Stanford CS229 ps0 sol

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1 Gradients and Hessians

(a) Note that A is symmetric, we have

$$\frac{\partial}{\partial x_i}\frac{1}{2}x^TAx = \frac{\partial}{\partial x_i}(\sum_{j\neq i}(a_{ij}+a_{ji})x_ix_j + a_{ii}x_i^2)/2 = \sum_{j=1}^{n} a_{ij}x_j,$$

$$\nabla f(x) = Ax + b.$$

(b)

$$\frac{\partial}{\partial x_i}g(h(x)) = \frac{\partial g(h(x))}{\partial h(x)}\frac{\partial h(x)}{\partial x_i} = g'(h(x))\frac{\partial h(x)}{\partial x_i},$$

$$\nabla f(x) = g'(h(x))\nabla h(x).$$

(c)

$$\frac{\partial^2}{\partial x_i^2}f(x) = a_{ii}, \frac{\partial^2}{\partial x_i x_j}f(x) = \frac{a_{ij}+a_{ji}}{2} = a_{ij}, \nabla^2 f(x) = A$$

(d)

$$\nabla f(x) = g'(a^Tx)a.$$

$$\frac{\partial^2}{\partial x_i^2}f(x) = \frac{\partial}{\partial x_i}g'(a^Tx)a_i = g''(a^Tx)a_i^2, \frac{\partial^2}{\partial x_j x_i}f(x) = \frac{\partial}{\partial x_j}g'(a^Tx)a_i = g''(a^Tx)a_ia_j,$$

$$\nabla^2 f(x) = g''(a^Tx)aa^T.$$

2 Positive definite matrices

(a) Since $x^Tzz^Tx = x^Tz(x^Tz)^T = (x^Tz)^2 \geq 0$ for any x, $A = zz^T$ is positive semi-definite.
(b) $\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\} = \{x \in \mathbb{R}^n : zz^Tx = 0\} = \{x \in \mathbb{R}^n : z^Tx = 0\}$,
$\mathrm{rank}(A) = n - \dim(\mathcal{N}(A)) = 1$
(c) Since $BAB^T = BA^TB^T = (BAB^T)^T$, B^TAB is symmetric. Then $x^TBAB^Tx = (B^Tx)^TA(B^Tx) \geq 0$ and BAB^T is PSD.

3 Eigenvectors, eigenvalues, and the spectral theorem

(a)
$$A[t^{(1)} \cdots t^{(n)}] = AT = T\Lambda = [t^{(1)} \cdots t^{(n)}]\mathrm{diag}(\lambda_1, \dots, \lambda_n),$$

So that $At^{(i)} = \lambda_i t^{(i)}, i = 1, \dots, n$ and they form eigenvalues/eigenvector pairs.
(b) Since $U^T U = I$, $U^{-1} = U^T$. Follow (a) we can derive the result.
(c) Since $Au^{(i)} = \lambda u^{(i)}$, A is PSD, $\lambda_i = \lambda_i (u^{(i)})^T u^{(i)} = (u^{(i)})^T A u^{(i)} \geq 0$.