UNIVERSITÀ DEGLI STUDI DI TRENTO

DEPARTMENT OF PHYSICS
BACHELOR DEGREE IN PHYSICS

An investigation of HURST EXPONENT

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Hurst exponent

The Hurst exponent is a dimensionless estimator used to evaluate self-similarity and long-range dependence properties of time series.

$$\mathbb{E}\left[\frac{R(N)}{S(N)}\right] \propto N^H \quad \text{for } N \to \infty$$

Detrended fluctuation analysis

- C.-K. Peng et al. "Mosaic organization of DNA nucleotides". In: *Phys. Rev. E* 49 (2 Feb. 1994), pp. 1685–1689. DOI: 10.1103/PhysRevE.49.1685. URL: https://link.aps.org/doi/10.1103/PhysRevE.49.1685
- Alessandro Foradori. *Hurst Exponent*. https://github.com/f3fora/hurst-exponent. 2020

Time series

A discrete weakly stationary stochastic process $\{X_n\}_{n\in\mathbb{Z}}$ is called **time series**.

- Vladas Pipiras and Murad S. Taqqu. *Long-Range Dependence and Self-Similarity*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2017. DOI: 10.1017/CBO9781139600347
- Beran Jan. *Statistics for long-memory processes*. Chapman and Hall/CRC monographs on statistics and applied probability. Boca Raton, FL: CRC Press, 1994. URL: https://cds.cern.ch/record/2304008

Long-range dependence

- $\gamma_X(k) \propto k^{2d-1}$
- $\sum_k |\gamma_X(k)| = \infty$
- $\mathrm{Var}(X_1 + \cdots + X_N) \propto N^{2d+1}$

with $d \in (0, 0.5)$

Short-range dependence

$\sum_k |\gamma_X(k)| < \infty$

Antipersistance

$\sum_k |\gamma_X(k)| = 0$ and $d < 0$

Self-similarity

Intuitively, self-similarity means that a stochastic process scaled in time (that is plotted with a different time scale) looks statistically the same as the original process when properly rescaled in space.

$$\{X(ct)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{c^H X(t)\}_{t\in\mathbb{R}} \quad \text{with } H > 0 \text{ and } c > 0$$

$$H = \frac{1}{2} + d$$

- $H \in (0.5, 1]$ long-range dependence
- $H = 0.5$ no time lag
- $H \in (0, 0.5)$ anti-persistent short-range dependence

The purpose of DFA is to estimate the variance of partial sums of the series $X = \{X_n\}_{n\in\mathbb{Z}}$. In this way, $\mathrm{Var}(X_1 + \cdots + X_N) \propto N^H$ allows to estimate the Hurst exponent

- Vladas Pipiras and Murad S. Taqqu. *Long-Range Dependence and Self-Similarity*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2017. DOI: 10.1017/CBO9781139600347
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Define the "profile" $\{Y(t)\}$ of $\{X(t)\}$.

$$Y(t) = \int_0^t dt'(X(t') - \bar{X})$$

where $\bar{X}$ is the mean of $\{X(t)\}$ computed on the whole time series.

Split the full time T in time windows τ.

Compute the linear regression $g^{(k)}(t, \tau, t')$ of order k of $\{Y(t')\}$ in range $t' \in [t, t+\tau]$.
Determinate the variance $f_2^{(k)}(t, \tau)$ between the profile $\{Y(t)\}$ and the regression $g^{(k)}(t, \tau, t')$.

$$f_2^{(k)}(t, \tau) \equiv \frac{1}{\tau} \int_t^{t+\tau} dt' \left(Y(t') - g^{(k)}(t, \tau, t') \right)^2$$

$F_q^{(k)}(\tau)$ is the momentum of order q over all time windows of length τ.

$$F_q^{(k)}(\tau) = \left(\int_0^{T-\tau} dt \left(f_2^{(k)}(t,\tau) \right)^{q/2} p(t,\tau) \right)^{1/q}$$

Determine the scaling behaviour of the fluctuation functions by analysing log–log plots of $F_q^{(k)}(\tau)$ versus τ.

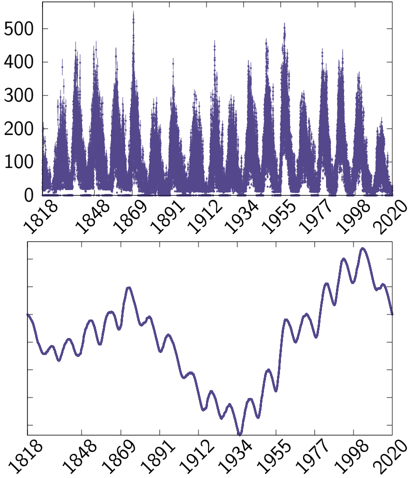
$$F_q^{(k)}(\tau) \propto \tau^{\alpha(q)}$$

- $\alpha(2) \in (0, 1]$ stationary process with $\alpha(2) = H$
- $\alpha(2) \in (1, 2]$ non-stationary process with $\alpha(2) = 1 + H$

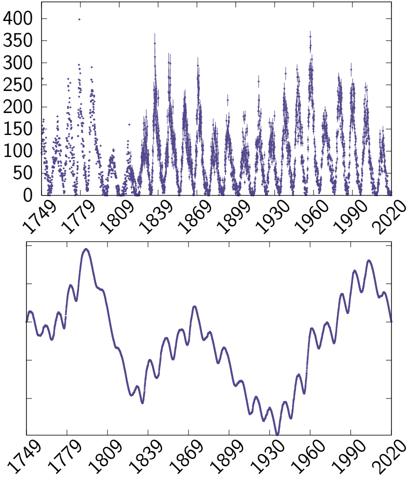
On the Sun's photosphere there is a strong magnetic field. However in some localized regions (called **sunspots**), the field is significant higher and the surface appear as spots darker than the surrounding areas. Their number varies according to the approximately 11-year solar cycle

- SILSO World Data Center. "The International Sunspot Number". In: *International Sunspot Number Monthly Bulletin and online catalogue* (1749-2020)
- M Sadegh Movahed et al. "Multifractal detrended fluctuation analysis of sunspot time series". In: *Journal of Statistical Mechanics: Theory and Experiment* 2006.02 (Feb. 2006), P02003–P02003. DOI: 10.1088/1742-5468/2006/02/p02003. URL: https://doi.org/10.1088%2F1742-5468%2F2006%2F02%2Fp02003

daily sunspot number

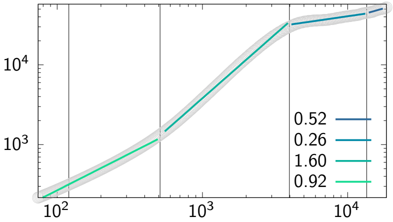


monthly mean sunspot number

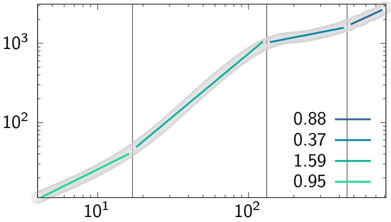


Vertical black lines at $\tau = 4, 17, 132, 450$ months.

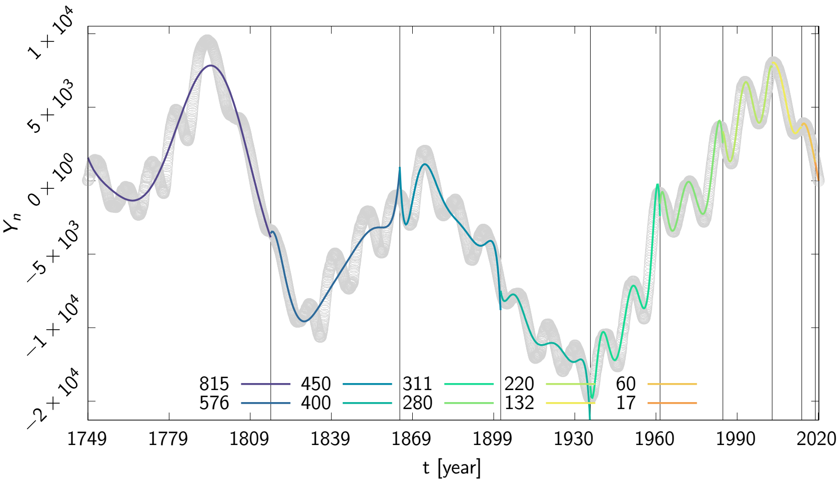
daily sunspot number

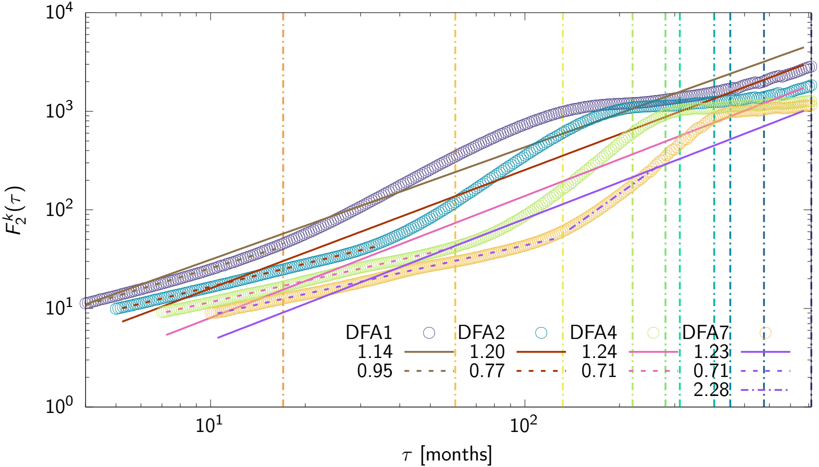


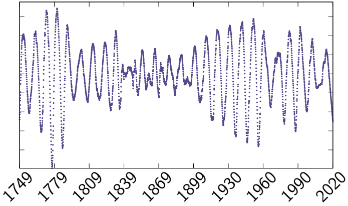
monthly mean sunspot number



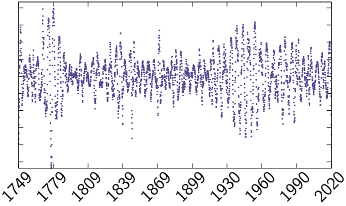
Profile fitted with polynomials of 7th order




10^4
10^3
10^2
10^1
10^0
$F_2^k(\tau)$
10^1
10^2
τ [months]
DFA1
DFA2
DFA4
DFA7
1.14
1.20
1.24
1.23
0.95
0.77
0.71
0.71
2.28



profile without 50 Fourier-terms



profile without 100 Fourier-terms