

The Weil conjecture for curves via the Jacobian variety

Fabio Neugebauer

February 3, 2023

Contents

1	Statement of the Weil Conjectures for Curves	1
2	Abelian varieties	2
2.1	A summary on the picard functor	5
2.1.1	The case when $X(k) \neq \emptyset$	7
2.1.2	The dual abelian variety	7
2.2	Endomorphisms of abelian varieties	9
3	The Jacobian variety	11
3.1	The canonical map from C to its Jacobian	12
3.2	Symmetric powers of a curve	13
3.3	The Jacobian as Albanese variety	17
3.4	Autoduality	18
3.5	The Rosati involution	20
3.6	The Lefschetz trace formula and positivity of the Rosati involution	21
3.7	The map induced on the Jacobian by an endomorphism of C	22

1 Statement of the Weil Conjectures for Curves

Let C be a proper non-singular curve of genus g over a finite field \mathbb{F}_q with q elements. Let k be an algebraic closure of \mathbb{F}_q and denote $\overline{C} := C \times_{\mathbb{F}_q} k$. Let $F : C \rightarrow C$ be the absolute Frobenius morphism of C , which is the identity on the underlying topological space and acts as the q -th power map on \mathcal{O}_C . Let $\overline{F} := F \times_{\mathbb{F}_q} 1_k$ be the k linear Frobenius morphism of \overline{C} .

Let $l \neq \text{char}(p)$ be a prime number. Then \overline{F} induces an endomorphism $V_l(\overline{F})$ of the \mathbb{Q} vector space $V_l J_k := T_l J_k \otimes_{\mathbb{Z}} \mathbb{Q}$. Here J_k denotes the Jacobian variety of \overline{C} and $T_l J_k$ its Tate module.

Theorem 1.1 (Weil Conjectures for Curves). $V_l J_k$ is $2g$ -dimensional. Let P be the characteristic polynomial of $V_l(\overline{F})$ and $\alpha_1, \dots, \alpha_{2g} \in \mathbb{C}$ its roots. The Weil conjectures for curves state:

1. (*Rationality of the zeta function*)

$$\exp \left(\sum_{n=1}^{\infty} \#C(\mathbb{F}_{q^n}) \frac{x^n}{n} \right) = \frac{P(x)}{(1-x)(1-qx)}$$

and P is an integer polynomial, i.e. $P \in \mathbb{Z}[x]$.

2. (*Riemann Hypothesis*) $|\alpha_i| = q^{\frac{1}{2}}$ for all $i = 1, \dots, 2g$.
3. (*Hesse-Weil Bound*)

$$\#C(\mathbb{F}_{q^n}) = 1 - \sum_{i=1}^{2g} \alpha_i^n + q^n$$

and in particular $|\#C(\mathbb{F}_{q^n}) - (q^n + 1)| \leq 2gq^{\frac{n}{2}}$.

Moreover, for $\text{Pic}^0(C)$ the group of isomorphism classes of degree zero line bundles on C

$$\#\text{Pic}^0(C \times_{\mathbb{F}_q} \mathbb{F}_{q^n}) \leq \prod_{i=1}^{2g} (1 - \alpha_i^n) \quad (1)$$

and equality holds whenever C has an \mathbb{F}_{q^n} valued point.

Example 1.2 (Elliptic Curves). Suppose the genus of C is equal to one. Then the Hesse-Weil bound gives us that $\#C(\mathbb{F}_q) = 1 - (\alpha_1 + \alpha_2) + q$ and by the Riemann-Hypothesis $|\alpha_1 + \alpha_2| \leq 2\sqrt{q}$. Therefore, $\#C(\mathbb{F}_q) \geq 1 - 2\sqrt{q} + q = (\sqrt{q} - 1)^2$. Hence C will admit an \mathbb{F}_q valued point, i.e. C is an elliptic curve.

2 Abelian varieties

For this section k will be a field, \bar{k} an algebraic closure of k and k_s the separable algebraic closure of k in \bar{k} .

A variety X will be a scheme, which is geometrically integral, separated and of finite type over k . Note that products of varieties will be varieties again. If $\dim X = 1$ we call X a curve.

Since we assume the variety X to be geometrically integral its smooth locus is nonempty and therefore the set of closed points in X with residue field a separable algebraic field extension of k is dense in X (see details at [1, Tag 04QM]).

We will use the notations $\mathcal{O}_X(D) = \mathcal{O}(D)$ for the sheaf associated to an effective Cartier divisor D on X , i.e. $\mathcal{O}(D) = I_D^{-1}$. $\mathcal{L}_X(D) = \mathcal{L}(D)$ to will denote the sheaf associated to a Weil-Divisor D on X .

The following Lemma is due to Mumford.

Lemma 2.1 (Rigidity Lemma). Let X, Y and Z be varieties. Suppose that X is proper. If $f : X \times Y \rightarrow Z$ is a morphism with the property that, for some $y \in Y(k)$, the fibre $X \times \{y\}$ is mapped to a point $z \in Z(k)$ then f factors through the projection $\text{pr}_Y : X \times Y \rightarrow Y$.

Proof. Suppose the theorem is true for the separable algebraic closure k_s of k . Then there exists $g : Y_{k_s} \rightarrow Z_{k_s}$ such that $f_{k_s} = g \circ \text{pr}_{Y_{k_s}}$. Let $\sigma \in \text{Aut}_k(k_s)$. Then

$$(1 \times \sigma^{-1}) \circ g \circ (1 \times \sigma) \circ \text{pr}_{Y_{k_s}} = (1 \times \sigma) \circ f_{k_s} \times (1 \times \sigma^{-1}) = f_{k_s} = g \circ \text{pr}_{Y_{k_s}}.$$

$\text{pr}_{Y_{k_s}}$ is an epimorphism because it can be obtained by base change from a faithfully flat morphism. Therefore g is Galois invariant and by Galois descent [2, Prop. 16.9] there exists a unique morphism $G : Y \rightarrow Z$ such that $G_{k_s} = g$. Therefore $f_{k_s} = (g \circ \text{pr}_Y)_{k_s}$ and by faithfully flat descent $f = g \circ \text{pr}_Y$.

By the above paragraph we can assume $k = k_s$. Choose a point $x_0 \in X(k)$, and we define $g : Y \rightarrow Z$ by $f \circ (x_0, \text{id}_Y)$. The goal is to show $f = g \circ \text{pr}_Y$.

Let U be an affine open neighborhood of z . Since X is proper over k , the projection $\text{pr}_Y : X \times Y \rightarrow Y$ is a closed map, so that $V := \text{pr}_Y(f^{-1}(Z \setminus U))$ is closed in Y (set theoretic preimage). Let $P \notin V$ be a k valued point Y . Then $f(X \times \{P\}) \subseteq U$ by construction of V .

Every morphism from an irreducible proper variety X to a affine variety is constant: The scheme-theoretic-image of the morphism is a closed subscheme of an affine variety and therefore an affine variety, say W . Now X is proper, $X \rightarrow W$ is surjective and W is separated of finite type over k , hence W is also a proper variety. Using Grothendiecks finiteness result on proper maps the global sections of W form a finite dimensional k -vector space. Hence W is zero-dimensional and by irreducibility W must be a point.

Applying the previous paragraph to $f|_{X \times \{P\}}$ (note: $X \cong X \times \{P\}$) we conclude that $f(X \times \{P\}) = g(P)$.

We have shown that the set of points, where $f = g \circ \text{pr}_Y$ contains $\bigcup_{P \in (X \setminus V)(k)} X \times \{P\}$. Since this set is dense in $X \times Y$ we are done by [3, Sect. 10.2.A]. \square

Recall that a group variety $(X, m_X, 0 = e_X, (-1)_X)$ is called abelian if it is proper. We denote its group operation additive.

Corollary 2.2. Let X and Y be abelian varieties and let $f : X \rightarrow Y$ be a morphism. Then f is the composition $f = t_{f(0)} \circ h$ of a homomorphism $h : X \rightarrow Y$ and a translation $t_{f(0)}$ by $f(0)$ on Y .

Proof. Let $y = -f(e_X)$ and let $h = t_y \circ f$. Define g to be the map that one closed points is given by $g(x, x') = h(x + x') - h(x) - h(x')$. Then

$$g(\{e_X\} \times X) = g(Y \times \{e_X\}) = -h(e_X) = \{e_Y\}$$

and by the Rigidity Lemma this implies that g factors both through the first and the second projection $X \times X \rightarrow X$. Hence g equals the constant map with value e_Y and h must be a homomorphism. \square

Remark 2.3. The above Lemma 2.2 applied to $(-1)_X$ shows that the group law on an abelian variety X is indeed commutative.

An application of Lemma 2.2 to the identity morphism $X \rightarrow X$ shows that there is at most one structure of an abelian variety on X such that $e \in X(k)$ is the identity element.

We define the kernel of a homomorphism $f : X \rightarrow Y$ of abelian varieties to be the fiber of f over $e_Y \in Y$.

Theorem 2.4 (Isogenies). For a homomorphism $f : X \rightarrow Y$ of abelian varieties the following are equivalent

- a) f is surjective and has finite kernel.
- b) $\dim X = \dim Y$ and f is surjective.
- c) $\dim X = \dim Y$ and f has finite kernel.
- d) f is finite and surjective.

If one of the above conditions is satisfied, we call f an isogeny.

Moreover, any isogeny f is flat and the following formula holds for all $q \in Y$

$$\deg f = \dim_{k(q)} H^0(f^{-1}(q), \mathcal{O}_{f^{-1}(q)}). \quad (2)$$

Proof. All nonempty fibers of f have the same dimension: Choose a point $p \in f^{-1}(q)(\bar{k})$. Then $(\ker f)_{\bar{k}} \xrightarrow{t_p \times_{\bar{k}} t_q} f^{-1}(q)_{\bar{k}}$ defines an isomorphism, where t_p is the translation of $X_{\bar{k}}$ by p and t_q is defined by mapping $\{e_y\}_{\bar{k}} \rightarrow \{q\}_{\bar{k}}$.

Assume that f is surjective. By [3, Thm. 11.4.1] there exists a nonempty open subset $U \subseteq Y$ such that for all $q \in U$ over q has pure dimension $\dim X - \dim Y$. By the above, $\dim \ker f = \dim X - \dim Y$. This proves a) \implies b) \implies c).

Note this dimension formula always holds if we replace Y by the scheme-theoretic image of f . Hence, if f has finite kernel, $\dim X = \dim Y$ implies that the scheme-theoretic image of f equals Y . Since f is closed, this proves c) \implies a).

Because quasi-finite, proper morphisms are finite, a) implies d). The converse follows because quasi-finite morphisms are finite.

Both X and Y have a nonempty smooth locus. By translations we see that X and Y are smooth over k . By c) and [3, thm. 26.2.11] any isogeny is flat.

Now let f be an isogeny. Because finitely generated, flat modules over Noetherian rings are locally free of finite rank, $f_* \mathcal{O}_X$ is a locally free quasi-coherent \mathcal{O}_Y module of finite rank. Since Y is connected this rank is constant, say $d \in \mathbb{N}$.

For any $q \in Y$ there is an affine open neighborhood $U = \operatorname{Spec} R$ such that $(f_* \mathcal{O}_X)|_U \cong \mathcal{O}_Y^d|_U$. f is finite and therefore affine, so $f^{-1}U = \operatorname{Spec} R'$ for some R' . Then $f_U^\# : R \rightarrow R'$ makes R' a free R module of rank d and $f^{-1}(q) \cong \operatorname{Spec}(R' \otimes_R k(q))$ proves that

$$\dim_{k(q)} H^0(f^{-1}(q), \mathcal{O}_{f^{-1}(q)}) = d. \quad (3)$$

For $q = \eta_Y$ the generic points, we have $f^{-1}(q) \cong \text{Spec}(R' \otimes_R \text{Quot}(R))$, and $R' \otimes_R \text{Quot}(R)$ is a finite $\text{Quot}(R)$ algebra and moreover an integral domain since X is assumed to be geometrically integral. Hence $R' \otimes_R \text{Quot}(R)$ is a field that contains R' and is contained in $\text{Quot}(R')$. Now, by the universal property of the residue field of R we have $R' \otimes_R \text{Quot}(R) = \text{Quot}(R')$. Applying (3) to η_Y therefore completes the prove of (2). \square

Theorem 2.5 (Theorem of the cube and the square). Let X, Y be abelian varieties.

1. For $f, g, h : X \rightarrow Y$ morphisms

$$(f + g + h)^* \mathcal{L} \cong (f + g)^* \mathcal{L} \otimes (g + h)^* \mathcal{L} \otimes (f + h)^* \mathcal{L} \otimes f^* \mathcal{L}^{-1} \otimes g^* \mathcal{L}^{-1} \otimes h^* \mathcal{L}^{-1} \quad (4)$$

2. (*theorem of the square*)

For an invertible sheaf \mathcal{L} on X , a k scheme T and pr_X, pr_T the projections of X_T , the map

$$\varphi_{\mathcal{L}} : X(T) \rightarrow \text{Pic}(X_T) : x \mapsto (m(1_X \times x))^* \mathcal{L} \otimes \text{pr}_X^* \mathcal{L}^{-1} \otimes \text{pr}_T^* x^* \mathcal{L}^{-1} \quad (5)$$

is a homomorphism. Note that $\varphi_{\mathcal{L}}(x) = t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$ for all $x \in X(k)$ (t_x the translation by x).

Proof. Both parts of the theorem can be proven as corollaries of the *theorem of the cube*, which is a theorem on proper varieties. References are [4, Chp. II §1] or [5, Chp. II.6]. \square

Remark 2.6. The theorem of the square can be used to prove that all abelian varieties are projective. References are for example [6, thm. 7.1] or [7, sect. 9.6].

For an abelian variety X and $n \in \mathbb{Z}$ we define $n_X : X \rightarrow X$ to be the homomorphism that one points is given by $x \mapsto nx$ and define $X[n] := \ker n_X \subseteq X$. Say we have $\dim X = g$.

Proposition 2.7 (Torsion Points of Abelian Varieties). For $n \neq 0$ the morphism n_X is an isogeny of degree $\deg n_X = n^{2g}$. If $\text{char}(k) \nmid n$ then n_X is étale and $X[n](k_s) = X[n](\bar{k}) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.

Proof. By remark 2.6 there exists a ample line bundle \mathcal{L} on X . We can assume \mathcal{L} to be symmetric, i.e. $(-1)^* \mathcal{L} \cong \mathcal{L}$, because when \mathcal{L} is ample then also $(-1)^* \mathcal{L} \otimes \mathcal{L}$ will be ample by [8, II Ex. 7.5 (c)]. Applying (4) to the maps $n_X, 1_X, (-1)_X$ we obtain $n_X^* \mathcal{L}^2 \otimes \mathcal{L}^2 \cong (n+1)_X^* \mathcal{L} \otimes (n-1)_X^* \mathcal{L}$ and by induction $n_X^* \mathcal{L} \cong \mathcal{L}^{n^2}$. In particular, $n_X^* \mathcal{L}$ is an ample line bundle provided that $n^2 > 0$. Its pullback along the closed immersion $\iota : X[n] \rightarrow X$ will also be an ample line bundle. But $n_X \circ \iota$ factors through the zero map and therefore $\iota^* n_X^* \mathcal{L}$ is a trivial bundle, which is ample. By [1, Tag 01QE] $X[n]$ is quasi-affine. Hence $X[n] \rightarrow \text{Spec}(\mathcal{O}_{X[n]}(X[n]))$ is an open immersion. But $X[n]$ is proper over k , so this open immersion is moreover proper and therefore a closed immersion. This proves that $X[n]$ is a proper and affine variety and therefore finite. By theorem 2.4 c) n_X is an isogeny.

Let D be an divisor such that $\mathcal{L} \cong \mathcal{L}(D)$, then $n_X^* D$ is linearly equivalent to $n^2 D$. We now invoke intersection theory on the smooth projective variety X to conclude

$$n^{2d}(D)^g = (n^2 D)^g = (n_X^* D)^g = \deg(n_X) \cdot (D)^g,$$

where we used [6, Lem. 8.3] for the last equality. Since D is ample, its self-intersection number is positive, and we can conclude $n^{2d} = \deg(n_X)$.

Now assume $\text{char}(k) \nmid n$. To prove that n_X is étale, we may assume that $k = k_s$. The locus U where n_X is étale is open in X , so, if we prove that its complement doesn't contain any k valued point, we win. A k -valued point is in P provided that the induced map on the tangent space at P is an isomorphism. Since $n_X \circ t_P = t_{nP} \circ n_X$, by the chain rule $d_P n_X \circ d_0 t_P = d_0 t_{nP} d_0 n_X$. Since the translations give isomorphism on the tangent spaces it suffices to prove that $d_0 n_X$ is bijective.

Recall that we can identify $T_{(0,0)}(X \times X)$ with $T_0 X \oplus T_0(X)$, when we set for $f : Y \rightarrow X \times X$ that $d_y f = d_y(\text{pr}_1 \circ f) \oplus d_y(\text{pr}_2 \circ f)$. We claim that for $x, x' \in T_0(X)$ the equality $d_{(0,0)} m(x, x') = x + x'$ holds. Let $a : X \cong \{0\} \times X \rightarrow X \times X$ the canonical map. Then

$$d_{(0,0)} m \circ (\text{id}_{T_0 X} \oplus 0) = d_{(0,0)} m \circ d_0 a = d_0(m \circ a) = \text{id}_{T_0 X}$$

yields that $d_{(0,0)}m$ restricted to the first factor is the identity. By symmetry and linearity we obtain our claim. Hence, for $f, g : X \rightarrow X$ homomorphisms we have

$$d_0(f + g) = d_0(m \circ (f, g)) = d_0(m) \circ (d_0(\text{pr}_1 \circ (f, g)) \oplus d_0(\text{pr}_1 \circ (f, g))) = d_0f + d_0g.$$

Hence, by induction $d_0n_X(x) = nx$ for all $x \in T_0X$, which defines an isomorphism since $n \in k^*$.

By equation (2) and n_X being unramified it directly follows that $G := X[n](k_s) = X[n](k)$ is an abelian group of order n^{2g} , which is killed by n . Further, for every divisor d of n the subgroup of elements that is killed by d is $X[d](k_s)$ and has order d^{2g} . An application of the structure theorem of finitely generated abelian groups shows $X[d](k_s) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$. \square

Note that since n_X is surjective $X(\bar{k})$ is a divisible group.

Proposition 2.8. If $f : X \rightarrow Y$ is an isogeny of degree d then there exists a unique isogeny $g : Y \rightarrow X$ such that $g \circ f = d_X$ and $f \circ g = d_Y$.

Proof. If $f : X \rightarrow Y$ is an isogeny of degree d , then $\ker f$ is a finite group scheme which is contained in the kernel of d_X by [4][Exerc.4.4]. Since X is quasi-projective, we can take the quotient $X/\ker f$ to get a factorization of d_X as $X \rightarrow X/\ker f \xrightarrow{g} X$. By [5, Sect. 12 Cor. 1] we can identify $X \rightarrow X/\ker f$ with $X \xrightarrow{f} Y$, so that we get $d_X = g \circ f$. By theorem 2.4 b) g is an isogeny. Then $g \circ d_Y = d_X \circ g = g \circ (f \circ g)$. Hence $h = d_Y - (f \circ g)$ maps into the finite k -scheme $\ker g$. The scheme-theoretic image of h is a closed irreducible subscheme of $\ker g$, so h is constant and $d_Y = f \circ g$ follows. \square

An non-zero abelian variety X is called *simple* if X has no other abelian subvarieties other than $\{e_X\}$ and X . Note that abelian subvarieties will be closed subschemes.

For any homomorphism of abelian varieties $f : X \rightarrow Y$ its scheme-theoretic image is an abelian subvariety of Y . Further by [4, 5.31] the reduced underlying scheme $(\ker f)_0^{\text{red}}$ of the identity component of $\ker f$ is an abelian subvariety of X .

Hence a non-constant homomorphism $f : X \rightarrow Y$ of simple abelian varieties is surjective and the identity component of $\ker f$ is $\{e_X\}$. All connected components of a \bar{k} -group scheme are isomorphic as \bar{k} -schemes by translating back and forth. In particular, all components of $\ker f$ have the same dimension and we see by theorem 2.4a) that f is an isogeny. It follows by Proposition 2.8 that for a simple abelian variety X

$$\text{End}_k^0(X) := \text{End}_k(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an associative division algebra over \mathbb{Q} . Our goal is to compose an arbitrary abelian variety into simple factors.

Theorem 2.9 (Poincaré Splitting Theorem). Let Y be an abelian subvariety of X , then there exists an abelian subvariety $Z \subseteq X$ such that the homomorphism $Y \times Z \rightarrow X$ given by $(y, z) \mapsto y + z$ is an isogeny.

For a finite dimensional vector space V admitting an inner product $V \rightarrow V^\vee, v \mapsto \langle \cdot, v \rangle$ and a subspace $W \subseteq V$ the subspace $\ker(V \rightarrow V^\vee \xrightarrow{\text{res}} W^\vee)$ constitutes a complement of W in V .

To mimic this prove we need the existence of a dual abelian variety and an isomorphism $X \rightarrow X^\vee$. This can be accomplished using results of the following subsection.

2.1 A summary on the picard functor

Given a smooth projective variety $X \rightarrow k$ over a field.

Note that the contravariant functor $\text{Sch}/k \rightarrow \text{Ab}, T \mapsto \text{Pic}(X_T)$ is not a Zariski sheaf:

We will denote $\text{pr}_T : X_T \rightarrow T$ to be the projection. Given $\mathcal{L} \in \text{Pic}(T)$ such that $\text{pr}_T^* \mathcal{L}$ is not trivial. Let $(U_i)_{i \in I}$ an open cover of T that trivializes \mathcal{L} . Then (X_{U_i}) constitutes an open cover of X_T and the pullback of $\text{pr}_T^* \mathcal{L}$ to X_{U_i} is trivial. Therefore \mathcal{L} is in the kernel of the map

$$\text{Pic}(X_T) \mapsto \prod_{i \in I} \text{Pic}(X_{U_i}),$$

while not being trivial.

In hope to get a representable functor we define the (relative) Picard functor of $X \rightarrow k$ by

$$T \mapsto \text{Pic}(X_T)/\text{pr}_T^* \text{Pic}(T). \quad (6)$$

It turns out that our assumptions on $X \rightarrow k$ suffice and that the picard functor is indeed representable by a separated scheme $\text{Pic}_{X/k}$ locally of finite type over k . Further, every closed subscheme $Z \hookrightarrow \text{Pic}_{X/k}$ which is of finite type over k is proper (in fact projective) over k . A proof is given in [9, Chapt. 8, thm. 3].

Let us denote the connected component of the identity in $\text{Pic}_{X/k}$ by $\text{Pic}_{X/k}^0$. Exploiting the properties of group schemes over fields as in [1, Tag 047J] it can be proven that $\text{Pic}_{X/k}^0 \hookrightarrow \text{Pic}_{X/k}$ is a flat closed immersion, $\text{Pic}_{X/k}^0$ is geometrically irreducible and quasi-compact over k .

Combining the last two paragraphs, we conclude that $\text{Pic}_{X/k}^0$ is a proper and geometrically irreducible group scheme over k .

$\text{Pic}_{X/k}^0$ need not necessarily be reduced, let alone geometrically reduced. The latter happens if and only if $\text{Pic}_{X/k}^0$ is smooth: If $\text{Pic}_{X/k}^0$ is geometrically reduced it is a variety and will have non-empty smooth-locus. Using the translation morphism of its group structure we see that it is smooth. Conversely, if $\text{Pic}_{X/k}^0$ is smooth then its base change to the algebraic closure will be regular. Any regular local ring is a domain and hence $\text{Pic}_{X/k}^0$ must be geometrically reduced.

Luckily, there is a criterion for when $\text{Pic}_{X/k}^0$ is smooth.

Theorem 2.10. The tangent space of $\text{Pic}_{X/S}$ at the identity element is isomorphic to $H^1(X, \mathcal{O}_X)$. Further, $\text{Pic}_{X/k}^0$ is smooth over k if and only if $\dim \text{Pic}_{X/k}^0 = \dim H^1(X, \mathcal{O}_X)$.

Proof. Let $S := \text{Spec}(k[\varepsilon])$ where $k[\varepsilon]$ is the ring of the dual numbers over k . For any k algebra A every element in $A \otimes_k k[\varepsilon]$ can be written as an product of element in A and a unit. Therefore the map $A \rightarrow A \otimes_k k[\varepsilon]$ induces a homeomorphism onto its image when passing to spectra. The map $A \rightarrow A \otimes_k k[\varepsilon]$ is also finite and injective, so it will actually induce a homeomorphism. Looking at affine patches as above, we can identify the topological spaces X and X_S .

On this space we have a short exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_X \xrightarrow{h} \mathcal{O}_{X_S}^* \xrightarrow{\text{res}} \mathcal{O}_X^* \rightarrow 1 \quad (7)$$

where h is given on sections by $f \mapsto 1 + \varepsilon f$ and res by $a + \varepsilon b \mapsto a$. Since this sequence also yields an exact sequence on global sections, we get an exact sequence on the first cohomology groups

$$0 \rightarrow H^1(X, \mathcal{O}_X) \rightarrow \text{Pic}(X_S) \xrightarrow{\text{res}} \text{Pic}(X). \quad (8)$$

(Cohomology in the category of sheaves of abelian groups on X .)

Let $s : \text{Spec}(k) \rightarrow S$ be the canonical morphism. Then $\text{Pic}(X_S) \xrightarrow{\text{res}} \text{Pic}(X)$ can be identified with the pull back along $X \xrightarrow{(1_X, s)} X_S$.

Since $\text{Pic}(S)$ and $\text{Pic}(k)$ are trivial, we have $\text{Pic}_{X/k}(k) = \text{Pic}(X)$ and $\text{Pic}_{X/k}(S) = \text{Pic}(X_S)$. Further, the pullback along $(1_X, s)$ is by definition of the contra-variant functor $\text{Pic}_{X/k}$ the induced map $\text{Pic}_{X/k}(S) \rightarrow \text{Pic}_{X/k}(k)$. Its kernel T consists of $f : S \rightarrow \text{Pic}_{X/k}$ such that $f \circ s = 0$, where 0 is the identity of the group scheme $\text{Pic}_{X/k}$. In [1, Tag 0B28] T is identified with the tangent space of $\text{Pic}_{X/k}$ at 0 , where the k action on T is induced by $k[\varepsilon] \rightarrow k[\varepsilon], \varepsilon \mapsto \lambda \varepsilon$. Therefore the sequence (8) identifies the underlying abelian group of the tangent space of $\text{Pic}_{X/S}$ at zero with the abelian group $H^1(X, \mathcal{O}_X)$. The k -vector space structure on $H^1(X, \mathcal{O}_X)$ is given by $\lambda \cdot \{f_{\alpha\beta} \in \mathcal{O}_X(U_\alpha \cap U_\beta)\} = \{\lambda f_{\alpha\beta} \in \mathcal{O}_X(U_\alpha \cap U_\beta)\}$ for any Čech 1-cocycle given a covering (U_α) .

The first map in the sequence (8), sends such a Čech 1-cocycle $\{f_{\alpha\beta} \in \mathcal{O}_X(U_\alpha \cap U_\beta)\}$ to a line bundle on X_S that trivializes on the U_α and has transition functions $1 + \varepsilon f_{\alpha\beta}$. Hence the k -vector space structure on T as tangent space exactly matches the k -action we obtain when identifying T via sequence (8) with the vector space $H^1(X, \mathcal{O}_X)$. This proves that $H^1(X, \mathcal{O}_X) \cong T_0(\text{Pic}_{X/k})$ as k -vector spaces.

The only if part of the second statement of the theorem follows from $H^1(X, \mathcal{O}_X) \cong T_0(\text{Pic}_{X/k}^0)$.

Conversely, assuming $\dim \text{Pic}_{X/k}^0 = \dim H^1(X, \mathcal{O}_X)$ we conclude that $\dim \text{Pic}_{X/k}^0 = \dim T_0(\text{Pic}_{X/k}^0)$. Hence, the stalk of $\Omega_{\text{Pic}_{X/k}^0}^1$ at 0 is generated by $\dim(\text{Pic}_{X/k}^0)$ elements. Therefore $\text{Pic}_{X/k}^0$ is smooth over k at 0 of relative dimension $\dim(\text{Pic}_{X/k}^0)$. The locus of smoothness of fixed relative dimension is open and by translating it on $\text{Pic}_{X/k}^0$ we win. \square

2.1.1 The case when $X(k) \neq \emptyset$

We assume there is $\varepsilon : k \rightarrow X$ a section to $X \rightarrow k$. Then for any k -scheme T the projection $\text{pr}_T : X \times T \rightarrow T$ admits a section $\varepsilon_T : T \cong k \times T \xrightarrow{\varepsilon \times 1} X \times T$.

Hence pr_T^* is a section to the pullback along $\varepsilon_T^* : \text{Pic}(X_T) \rightarrow \text{Pic}(T)$, and therefore the maps

$$\ker(\varepsilon_T^*) \hookrightarrow \text{Pic}(X_T) \rightarrow \text{Pic}(X_T)/\text{pr}_T^* \text{Pic}(T) \quad (9)$$

compose to an isomorphism with inverse $\mathcal{L} \mapsto \mathcal{L} \otimes \text{pr}_T^* \varepsilon_T^* \mathcal{L}^{-1}$. If we consider both left and right hand side of (9) as contravariant functors in T then (9) defines a natural isomorphism between those and we obtain that $\text{Pic}_{X/k}$ also represents the functor

$$T \mapsto \{\mathcal{L} \in \text{Pic}(X_T) : \varepsilon_T^* \mathcal{L} \text{ is trivial}\}. \quad (10)$$

Proposition 2.11 (The Poincaré Bundle). There is an isomorphism class of line bundles \mathcal{P} on $X \times_k \text{Pic}_{X/k}$ such that $\varepsilon_{\text{Pic}_{X/k}}^* \mathcal{P}$ is trivial, that satisfies the following universal property: For any $\mathcal{L} \in \text{Pic}(X_T)$ with $\varepsilon_T^* \mathcal{L}$ trivial, there exists a unique $g : T \rightarrow \text{Pic}_{X/k}$ such that $(1_X \times g)^* \mathcal{P} = \mathcal{L}$.

Moreover, $\mathcal{P}|_{X \times 0}$ is trivial for $0 \in \text{Pic}_{X/k}(k)$ representing the identity in $\text{Pic}(X)$.

Proof. This is the contravariant Yoneda Lemma applied to (10). See the diagram below. The last assertion is clear from the first statement by taking $\mathcal{L} = \mathcal{O}_X \in \text{Pic}(X)$. \square

$$\begin{array}{ccc} \text{Hom}_k(\text{Pic}_{X/k}, \text{Pic}_{X/k}) & \xrightarrow{\text{Hom}(g, \text{Pic}_{X/k})} & \text{Hom}_k(T, \text{Pic}_{X/k}) \\ \downarrow \wr & \begin{array}{ccc} \text{id}_{\text{Pic}_{X/k}} & \xrightarrow{\quad} & g \\ \downarrow & & \downarrow \\ \mathcal{P} & \xrightarrow{\quad} & \mathcal{L} \end{array} & \downarrow \wr \\ \{\mathcal{L} \in \text{Pic}(X \times_k \text{Pic}_{X/k}) : \varepsilon_{\text{Pic}_{X/k}}^* \mathcal{L} \text{ is trivial}\} & \xrightarrow{(1 \times g)^*} & \{\mathcal{L} \in \text{Pic}(X_T) : \varepsilon_T^* \mathcal{L} \text{ is trivial}\} \end{array} \quad (11)$$

We will call $\mathcal{P}_X := \mathcal{P}$ from Proposition (2.11) the *Poincaré bundle*.

2.1.2 The dual abelian variety

In the case that X is an abelian variety, we will always take ε to be the inclusion of the identity, which we denote 0, into X . Let \mathcal{L} be a line bundle on X . On $X \times X$ we define the *Mumford line bundle* $\Lambda(\mathcal{L})$ by

$$\Lambda(\mathcal{L}) := m^* \mathcal{L} \otimes \text{pr}_1^* \mathcal{L}^{-1} \otimes \text{pr}_2^* \mathcal{L}^{-1}.$$

Then $\varepsilon_X^* \Lambda(\mathcal{L})$ is trivial and by proposition (2.11) there is a unique $\varphi_{\mathcal{L}} : X \rightarrow \text{Pic}_{X/k}$ such that $(1 \times \varphi_{\mathcal{L}})^* \mathcal{P} = \Lambda(\mathcal{L})$.

On T valued-points this map is given by mapping $x : T \rightarrow X$ to $\varphi_{\mathcal{L}} \circ x$ and diagram (11) tells us that this point represents $(1 \times \varphi_{\mathcal{L}} \circ x)^* \mathcal{P} \in \text{Pic}(X_T)$. Moreover, since

$$(1 \times \varphi_{\mathcal{L}} \circ x)^* \mathcal{P} = (1 \times x)^* \Lambda(\mathcal{L}) = (m \circ (1 \times x))^* \mathcal{L} \otimes \text{pr}_X^* \mathcal{L}^{-1} \otimes \text{pr}_T^* x^* \mathcal{L}^{-1} \quad (12)$$

we can identify $\varphi_{\mathcal{L}}$ on T -valued points with the map from the theorem of the square (2.5). Now theorem (2.5) part b) proves that $\varphi_{\mathcal{L}}$ is a homomorphism. In particular, $\varphi_{\mathcal{L}}(0) = 0$ and because X is connected $\varphi_{\mathcal{L}}$ factors through $\text{Pic}_{X/k}^0$.

Lemma 2.12. Let us denote the kernel of $\varphi_{\mathcal{L}} : X \rightarrow \text{Pic}_{X/k}^0$ by $K(\mathcal{L})$.

- (i) We have $\Lambda(\mathcal{L})|_{X \times K(\mathcal{L})} \cong \mathcal{O}_{X \times K(\mathcal{L})}$,
- (ii) If \mathcal{L} is ample, then $K(\mathcal{L})$ is finite. Conversely, if \mathcal{L} has a non-zero global section and $K(\mathcal{L})$ is finite, then \mathcal{L} is ample.

Proof. Let $T = K(\mathcal{L})$ and $x : K(\mathcal{L}) \rightarrow X$ be the inclusion. Then $\Lambda(\mathcal{L})|_{X \times K(\mathcal{L})} = (1 \times x)^* \Lambda(\mathcal{L}) = \varphi_{\mathcal{L}}(K(\mathcal{L}))$, which is trivial by definition of $K(\mathcal{L})$.

For (ii) let \mathcal{L} be an ample line bundle on X . Then its pullback \mathcal{L}' to $K(\mathcal{L})$ is ample because $\iota : K(\mathcal{L}) \rightarrow X$ is a closed immersion. By (i) the bundle $\Lambda(\mathcal{L})$ is trivial pulled back to $X \times K(\mathcal{L})$. Pulling this line bundle back to $K(\mathcal{L})$ via $(\iota, -1)$ gives that $\mathcal{L}'^{-1} \times (-1)^* \mathcal{L}'$ is trivial on $K(\mathcal{L})$. This yields a ample and trivial sheaf on the closed subscheme $K(\mathcal{L})$ of the proper scheme X .

In the first paragraph of the proof of proposition (2.7) we showed that if the structure sheaf of a proper scheme over k is ample, then the scheme is finite over k . This proves the first assertion of statement (ii). The converse statement is proposition 2.2 in [4]. \square

Theorem 2.13. For an abelian variety X over k the *dual abelian variety* $X^\vee := \text{Pic}_{X/k}^0$ is an abelian variety over k . If $\mathcal{L} \in \text{Pic}(X)$ is ample, then $\varphi_{\mathcal{L}} : X \rightarrow X^\vee$ is an isogeny and, further, $\dim X = \dim_k H^1(X, \mathcal{O}_X) = \dim X^\vee$. In particular, if X is a curve, it's a curve of genus one.

Proof. Choose an ample line bundle $\mathcal{L} \in \text{Pic}(X)$, which exists by (2.6). Then the map $\varphi_{\mathcal{L}} : X \rightarrow \text{Pic}_{X/k}^0$ has finite fiber over 0 by (2.12) and we conclude $\dim X \leq \dim \text{Pic}_{X/k}^0$.

It can be shown that for any group variety over a field $\dim H^1(X, \mathcal{O}_X) \leq \dim X$, see [4, Cor. 6.15] and therefore

$$\dim X \leq \dim \text{Pic}_{X/k}^0 \leq \dim T_0(\text{Pic}_{X/k}^0) \stackrel{2.10}{=} \dim_k H^1(X, \mathcal{O}_X) \leq \dim X.$$

Hence $\text{Pic}_{X/k}$ is an abelian variety by (2.10) and the discussion above (2.10). $\varphi_{\mathcal{L}}$ will be an isogeny by theorem (2.4). \square

Consider two line bundles $\mathcal{L}, \mathcal{L}'$ on X . If $\mathcal{L} \cong \mathcal{L}'$, then $\Lambda(\mathcal{L}) \cong \Lambda(\mathcal{L}')$ and therefore $\varphi_{\mathcal{L}} = \varphi_{\mathcal{L}'}$. Hence we obtain a morphism

$$\varphi : \text{Pic}(X) \rightarrow \text{Hom}_k(X, X^\vee), \mathcal{L} \mapsto \varphi_{\mathcal{L}}. \quad (13)$$

Further $\Lambda(\mathcal{L} \otimes \mathcal{L}') \cong \Lambda(\mathcal{L}) \otimes \Lambda(\mathcal{L}')$ and therefore $\varphi_{\mathcal{L} \otimes \mathcal{L}'} = \varphi_{\mathcal{L}} + \varphi_{\mathcal{L}'}$, i.e. φ is a homomorphism.

An isogeny $\lambda : X \rightarrow X^\vee$ will be called *polarization*, if there exists some invertible ample sheaf \mathcal{L} on $X_{\bar{k}}$ such that $\lambda_{\bar{k}} = \varphi_{\mathcal{L}}$. By theorem (2.13) and remark (2.6) there always exists at least one polarization.

If $f : X \rightarrow Y$ is a homomorphism of abelian varieties over k then $(f \times 1)^* \mathcal{P}_Y$ is trivial when pulled back to $\{0\} \times Y^\vee$. Therefore, by proposition (2.11) there exists a unique $f^\vee : Y^\vee \rightarrow X^\vee$ such that

$$(1 \times f^\vee)^* \mathcal{P}_X \cong (f \times 1)^* \mathcal{P}_Y. \quad (14)$$

Note that $f \mapsto f^\vee$ is a contravariant functor. Moreover, it can be shown that $(f + g)^\vee = f^\vee + g^\vee$ for $f, g : X \rightarrow Y$ homomorphisms, see [4, Chap. 7]. In particular, $n_X^\vee = n_{X^\vee}$ and proposition (2.8) shows that the dual of an isogeny of degree d is again an isogeny of degree d . The existence of such dual homomorphisms justifies the name *dual abelian variety*.

2.2 Endomorphisms of abelian varieties

In this chapter X and Y will be abelian varieties over the field k , X will have dimension g and l will be a prime number not equal to $\text{char}(k)$. We give a proof of the Poincaré Splitting Theorem (2.9).

Proof. Let $\iota : Y \rightarrow X$ be the inclusion and $\lambda : X \rightarrow X^\vee$ a polarization.

For $K := \ker(X \xrightarrow{\lambda} X^\vee \xrightarrow{\iota^\vee} Y^\vee)$ define Z to be the connected components of K with its reduced subscheme structure. Then Z is an abelian variety of dimension $\dim X - \dim Y$. By [4, Exerc. 11.1] $\iota^\vee \circ \lambda \circ \iota$ is a polarization of Y . In particular, $Z \cap Y$ is finite. Now the kernel of the homomorphism $Y \times Z \rightarrow X$ is contained in $(Y \cap Z) \times (Y \cap Z)$ and therefore finite. The proposition follows from theorem (2.4) part c). \square

Corollary 2.14. There exist simple abelian varieties Y_1, \dots, Y_n , non two of which are k -isogenous, and there are positive integers m_1, \dots, m_n such that X is isogenous to $Y_1^{m_1} \times Y_2^{m_2} \times \dots \times Y_n^{m_n}$. The factors are unique up to k -isogeny and permutation.

Proof. This follows from the Poincaré Splitting Theorem (2.9) and the fact that any homomorphism of simple abelian varieties is constant or an isogeny. \square

Definition 2.15 (The Tate module). We define the *Tate module* of X by

$$T_l X := \lim \left(\{0\} \xleftarrow{l} X[l](k_s) \xleftarrow{l} X[l^2](k_s) \xleftarrow{l} \dots \right). \quad (15)$$

It follows from theorem (2.4) that $T_l X$ is (non-canonically) isomorphic to \mathbb{Z}_l^{2g} and we introduce the $2g$ dimensional \mathbb{Q}_l vector space $V_l X := T_l X \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$.

A homomorphism of abelian varieties $f : X \rightarrow Y$ induces a homomorphism $T_l f : T_l X \rightarrow T_l Y$. It sends a point $(0, x_1, x_2, \dots) \in T_l X$ to $(0, f(x_1), f(x_2), \dots) \in T_l Y$.

Remark 2.16. $\mathbb{Q}_l/\mathbb{Z}_l$ is the union of its subgroups $l^{-n}\mathbb{Z}_l/\mathbb{Z}_l$, which we identify with $\mathbb{Z}/l^n\mathbb{Z}$. Therefore $\mathbb{Q}_l/\mathbb{Z}_l = \text{colim } \mathbb{Z}/l^n\mathbb{Z}$, where the colimit is taken over the homomorphisms $\mathbb{Z}/l^n\mathbb{Z} \hookrightarrow \mathbb{Z}/l^{n+1}\mathbb{Z}$ given by $(1 \bmod l^n) \mapsto (l \bmod l^{n+1})$ and we see that $T_l X = \text{Hom}(\mathbb{Q}_l/\mathbb{Z}_l, X(k_s))$. Using this characterization of the Tate module and the long exact sequence of $\text{Ext}_{\mathbb{Z}}(\mathbb{Q}_l/\mathbb{Z}_l, \cdot)$ modules it can be shown that for any isogeny $f : X \rightarrow Y$ the induced map $T_l f : T_l X \rightarrow T_l Y$ is injective with cokernel the l -Sylow group of $\ker f(k_s)$ and that $V_l f : V_l X \rightarrow V_l Y$ is an isomorphism, see [4, Cor. 10.7].

Lemma 2.17. Let $f : X \rightarrow Y$ be a homomorphism. If $T_l f \in \text{Hom}_{\mathbb{Z}_l}(T_l X, T_l Y)$ is divisible by l^n then f is divisible by l^n in $\text{Hom}(X, Y)$.

Proof. The divisibility of $T_l(f)$ means that f vanishes on $X[l^n](k_s)$. $X[l^n]$ is étale over k and therefore f vanishes on $X[l^n]$. By [5, Sect. 12 Cor. 1] the isogeny $l_X^n : X \rightarrow X$ gives X the structure of the quotient $X/X[l^n]$. Therefore f factors through l_X^n and hence is divisible by l_X^n . \square

If $f \in \text{Hom}_k(X, Y)$ and $n \in \mathbb{Z} \setminus \{0\}$ then $n \cdot f = 0 \implies n_Y \circ f = f \circ n_X = 0$, but $[n_X]$ is surjective, so $f = 0$. Hence, $\text{Hom}_k(X, Y)$ is a torsion-free abelian group. In particular, the canonical map $\text{End}(X) \rightarrow \text{End}^0(X)$ is injective.

For $f \in \text{End}(X)$ we define $\deg f$ to be the degree of f if f is an isogeny and zero otherwise. Because the degree is multiplicative we can extend this to $\deg : \text{End}^0(X) \rightarrow \mathbb{Q}$ via $\deg(\frac{f}{n}) := n^{-2g} \deg f$.

Theorem 2.18. The map $\deg : \text{End}_k^0(X) \rightarrow \mathbb{Q}$ is a homogeneous polynomial mapping of degree $2g$, i.e. if e_1, \dots, e_n are independent elements of $\text{End}_k^0(X)$ then there is a homogeneous polynomial $P \in \mathbb{Q}[x_1, \dots, x_n]$ of degree $2g$ such that $\deg(x_1 e_1 + \dots + x_n e_n) = P(x_1, \dots, x_n)$ for all $x_1, \dots, x_n \in \mathbb{Q}$.

Proof. By corollary 2.14 and proposition 2.8 we may assume that X is simple. Note that $\deg(nf) = n^{2g} \deg(f)$ for all $n \in \mathbb{Q}$ and all f . So, if P is a polynomial mapping, it must be homogeneous of degree $2g$. [6, Lem. 12.3] shows via an induction argument that it suffices to prove that for all $f, g \in \text{End}_k^0(X)$ there exists $P \in \mathbb{Q}[x]$ of degree $\leq 2g$ such that $\deg(nf + g) = P(n)$ for all $n \in \mathbb{Q}$.

By multiplying with a big enough integer and using that $\deg(nf) = n^{2g} \deg(f)$ we may assume that $f, g \in \text{End}(X)$ and that $n \in \mathbb{Z}$.

Let D be a very ample divisor on X and let $D_n := (nf + g)^*D$. Then by [6, Lem. 8.3] $\deg(nf + g)(D)^g = (D_n^g)$, since $nf + g$ is either an isogeny or the zero map. Hence, it suffices to prove that (D_n^g) is a polynomial in n of degree $\leq 2g$.

By theorem 2.5 part a) applied to the maps $nf + g, f, f : X \rightarrow X$ and $\mathcal{L} = \mathcal{L}(D)$ shows that $D_{n+2} - 2D_{n+1} + D_n$ is linearly equivalent to $D' := (2f)^*D - 2(f^*D)$. So by induction D_n is linearly equivalent to $\frac{n(n-1)}{2}D' + nD_1 - (n-1)D_0$. By the multilinearity of the g -fold intersecion number we conclude that $(D_n)^g = \left(\frac{n(n-1)}{2}\right)^g (D')^g + \dots$ is a polynomial in n . \square

Theorem 2.19. The \mathbb{Z}_l -linear map $\text{Hom}_k(X, Y) \otimes \mathbb{Z}_l \rightarrow \text{Hom}_{\mathbb{Z}_l}(T_l X, T_l Y)$ given by $f \otimes c \mapsto c \cdot T_l(f)$ is injective.

Proof. Claim: If X is simple, then the map $\text{End}(X) \otimes \mathbb{Z}_l \rightarrow \text{End}(T_l X)$ is injective:

Suppose the map is not injective. Then there exist $f_1, \dots, f_n \in \text{End}(X)$ and l -adic integers c_1, \dots, c_n such that $c_1 T_l f_1 + c_2 T_l f_2 + \dots + c_n T_l f_n = 0$.

Let M be the \mathbb{Z} submodule of $\text{End}^0(X)$ generated by the $\{f_1, \dots, f_n\}$. By theorem 2.18 the map $\deg : \mathbb{Q}M := \mathbb{Q} \otimes M \rightarrow \mathbb{Q}$ is continuous for the real topology and so $U := \{v \in \mathbb{Q}M \mid \deg(v) < 1\}$ is an open neighborhood of 0. Since X is simple, every nonzero endomorphism of X has degree a positive integer and therefore $(\mathbb{Q}M \cap \text{End}(X)) \cap U = \{0\}$ and we see that $\mathbb{Q}M \cap \text{End}(X)$ is discrete in $\mathbb{Q}M$. By [10, Prop. 4.15] this equivalent to $\mathbb{Q}M \cap \text{End}(X)$ being a finitely generated \mathbb{Z} -module. Since $\text{End}(X)$ is torsion-free there is $r > 0$ such that $\mathbb{Q}M \cap \text{End}(X) = e_1 \mathbb{Z} \oplus \dots \oplus e_r \mathbb{Z}$ for certain $e_i \in \text{End}(X)$. Moreover, there are $a_1, \dots, a_r \in \mathbb{Z}_l$ such that $\sum_{i=1}^r a_i T_l(e_i) = 0$.

Since the integers are dense in the l -adic integers, for any $m \in \mathbb{N}$ there exists $n_1(m), \dots, n_r(m) \in \mathbb{Z}$ such that for all $i = 1, \dots, r$ we have $n_i(m) - a_i$ is divisible through l^m . Then also

$$T_l \left(\sum_{i=1}^r n_i(K) e_i \right) = \sum_{i=1}^r n_i(K) T_l(e_i) = \sum_{i=1}^r (n_i - a_i) T_l(e_i)$$

is divisible through l^m and by Lemma 2.17 $\sum_{i=1}^r n_i(m) e_i \in \text{End}(X)$ is divisible by l^m in $\text{End}(X)$ and therefore in $\mathbb{Q}M \cap \text{End}(X)$ by definition of $\mathbb{Q}M$. On the other hand, since $|n_i(m) - a_i|_l \leq l^{-m}$ there exist $M_i, K_i \in \mathbb{Z}$ such that $v_l(n_i(m)) = K_i$ for all $m \geq M_i$. Let $M = \max M_i$ and $K = \max K_i$. Then $\sum_{i=1}^r n_i(m) e_i$ is not divisible by a power of l higher than K for all $m \geq M$ in $\mathbb{Q}M \cap \text{End}(X)$ since the e_1, \dots, e_r form a free generating system. This contradicts the earlier statement.

Now we prove the general case. Note that since ‘limits commute’ $T_l(X \times Y) = T_l(X) \times T_l(Y)$.

There exists isogenies $X \rightarrow \prod_{i=1}^r X_i$ and $Y \rightarrow \prod_{j=1}^r Y_j$, where the X_i, Y_j are simple abelian varieties. Proposition 2.8 lets us map $\text{Hom}(X, Y)$ into $\text{Hom}(\prod_{i=1}^r X_i^{m_i}, \prod_{j=1}^r Y_j^{n_j})$ and since n_X is an epimorphism for all $n \in \mathbb{Z}$ this is injective. Since every nonzero homomorphisms of simple abelian varieties is an isogeny, $\text{Hom}(\prod_{i=1}^r X_i^{m_i}, \prod_{j=1}^r Y_j^{n_j}) = \prod_{i,j} \text{Hom}(X_i, Y_j)$ and if X_i and Y_i are isogenous then $\text{Hom}(X_i, Y_i)$ embeds into $\text{End}(X_i)$, else $\text{Hom}(X_i, Y_i) = 0$. So, the theorem follows from the special case proven above. \square

Given $f \in \text{End}_k^0(X)$ there is a necessarily unique polynomial $P_f \in \mathbb{Q}[x]$ of degree $2d$ such that $P_f(n) = \deg(n_X - f)$ for all $n \in \mathbb{N}$ by theorem 2.18. The next theorem justifies that we will refer to P_f as the *characteristic polynomial* of f .

Theorem 2.20. For $f \in \text{End}^0(X)$ let $P_{f,l} \in \mathbb{Q}_l[x]$ be the characteristic polynomial of $V_l f \in \text{End}_{\mathbb{Q}}(V_l X)$. Then $P_{f,l} = P_f$ is independent of l and has integer coefficients whenever $f \in \text{End}(X)$.

Proof. We only give a sketch, whereas a detailed proof can be found in [6, Chapt. 12] or [4, thm. 12.8]. It can be assumed that $f \in \text{End}(X)$ and, further, using corollary 2.14 that X is simple.

We start with the notation of the proof of theorem 2.18, $g = \text{id}$ and a chosen ample symmetric divisor D . A computation as in the proof of theorem 2.7 shows that $D' \sim -2D$ and we conclude from the last equation in the proof of theorem 2.18 that P_f has integer coefficients and leading coefficient 1.

Let $P_f = \prod_{i=1}^{2g} (x - a_i)$ and let $P_{f,l} = \prod_{i=1}^{2g} (x - b_i)$. Let $F \in \mathbb{Z}[t]$. Using the properties of the determinant it can be proven that $\det V_l(F(f)) = \pm \prod_{i=1}^{2g} F(b_i)$ and similarly using the multiplicativity of the degree it can be shown that $\deg(F(f)) = \pm \prod_{i=1}^{2g} F(a_i)$.

Let $\alpha := F(f)$. Using the Smith-Normal form on $T_l \alpha$ to assume it in diagonal form, we can see that $\frac{1}{\#(\text{coker}(T_l \alpha))} = |\det(T_l \alpha)|_l$. Further by remark 2.16 $\text{coker}(T_l \alpha)$ is isomorphic to the l -Sylow-group N_l of $(\ker \alpha)(k_s)$. N_l is an étale group scheme over k by [4, Cor. 4.48] provided that l is relatively prime to $\text{char}(p)$ and hence $\#N_l = |\deg(\alpha)|_l^{-1}$ by equation (2). Summarized we have,

$$\left| \prod_{i=1}^{2g} F(a_i) \right|_l = |\deg(\alpha)|_l = \frac{1}{\#N_l} = \frac{1}{\#(\text{coker}(T_l \alpha))} = |\det(T_l \alpha)|_l = \left| \prod_{i=1}^{2g} F(b_i) \right|_l$$

for all $F \in \mathbb{Z}[t]$. By lemma 1 in [11, lem. VII 1.], this implies that $P_{f,l} = P_f$ as elements of $\mathbb{Q}_l[x]$. (The proof of the cited lemma relies on the denseness of the integers in the l -adic integers and the continuity of the given polynomials with respect to the l -adic topology). \square

We define the trace of $f \in \text{End}^0(X)$ via the following equation $P_f(x) = x^{2g} - \text{tr}(f)x^{2g-1} + \dots + \deg(f)$.

3 The Jacobian variety

In this section C shall be a non-singular proper curve of genus g over a field k .

Proposition 3.1. $\text{Pic}_{C/k}$ is smooth over k .

Proof. We already know that $\text{Pic}_{C/k}$ is locally of finite type over k and therefore it suffices to proof that $\text{Pic}_{C/k}$ is formally smooth. To show this let Z be an affine scheme over k and $i : Z_0 \hookrightarrow Z$ a closed subscheme cut out by an ideal $I \subseteq \mathcal{O}_Z$ that satisfies $I^2 = 0$. Passing to the functor that the scheme $\text{Pic}_{C/k}$ represents we have to proof the following: The pullback along $(1 \times i)$ induces a surjection $\text{Pic}(C \times Z)/\text{pr}_Z^* \text{Pic}(Z) \rightarrow \text{Pic}(C \times Z_0)/\text{pr}_{Z_0}^* \text{Pic}(Z_0)$.

Note that $b + I \in \mathcal{O}_{Z_0}(Z_0)$ is invertible if and only if $b \in \mathcal{O}_Z(Z)^\times$: If $b = 1 + c$ for $c \in I$ then $b^{-1} = (1 - c)$. Hence we obtain an exact sequence of sheaves of abelian groups on the topological space $|Z| = |Z_0|$ given by $0 \rightarrow I \rightarrow \mathcal{O}_Z^\times \xrightarrow{i^\#} \mathcal{O}_{Z_0}^\times \rightarrow 1$, where the first map sends s to $1 + s$.

This gives the following short exact sequence on the topological space $|C \times Z| = |C \times Z_0|$

$$0 \rightarrow \text{pr}_Z^* I = \mathcal{O}_C \otimes_k I \xrightarrow{n \mapsto 1+n} \mathcal{O}_{C \times Z}^\times \xrightarrow{(1 \times i)^\#} \mathcal{O}_{C \times Z_0}^\times \rightarrow 1.$$

We apply the pushforward along pr_Z to obtain a long exact sequence

$$\begin{aligned} 0 \rightarrow R^0(\text{pr}_Z)_*(\mathcal{O}_C \otimes_k I) &\rightarrow R^0(\text{pr}_Z)_*\mathcal{O}_{C \times Z}^\times \rightarrow R^0(\text{pr}_Z)_*\mathcal{O}_{C \times Z_0}^\times \\ &\rightarrow R^1(\text{pr}_Z)_*(\mathcal{O}_C \otimes_k I) \rightarrow R^1(\text{pr}_Z)_*\mathcal{O}_{C \times Z}^\times \rightarrow R^1(\text{pr}_Z)_*\mathcal{O}_{C \times Z_0}^\times \rightarrow \dots \end{aligned}$$

The map $(\text{pr}_Z)_*\mathcal{O}_{C \times Z}^\times \rightarrow (\text{pr}_Z)_*\mathcal{O}_{C \times Z_0}^\times$ is a surjective map of sheaves on Z and therefore $R^1(\text{pr}_Z)_*(\mathcal{O}_C \otimes_k I) \rightarrow R^1(\text{pr}_Z)_*\mathcal{O}_{C \times Z}^\times$ is injective. Further, $R^2(\text{pr}_Z)_*(\mathcal{O}_C \otimes_k I)$ vanishes because pr_Z is proper, I is quasi-coherent and $H^2(C, \mathcal{O}_C) = 0$, see [12, 7.7.10 and 7.7.5 (II)]. Therefore we obtain an exact sequence

$$0 \rightarrow R^1(\text{pr}_Z)_*(\mathcal{O}_C \otimes_k I) \rightarrow R^1(\text{pr}_Z)_*\mathcal{O}_{C \times Z}^\times \rightarrow R^1(\text{pr}_Z)_*\mathcal{O}_{C \times Z_0}^\times \rightarrow 1.$$

We apply the global section functor $H^0(Z, \cdot)$ to see that the obstruction for

$$\text{Pic}_{X/k}(Z) = H^0(Z, R^1(\text{pr}_Z)_*\mathcal{O}_{C \times Z}^\times) \rightarrow H^0(Z_0, R^1((\text{pr}_Z))_\times \mathcal{O}_{C \times Z_0}^*) = \text{Pic}_{C/k}(Z_0)$$

being surjective is $H^1(Z, R^1(\text{pr}_Z)_*(\mathcal{O}_C \otimes_k I))$, which vanishes because Z is affine and $(\text{pr}_Z)_*(\mathcal{O}_C \otimes_k I)$ is quasi-coherent by properness of pr_Z . \square

The given proof that $\text{Pic}_{C/k}$ is formally smooth can be found in [9, Prop. 8.4.2] and relies on $H^2(C, \mathcal{O}_C)$ vanishing. So, our assumption that C is a curve plays a crucial role in this proof of smoothness of $\text{Pic}_{C/k}$.

A regular local ring is reduced and therefore $\text{Pic}_{C/k}^0$ is geometrically reduced over k . Moreover, we have seen in section 2.1 that $\text{Pic}_{C/k}^0$ is also proper and geometrically irreducible, i.e. $\text{Pic}_{C/k}^0$ is an abelian variety. We will refer to $J := \text{Pic}_{C/k}^0$ as *Jacobian variety* or short *Jacobian* of C . By theorem 2.10 J has dimension g and its tangent space at the zero is isomorphic to $H^1(C, \mathcal{O}_C)$. In particular, if $g = 0$ then $J = \text{Spec}(k)$.

3.1 The canonical map from C to its Jacobian

In this subsection we assume C to have a k -rational point $P \in C(k)$ corresponding to a k -morphism $\varepsilon : k \rightarrow C$. By [1, Tag 0C6U], if $g = 0$ then $C \cong \mathbb{P}_k^1$ and we will assume in the following subsection that $g > 0$.

Further, we will denote the canonical line bundle on $C \times J$ from Proposition 2.11 by \mathcal{M}^P .

Since $C \times C$ is regular, we can associate an invertible sheaf \mathcal{L}^P to the Weil-Divisor

$$\Delta - C \times \{P\} - \{P\} \times C \quad (16)$$

on $C \times C$. Then $\varepsilon_C^* \mathcal{L}^P \cong \mathcal{L}(P) \otimes \mathcal{L}(P)^{-1} \otimes \varepsilon_C^*((\varepsilon_C)_* \mathcal{O}_C)^{-1}$ is trivial, since ε_C is a closed immersion.

By proposition 2.11 there exists a unique map $f : C \rightarrow \text{Pic}_{C/k}$ such that $(1 \times f)^* \mathcal{M}^P = \mathcal{L}^P$.

For K/k a field extension and $Q \in C(K) \setminus P$ with corresponding map $x : K \rightarrow C$ we have $(1 \times x)^* \mathcal{L}^P = \mathcal{L}_{C_K}(Q) \otimes \mathcal{L}_{C_K}(P)^{-1}$.

Consulting diagram 11 we deduce that f is given on K -valued by

$$f(Q) = \mathcal{L}_{C_K}(Q) \otimes \mathcal{L}_{C_K}(P)^{-1}. \quad (17)$$

Since C is connected and $f(P) = \mathcal{O}_X$ the map f factors through $\text{Pic}_{X/k}^0 = J$.

The canonical map $h_J : \Gamma(J, \Omega_J^1) \rightarrow \Omega_{J,0}^1 = (T_0 J)^\vee$ is an isomorphism for any group variety over a field, see [9, 4.2 Prop. 2]. Serre-duality gives a canonical isomorphism $\text{ser} : \Gamma(C, \Omega_C^1) \rightarrow H^1(C, \mathcal{O}_C)^\vee$. These isomorphisms are related via the pullback along f . This is encoded in the next proposition, whose proof can be found in [4, Thm. 14.4].

Proposition 3.2. For $\nu : H^1(C, \mathcal{O}_C) \rightarrow T_0 J$ the isomorphism from theorem 2.10 and $f^* : \Gamma(J, \Omega_J^1) \rightarrow \Gamma(C, \Omega_C^1)$ the canonical map the diagram

$$\begin{array}{ccc} \Gamma(J, \Omega_J^1) & \xrightarrow{f^*} & \Gamma(C, \Omega_C^1) \\ \downarrow h_J & & \downarrow \text{ser} \\ T_0(J)^\vee & \xrightarrow{\nu^\vee} & H^1(C, \mathcal{O}_C)^\vee \end{array}$$

commutes. In particular, $f^* : \Gamma(J, \Omega_J^1) \rightarrow \Gamma(C, \Omega_C^1)$ is an isomorphism.

Theorem 3.3. $f : C \rightarrow J$ is a closed immersion. If C has genus $g = 1$ then f is an isomorphism.

Proof. Whether a morphism is a closed immersion, can be checked after faithfully flat base change, so we may assume $k = \bar{k}$. Since f is a morphism of smooth, projective k -varieties, it is a closed immersion if it separates points and tangent vectors. (The proof is the same as the “if” part of [8, II 7.3]). To see that f separates points, assume that $Q_1, Q_2 \in C(k)$ have the same image under f . Then $\mathcal{L}(Q_1) \otimes \mathcal{L}(Q_2)^{-1}$ is trivial, i.e. $Q_1 - Q_2$ is the divisor of a function f . But then f defines an isomorphism $C \rightarrow \mathbb{P}_C^1$, contradicting our assumption $g > 0$.

We will only sketch the proof of f separating tangent vectors. To see that $(df_Q) : T_Q C \rightarrow T_Q J$ is injective, we may assume that $Q = P$. It can be shown that the dual map of df_P is $\Gamma(J, \Omega_J^1) \xrightarrow{f^*} \Gamma(C, \Omega_C^1) \xrightarrow{\text{can}} (T_P C)^\vee$. We have seen in Proposition 3.2 that the first of these maps is an isomorphism. Therefore it suffices to proof that $\Gamma(C, \Omega_C^1) \xrightarrow{\text{can}} (T_P C)^\vee$ is surjective. The

kernel of this map can be identified with $\{\omega \in \Gamma(C, \Omega_C^1) \mid \omega(P) = 0\}$ and by Serre duality the latter is dual to $H^1(C, \mathcal{L}(P))$. Since $T_P C$ is one-dimensional, we now only have to proof that $\dim H^1(C, \mathcal{L}(P)) < \dim \Gamma(C, \Omega_C^1)$. Moreover, we know that $\dim \Gamma(C, \Omega_C^1) = g$ by Serre duality and $h^1(C, \mathcal{L}(P)) = h^0(C, \mathcal{L}(P)) + g - 2$ by the Riemann-Roch theorem. Because we assumed $g > 0$, there exist no meromorphic functions on C that only have one simple pole and are regular elsewhere, as such define an isomorphism $C \rightarrow \mathbb{P}_k^1$. We conclude that $H^0(C, \mathcal{L}(P)) = H^0(C, \mathcal{O}_C) \cong k$ and hence $h^1(C, \mathcal{L}(P)) = g - 1 < g = \dim \Gamma(C, \Omega_C^1)$. In summary, we have shown that f also separates tangent vectors and hence must be a closed immersion.

In the case $g = 1$ both varieties J, C are proper, smooth curves and hence f must be an isomorphism. \square

Remark 3.4 (*Elliptic Curves*). Due to theorem 2.13 abelian varieties of dimension one have genus one. By the last theorem 3.3, a nonsingular, proper curves of genus one, which admits a k -valued point, is isomorphic to its own Jacobian variety. We conclude that these notions coincide and refer to abelian varieties of dimension one as *elliptic curves*. Let C be an elliptic curve and $Q_1, Q_2 \in C(\bar{k})$. Then we can read from f 's action on closed points, that there exists a unique $Q_3 \in C(\bar{k})$ such that $\mathcal{L}(Q_1 + Q_2 - 2P) \cong \mathcal{L}(Q_3 - P)$. Further, $(Q_1, Q_2) \mapsto Q_3$ defines the unique group law on C such that f is a homomorphism of abelian varieties.

3.2 Symmetric powers of a curve

In this subsection we assume there exists $P \in C(k)$ and that $g > 0$. We will write f for the canonical closed immersion $C \rightarrow J$ from theorem 3.3.

For $n > 0$ let S_n be the symmetric group on n letters. S_n acts on C^n by permuting the factors. A morphism $\varphi : C^n \rightarrow T$ is said to be symmetric if $\varphi \circ \sigma = \varphi$ for all $\sigma \in S_n$.

Since quasi-projective schemes admit quotients by finite groups, see [5, p. 66], there exists a variety $C^{(n)}$ and a symmetric morphism $\pi : C^n \rightarrow C^{(n)}$, such that

1. as topological space $(C^{(n)}, \pi)$ is the quotient of C^n by S_n .
2. for any open affine subset U of C , $U^{(n)}$ is an open affine subset of $C^{(n)}$ and $\mathcal{O}_{C^{(n)}}(U^{(n)})$ is the subring $\mathcal{O}_{C^n}(U^n)^{S_n}$ of $\mathcal{O}_{C^n}(U^n)$ given by elements fixed by the action of S_n .

The pair $(C^{(n)}, \pi)$ has the following universal property: every symmetric k -morphism $\varphi : C^n \rightarrow T$ factors uniquely through π . Moreover, the map π is finite and surjective. Since C^n is proper this implies that $C^{(n)}$ is proper over k .

For $m_1, \dots, m_k \in \mathbb{N}_0$ a partition $n = m_1 + \dots + m_r$ the natural isomorphism $C^{m_1} \times \dots \times C^{m_r} \rightarrow C^n$ induces a natural morphism $s = s_{m_1, \dots, m_r} : C^{(m_1)} \times \dots \times C^{(m_r)} \rightarrow C^{(n)}$ that we will refer to the sum map.

Proposition 3.5. Suppose given a partition $n = m_1 + \dots + m_r$ and points $P_1, \dots, P_r \in C(k)$ with $P_i \neq P_j$ if $i \neq j$. Write $m_i P_i \in C^{(m_i)}(k)$ for the image of the point $(P_i, \dots, P_i) \in C^{m_i}$ under the quotient map $C^{m_i} \rightarrow C^{(m_i)}$.

- (i) Then the sum morphism $C^{(m_1)} \times \dots \times C^{(m_r)} \rightarrow C^{(n)}$ is étale at the point $(m_1 P_1, \dots, m_r P_r)$.
- (ii) The symmetric power $C^{(n)}$ of a non-singular curve is regular of dimension n for any $n > 0$.

In particular, $\pi : C^n \rightarrow C^{(n)}$ is finite and flat of degree $n!$.

Proof. We won't give a proof of part (i) here, a proof can be found in [4, Lem. 14.7].

For the proof of part (ii) we may assume that k is algebraically closed. It suffices to check that for all k -valued points Q of $C^{(n)}$ the stalks at Q is regular. By part (i) we only have to check this on points of the form $Q := np$ for given $p \in C(k)$. Let us denote $P := (p, \dots, p) \in C^n(k)$. Note that the formation of the fixed ring under the action of S_n is a finite categorical limit. Finite limits commute with filtered colimits, e.g. localization, as well as, all categorical limits. In particular,

$\mathcal{O}_{C^{(n)},Q} = (\mathcal{O}_{C^n,P})^{S_n}$. The ideal \mathfrak{m} cutting out the closed point P in $\mathcal{O}_{C^n,P}$ is invariant under the action of S_n and equals the ideal cutting out the closed point Q in $\mathcal{O}_{C^{(n)},P}$. Therefore

$$\begin{aligned} (\widehat{\mathcal{O}_{C^n,P}})^{S_n} &= (\lim_m \mathcal{O}_{C^n,P}/\mathfrak{m}^m)^{S_n} = \lim_m (\mathcal{O}_{C^n,P}/\mathfrak{m}^m)^{S_n} = \lim_m (\mathcal{O}_{C^n,P})^{S_n}/\mathfrak{m}^m \\ &= \lim_m \mathcal{O}_{C^{(n)},Q}/\mathfrak{m}^m = \widehat{\mathcal{O}_{C^{(n)},Q}}. \end{aligned}$$

Since C^n is regular, $\widehat{\mathcal{O}_{C^n,P}} \cong k[[x_1, \dots, x_n]]$, where S_n acts on $\widehat{\mathcal{O}_{C^n,P}}$ by permuting the variables. By the fundamental theorem on symmetric polynomials $k[[x_1, \dots, x_n]] \rightarrow k[[x_1, \dots, x_n]]^{S_n}, x_i \rightarrow \sigma_i$, for σ_i being the i -th symmetric polynomial in n variables, is an isomorphism. Since a local Noetherian ring is regular if and only if its completion is regular, we have proven that $C^{(n)}$ is regular at Q .

For the last assertion note that a quasi-finite morphism of regular varieties is flat by [3, 26.2.11.]. As in the proof of theorem 2.4, we can compute the degree of π as the dimension of the k -vector space of global sections of any fiber. Choosing a fiber containing a point (P_1, \dots, P_n) with $P_i \neq P_j$ for all $i \neq j$, we see by (i) that the fiber is étale over k . So, $\deg \pi$ equals the number of closed points of this fiber, which is the cardinality of the S_n orbit of (P_1, \dots, P_n) in C^n . Hence, $\deg \pi = n!$. \square

Recall that for $C \rightarrow T$ a morphism of k -schemes a *relative effective Cartier divisor* D on $C_T := C \times T$ over T is a closed subscheme $D \subseteq C_T$, which is flat over T and such that the ideal sheaf $\mathcal{I}_D \subseteq \mathcal{O}_{C_T}$ is an invertible \mathcal{O}_{C_T} module.

When we tensor the inclusion $\mathcal{I}_D \hookrightarrow \mathcal{O}_{C_T}$ with $\mathcal{L}(D)$ we obtain an inclusion $\mathcal{O}_{C_T} \hookrightarrow \mathcal{L}(D)$ and hence a canonical global section s_D of $\mathcal{L}(D)$. The map $D \mapsto (\mathcal{L}(D), s_D)$ defines a bijection between relative effective divisors on C_T over T and isomorphism classes of pairs (\mathcal{L}, s) where \mathcal{L} is an invertible sheaf on C_T and $s \in \Gamma(C_T, \mathcal{L})$ is such that

$$\mathcal{L}/s\mathcal{O}_{C_T} := \text{Coker}(\mathcal{O}_{C_T} \xrightarrow{s} \mathcal{L})$$

is flat over T . Here two pairs (\mathcal{L}, s) and (\mathcal{L}', s') are considered to be isomorphic if there is an isomorphism of \mathcal{O}_{C_T} -modules $h : \mathcal{L} \rightarrow \mathcal{L}'$ with $h(s) = s'$. The inverse of the above bijection associates to (\mathcal{L}, s) the zero scheme $D = Z(s) \subseteq C_T$ of the section s .

Relative effective Cartier divisors on C_T over T can be added. If D corresponds to the pair (\mathcal{L}, s) and D' to the pair (\mathcal{L}', s') then $D + D'$ is cut out by $\mathcal{I}_D \cdot \mathcal{I}_{D'}$ and corresponds to $(\mathcal{L} \otimes \mathcal{L}', s \otimes s')$. To see that $D + D'$ is again flat over T consult [1, Tag 0B8U].

While the pullback of an effective Cartier divisor might not be effective, the charm of relative effective Cartier divisors is that they behave nicely with respect to base-changes:

If $D \subseteq C_T$ is an relative effective Cartier divisor over T and $h : T' \rightarrow T$ is a morphism of k -schemes then we can pull D back to an relative effective Cartier divisor $D_{T'} = h^*D \subseteq C_{T'}$ on $C_{T'}$ over T' . A proof of this property can be found at [1, Tag 056Q].

Consider an relative effective Cartier divisor D on C_T over T . Then for any $t \in T$ the pullback D_t of D along $t \rightarrow T$ is a Cartier Divisor on the curve $C_{k(t)}$ and therefore finite. We conclude that $D \rightarrow T$ is quasi-finite. As C is proper over k , D is proper over T too, and quasi-finite + proper implies finite. So, D is finite and flat over T and hence \mathcal{O}_D is finite locally free as an \mathcal{O}_T module. The rank of \mathcal{O}_D as an \mathcal{O}_T module (which is a locally constant function on T) is called the degree of D and denoted $\deg D$. It is straightforward to check that, if D has constant degree n over T then the same holds for $D_{T'}$ over T' for any $h : T' \rightarrow T$. It is proven in [13, Lem. 1.2.6] that for two relative effective Cartier divisors D_1, D_2 on C_T over T their sum $D_1 + D_2$ has degree $\deg(D_1) + \deg(D_2)$ over T .

We obtain a contravariant functor $\text{Div}_{C/k}^{\text{eff},n} : \text{Sch}/k \rightarrow \text{Sets}$ with

$$\text{Div}_{C/k}^{\text{eff},n}(T) = \{\text{relative effective Cartier divisors } D \subseteq C_T \text{ of constant degree } n \text{ over } T\}. \quad (18)$$

If $P \in C(T)$ is a T -valued point of C then this gives a section $T \rightarrow C_T$ of the structural morphism, whose image is a relative effective Cartier divisor $P \subseteq C_T$ of constant degree 1 over T by [13, Lem. 1.2.2]. More generally, for $P_1, \dots, P_n \in C(T)$ we get an relative effective Cartier divisor $P_1 + \dots + P_n$ on C_T of constant degree n over T . In this way we obtain a morphism $C^n \rightarrow \text{Div}_{C/k}^{\text{eff},n}$. Since this

morphism is S_n invariant, it factors through a morphism $h : C^{(n)} \rightarrow \text{Div}_{C/k}^{\text{eff},n}$. Checking on closed points motivates that h defines an isomorphism. This is proven in [6, Thm. 3.13].

Remark 3.6. We will henceforth identify $C^{(n)}$ with $\text{Div}_{C/k}^{\text{eff},n}$ via the above isomorphism h .

Let f^n be the map $C^n \rightarrow J$ sending (P_1, \dots, P_n) to $f(P_1) + \dots + f(P_n)$. Here f is the canonical closed immersion from theorem 3.3. On k -valued points f^n is given by $(P_1, \dots, P_n) \mapsto \mathcal{L}(P_1) \otimes \dots \otimes \mathcal{L}(P_n) \otimes \mathcal{L}(P)^{-n}$. Since f^n is symmetric, it induces a map $f^{(n)} : C^{(n)} \rightarrow J$.

Given a k -scheme T . We can pull back $P \rightarrow C$ along $C_T \rightarrow C$ to obtain a relative effective divisor on C_T of degree 1 over T .

We claim that, in terms of Cartier divisors $f^{(n)}$ sends a relative effective Cartier divisor D on C_T of degree n over T to the class in $J(T)$ represented by $\mathcal{O}_{C_T}(D) \otimes \mathcal{O}_{C_T}(P_T)^{-n}$, in short

$$f^{(n)}(D) = \mathcal{O}_{C_T}(D) \otimes \mathcal{O}_{C_T}(P_T)^{-n} \quad \text{for all } D \in C^{(n)}(T). \quad (19)$$

To see this, note that (19) defines a natural transformation between the functors represented by $C^{(n)}$ and $\text{Pic}_{C/k}$. Thus, there exists a morphism $\widetilde{f}^{(n)} : C^{(n)} \rightarrow \text{Pic}_{C/k}$ that on T -valued points is given by (19). Since $C^{(n)}$ is connected and $\widetilde{f}^{(n)}$ sends $nP \in C^{(n)}(k)$ to zero, we can consider $\widetilde{f}^{(n)}$ as a map to $J = \text{Pic}_{C/k}^0$. It remains to be proven that $\widetilde{f}^{(n)} = f^{(n)}$. We know by proposition 3.5 that $C^{(n)}$ is reduced, so looking at the locus where $\widetilde{f}^{(n)}$ and $f^{(n)}$ agree, as in [3, 10.2. A], it suffices to proof that they induce the same map on \bar{k} valued points. But this readily follows from equation (17).

Via the description of $f^{(n)}$ in 19 in the case that $T = k$, we see that the k -valued points of the fibre of $f^{(n)}$ containing $D \in C^{(n)}(k)$ will correspond to the complete linear system $|D|$. The set $|D|$ is in natural bijection with $(\Gamma(X, \mathcal{O}(D)) \setminus \{0\})/k^\times$ via $D + (f) \mapsto \{\lambda f \mid \lambda \in k^\times\}$. This observation on k -valued points has a scheme-theoretic reformulation.

Theorem 3.7 (Abel's theorem). Let \mathcal{L} be a line bundle of degree n on C . Then the scheme-theoretic fibre of $f^{(n)} : C^{(n)} \rightarrow J$ over the point $p \in J(k)$ represented by $\mathcal{L} \otimes \mathcal{L}(P)^{-n} \in \text{Pic}^0(C)$ is

$$f^{(n)-1}(p) = \mathbb{P}(H^0(C, \mathcal{L})) := \text{Proj}(\text{Sym}(H^0(C, \mathcal{L}))) \cong \mathbb{P}_k^m,$$

for $m = h^0(C, \mathcal{L}) - 1$.

Proof. Write $\Phi \subseteq C^{(n)}$ for the scheme-theoretic fibre of $f^{(n)}$ over p and let $\mathbb{P} := \mathbb{P}(H^0(C, \mathcal{L}))$. Let $g : T \rightarrow \text{Spec}(k)$ be a k -scheme and consider the cartesian diagram

$$\begin{array}{ccc} C_T & \xrightarrow{\text{pr}_C} & C \\ \downarrow \text{pr}_T & & \downarrow h \\ T & \xrightarrow{g} & \text{Spec}(k) \end{array} \quad .$$

Considering the functors represented by J and $C^{(n)}$ we get natural isomorphisms

$$\begin{aligned} \Phi(T) &\cong \{D \subseteq C_T \text{ rel. eff. divisor of degree } n \text{ over } T \text{ with } \mathcal{O}_{C_T}(D) \cong \text{pr}_C^* \mathcal{L} \mod \text{pr}_T^* \text{Pic}(T)\} \\ &\cong \left\{ \begin{array}{l} \text{isomorphism classes } (\mathcal{L}', s) \text{ with } s \in H^0(C_T, \mathcal{L}') \text{ such that} \\ \mathcal{L}'/s\mathcal{O}_{C_T} \text{ is flat over } \mathcal{O}_T \text{ and } \exists M \in \text{Pic}(T) \text{ with } \mathcal{L}' = \text{pr}_C^* \mathcal{L} \otimes \text{pr}_T^* M \end{array} \right\} \end{aligned} \quad (20)$$

By definition, $\mathbb{P} = \text{Proj}(\text{Sym}((h_* \mathcal{L})))$, which is isomorphic to \mathbb{P}_k^m . T -valued points of such a projective space can be described as follows:

A map $T \rightarrow \mathbb{P}$ is given by a line bundle M on T together with a surjective homomorphism of \mathcal{O}_T modules $t : g^*((h_* \mathcal{L})) \rightarrow M$, where two such pairs (M, t) and (M, t') are considered equivalent if there exists an isomorphism $\alpha : M \xrightarrow{\sim} M'$ with $\alpha \circ t = t'$.

Such a map t determines and is determined by an element $t \in H^0(T, (g^* h_* \mathcal{L}) \otimes M)$ such that $t(x) \neq 0$ for all $x \in T$:

$g^*(h_*\mathcal{L})$ is non-canonically isomorphic to $\mathcal{O}_T^{\oplus(m+1)}$, therefore $(g^*h_*\mathcal{L}) \otimes M$ is isomorphic to $M^{\oplus(m+1)}$. Since M is a line bundle, a homomorphism $\mathcal{O}_T^{\oplus(m+1)}$ is determined by $(m+1)$ sections in M and the map being surjective translates to the non-vanishing condition via Nakayama's Lemma.

By flat base change along g we have a canonical isomorphism $g^*h_*\mathcal{L} \cong \mathrm{pr}_{T*}\mathrm{pr}_C^*\mathcal{L}$.

By the projection formula the canonical map $\mathrm{pr}_{T*}(\mathrm{pr}_C^*\mathcal{L} \otimes \mathrm{pr}_T^*M) \rightarrow (\mathrm{pr}_{T*}\mathrm{pr}_C^*\mathcal{L}) \otimes M$ is an isomorphism.

We get the following isomorphism that is natural in T :

$$\begin{aligned} H^0(T, (g^*h_*\mathcal{L}) \otimes M) &\cong H^0(T, (\mathrm{pr}_{T*}\mathrm{pr}_C^*\mathcal{L}) \otimes M) \cong H^0(T, \mathrm{pr}_{T*}(\mathrm{pr}_C^*\mathcal{L} \otimes \mathrm{pr}_T^*M)) \\ &= H^0(C_T, \mathrm{pr}_C^*\mathcal{L} \otimes \mathrm{pr}_T^*M) \end{aligned} \quad (21)$$

And we conclude

$$\mathbb{P}(T) \cong \left\{ \begin{array}{l} \text{isomorphism classes } (M, \mathrm{pr}_{T*}s) \text{ with } s \in H^0(C_T, \mathrm{pr}_C^*\mathcal{L} \otimes \mathrm{pr}_T^*M) \\ \text{and } (\mathrm{pr}_{T*}s)(x) \neq 0 \text{ for all } x \in T \end{array} \right\} \quad (22)$$

It can be checked that the isomorphism in (21) lets the notion of isomorphism classes of pairs in (20) and (22) coincide, if one identifies the appearing sets $\mathbb{P}(T)$ and $\Phi(T)$ in the canonical way. We omit the proof that the notion of flatness in (20) matches up with the notion of non vanishing in (22).

We have proved that $\Phi(T)$ and $\mathbb{P}(T)$ are isomorphic, naturally in T , and conclude $\Phi \cong \mathbb{P}$ by the Yoneda-Lemma. \square

Theorem 3.8 (Jacobi's inversion theorem). For $0 \leq n \leq g$ the morphism $f^{(n)} : C^{(n)} \rightarrow J$ is birational onto its scheme-theoretic image, denoted W^n , which is irreducible. For $n \geq g$ the morphism $f^{(n)}$ is surjective. In particular, $f^{(g)} : C^{(g)} \rightarrow J$ is a birational equivalence.

Proof. Note that since $f^{(n)}$ is proper the scheme theoretic image agrees with the set-theoretic image on the level of sets, so the assertion for the case $n = g$ follows from the other two statements. Further, W^n is irreducible as image of the irreducible topological space $C^{(n)}$ under the continuous map $f^{(n)}$.

Whether a morphism is surjective or birational can be detected after quasi-compact, faithfully flat base change, see [14, B.2]. $(C^{(n)})_{\bar{k}}$ represents the functor $\mathrm{Div}_{C_{\bar{k}}/\bar{k}}^{\mathrm{eff}, n}$ and $J_{\bar{k}}$ represents the functor $\mathrm{Pic}_{C_{\bar{k}}/\bar{k}}^0$ and moreover the formation of $f^{(n)}$ commutes with base change to \bar{k} . This can be seen by checking the given definitions of these functors. Hence, we may assume that k is algebraically closed.

For proving surjectivity, it suffices to show that the map is surjective on k valued points. Let $n \geq g$. For any $\mathcal{L} \in \mathrm{Pic}^0(C) = J(k)$ the Riemann-Roch theorem implies that $\mathcal{L} \otimes \mathcal{L}(P)^n$ is effective, and therefore \mathcal{L} is in the image of $f^{(n)}$.

Now assume $0 \leq n \leq g$. We try to find a non-empty open set $U \subseteq C^{(n)}$ where the fibers of $f^{(n)}$ are zero-dimensional. Since the dimension of the fibers change in an upper-semicontinuous manner on the domain, it suffices by Abel's theorem 3.7 to find an effective divisor D of degree n on C such that $h^0(C, \mathcal{O}_C(D)) = 1$. We proceed by induction on $n \leq g$. For $n = 1$ the assertion follows, because $h^0(C, \mathcal{L}(P)) = 1$ using that $g > 0$ and a meromorphic function with exactly one zero would define an isomorphism $C \rightarrow \mathbb{P}_k^1$. Suppose then that $2 \leq n \leq g$ and that we have an effective divisor E of degree $n - 1$ with $h^0(E) = 1$. Let $K = \Omega_C^1$ be the canonical divisor on C . By Serre duality $h^1(K - E) = 1$ and so by the Riemann-Roch theorem

$$h^0(K - E) - 1 = 1 - g + \deg(K - E) = 1 - g + (2g - 2) - (n - 1) = g - n \geq 0. \quad (23)$$

Thus $[K - E]$ is effective. Choose a point $Q \in C(k)$ which is not a base point of the linear system $|K - E|$. Then $h^0(K - E - Q) = h^0(K - E) - 1 \stackrel{23}{=} g - n$, where the first equality follows from $[K - E]$ being effective because there is no meromorphic function $f \in K(C)$, whose only pole is Q , since else f would define an isomorphism $C \rightarrow \mathbb{P}^1$.

Thus, by the Riemann-Roch theorem and then Serre duality

$$h^0(E + Q) = 1 - g + n + h^1(E + Q) = 1 - g + n + h^0(K - E - Q) = 1.$$

This proves there exists $\emptyset \neq U \subseteq C^{(n)}$ such that $f^{(n)}|_U : U \rightarrow J$ has only zero-dimensional fibers. By Abel's theorem 3.7 the non-empty fibers of $f^{(n)}|_U$ over k -valued points are isomorphic to $\mathbb{P}_k^0 \cong \text{Spec}(k)$.

By [3, 10.1.P] a morphism of finite type schemes over an algebraically closed field k is universally injective if and only if the induced map on k valued points is injective. We conclude from the above paragraph that that $f|'_U : U' \rightarrow W^n$ is indeed universally injective. In particular, the field extension $k(C^{(n)})/k(W^n) = k(U)/k(W^n)$ is purely inseparable, see [1, Tag 01S2].

The morphism $f^{(n)} : C^{(n)} \rightarrow W^n$ is surjective and closed. Hence $\dim C^{(n)} \geq \dim W^n$. But on the other hand, taking $p \in U$ and $q = f^{(n)}(p)$, we see that $\dim C^{(n)} \leq \dim W^n + \dim(f^{(n)})^{-1}(Q) = \dim W^n$.

So, $\dim W^n = \dim C^{(n)}$, and the residue field extension $k(C^{(n)})/k(W^n)$ is algebraic.

Since W^n is reduced and K algebraically closed, the field extension $k(W^n)/k$ is separable. Similarly, the field extension $k(C^{(n)})/k$ is separable. But then $k(C^{(n)})/k(W^n)$ must be separable, too:

To see this take K a purely transcendental extension K/k such that $K \subseteq k(W^n)$ and $k(W^n)/K$ is separable algebraic. Then also $k(C^{(n)})/K$ is separable, since $k(C^{(n)})/k$ is. Hence

$$\begin{aligned} [k(C^{(n)}) : k(W^n)]_s [k(W^n) : K]_s &= [k(C^{(n)}) : K]_s = [k(C^{(n)}) : K] = [k(C^{(n)}) : k(W^n)] [k(W^n) : K] \\ &= [k(C^{(n)}) : k(W^n)] [k(W^n) : K]_s, \end{aligned}$$

therefore $k(C^{(n)})/k(W^n)$ is separable and algebraic. We have already proven that this field extension is purely inseparable and the only way to not obtain a contradiction is $[k(C^{(n)}) : k(W^n)] = 1$, i.e. $f^{(n)} : C^{(n)} \rightarrow W^n$ is birational. □

We define the theta divisor by $\Theta := W^{g-1} \subseteq J$. By theorem 3.8 Θ is indeed a divisor on J .

3.3 The Jacobian as Albanese variety

Throughout this section C will again be a proper nonsingular curve of genus $g > 0$ over a field k , J will be its Jacobian variety and $P \in C(k)$ will be a k -rational point. We will continue with all notations from the previous section. In particular, the definition of f from (3.1).

Proposition 3.9 (Universal property of the canonical map $f : C \rightarrow J$). For any map $g : C \rightarrow X$ from C into an abelian variety X sending P to 0, there is a unique homomorphism $h : J \rightarrow X$ such that $g = h \circ f$.

Proof. Consider the map $g^g : C^g \rightarrow X$ that on points is given by $(P_1, \dots, P_g) \mapsto \sum_{i=1}^g g(P_i)$. Since this is symmetric, it factors as $g^{(g)} \circ \pi = g^g$ for $g^{(g)} : C^{(g)} \rightarrow X$ and $\pi : C^g \rightarrow C^{(g)}$ the canonical morphism. Now by Jacobi's Inversion theorem 3.8 we obtain a rational map $h : J \rightarrow X$ such that $h \circ f^{(g)} = g^{(g)}$, where this expression is defined. But a rational map from a smooth variety J to an abelian variety X is defined on the whole of J , by [6, Thm. 3.1]. Let $\varphi : C \rightarrow C^{(g)}$ on closed points be given by $Q \mapsto \pi(Q, P, \dots, P)$. Then $f = f^{(g)} \circ \varphi$ and therefore $h \circ f = h \circ f^{(g)} \circ \varphi = g^{(g)} \circ \varphi = g$. In particular, h sends 0 to 0, and corollary 2.2 shows it is a homomorphism.

If h' is another such homomorphism, then $h' \circ f^g = h \circ f^g$. Since X is separated, J is reduced and f^g is surjective by theorem 3.8, we must have $h = h'$. (This is because the "coincidence scheme" of h and h' , as in [7, 7.4 ex. 3], equals J .) □

Corollary 3.10. Let C_1 and C_2 be nonsingular, proper curves over k , $P_1 \in C_1(k)$ and $P_2 \in C_2(k)$ be their Jacobians. Let $f^{P_i} : C_i \rightarrow J_i$ be the canonical maps from section 3.1. The map

$$\text{Hom}_k(J_1, J_2) \rightarrow \{\mathcal{L} \in \text{Pic}(C_2 \times C_1) : \mathcal{L}|_{C_2 \times \{P_1\}} \text{ and } \mathcal{L}|_{\{P_2\} \times C_1} \text{ are trivial}\}, \quad h \mapsto (1_{C_2} \times (h \circ f^{P_1}))^* \mathcal{M}^{P_2}$$

is an isomorphism.

Proof. The map is well-defined because $\mathcal{M}^{P_2}|_{\{P_2\} \times C_2}$ and $\mathcal{M}^{P_2}|_{C_2 \times \{0\}}$ is trivial by definition of \mathcal{M}^{P_2} in section 3.1, (also see proposition 2.11).

Now given $\mathcal{L} \in \text{Pic}(C_2 \times C_1)$ such that both $\mathcal{L}|_{C_2 \times \{P_1\}}$ and $\mathcal{L}|_{\{P_2\} \times C_1}$ are trivial. Since \mathcal{M}^{P_2} is the universal bundle on $C_2 \times J_2$ from proposition 2.11, there is a unique map $g : C_1 \rightarrow J_2$ such that $(1_{C_2} \times g)^* \mathcal{M}^{P_2} \cong \mathcal{L}$. It follows from diagram 11 that $g(P_1)$ is represented by $\mathcal{L}|_{C_2 \times P_1}$ which is trivial. Hence g sends P_1 to 0 and by proposition 3.9 there exists a unique homomorphism $h : J_1 \rightarrow J_2$ such that $g = h \circ f^{P_1}$. \square

3.4 Autoduality

Let \mathcal{P} denote the Poincaré bundle on $J \times J^\vee$.

Consider the Mumford line bundle $\Lambda(\mathcal{L}(\Theta)) = m^* \mathcal{L}(\Theta) \otimes \text{pr}_1^* \mathcal{L}(\Theta)^{-1} \otimes \text{pr}_2^* \mathcal{L}(\Theta)^{-1}$ on $J \times J$ from section 2.1.2. We obtain a polarization $\varphi_{\mathcal{L}(\Theta)} : J \rightarrow J^\vee$ with $(1 \times \varphi_{\mathcal{L}(\Theta)})^* \mathcal{P} \cong \Lambda(\mathcal{L}(\Theta))$.

Write Θ^- for the pullback of Θ along $(-1)_J : J \rightarrow J$ and Θ_a for $t_{-a}^* \Theta = \Theta + a$, $a \in J(k)$.

Remark 3.11. It is shown in [4, 14.22 and 14.28 (ii)] that Θ , Θ^- and Θ_a are numerically equivalent and this implies that $\varphi_{\mathcal{L}(\Theta)} = \varphi_{\mathcal{L}(\Theta^-)} = \varphi_{\mathcal{L}(\Theta_a)}$ for all $a \in J(k)$ by [4, ??].

We abbreviate $(\Theta^-)_a$ as Θ_a^- .

Consider the invertible sheaf $(f \times 1)^* \mathcal{P}$ on $C \times J^\vee$. Its restriction to $\{P\} \times J^\vee$ is trivial because $f(P) = 0$ and \mathcal{P} restricted to $\{0\} \times J^\vee$ is trivial. Applying proposition 2.11 to the universal bundle \mathcal{M}^P on $C \times J$ we obtain a unique morphism $f^\vee : J^\vee \rightarrow J$ such that $(f \times 1)^* \mathcal{P} \cong (1 \times f^\vee)^* \mathcal{M}^P$.

Theorem 3.12. The maps $-f^\vee : J^\vee \rightarrow J$ and $\varphi_{\mathcal{L}(\Theta)} : J \rightarrow J^\vee$ are inverses.

Proof. $(J^\vee)_{\bar{k}}$ represents $(J_{\bar{k}})^\vee$ and $J_{\bar{k}}$ represents $\text{Pic}_{C_{\bar{k}}/\bar{k}}^0$, moreover the formation of f and therefore also the formation of Θ , $\varphi_{\mathcal{L}(\Theta)}$ and f^\vee commutes with base change to \bar{k} . Whether a morphism is an isomorphism can be detected after faithfully flat, quasi-compact base change by [14, B.2].

Therefore we may assume that k is algebraically closed.

Let U be the largest open subset of J such that:

- (i) the fiber of $f^{(g)} : C^{(g)} \rightarrow J$ at any point of U has dimension zero; and
- (ii) if $a \in U(k)$ and $D(a)$ is the, by Abel's theorem necessarily unique, element of $C^{(g)}(k)$ mapping to a ; then $D(a)$ is a sum of g distinct points of $C(k)$.

Note that U can be obtained in two steps: First, by removing the subset over which the fibers have dimension > 0 , which is closed because the fiber dimension changes upper-semi-continuously on the target. (Note that $f^{(g)}$ is proper and see [3, 11.4.2]). Secondly, by removing images of certain closed sets of the form $\Delta_C \times C^{g-2}$ under the proper map f^g . The first step yields a nonempty open set by (the proof of) Jacobi's inversion theorem 3.8. In the second step a proper closed subset of J gets removed, so, by irreducibility of J the set U is a nonempty open dense subset of J .

$$\textbf{Claim: } f^{-1}(\Theta_a^-) = D(a) \text{ for all } a \in U(k) \quad (24)$$

Let $a \in U(k)$ and let $D(a) = \sum_{i=1}^g P_i$ with $P_i \neq P_j$ for all $i \neq j$. A point Q_1 maps to Θ_a^- if and only if there exists a divisor $\sum_{i=2}^g Q_i$ on C such that $f(Q_1) = -\sum_{i=2}^g Q_i + a$. This equality implies $\sum_{i=1}^g Q_i \sim D$, and the fact that $|D|$ has dimension 0 by Abel's theorem 3.7 implies that $\sum_{i=1}^g Q_i = D$. It follows that the support of $f^{-1}(\Theta_a^-)$ is $\{P_1, \dots, P_g\}$, and it remains to show that $f^{-1}(\Theta_a^-)$ has degree $\leq g$ for all $a \in U(k)$.

Consider the map $\Psi : C \times \Theta \rightarrow J$ sending (Q, b) to $f(Q) + b$. By Jacobi's inversion theorem 3.8 and proposition 3.5, the maps $f^{g-1} : C^{g-1} \rightarrow \Theta$ and $f^g : C^g \rightarrow J$ have degree $(g-1)!$ and $g!$ respectively. As, Ψ composed with $1 \times f^{g-1} : C \times C^{g-1} \rightarrow C \times \Theta$ is f^g , we conclude that Ψ has degree g . Also, Ψ is proper as $C \times \Theta$ is a proper variety and therefore $\Psi' := \Psi|_{\Psi^{-1}(V)}$ is proper and quasi-finite, hence finite. Moreover, Ψ' is flat by [3, 26.2.11] using that $C \times \Theta$ and J are regular of dimension g . It follows as in the proof of theorem 2.4 that all fibers of Ψ' have global sections a g

dimensional k vector space. In particular, all fibers have less than g points. But for $a \in U$ the k valued points of $\Psi^{-1}(a)$ are exactly the k valued points of $f^{-1}(\Theta_a^-)$ and the claim follows.

Claim: (i) Let $a \in J(k)$, and let $f^{(g)}(D) = a$; then $f^*\mathcal{L}(\Theta_a^-) \cong \mathcal{L}(D)$. (25)

(ii) The sheaves $(f \times (-1)_J)^*\Lambda(\mathcal{L}(\Theta^-))$ and \mathcal{M}^P on $C \times J$ are isomorphic. (26)

Note that the map

$$C \xrightarrow{Q \mapsto (Q, a)} C \times \{a\} \xrightarrow{f \times (-1)} J \times J \xrightarrow{m} J$$

equals $t_{-a} \circ f$, where t_{-a} is the translation on J by a . Therefore

$$(f \times (-1))^*m^*\mathcal{L}(\Theta^-)|_{C \times \{a\}} \cong (t_{-a} \circ f)^*\mathcal{L}(\Theta^-) \cong f^*\mathcal{L}(\Theta_a^-)$$

On the other hand, \mathcal{M}^P is an invertible sheaf on $C \times J$ such that

a) $\mathcal{M}^P|_{C \times \{a\}} \cong \mathcal{L}(D - gP)$ if D is an effective divisor of degree g on C such that $f^{(g)}(D) = a$ (see the definition of f).

b) $\mathcal{M}^P|_{\{P\} \times J}$ is trivial (see the definition of \mathcal{M}^P).

Hence, $\mathcal{M}^P \otimes \text{pr}_1^*\mathcal{L}(gP)|_{C \times \{a\}}$ is isomorphic to $\mathcal{L}(D)$, whenever $f^{(g)}(D) = a$ for $D \in C^{(g)}(k)$ an effective divisor of degree g on C . Hence (i) is equivalent to $(f \times (-1))^*m^*\mathcal{L}(\Theta^-)|_{C \times \{a\}}$ being isomorphic to $\mathcal{M}^P \otimes \text{pr}_1^*\mathcal{L}(gP)|_{C \times \{a\}}$ for all $a \in J(k)$. By claim 24 we know that (i) holds on a nonempty dense open and therefore

$$\mathcal{N} := ((f \times (-1))^*m^*\mathcal{L}(\Theta^-)) \otimes (\mathcal{M}^P \otimes \text{pr}_1^*\mathcal{L}(gP))^{-1}$$

is trivial, when restricted to sets of the form $C \times \{a\}$ for all a in a dense open subset of J .

The set of all $a \in J$ such that \mathcal{N} restricted to $C \times \{a\}$ is trivial is closed in J , by [6, 5.3]. (This is because on the proper variety C an invertible sheaf is trivial if and only if \mathcal{N} and its dual \mathcal{N}^{-1} have nonzero global sections; but the dimensions of global sections of $\mathcal{N}|_{C \times \{a\}}$ and $\mathcal{N}^{-1}|_{C \times \{a\}}$ vary in an upper-semicontinuous manner on J by [3, 28.1.1].) Because a closed set in J that contains a dense set is equal to J , we obtain that claim (i) holds.

Taking $a = 0$ we obtain $f^*\mathcal{L}(\Theta^-) \cong \mathcal{L}(gP)$ and therefore

$$(f \times (-1))^*\text{pr}_1^*\mathcal{L}(\Theta^-) \cong (f \circ \text{pr}_1)^*\mathcal{L}(\Theta^-) \cong \text{pr}_1^*\mathcal{L}(gP).$$

Now for all $a \in J(k)$ the sheaves

$$(f \times (-1))^* \left(m^*\mathcal{L}(\Theta^-) \otimes (\text{pr}_1^*\mathcal{L}(\Theta^-))^{-1} \right) |_{C \times \{a\}}$$

and $\mathcal{M}^P|_{C \times \{a\}}$ are isomorphic on C . Now by the so-called Seesaw principle [6, 5.1], which is a theorem on proper varieties, there exists an invertible sheaf \mathcal{F} on J such that

$$(f \times (-1))^* \left(m^*\mathcal{L}(\Theta^-) \otimes (\text{pr}_1^*\mathcal{L}(\Theta^-))^{-1} \right) \cong \mathcal{M}^P \otimes \text{pr}_2^*\mathcal{F}$$

On computing the restriction to $\{P\} \times J$ of the above equation, we obtain

$$\mathcal{F} \cong (f \times (-1))^* \left(m^*\mathcal{L}(\Theta^-) \otimes (\text{pr}_1^*\mathcal{L}(\Theta^-))^{-1} \right) |_{\{P\} \times J} \cong (-1)^*\mathcal{L}(\Theta^-).$$

and therefore

$$\mathcal{M}^P \cong (f \times (-1))^* \left(m^*\mathcal{L}(\Theta^-) \otimes \text{pr}_1^*\mathcal{L}(\Theta^-)^{-1} \right) \otimes \text{pr}_2^*(-1)^*\mathcal{L}(\Theta^-)^{-1}$$

But $(f \times (-1))^* \text{pr}_2^* \mathcal{L}(\Theta^-)^{-1} \cong \text{pr}_2^*(-1)^* \mathcal{L}(\Theta^-)^{-1}$ and therefore claim (ii) in equation 26 follows from the definition of $\Lambda(\Theta^-)$

Now we are ready to proof the theorem: We have $\varphi_{\mathcal{L}(\Theta)} = \varphi_{\mathcal{L}(\Theta^-)}$ and

$$\begin{aligned} (1 \times -\varphi_{\mathcal{L}(\Theta)})^* (1 \times f^\vee)^* \mathcal{M}^P &\cong (1 \times -\varphi_{\mathcal{L}(\Theta)})^* (f \times 1)^* \mathcal{P} \cong (f \times (-1))^* (1 \times \varphi_{\mathcal{L}(\Theta^-)})^* \mathcal{P} \\ &\cong (f \times (-1)^*) \Lambda(\mathcal{L}(\Theta^-)) \end{aligned}$$

and therefore by the claim in (26) we have $(1 \times (-\varphi_{\mathcal{L}(\Theta)} \circ f^\vee))^* \mathcal{M}^P \cong \mathcal{M}^P$.

Hence $-\varphi_{\mathcal{L}(\Theta)} \circ f^\vee = \text{id}_{J^\vee}$ by definition of \mathcal{M}^P as the universal line bundle on $C \times J$ and the uniqueness assertion in proposition 2.11. By theorem 2.4 both $\varphi_{\mathcal{L}(\Theta)}$ and f^\vee are isogenies. Now their degree must be equal to one and the theorem follows from proposition 2.8. \square

Corollary 3.13. a) $(f \times (-1)_J)^* \Lambda(\mathcal{L}(\Theta)) \cong (f \times 1_J)^* \Lambda(\mathcal{L}(\Theta)^{-1}) \cong \mathcal{M}^P$ on $C \times J$.

b) For \mathcal{L}^P the sheaf on $C \times C$ from equation (16) we have an isomorphism $\mathcal{L}^P \cong (f \times f)^* \Lambda(\mathcal{L}(\Theta)^{-1})$.

c) The divisor Θ on J is ample and has self-intersection number $(\Theta)^g = g!$. Moreover, $H^0(J, \mathcal{L}(\Theta)) = k$ and $H^i(J, \mathcal{L}(\Theta)) = 0$ for $i \geq 1$.

Proof. We have

$$\begin{aligned} (f \times (-1)_J)^* \Lambda(\mathcal{L}(\Theta)) &\cong (f \times (-1)_J)^* (1_J \times \varphi_{\mathcal{L}(\Theta)})^* \mathcal{P} \cong (1_J \times -\varphi_{\mathcal{L}(\Theta)})^* (f \times 1_J)^* \mathcal{P} \\ &\cong (1_J \times -\varphi_{\mathcal{L}(\Theta)})^* (1 \times f^\vee)^* \mathcal{M}^P \cong (1_J \times (f^\vee \circ (-\varphi_{\mathcal{L}(\Theta)})))^* \mathcal{M}^P \stackrel{3.12}{=} \mathcal{M}^P \end{aligned}$$

Since $\mathcal{L} \rightarrow \varphi_{\mathcal{L}}$, as in equation (13), is a homomorphism, we have $\varphi_{\mathcal{L}(\Theta)^{-1}} = -\varphi_{\mathcal{L}(\Theta)}$ and so

$$\mathcal{M}^P \cong (f \times 1_J)^* (1_J \times -\varphi_{\mathcal{L}(\Theta)})^* \mathcal{P} = (f \times 1_J)^* (1_J \times \varphi_{\mathcal{L}(\Theta)^{-1}})^* \mathcal{P} \cong (f \times 1_J)^* \Lambda(\mathcal{L}(\Theta)^{-1}).$$

Now b) follows from a) because $(f \times f)^* \Lambda(\mathcal{L}(\Theta)^{-1}) \cong (1_C \times f)^* \mathcal{M}^P \cong \mathcal{L}^P$ by definition of f .

Θ is ample by lemma 2.12. By the vanishing theorem for line bundles on abelian varieties from [4, prop. 9.14] we have $H^i(J, \Theta) \neq 0$ only for $i = 0$. By the Riemann-Roch theorem for abelian varieties as in [4, thm. 9.11] $\chi(\mathcal{L}(\Theta))^2 = \deg(\varphi_{\mathcal{L}(\Theta)}) = 1$ and $(\Theta)^g = g! \cdot \chi(\mathcal{L}(\Theta)) = g!$, so c) follows. \square

3.5 The Rosati involution

Definition 3.14. The Rosati involution corresponding to $\varphi_{\mathcal{L}(\Theta)}$ is defined as the involution on $\text{End}^0(J)$ given by

$$h \mapsto h^\dagger := \varphi_{\mathcal{L}(\Theta)}^{-1} \circ h^\vee \circ \varphi_{\mathcal{L}(\Theta)} = f^\vee \circ h^\vee \circ \varphi_{\mathcal{L}(\Theta)^{-1}}.$$

Let $g, h \in \text{End}^0(J)$. It is clear from the definition that $(hg)^\dagger = g^\dagger h^\dagger$ and because $(h + g)^\vee = h^\vee + g^\vee$ also $(h + g)^\dagger = h^\dagger + g^\dagger$. Moreover, $g^\dagger = g$ if $g \in \mathbb{Q}$.

Let $\sigma : C \times C \rightarrow C \times C$ be the k morphism that switches the above factors. Then σ acts on

$$F := \{ \mathcal{L} \in \text{Pic}(C \times C) : \mathcal{L}|_{C \times \{P\}} \text{ and } \mathcal{L}|_{C \times \{P\}} \text{ is trivial} \}$$

by pullback. By corollary 3.10 this corresponds to an action on $\text{End}(J)$.

Lemma 3.15. The action on F by σ agrees with the Rosati involution when F is identified with $\text{End}(J)$ via the isomorphism in corollary 3.10.

Proof. Let $h \in \text{End}(J)$. Since

$$\begin{aligned} (1_C \times (h^\dagger \circ f))^* \mathcal{M}^P &= (1_C \times (f^\vee \circ h^\vee \circ \varphi_{\mathcal{L}(\Theta)^{-1}} \circ f))^* \mathcal{M}^P \cong (1 \times h^\vee \circ \varphi_{\mathcal{L}(\Theta)^{-1}} \circ f)^* (1 \times f^\vee)^* \mathcal{M}^P \\ &\cong (1 \times h^\vee \circ \varphi_{\mathcal{L}(\Theta)^{-1}} \circ f)^* (f \times 1)^* \mathcal{P} \cong (f \times 1)^* (1 \times \varphi_{\mathcal{L}(\Theta)^{-1}} \circ f)^* (1 \times h^\vee)^* \mathcal{P} \\ &\cong (f \times 1)^* (1 \times \varphi_{\mathcal{L}(\Theta)^{-1}} \circ f)^* (h \times 1)^* \mathcal{P} \\ &\cong ((h \circ f) \times 1)^* (1 \times f)^* (1 \times \varphi_{\mathcal{L}(\Theta)^{-1}})^* \mathcal{P} \cong ((h \circ f) \times 1)^* (1 \times f)^* \Lambda(\mathcal{L}(\Theta)^{-1}) \\ &\stackrel{3.13}{=} ((h \circ f) \times 1)^* \mathcal{M}^P \cong \sigma^* (1_C \times (h \circ f))^* \mathcal{M}^P, \end{aligned}$$

the assertions follows from corollary 3.10. \square

Since $\sigma^2 = \text{id}$ we conclude from the previous lemma 3.15 that $(h^\dagger)^\dagger = h$ for all $h \in \text{End}(J)$. Since $g^\dagger = g$ for all $g \in \mathbb{Q}$, this result extends to $\text{End}^0(J)$.

3.6 The Lefschetz trace formula and positivity of the Rosati involution

We invoke intersection theory on the surface $C \times C$. Notation will be as in Hartshorne chapter V.1.

Theorem 3.16 (Lefschetz trace formula). Let $h \in \text{End}(J)$ and let X be a divisor on $C \times C$ such that $\mathcal{L}(X) \cong (1_C \times (h \circ f))^* \mathcal{M}^P$. Then the negative intersection number of the diagonal divisor $\Delta_C \subseteq C \times C$ with X equals $\text{tr}(h)$, i.e. $-\Delta_C \cdot X = \text{tr}(h)$.

Before we can proof the theorem we need the following relation between trace and intersection theory.

Lemma 3.17. Let $h \in \text{End}(J)$. Let $D_\Theta(h) := (h+1)^* \Theta - h^* \Theta - \Theta$. Then

$$\text{tr}(h) = \frac{g}{(\Theta^g)} (\Theta^{g-1} \cdot D_\Theta(h)) = \frac{1}{(g-1)!} (\Theta^{g-1} \cdot D_\Theta(h)) = (f(C) \cdot D_\Theta(h)) = \deg f^* \mathcal{L}(D_\Theta(h)).$$

Sketch of a proof. In the last paragraph of theorem 2.18 we computed, that for all $n \in \mathbb{N}$ we have $\deg(h+n) = \frac{(D_n)^g}{(D)^g}$, where we can choose

$$D = \Theta, \quad D' = 2^* D - 2D, \quad D_n = (n+h)^* D = \frac{n(n-1)}{2} D' + n(h+1)^* D - (n-1)h^* D.$$

It is a consequence of theorem 2.5 that $2^* D$ is linearly equivalent to $3D + (-1)^* D$, see [4, cor. 2.12] or convince yourself by proceeding as in the proof of proposition 2.7. By remark 3.11 $D = \Theta$ and $(-1)^* D = \Theta^-$ are numerically equivalent. Therefore, D' is numerically equivalent to $2D$ and, so, D_n is numerically equivalent to $n^2 D + n D_\Theta(h) + h^* D$.

Since by definition $P_h(-n) = \deg(h+n) = \frac{(D_n)^g}{(D)^g}$ for all $n \in \mathbb{N}$ we have that $\text{tr}(h)$ is the coefficient in front of n^{2g-1} in the expression $\frac{(D_n)^g}{(D)^g}$, which we can identify with $\frac{g}{(D)^g} (\Theta^{g-1} \cdot D_\Theta(h))$ by using the linearity of the intersection number.

We have proven the first equality of the assertion and the second follows from corollary 3.13.

To show that $(\Theta^{g-1} \cdot D_\Theta(h)) = (g-1)! (f(C) \cdot D_\Theta(h))$ one proceeds as in [11, IV §3 Thm 5] to relate intersection numbers with taking sums of divisors via the addition of J . For this one considers the so-called Pontrjagin product $*$ on the Chow ring of J ; its definition can be found in [11, p. 8]. It is shown in [11, II §3 prop. 4] that for the r -fold Pontrjagin product of $f(C)$ one has $f(C)^{*r} = r! W^r$.

Further one shows that taking images (in the sense of intersection theory) under endomorphisms $h : J \rightarrow J$ induces a endomorphism of the group of cycles of J with the Pontrjagin product. Whereas taking inverses images under h induces an endomorphism of the chow ring with the intersection product. These two operations are adjoint to each other: $(h(\xi) \cdot \nu) = (\xi \cdot h^{-1}(\nu))$. (This is [11, IV §3 Thm 5]).

Using the above two properties it can be computed that $(\Theta^{g-1} \cdot D_\Theta(h)) = (g-1)! (W^1 \cdot D_\Theta(h))$. This is done here [11, p.112].

Now it only remains to justify $(f(C) \cdot D_\Theta(h)) = \deg f^* \mathcal{L}(D_\Theta(h))$. The proof of this is similar to [8, V 1. Lem. 1.3], see [15, Exmp. 7.1.17]. \square

Proof of theorem 3.16. By Corollary 3.13 we have that

$$\begin{aligned} \Delta_C^* \mathcal{L}(X) &\cong \Delta_C^* (1_C \times (h \circ f)^*) \mathcal{M}^P \cong \Delta^* (1_C \times (h \circ f))^* (f \times 1_J)^* \Lambda(\mathcal{L}(\Theta)^{-1}) \\ &\cong ((1_J \times h) \circ (f \times f) \circ \Delta_C)^* \Lambda(\mathcal{L}(\Theta)^{-1}) \cong f^* (1_J, h)^* \Lambda(\mathcal{L}(\Theta)^{-1}) \\ &= f^* (1_J, h)^* (m^* \mathcal{L}(\Theta)^{-1} \otimes \text{pr}_1^* \mathcal{L}(\Theta) \circ \text{pr}_2^* \mathcal{L}(\Theta)) = f^* D_\Theta(h)^{-1}. \end{aligned}$$

So, by the previous lemma 3.17 we conclude $\text{tr}(h) = \deg f^* D_\Theta(h) = \deg \Delta_C^* \mathcal{L}(X)^{-1}$. By [8, V.1 Lem. 1.3] $\deg \Delta_C^* \mathcal{L}(X)^{-1} = \Delta \cdot (-X)$ and the theorem follows from additivity of the intersection pairing. \square

3.7 The map induced on the Jacobian by an endomorphism of C

Throughout this section C will again be a proper non-singular curve of genus $g > 0$ over a field k , J will be its Jacobian variety and $P \in C(k)$ will be a k -rational point. f will be defined as in section 3.1.

Let $\alpha : C \rightarrow C$ be a non-constant k -morphism. Note that α will necessarily be finite and flat. There are two approaches to obtain a homomorphism $J \rightarrow J$ induced by α .

1. Let $t_{-f(\alpha(P))}$ be the translation on J by $-f(\alpha(P))$. Then $t_{-f(\alpha(P))} \circ f \circ \alpha : C \rightarrow J$ maps P to 0 and by proposition 3.9 there is a unique homomorphism $\alpha' : J \rightarrow J$ such that

$$t_{-f(\alpha(P))} \circ f \circ \alpha = \alpha' \circ f.$$

2. For a given k -scheme T and $\mathcal{L} \in \text{Pic}(C \times T)$ the map $\mathcal{L} \mapsto (\alpha \times 1_T)^* \mathcal{L}$ is natural in T and therefore defines a map $\text{Pic}_{C/k} \rightarrow \text{Pic}_{C/k}$. Since the trivial line bundle in $\text{Pic}_{C/k}(k)$ is sent to itself, this defines a homomorphism $\alpha^* : J \rightarrow J$.

Using that for $\mathcal{L} \in \text{Pic}(C \times T)$ the degree function $T \ni t \mapsto \deg(\mathcal{L}|_{C \times \{t\}})$ is locally constant, it can be shown that $J(k)$ can be identified with $\text{Pic}^0(C)$, the degree 0 line bundles on C , see [4, 14.1]. So, α^* is on k -valued points literally given by the pullback of degree 0 line bundles along α .

The following lemma says that the Rosati involution translates one approach into the other. In particular, α' is independent of the choice of P .

Lemma 3.18. We have $(\alpha')^\dagger = \alpha^*$ as k -morphisms $J \rightarrow J$ and

$$(1 \times (\alpha' \circ f))^* \mathcal{M}^P \cong \sigma^* \mathcal{L}(\Gamma_\alpha - C \times \{\alpha(P)\} - \alpha^{-1}(P) \times C),$$

for $\sigma : C \times C \rightarrow C \times C$ the morphism that switches the factors.

Proof. The sheaf $\mathcal{C} = \mathcal{L}(\Gamma_\alpha - C \times \{\alpha(P)\} - \alpha^{-1}(P) \times C)$ on $C \times C$ is trivial, when restricted to $C \times \{P\}$, as well as, when restricted to $\{P\} \times C$.

By proposition 2.11 applied to \mathcal{M}^P we obtain a unique k -morphism $g : C \rightarrow J$ such that $(1_C \times g)^* \mathcal{M}^P \cong \mathcal{C}$. Let K be a field extension of k . By diagram 11 for K -valued point R of C with inclusion $R \xrightarrow{x} C$ we have $g(R)$ is represented by $(1 \times x)^* \mathcal{C} \cong \mathcal{L}_{C_K}(\alpha^{-1}(R)) \otimes \mathcal{L}_{C_K}(\alpha^{-1}(P))^{-1} \cong \alpha^* \mathcal{L}_{C_K}(R - P)$. Since $g(P) = 0$ there is a homomorphism $h : J \rightarrow J$ such that $g = h \circ f$. Now $f(R)$ is represented by $\mathcal{L}_{C_K}(R - P)^{-1}$ by equation (17). We conclude that $h \circ f^g$ and $\alpha^* \circ f^g$ agree on K valued points. It follows as in the last paragraph of the proof of proposition 3.9 that $h = \alpha^*$. By corollary 3.10 it therefore suffices to proof that $(1_C \times (\alpha' \circ f))^* \mathcal{M}^P \cong \sigma^* \mathcal{C}$, we win by direct computation:

$$\begin{aligned} (1_C \times (\alpha' \circ f))^* \mathcal{M}^P &= (1_C \times (t_{-f(\alpha(P))} \circ f \circ \alpha))^* \mathcal{M}^P \cong (1_C \times \alpha)^*(1 \times (t_{-f(\alpha(P))} \circ f))^* \mathcal{M}^P \\ &\cong (1_C \times \alpha)^* \mathcal{L}(\Delta - \{\alpha(P)\} \times C - C \times \{P\}) \\ &\cong \mathcal{L}(\sigma^{-1} \Gamma_\alpha - \{\alpha(P)\} \times C - C \times \alpha^{-1}(P)) \\ &\cong \sigma^* \mathcal{L}(\Gamma_\alpha - C \times \{\alpha(P)\} - \alpha^{-1}(P) \times C), \end{aligned}$$

where $(t_{-f(\alpha(P))} \circ f)^* \mathcal{M}^P \cong \mathcal{L}(\Delta - \{\alpha(P)\} \times C - C \times \{P\})$ because the unique map $\varphi : C \rightarrow J$ such that $(1 \times \varphi)^* \mathcal{M}^P \cong \mathcal{L}(\Delta - \{\alpha(P)\} \times C - C \times \{P\})$ agrees with $t_{-f(\alpha(P))} \circ f$. Note that this can be checked on \bar{K} -valued points and φ can be computed on these points via diagram 11. \square

By theorem 2.20 α' and α^* must have the same characteristic polynomial. We now try to relate $\text{tr}(\alpha') = \text{tr}(\alpha^*)$ with the fixed points of α .

Theorem 3.19 (Lefschetz fixed point formula). We have $(\Gamma_\alpha \cdot \Delta) = 1 - \text{tr}(\alpha^*) + \deg(\alpha)$.

Proof. content... \square

References

- [1] T. Stacks project authors, “The stacks project.” <https://stacks.math.columbia.edu>, 2023.
- [2] J. S. Milne, *Algebraic geometry*. Allied Publishers, 2012.
- [3] R. Vakil, “The rising sea: Foundations of algebraic geometry,” *preprint*, 2017.
- [4] G. van der Geer and B. Moonen, “Abelian varieties,” *Book in preparation*, 2007.
- [5] D. Mumford, C. P. Ramanujam, and J. I. Manin, *Abelian varieties*, vol. 5. Oxford university press Oxford, 1974.
- [6] J. S. Milne, “Abelian varieties,” *Arithmetic geometry*, pp. 103–150, 1986.
- [7] S. Bosch, *Algebraic geometry and commutative algebra*. Springer, 2013.
- [8] R. Hartshorne, *Algebraic geometry*, vol. 52. Springer Science & Business Media, 2013.
- [9] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, vol. 21. Springer Science & Business Media, 2012.
- [10] J. S. Milne, *Algebraic number theory*. JS Milne, 2008.
- [11] S. Lang, *Abelian varieties*. Dover Publications, 2019.
- [12] A. Grothendieck, “Éléments de géométrie algébrique: Iii. étude cohomologique des faisceaux cohérents, première partie,” *Publications Mathématiques de l’IHÉS*, vol. 11, pp. 5–167, 1961.
- [13] N. M. Katz and B. Mazur, “Arithmetic moduli of elliptic curves.(am-108), volume 108,” in *Arithmetic Moduli of Elliptic Curves.(AM-108), Volume 108*, Princeton University Press, 2016.
- [14] A. Grothendieck, “Technique de descente et théorèmes d’existence en géométrie algébrique. i. généralités. descente par morphismes fidèlement plats,” *Séminaire Bourbaki*, vol. 5, pp. 299–327, 1959.
- [15] W. Fulton, *Intersection theory*, vol. 2. Springer Science & Business Media, 2013.