PROVA RISERVATA 2 - PAGINE 1-9

CONSEGNA: MERCOLEDÌ 26 DICEMBRE 2018 ORE 18.00

`machaon.jpg`

Cognome e Nome: Andreuzzi Francesco

Numero di Matricola:

Laurea in Ingegneria Informatica (applicazioni informatiche)

Valutazione PR1: 26

Bonus: 0

Soluzione Esercizio 1

$$\text{Osserviamo che} \quad \frac{\left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right|}{|a_{n+1}-a_n|} = \frac{\frac{\cancel{|a_{n+1}-a_n|}}{a_n a_{n+1}}}{\cancel{|a_{n+1}-a_n|}} = \frac{1}{|a_n a_{n+1}|} \tag{1}$$

- $\sum |a_{n+1}-a_n| < \infty$

$$\lim_{n\to\infty} a_n a_{n+1} = a^2 \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{1}{a_n a_{n+1}} = \frac{1}{a^2}$$

$$\Longrightarrow \quad \forall\,\epsilon > 0 \quad \exists\,\overline{n}: \quad \forall\, n \geq \overline{n} \quad \left|\frac{1}{|a_n a_{n+1}|} - \frac{1}{a^2}\right| < \epsilon \quad \Longrightarrow \quad \frac{1}{a^2} - \epsilon < \frac{1}{|a_n a_{n+1}|} < \frac{1}{a^2} + \epsilon$$

$$\epsilon = \frac{1}{2a^2} \quad \Longrightarrow \quad \frac{1}{2a^2} < \frac{1}{|a_n a_{n+1}|} < \frac{3}{2a^2}$$

$$\Longrightarrow \quad \left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right| < \frac{3}{2a^2}|a_{n+1}-a_n| \quad \forall\, n \geq \overline{n}_\epsilon \qquad \text{per } \mathbf{(1)}$$

$$\Longrightarrow \quad \sum \left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right| < \infty \qquad (\text{punto } \mathbf{(1.1)} \text{ in appendice})$$

- $\sum \left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right| < \infty$

$$\lim_{n\to\infty} a_n a_{n+1} = a^2 \quad \Longrightarrow \quad \forall\,\epsilon > 0 \quad \exists\,\overline{n}: \quad \forall\, n \geq \overline{n} \quad \big||a_n a_{n+1}| - a^2\big| < \epsilon$$

$$\Longrightarrow a^2 - \epsilon < |a_n a_{n+1}| < a^2 + \epsilon$$

$$\epsilon = \frac{1}{2}a^2 \quad \Longrightarrow \quad \frac{1}{2}a^2 < |a_n a_{n+1}| < \frac{3}{2}a^2$$

$$\Longrightarrow \quad |a_{n+1}-a_n| < \frac{3}{2}a^2 \left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right| \quad \forall\, n \geq \overline{n}_\epsilon \qquad \text{per } \mathbf{(1)}$$

$$\Longrightarrow \quad \sum |a_{n+1}-a_n| < \infty \qquad (\text{punto } \mathbf{(1.1)} \text{ in appendice})$$

Soluzione Esercizio 2

i)

$$\int_{(n-1)h}^{nh} f(x)dx \geq \int_{(n-1)h}^{nh} f(nh) \geq \int_{nh}^{(n+1)h} f(x)dx \qquad (f \text{ è } \textbf{decrescente})$$

$$\int_{(n-1)h}^{nh} f(nh) = f(nh)(nh - nh + h) = hf(nh)$$

Sommando per $n = 1, 2, ..., \infty$: $\displaystyle\sum_{n=1}^{\infty} \int_{(n-1)h}^{nh} f(x)dx \geq \sum_{n=1}^{\infty} hf(nh) \geq \sum_{n=1}^{\infty} \int_{nh}^{(n+1)h} f(x)dx$

$$\int_{0}^{\infty} f(x)dx \geq h\sum_{n=1}^{\infty} f(nh) \geq \sum_{n=1}^{\infty} \int_{h}^{\infty} f(x)dx$$

Se $h \to 0^+ \quad \Longrightarrow \quad \displaystyle\int_{0}^{\infty} f(x)dx \geq h\sum_{n=1}^{\infty} f(nh) \geq \int_{0}^{\infty} f(x)dx$

Per il teorema dei **carabinieri** : $\displaystyle\int_{0}^{\infty} f(x)dx = \lim_{h\to 0^+} h\sum_{n=1}^{\infty} f(nh)$

$$\lim_{h\to 0^+} \sum_{n=1}^{\infty} \frac{h}{1+h^2x^2} = \lim_{h\to 0^+} h\sum_{n=1}^{\infty} \frac{1}{1+h^2x^2} = \int_{0}^{\infty} \frac{1}{1+x^2}dx = [\arctan(x)]_{x=0}^{x\to\infty} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Soluzione Esercizio 2

ii)

Soluzione Esercizio 3

Dimostriamo che $g(x)$ **non è** uniformemente continua in $\mathbb{R}$

$$a_n := \frac{2}{3}n^2\pi + \frac{1}{3n} \quad b_n := \frac{2}{3}n^2\pi$$
$$|a_n - b_n| = \left|\frac{2}{3}n^2\pi + \frac{1}{3n} - \frac{2}{3}n^2\pi\right| = \left|\frac{1}{3n}\right| \to 0$$

$$|g(a_n) - g(b_n)| = \left|\left|\frac{2}{3}n^2\pi + \frac{1}{3n}\right| \sin(2n^2\pi + \frac{1}{n}) - \cancel{\left|\frac{2}{3}n^2\pi\right| \sin(2n^2\pi)}\right|$$
$$= \left|\left(\frac{2}{3}n^2\pi + \frac{1}{3n}\right)\left[\cancel{\sin(2n^2\pi)\cos\left(\frac{1}{n}\right)} + \cos(2n^2\pi)\sin\left(\frac{1}{n}\right)\right]\right|$$
$$= \left|\left(\frac{2}{3}n^2\pi + \frac{1}{3n}\right)\sin\left(\frac{1}{n}\right)\right| \quad \not\to \quad 0$$

La successione $\left(\frac{2}{3}n^2\pi + \frac{1}{3n}\right)\sin\left(\frac{1}{n}\right)$ **diverge**, dato che

$$\left(\frac{2}{3}n^2\pi + \frac{1}{3n}\right)\sin\left(\frac{1}{n}\right) > n^2\sin\left(\frac{1}{n}\right) \quad \forall\, n$$

e la successione $n^2\sin\left(\frac{1}{n}\right)$ **diverge** (punto **(3)** in appendice)

APPENDICE

1.1)

$$\sum a_n < +\infty$$
$$\exists\, C > 0, \quad \exists\, \overline{n} : \quad b_n < C a_n \quad \forall\, n \geq \overline{n}$$
$$\implies \quad \sum b_n < +\infty$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\overline{n}} b_n + \sum_{n=\overline{n}+1}^{\infty} b_n \qquad (\text{con } \sum_{n=1}^{\overline{n}} b_n < \infty)$$
$$\sum_{n=\overline{n}+1}^{\infty} b_n \leq \sum_{n=\overline{n}+1}^{\infty} C a_n = C \sum_{n=\overline{n}+1}^{\infty} a_n < \infty$$
$$\implies \quad \sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\overline{n}} b_n + C \sum_{n=\overline{n}+1}^{\infty} a_n < \infty$$

2.1)

$$\int_0^{\infty} f(x)dx < \infty$$
$$\implies \lim_{x\to\infty} (x-1) \int_{x-1}^{x} f(t)dt = 0$$

Supponiamo per **assurdo** che $\lim_{x\to\infty} (x-1) \int_{x-1}^{x} f(t)dt \neq 0$

$$\implies \quad \exists\, \epsilon > 0 : \quad \forall\, \overline{x} \quad \exists\, x \geq \overline{x} : \quad (x-1)\int_{x-1}^{x} f(t)dt \geq \epsilon \quad \implies \quad \int_{x-1}^{x} f(t)dt \geq \frac{\epsilon}{x-1}$$

Esistono **infiniti** valori di x in cui la relazione sopra risulta valida. Chiamiamo $x_1, x_2, ..., x_n$ questi valori

$$\implies \int_0^{\infty} f(x)dx > \int_0^{\infty} f(x)dx - \sum_{i=1}^{\infty} \int_{x_i-1}^{x_i} f(t)dt + \sum_{i=1}^{\infty} \frac{\epsilon}{x_i - 1} = \infty$$

Ma per ipotesi $\int_0^{\infty} f(x)dx < \infty \quad \implies \quad$ abbiamo ottenuto una **contraddizione**

APPENDICE

3)

$$\lim_{n\to\infty} n^2 \sin\left(\frac{1}{n}\right) = +\infty$$

$$\lim_{n\to\infty} n^2 \sin(\frac{1}{n}) \overset{t=\frac{1}{n}}{=} \lim_{t\to 0} \frac{\sin(t)}{t^2}$$

$$\frac{\sin(t)}{t^2} \overset{t\to 0}{\sim} \frac{t}{t^2} = \frac{1}{t}$$

$$\lim_{t\to 0} \frac{1}{t} = +\infty \quad \Longrightarrow \quad \lim_{n\to\infty} n^2 \sin(\frac{1}{n}) = +\infty$$