Mini-Project I

Unsupervised and reinforcement learning in neural networks

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Introduction

In the first part of the project, the task was to analyze the main properties of the BCM plasticity rules, using both computational and analytical techniques. The learning rule is given by:

$$\frac{d}{dt}w_i = \eta x_i(y^2 - y\theta)$$

1 Exercise: Stability

The goal of this exercise was to understand the importance of having a dynamic theshold. For that we consider a simple model $y = wx$ of a postsynaptic neuron receiving inputs only from a single presynaptic neuron, where x is the activity of said neuron.

1.1 Question

- First, we analyze the dynamics of the synaptic weight w according to the differential equation for the case of constant input $x(t) = x_0$ and constant threshold $\theta(t) = \theta^*$.

$$\begin{aligned}\frac{d}{dt}w_i &= \eta x_0(y^2 - y\theta^*) \\ &= \eta x_0(w^2 x_0^2 - w x_0 \theta^*) \\ &= \eta w x_0^2(w x_0 - \theta^*)\end{aligned}$$

For which cases does w not change?

$$\begin{aligned}\frac{d}{dt}w_i &= \eta x_0(y^2 - y\theta^*) = 0\\ &\Leftrightarrow \eta w x_0^2(wx - \theta^*) = 0\\ &\Rightarrow w_1^* = 0\\ &\Rightarrow y = w_2^* x_0 = \theta^*\end{aligned}$$

Here w_1^* and w_2^* are the fixed points. If the threshold is equal to the weighted input, there is no increase in w. To examine the stability of the fixed points, we first calculate $G'(w)$ where $\frac{d}{dt}w_i = G(w)$. We get:

$$\frac{dF}{dw} = G'(w) = \eta x_0^2(2wx_0 - \theta^*)$$

Evaluating this function in the fixed points yields:

$$\begin{aligned}G'(w_1^*) &= -\eta x_0^2\theta^* < 0\\ G'(w_2^*) &= \eta x_0^2\theta^* > 0\end{aligned}$$

We see that only the fixed point $w_1^* = 0$ is a stable one.

- In the next step, we examine the case where the threshold θ is dynamic according to the equation

  $$\theta(t) = \frac{1}{\tau}\int_{-\infty}^{t} y^p(s)e^{(-\frac{t-s}{\tau})}ds \approx \langle y^p\rangle_t \approx \langle y^p\rangle_x = \sum_i y^p p(\vec{x}^{(i)})$$

  In this case the learning rule can be written as a system of two differential equations:

  $$\dot{w} = \eta x(y^2 - y\theta) \tag{1}$$
  $$\tau\dot{\theta}_M = -\theta_M + y^p \tag{2}$$

  If we assume that $\tau \ll \eta^{-1}$, (2) will converge much faster than (1). This means that we can assume that (2) is always in equilibrium: $\theta_M = y^p$. With this assumption, we can calculate the fixed points of (1) in function of p by setting (1) equal to 0. We easily find two fixed points $\tilde{w}_1 = 0$ and $\tilde{w}_2 = \sqrt[p-1]{x^{1-p}} = 1/x$. Note that this expression is only valid for $p > 1$. For $p = 1$, $\dot{w} = 0$ and the system is stationary, hence always stable.

  To analyze the stability of these points, we again substitute them in the derivative of (1). Defining $\dot{w} = \tilde{F}(w)$, we first differentiate $\tilde{F}$ with respect to w:

  $$\frac{d\tilde{F}}{dw} = \eta x^3\left(2w - (p+1)x^{p-1}w^p\right)$$

  Evaluating this in the found fixed points, we find that $\dot{\tilde{F}}(\tilde{w}_1) = 0$ and $\dot{\tilde{F}}(\tilde{w}_2) < 0$. Hence, $\tilde{w}_2$ is a stable fixed point. To investigate whether $\tilde{w}_2$ is stable, we would have to compute F'', but let's refrain from that, it is not really an interesting solution anyway.

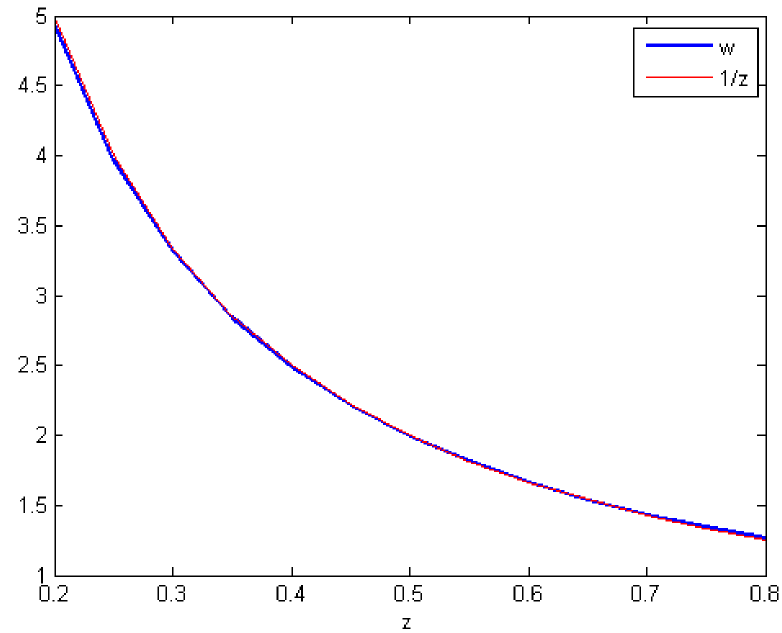


Figure 1: Result of simulation 1 with a Bernoulli distributed input.

1.2 Question: Simulation

The power exponent is now fixed to p=2. We implemented the BCM rule as expressed in (1) and (2). As input, we considered a Bernoulli distributed random variable x, i.e. $x = x_0$ with probability z and $x = 0$ with probability $1 - z$. We simulated the process for 13 different values of z, ranging from 0.2 to 0.8. After convergence, we measured the firing rate y, which, since we took $x_0 = 1$, is coincidentally equal to the weight w. We then plotted the average y over many simulations as a function of z, the result can be found in Figure 1. In the figure, we also plotted the function $1/z$ and we see that the weights very closely resemble this function. This makes sense, since in the theoretical part of this question, we saw that $w = \frac{1}{x}$ is a stable fixed point of the system. Now, if the input 1 is presented with probability z, then on average the input firing rate is given by z.

2 Exercise: Selectivity and optimization principle

2.1 Question

Show that the BCM-like learning rule given by

$$\dot{w} \propto xy^2 - xy\theta$$

can be derived by gradient descent on the objective function

$$F(w) = \left\langle \left(\frac{y}{\sigma_y}\right)^3 \right\rangle$$

In gradient descent, we go in the negative direction of the objective function:

$$\dot{w} = -\eta \frac{dF}{dw}(w)$$

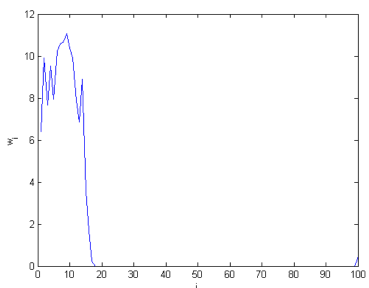
Now if we calculate $F'(w)$:

$$\begin{aligned} F'(w) = fracdFdw(w) &= \left\langle \frac{d}{dy}\left(\frac{y^3}{\sqrt{\langle y^2\rangle}^3}\right)\frac{dy}{dw} \right\rangle \\ &= \left\langle 3xy^2\sqrt{\langle y^2\rangle}^{-3} - \frac{3xy^3\langle y\rangle}{\langle y^2\rangle^2\sigma_y} \right\rangle \\ &= \frac{3}{\sigma_y^3}\left\langle xy^2 - xy\theta \right\rangle \end{aligned}$$

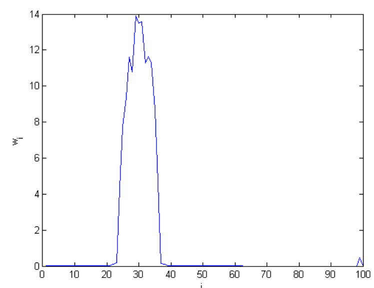
with $\theta = \frac{\langle y^3\rangle}{\langle y^2\rangle}$. By converting this offline rule to an online rule $\eta(xy^2 - xy\theta)$, we see that indeed, $\dot{w} \propto xy^2 - xy\theta$.

2.2 Simulation

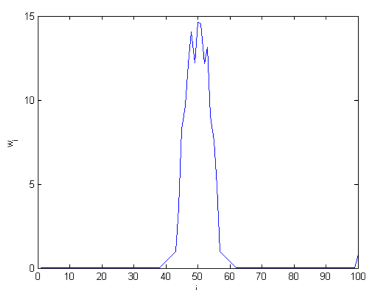
We simulated a situation in which one of five Gaussian input patterns (for 100 presynaptic inputs) is offered randomly to a node. The patterns have centers on 10, 30, 50, 70 or 90. In Figure 2, the resulting weights after convergence of the set-up are shown for four different experiments. We can see a very clear spike at one of the means, while the other weights are zero. This indicates that the output neuron becomes selective to a single input (pattern). As the figure shows, which input it becomes selective to varies and is determined randomly, hence we get different results for different simulations. In Figure 3, the temporal evolution of the threshold θ, the responses to the different inputs $y^{(j)}$ and of the objective function F are shown. We see that F very quickly converges to its approximate final value. $y^{(j)}$ also converges very fast to zero for four of the five means, but keeps oscillating around the threshold value. Theta also keeps oscillating. So we see that indeed, the neuron becomes selective to one single input, and very fast at that.



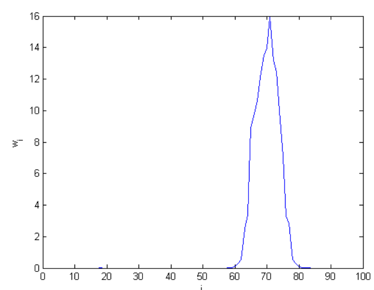
(a) Selective to Gaussian with $\mu = 10$



(b) Selective to Gaussian with $\mu = 30$

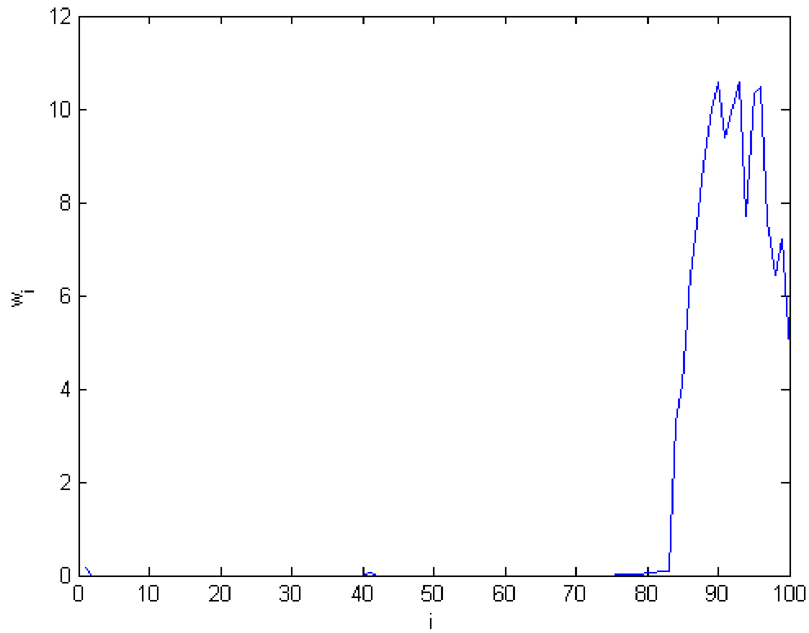


(c) Selective to Gaussian with $\mu = 50$

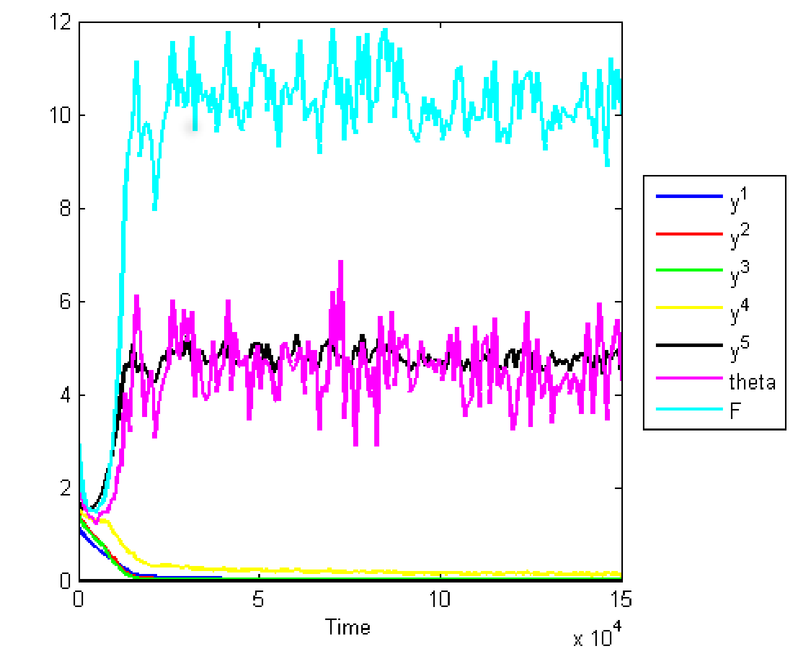


(d) Selective to Gaussian with $\mu = 70$

Figure 2: Weights after convergence for four different experiments



(a) Weights w_i

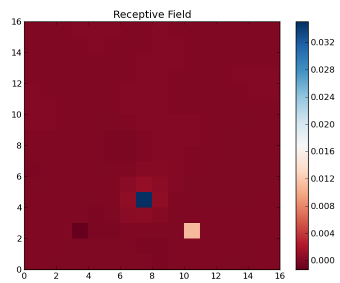


(b) Temporal evolution

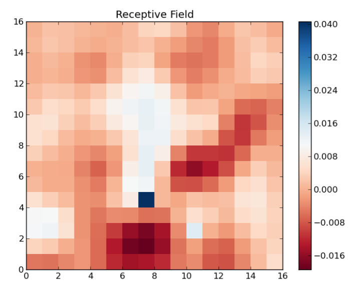
Figure 3: Final weights w_i and temporal evolution of θ, $y^{(j)}$ and F

3 Exercise: Explaining V1 Receptive Fields Formation

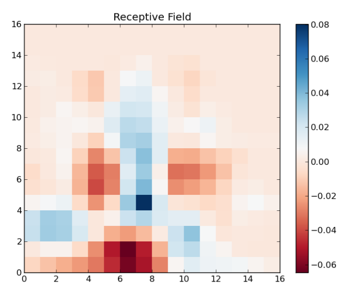
In the last exercise we fed a postynaptic neuron with elements of a set of 100'000 16x16 patches used for the activity of 512 presynaptic neurons. The patches were extracted from 10 whitened natural images. In each iteration we took a random patch from the set of patches and used its positive part x_{ON} as the activity of the first 256 neurons and the absolute value of the negative part x_{OFF} as the activity of the second bunch of 256 neurons.



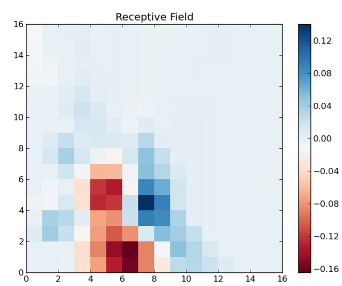
(a) 75000 iterations



(b) 100000 iterations



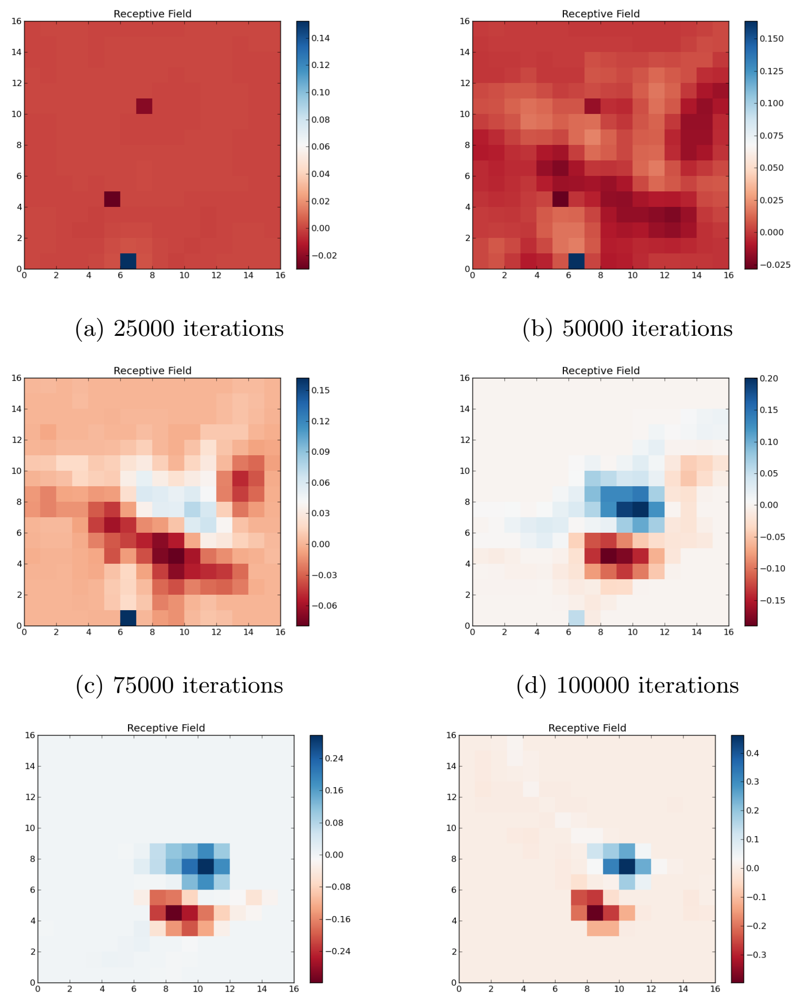
(c) 125000 iterations



(d) 149999 iterations

Figure 4: development of the a receptive field filter W ($W^+ - W^-$, where W^+ is the 16x16 matrix of weights for x_ON and W^- respectively for X_OFF) over the duration of the first simulation. The color map is applied with a dynamic scale, so as to allow for accurate analysis of the results of each iteration.

Discussion The results of two simulations are displayed in Figures 4d and 5. As can be made out very well in Figure 5 after 150000 iterations the W-filter shows a bimodal receptive field, to which both w_{ON} and w_{OFF} contribute a localized unimodal distribution. In the case w_{ON} the response is towards the positive inputs x_{ON} of a patch and in the case of w_{OFF} the response is towards the absolute values of negative inputs x_{OFF} of a patch. The resulting bimodal filter in Figure 5 resembles a gabor filter used for edge detection with the edge lying between the centers and perpendicular to the line connecting them. If we were to add patches with high responses, the edge would become visible. We note that in both figures, the receptive field is different and randomly determined. However, we do not get a receptive field for every simulation, because there might not be a feature present in the presented patches.



(a) 25000 iterations

(b) 50000 iterations

(c) 75000 iterations

(d) 100000 iterations

(e) 125000 iterations

(f) 150000 iterations

Figure 5: Typical development of a receptive field for 150000 simulations. The resulting plot after the last iteration shows a perfect localized bimodal W-filter with both w_{ON} contributing the modality depicted blue and w_{OFF} contributing the modality shown in red