,T]}$ and then applied to a time series $\{X_n\}_{n=1,\dots,N}$ sampled at time t_n.

Notice that the symbology of $\{X_n\}_{n=1,\dots,N}$ is assumed as convention and, to be consistent with the previous one, should be $\{X(t)\}_{t\in T_N}$ where T_N is the set of the sampling times $T_N \equiv \{t_n\}_{n=1,\dots,N}$ with $t_1 = 0$ and $t_N = T$.
Moreover suppose that this series $\{X_n\}$ is of compact support, i.e. $X_n = 0$ or for an insignificant fraction of the values only.

Step 1. Define the "profile" $\{Y(t)\}$ of $\{X(t)\}$.

$$Y(t) = \int_0^t dt'(X(t') - \bar{X})$$

where $\bar{X}$ is the mean of $\{X(t)\}$ computed on the whole time series.

$$\bar{X} = \frac{1}{T}\int_0^T dt' X(t')$$

Subtracting the mean is actually not necessary, because it will be remove anyway in linear detrending in Step 3.

For $\{X_n\}$, instead, the profile is defined as

$$Y_n = \sum_{i=1}^{n} \left(X_i - \bar{X}\right)$$

and $\bar{X}$ as

$$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Step 2. Split the full time T in time windows τ.

$$k + 2 \leq \tau \leq \frac{T}{4}$$

Each window contains $l_n(\tau)$ points of the $\{X_n\}$ series.
If $\{X_n\}$ is sampled at constant frequency $\nu = {}^{1}/_{\Delta T}$, so $T = N\Delta T$, $l(\tau) = l_n(\tau)$ and $l(\tau)\Delta T = \tau$.

Step 3. Compute the linear regression $g^{(k)}(t, \tau, t')$ of order k of $\{Y(t')\}$ in range $t' \in [t, t+\tau]$.

$$g^{(k)}(t, \tau, t') = \sum_{i=0}^{k} a_i t'^i \quad \text{with } a_i \text{ parameters}$$

Determinate the variance $f_2^{(k)}(t, \tau)$ between the profile $\{Y(t)\}$ and the regression $g^{(k)}(t, \tau, t')$.

$$f_{2\ +}^{(k)}(t, \tau) \equiv \frac{1}{\tau}\int_t^{t+\tau} dt' \left(Y(t') - g^{(k)}(t, \tau, t')\right)^2$$

Since the detrending of time series is done by the subtraction of polynomial fits from the profile $\{Y(t)\}$, different order DFA differ in their capability of eliminating trends in the series.
In (MF-)DFAk (kth-order (MF-)DFA) trends of order k in the profile $\{Y(t)\}$ (or, equivalently, of order $k-1$ in the original series $\{X(t)\}$) are eliminated.
If the time series is correctly detrended for k_{min}, its behaviour is predicted by all $k \geq k_{min}$. There may be, however, some strange effects for $k < k_{min}$.

Analogously, introduce $g_n^{(k)}(\tau, t_i)$ with $t_i \in [t_n, t_n + \tau]$, which contains $l_n(\tau)$ points, and $f_{2\ n}^{(k)}(\tau)$.

$$f_{2\ n\ +}^{(k)}(\tau) \equiv \frac{1}{l_n(\tau) - k - 1}\sum_{i=n}^{n+l_n(\tau)} \left(Y_i - g_n^k(\tau, t_i)\right)$$

$f_{2\ n}^{(k)}(\tau)$ is the variance of $\{Y_i^{(l_n(\tau))}\}_{i=n,\dots,n+l_n(\tau)}$ with the correct normalization $l_n(\tau) - k - 1$. In fact, for kth order detrending, a time window containing $l_n(\tau) = k+1$ sample points yields exactly zero fluctuation. This also justify the condition $\tau \geq k+2$ in Step 2.
Moreover, the results for large τ become unreliable, because of the few time windows who can be chosen. So often only $\tau \leq {}^{T}/_{4}$ is used.

$f_{2\ +}^{(k)}(t, \tau)$ and $f_{2\ n\ +}^{(k)}(\tau)$ are defined in an asymmetric way. Indeed $t \in [0, T-\tau]$ and $n = 1\dots, N - l_n(\tau)$. To remove this asymmetry, integral and summary can be also computed "starting from the opposite end", so $t \in [\tau, T]$ and $n = 1 + l_n(\tau), \dots, N$

$$f_{2\ -}^{(k)}(t,\tau) \equiv \frac{1}{\tau}\int_{t-\tau}^{t} dt' \left(Y(t') - g^{(k)}(t,\tau,t')\right)^2$$

$$f_{2\ n\ -}^{(k)}(\tau) \equiv \frac{1}{l_n(\tau)-k-1}\sum_{i=n-l_n(\tau)}^{n}\left(Y_i - g_n^k(\tau,t_i)\right)$$

Practically, calculating $f_{2\ n}^{(k)}(\tau)$ for each n is useless and demands a lot of time. Instead computing $f_{2\ n}^{(k)}(\tau)$ only for n, where t_n is the closest to a multiply of $\Delta\tau = {}^{\tau}/_{m}$ with $m \in \mathbb{N}$, is a good approximation.

Step 4. $F_q^{(k)}(\tau)$ is the momentum of order q over all time windows of length τ.

$$F_q^{(k)}(\tau) = \left(\int_0^{T-\tau} dt \left(f_2^{(k)}(t,\tau)\right)^{q/2} p(t,\tau)\right)^{1/q}$$

For a given τ there are N_τ functions $f_{2\ n}^{(k)}(\tau)$, $F_q^{(k)}(\tau)$. Assume that $\{t_n\}$ are uniform distributed.

$$F_q^{(k)}(\tau) = \left(\frac{1}{N_\tau}\sum_{n=1}^{N_\tau}\left(f_{2\ n}^{(k)}(\tau)\right)^{q/2}\right)^{1/q}$$

Step 5. Determine the scaling behaviour of the fluctuation functions by analysing log–log plots of $F_q^{(k)}(\tau)$ versus τ.

$$F_q^{(k)}(\tau) \propto \tau^{\alpha(q)}$$

In general, the exponent $\alpha(q)$ may depend on q.

A tip is to choose equally spaced $\log(\tau)$ and calculate $F_q^{(k)}(\tau)$ for each τ. Then fit the following equation.

$$\log\left(F_q^{(k)}(\tau)\right) = C + \alpha(q)\log(\tau)$$

This construction do not assume that $l_n(\tau)$ is the same for each n, but it is not a problem. Let be $\mathbb{N} \ni l_n(\tau) = l(\tau) + \varepsilon_n$, where $\tau = \langle\Delta T\rangle\, l(\tau)$, $\langle\Delta T\rangle$ is the average as sampling time and $l(\tau)$ can be real.

For big τ, ε_n is negligible, and so $l_n(\tau) \approx l(\tau)$. For small τ, instead, ε_n can be important, but there are $N_\tau \propto \tau^{-1}$ windows, and so the average $\langle l_n(\tau)\rangle_{n=1,\dots N_\tau} = l(\tau)$.

3.2 Hurst exponent and DFAk

The $\alpha(q)$ parameter can be interpreter in term of Hurst exponent H. There's a linear relationship between $\alpha(q=2)$ and H. As said before, the DFAk is equivalent to MF-DFA$k(q=2)$.

If X is a stationary time series (e.g. fractional Gaussian noise), Y is a self-similarity process, and so $0 < \alpha(q=2) < 1$. For Condition 2.7.3, the exponent $\alpha(2)$ is identical to the Hurst exponent H.

For a non-stationary signal (e.g. fractional Brownian motion), instead $\alpha(q=2) > 1$. In this case the relation between exponents $\alpha(2)$ and H is $H = \alpha(q=2) - 1$[5].

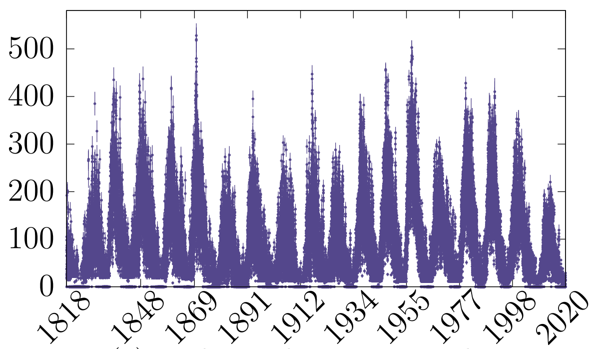
The parameter $\alpha(q)$ is known as the generalized Hurst exponent.

4 Sunspots

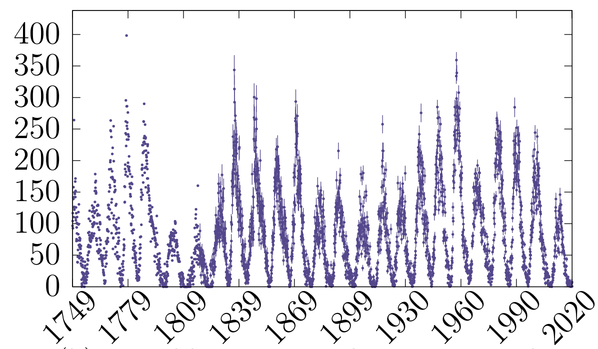
On the Sun's photosphere there is a strong magnetic field. However in some localized regions (called **sunspots**), the field is significant higher and the surface appear as spots darker than the surrounding areas. Their number varies according to the approximately 11-year solar cycle.

4.1 Raw data

The SIDC, a department of the Royal Observatory of Belgium, supplies, through one of its project, two databases of the sunspot number [6].
The first one provides daily measures from 01-01-1818 and the other one the monthly means from 01-01-1749. The relative plots can be found in Figure 1.



(a) Daily total sunspot number



(b) Monthly mean total sunspot number

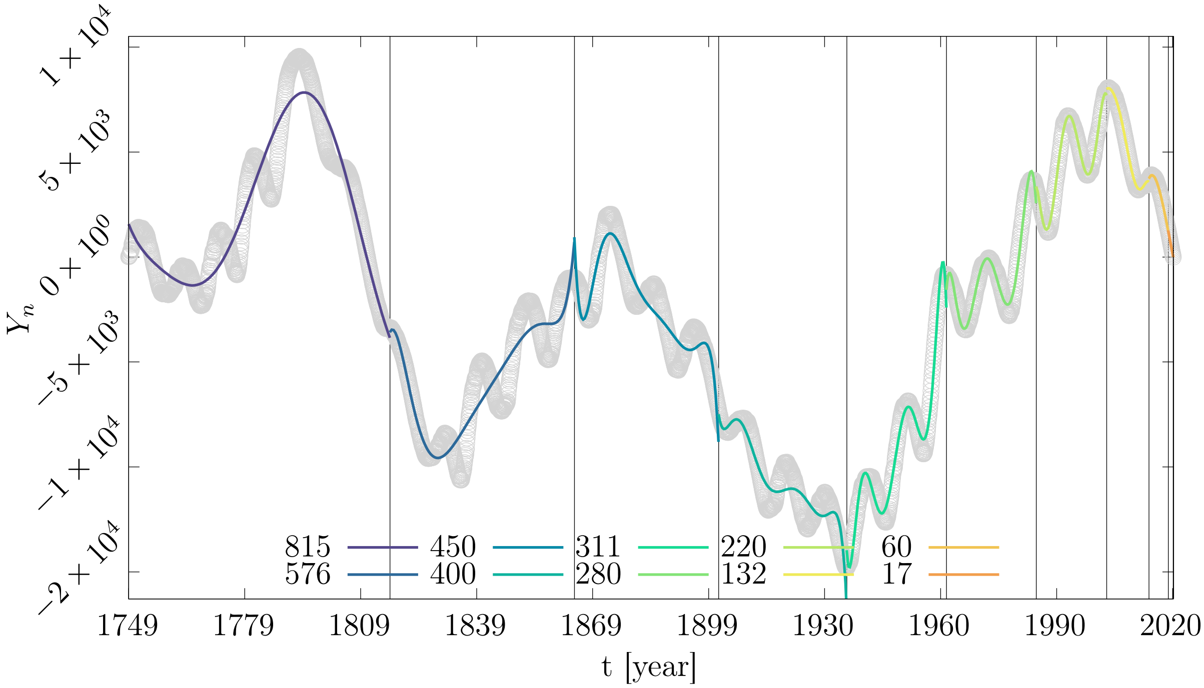
Figure 1: Time series $\{X_n\}$ of sunspot number, sampled at different frequency

4.2 Data Analysis

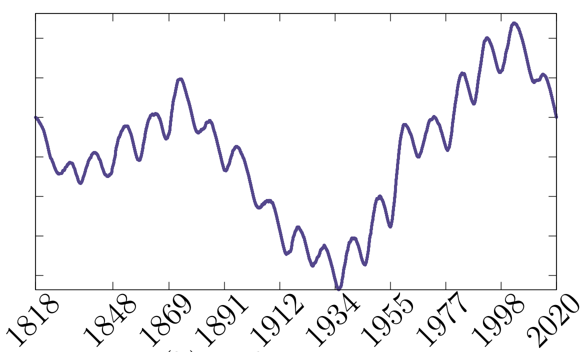
This section is inspired by the work of Movahed *et al.*[5]. However, I choose to developed a slightly different DFAk, which is illustrated in section 3, because the sampling frequency is not constant, especially "in the old days" and in the daily measured time series.

Profile of sunspot number Figure 2b and Figure 2c show the profile $\{Y_n\}$ defined in Step 1. The profile is then split in equal time windows τ, as illustrated in Step 2 and each division is fitted with a polynomial of kth order, as in Step 3.

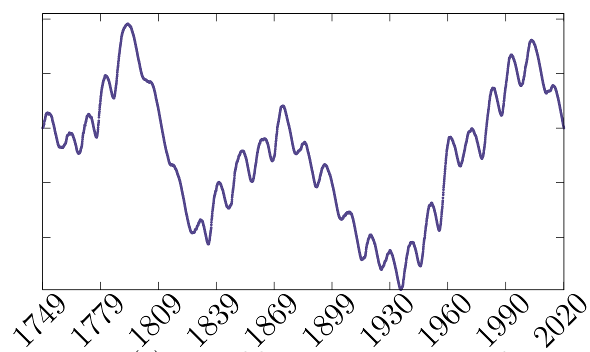
Comparison of two databases As describe in Step 4, the total variance $F_2^{(k)}(\tau)$ is estimated for 300 equally spaced $\log(\tau)$ and using $m = 10$.
Neglecting the section with the biggest τs (and also few points), $\alpha(2)$ parameter estimated by DFA1 describes the same behaviour at the same τ, independently from the frequency of sampling, in the comparison between Figure 3b and Figure 3c.
Also for higher k, the estimation of Hurst exponent provides the same results between 4 and 450 months. For convenience, the following work is presented for monthly mean sunspots data only.



(a) Monthly mean sunspots 1b and example of fits.
Coloured lines represent polynomial fits of 7th order at different time intervals, indicated in the legend and with months as unit.



(b) Daily sunspots 1a



(c) Monthly mean sunspots 1b

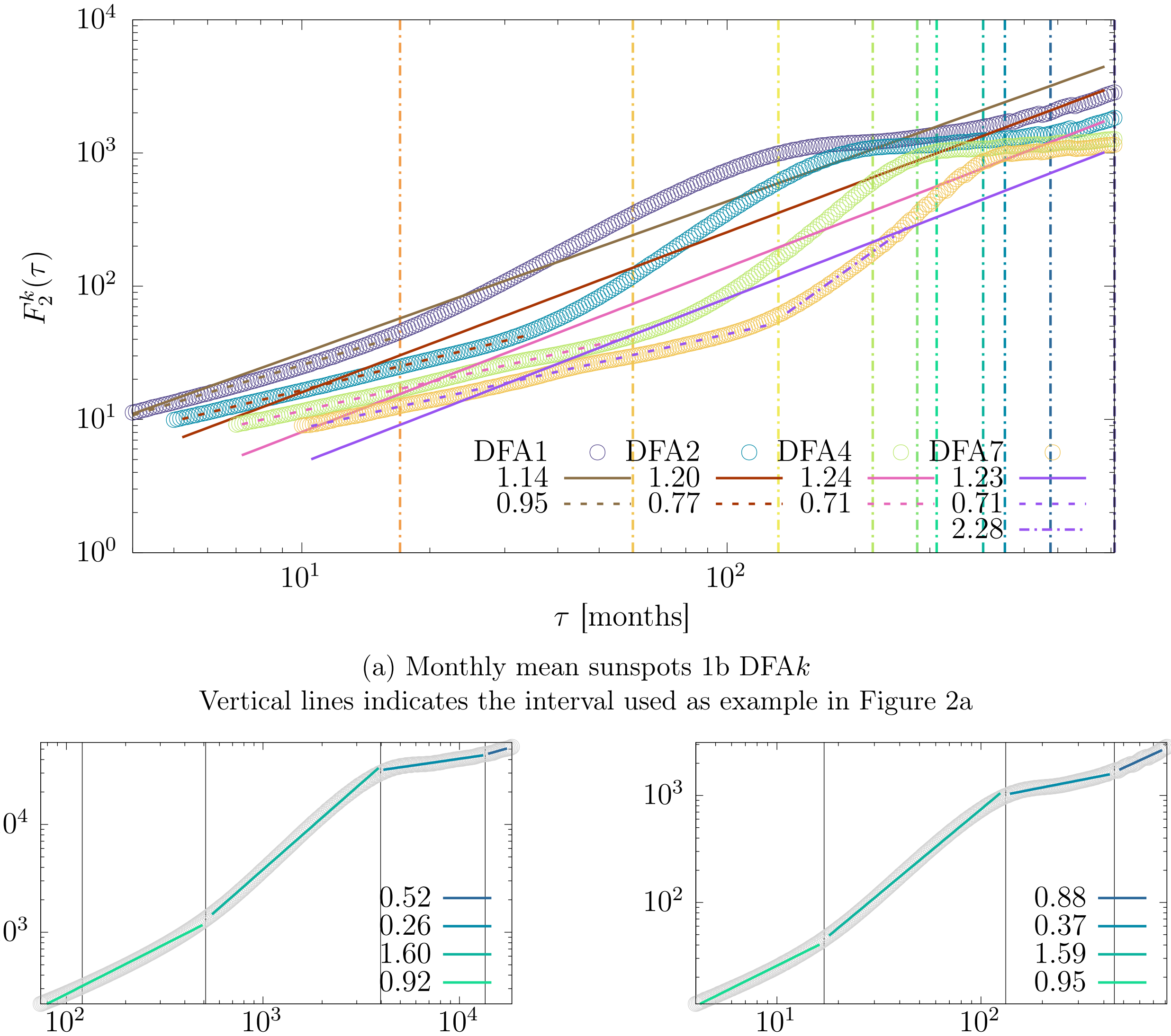
Figure 2: The profile $\{Y_n\}$, defined as Step 1 and relative to data in Figure 1

Considerations on DFA1 The plots in Figure 3 show that time series have different slopes at different time scales. Ideally, however, if a process is self-affinal, the log-log plot of $F_2^{(k)}(\tau)$ should be a straight line. This could mean that the process has different behaviour at different time scales or that the order of detrend k is too low.

Problems of DFAk A comparison of DFAk with different k is plotted in Figure 3a. The behaviour of time series is the same in windows where it is well detrended (dashed line fits in Figure 3a).

As illustrated for example in Figure 2a, the polynomial of order $k = 7$ cannot fit properly a time window $\tau > 280$ months. Analogously, lower k capability of fitting is limited.

OLS cannot fit the sunspots $\{Y_n\}$ with a polynomial of order 8 or bigger, because of the overflow of double in C program.



(a) Monthly mean sunspots 1b DFAk
Vertical lines indicates the interval used as example in Figure 2a

(b) Daily sunspot 1a DFA1

(c) Monthly mean sunspots 1b DFA1

Figure 3: Detrended fluctuation analysis
The estimation of the $\alpha(2)$ parameter is reported in the legend of graphs.
In Figure 3b the unit of τ is [days] and in Figure 3c is [months]. The vertical black lines are placed at same $\tau = 4, 17, 132, 450$ months.

Considerations on DFA7 The series, through DFA7, shows at least two behaviour, and so is multi-fractal. In particular:

- for $\tau \leq 132$ months, $\alpha(2) \in (0.5, 1)$ and so the process is a stationary long-range memory one.
- for 132 months $< \tau \leq 280$ months, $\alpha(2) \in (2, 2.5)$
- for $\tau > 280$ months, the series is not correctly detrended.

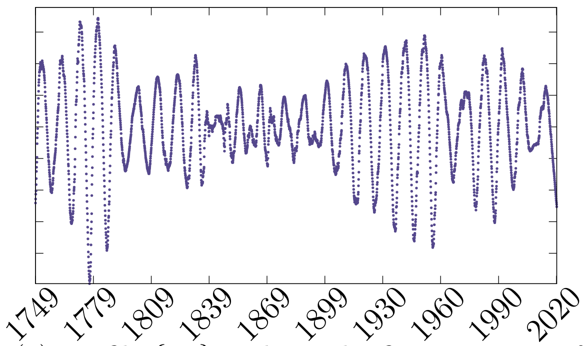
In subsection 3.2 nothing is said about process with $\alpha(2) > 2$. Actually the literature is not so rich and in Movahed *et al.*[5] it is given a proof that $\alpha(2) \in (1, 2)$ only for fractional Brownian

motion.
Assuming I have made no mistake, my hypothesis is that time series has a multi-dimensional noise, like a bifractional Brownian motion.

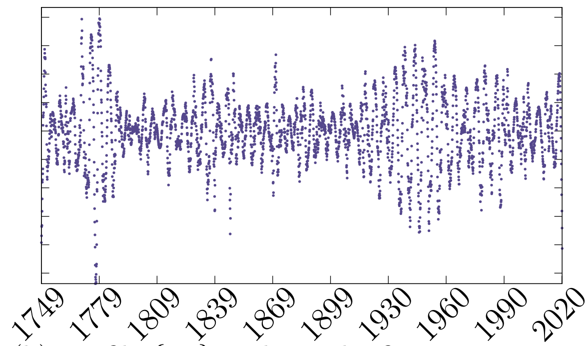
Anyway, 132 months is not a random time scale, because is the period of solar cycle.

Comparison with other authors

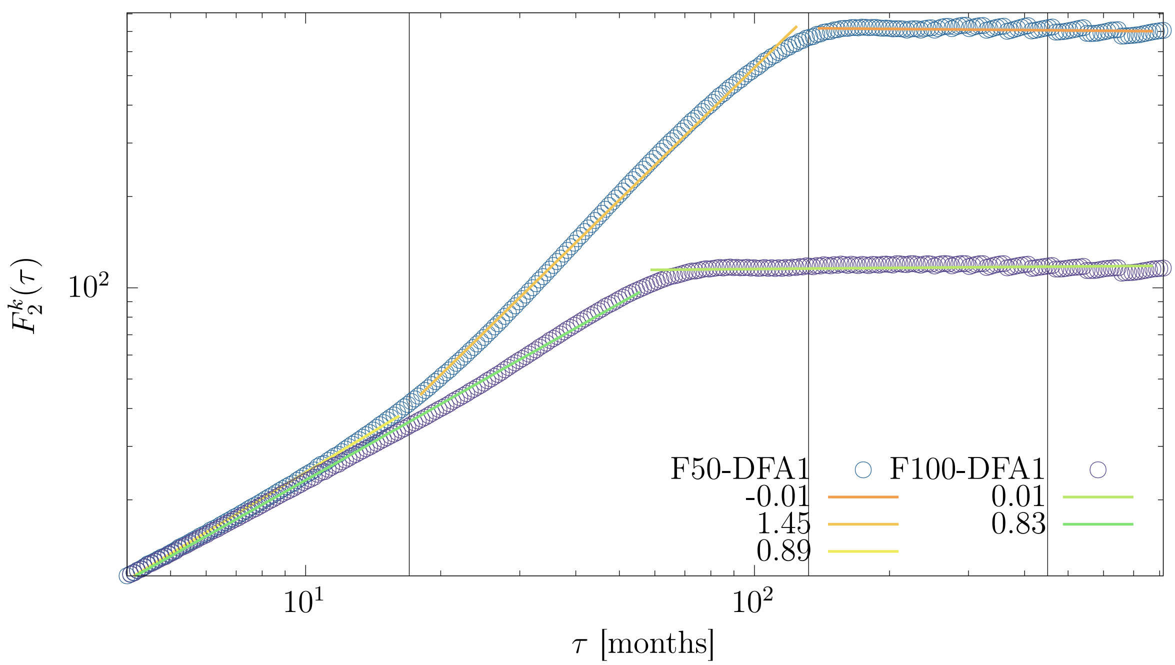
Comparing with the same work done by Movahed *et al.*[5], the DFA1 applied on every dataset gives the same result. However their estimation on $F_2^{(k)}(\tau)$ is more unstable, and so the residuals are much bigger and the uncertainty on α parameters are higher.



(a) Profile $\{Y_n\}$ without the first 50 terms of Fourier transform



(b) Profile $\{Y_n\}$ without the first 100 terms of Fourier transform



(c) Fn-DFA1

Figure 4: Applying the Fourier-DFA method to the monthly mean sunspots 1b

Fourier DFA To remove the sinusoidal trend, Movahed *et al.* apply a F50-DFA1. It consists in Fourier transforming the profile $\{Y_n\}$, setting to zero the first 50 terms, anti-transforming, and

then applying the DFA1.
In general, Fn-DFAk indicates the DFAk, where the first n terms of real Fourier transform are set to 0.

In Figure 4a and Figure 4b the first 50 and 100 terms are removed from the profile. However, the graph of Figure 4a still contains a trend. This is also evident because F50-DFA1 and DFA1 (Figure 3c) for sufficient small τ gives the same $\alpha(2)$.
For small τ, F100-DFA works, because the estimation of time series behaviour is compatible with DFAk with $k > 1$.

However, F-DFA do not provides information for big τ, because the lowest frequencies are removed and so the variance for long period become constant.

In this case, my observations are different from Movahed and coworkers' ones. They assert that a F50-DFA1 is sufficient. In my opinion instead, a sinusoidal trend is still evident.

4.3 Final considerations

DFAk is a powerful method for the analysis of time series. However, in the studies about sunspots data, there are several problems, linked with $\sin^2$ trend of the series.
In particular, the approximation with polynomials does not allow to evaluate the behaviour for big periods, and also alternative ways, like Fourier-DFA, can lead to inconclusive results.

Moreover, the theory about long-range dependency, self-similarity and multifractal processes is relative new and needs to be explored deeper.
In general, anyway, non-linear systems can cause errors and misunderstandings in numeric analysis also if the approach is only slightly different. They must be treated carefully. I encourage to check and compare Movahed and coworkers' paper and this essay.

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