Mechanics of n-covered circles

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This is work in progress on the mechanics of an n-covered filament of circular rest shape.

I. INTRODUCTION

The objective of the project is to study and compute the configurations of an n-covered circular filament hanging from a fixed joint under gravity. Since we use a fixed joint, the position and slope of one end of the filament are fixed. By an n-covered filament we mean a filament whose unloaded or unstressed configuration, also referred to as the rest shape in this report, is a circle wound n times over itself. We model the filament as a planar Kirchhoff elastic rod with a constant rest curvature, and derive a boundary value problem that governs the deformations of the filament. We show that the equilibria of our system are a one-parameter family of curves, where the parameter is a combination of linear mass density (ρ), gravity (g), length (L) and stiffness (K). We then solve the boundary value problem numerically by using pseudo arc-length continuation implemented in the program AUTO-07p. We validate our results by monitoring an integral called the Hamiltonian of the system. Finally, we validate our theory by comparing the theoretical results with the experiments.

II. THEORY

Consider a segment of an inextensible and unshearable elastic rod with its centerline curve defined as $\boldsymbol{r}(s)$ where the parameter s is along the arc length of the centerline [FIG. 1]. Let the rod segment start at s_1 and end at s_2. The internal forces and internal moments applied due to the other segments of the rod are $-\boldsymbol{n}(s_1)$, $\boldsymbol{n}(s_2)$ and $-\boldsymbol{m}(s_1)$, $\boldsymbol{m}(s_2)$, respectively. The positive and negative signs of the internal forces and moments are due to the sign convention, that is, the internal forces are defined as the forces applied by the '+' side on the'−' side. $\boldsymbol{p}(s)$ and $\boldsymbol{l}(s)$ are the external forces and moments distributed along the length of the material. $\{\boldsymbol{d_1}(s), \boldsymbol{d_2}(s), \boldsymbol{d_3}(s)\}$ is orthogonal frame attached so that, $\boldsymbol{d_3}(s)$ is tangential to the centerline. We now formulate the kinematics, equilibrium, and constitutive relation for the above configuration.

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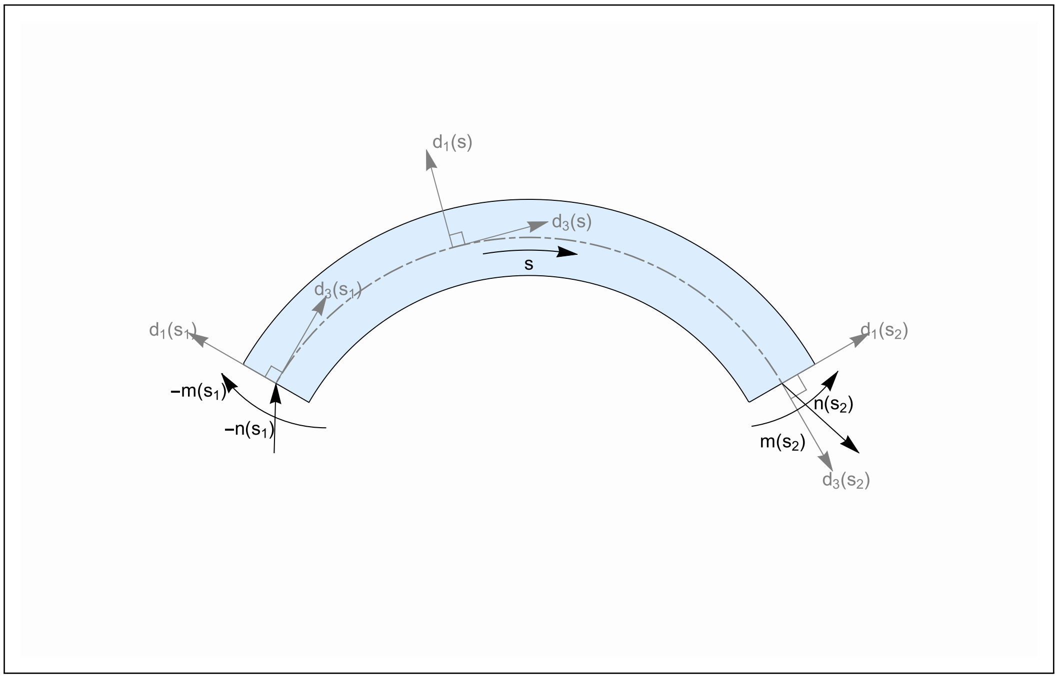


FIG. 1: Segment of elastic rod

A. Kinematics

Since $\boldsymbol{d_3}$ is tangent to the centre line and the rod is inextensible and unshearable, the tangent to $\boldsymbol{r}(s)$ will be along $\boldsymbol{d_3}$. Thus, we get the following relation:

$$\boldsymbol{r}'(s) = \boldsymbol{d_3}(s) \tag{1}$$

The orthonormality of the direction vectors can be expressed as:

$$\boldsymbol{d_i'} = \boldsymbol{u} \times \boldsymbol{d_i} \tag{2}$$

Where, $\boldsymbol{u}(s)$ is Darboux vector associated with the director frames and prime denotes the first derivative w.r.t s, The director components $u_i \coloneqq \boldsymbol{u}.\boldsymbol{d_i}$, $i \in \{\ 1, 2, 3\}$ of Darboux vector measure the bending strains of rod about directors $\boldsymbol{d}_i$ [1]. Since the we are considering a 2D problem, the bending is about $\boldsymbol{d_2}$ only thus we get:

$$\boldsymbol{u} = u_2 \boldsymbol{d_2} \tag{3}$$

B. Equilibrium

The equilibrium relations obtained from force and moment balance, respectively, are:

$$\boldsymbol{n}' + \boldsymbol{p} = \boldsymbol{0}\,, \tag{4}$$

$$\boldsymbol{m}' + \boldsymbol{r}' \times \boldsymbol{n} + \boldsymbol{l} = \boldsymbol{0}\,. \tag{5}$$

Refer to Appendix A for derivations.

C. Constitutive relations

We introduce a constitutive relation to obtain a closed system of equations by assuming that the rod is hyperelastic with straight and uniform configuration [1]. The relation that we obtain is:

$$\boldsymbol{m} = K u_2 \boldsymbol{d_2} \tag{6}$$

For derivations, refer to Appendix A.
However, for the n-covered circular filament, the constitutive relation will take a different form. $\boldsymbol{m} = Ku_2\boldsymbol{d_2}$ is true for a linearly elastic material with a straight unstretched configuration (rest shape).
For a shape with a different rest shape, the constitutive relation is given by:

$$\boldsymbol{m} = K(u_2 - u_2^o)\boldsymbol{d_2} \tag{7}$$

where u_2^o is the curvature of the rest shape.

The vector equations obtained for the n-covered circular filament are:

$$\begin{aligned}
\boldsymbol{r}'(s) &= \boldsymbol{d_3}(s) \\
\boldsymbol{n}'(s) &= -\boldsymbol{p}(s) \\
\boldsymbol{m}'(s) &= -\boldsymbol{r}'(s) \times \boldsymbol{n}(s) - \boldsymbol{l}(s) \\
\boldsymbol{d}_i'(s) &= \boldsymbol{u}(s) \times \boldsymbol{d}_i(s) \\
\boldsymbol{u}(s) &= u_2\boldsymbol{d_2}(s) \\
\boldsymbol{m} &= K(u_2 - u_2^o)\boldsymbol{d_2}
\end{aligned}$$

III. BOUNDARY VALUE PROBLEM

Here, we use the kinematics, equilibrium equations, and the constitutive relations stated in the previous section to derive a boundary value problem describing an n-covered circular filament. From here onwards, $\boldsymbol{l}$ is assumed to be zero.

To derive the scalar equations, we introduce a global coordinate system $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ such that $\boldsymbol{e_3}$ points towards right making and angle of θ with $\boldsymbol{d_3}$, $\boldsymbol{e_1}$ is vertically upward and $\boldsymbol{e_2}$ is out of the plane of the paper. Thus, $\boldsymbol{d_1}, \boldsymbol{d_2}, \boldsymbol{d_3}$ and $\boldsymbol{d_3'}$ can be written as:

$$\boldsymbol{d_1} = -\sin\theta\boldsymbol{e_3} + \cos\theta\boldsymbol{e_1} \tag{8}$$
$$\boldsymbol{d_2} = \boldsymbol{e_2} \tag{9}$$
$$\boldsymbol{d_3} = \cos\theta\boldsymbol{e_3} + \sin\theta\boldsymbol{e_1} \tag{10}$$
$$\boldsymbol{d_3'} = -\theta'\sin\theta\boldsymbol{e_3} + \theta'\cos\theta\boldsymbol{e_1} \tag{11}$$

A. Kinematics

Now $\boldsymbol{r}(s)$ can be written as:

$$\boldsymbol{r} = x\boldsymbol{e_3} + y\boldsymbol{e_1} \tag{12}$$

differentiating (12) w.r.t s we get:

$$\boldsymbol{r}' = x'\boldsymbol{e_3} + y'\boldsymbol{e_1} \tag{13}$$

Substituting (13) and (10) in (1) we get:

$$x'\boldsymbol{e_3} + y'\boldsymbol{e_1} = cos\theta\boldsymbol{e_3} + \sin\theta\boldsymbol{e_1} \tag{14}$$

Due to the linear independence of the unit vectors we get,

$$x' = \cos\theta \tag{15}$$
$$y' = \sin\theta \tag{16}$$

from (2) we get:

$$\boldsymbol{d_1'} = -u_1\boldsymbol{d_3} \tag{17}$$
$$\boldsymbol{d_2'} = \boldsymbol{0} \tag{18}$$
$$\boldsymbol{d_3'} = u_2\boldsymbol{d_1} \tag{19}$$

Substituting (8), (9) and (11) in (19) we get:

$$u_2 \cos\theta \boldsymbol{e_1} - u_2 \sin\theta \boldsymbol{e_3} = \theta' \cos\theta \boldsymbol{e_1} - \theta' \sin\theta \boldsymbol{e_3} \tag{20}$$

Due to the linear independence of the unit vectors we get,

$$u_2 = \theta' \tag{21}$$
$$\sin\theta = 0 \tag{22}$$
$$\cos\theta = 0 \tag{23}$$

Solution (22) and (23) are contradict each other thus the only acceptable solution is (21) that is:

$$u_2 = \theta'$$

B. Equilibrium

Since it is a 2D problem, $\boldsymbol{n}, \boldsymbol{m}$ and $\boldsymbol{p}$ can be written as:

$$\boldsymbol{n} = n_1 \boldsymbol{d_1} + n_3 \boldsymbol{d_3} \tag{24}$$
$$\boldsymbol{m} = m_2 \boldsymbol{d_2} \tag{25}$$
$$\boldsymbol{p} = p_1 \boldsymbol{d_1} + p_3 \boldsymbol{d_3} \tag{26}$$

differentiating (24) and (25) w.r.t s we get:

$$\boldsymbol{n}' = n_1' \boldsymbol{d_1} + n_1 \boldsymbol{d_1'} + n_3' \boldsymbol{d_3} + n_3 \boldsymbol{d_3'} \tag{27}$$
$$\boldsymbol{m}' = m_2' \boldsymbol{d_2} + m_2 \boldsymbol{d_2'} \tag{28}$$

Substituting (17), (18) and (19) in (27) and (28) we get:

$$\boldsymbol{n}' = (n_1' + n_3 u_2)\boldsymbol{d_1} + (n_3' - n_1 u_2)\boldsymbol{d_3} \tag{29}$$
$$\boldsymbol{m}' = m_2' \boldsymbol{d_2} \tag{30}$$

Substituting (29), (30), (26) and (1) in (4) and (5) we get,

$$(n_1' + n_3 u_2 + p1)\boldsymbol{d_1} + (n_3' - u_2 n_1 + p_3)\boldsymbol{d_3} = \boldsymbol{0} \tag{31}$$
$$(m_2' + n_1)\boldsymbol{d_2} = \boldsymbol{0} \tag{32}$$

Due to the linear independence of the directors, we get,

$$n_1' = -p_1 - n_3 u_2 \tag{33}$$
$$n_3' = n_1 u_2 - p_3 \tag{34}$$
$$m_2' = -n_1 \tag{35}$$

Now, gravity is acting along $-\boldsymbol{e_1}$, thus, $\boldsymbol{p} = -\rho g \boldsymbol{e_1}$.

$$p_1 = \boldsymbol{p} \cdot \boldsymbol{d_1}$$

From (8) we get:

$$p_1 = -\rho g \boldsymbol{e_1} \cdot (-\sin\theta \boldsymbol{e_3} + \cos\theta \boldsymbol{e_1})$$
$$p_1 = -\rho g \sin\theta \tag{36}$$

$$p_3 = \boldsymbol{p} \cdot \boldsymbol{d_3}$$

From (10) we get:

$$p_3 = -\rho g \boldsymbol{e_1} \cdot (\cos\theta \boldsymbol{e_3} + \sin\theta \boldsymbol{e_1})$$
$$p_3 = -\rho g \cos\theta \tag{37}$$

Thus, from (33), (34), (36) and (37) we get:

$$n_1' = -n_3 u_2 + \rho g \cos\theta \tag{38}$$
$$n_3' = n_1 u_2 + \rho g \sin\theta \tag{39}$$

C. Constitutive relations

For an n-covered circular filament, the rest shape is a circle. Thus, the magnitude of u_2^o is $\frac{1}{R}$ where R is the radius of the loop.
From (2) we get:

$$\boldsymbol{d_3'} = u_2 \boldsymbol{d_1}$$

Thus, u_2 measures the rate of change of $\boldsymbol{d_3}$.
From the [FIG. 1], the rate of change of $\boldsymbol{d_3}$ is in the opposite direction of $\boldsymbol{d_1}$. Thus, u_2^o will have a negative sign.

$$\therefore u_2^o = -\frac{1}{R} \tag{40}$$

$$\boldsymbol{m} = K(u_2 + \frac{1}{R})\boldsymbol{d_2} \tag{41}$$

Substituting (28) and (18) in (41) we get,

$$m_2' = K u_2' \tag{42}$$

Now, from (42) and (35) and we get:

$$K u_2' = -n_1 \tag{43}$$

Thus, the differential equations for an n-covered filament are:

$$x' = \cos\theta \tag{44}$$
$$y' = \sin\theta \tag{45}$$
$$\theta' = u_2 \tag{46}$$
$$n_1' = -n_3 u_2 + \rho g \cos\theta \tag{47}$$
$$n_3' = n_1 u_2 + \rho g \sin\theta \tag{48}$$
$$K u_2' = -n_1 \tag{49}$$

D. Non-dimensionalisation of the differential equation

The above differential equations have 5 parameters, i.e F, K, ρ, L, g. Thus, we now non-dimensionalise the equations. We choose L as the characteristic length. Thus, we get:

$$\therefore \bar{x} = \frac{x}{L} \tag{50}$$
$$\bar{y} = \frac{y}{L} \tag{51}$$
$$\bar{s} = \frac{s}{L} \tag{52}$$

Let F be the force scale.

$$\bar{n}_1 = \frac{n_1}{F} \tag{53}$$
$$\bar{n}_3 = \frac{n_3}{F} \tag{54}$$

Also,

$$\bar{u}_2 = u_2 L \tag{55}$$

Now,

$$\frac{d}{ds} = \frac{d\bar{s}}{ds}\frac{d}{d\bar{s}}$$

From (52) we get:

$$\frac{d}{ds} = \frac{1}{L}\frac{d}{d\bar{s}} \tag{56}$$

From here onwards prime denotes derivative w.r.t $\bar{s}$ Substituting (50) and (56) in (44) and (45) we get:

$$\bar{x}' = \cos\theta \tag{57}$$

$$\bar{y}' = \sin\theta \tag{58}$$

Substituting (55) and (56) in (46) we get:

$$\theta' = \bar{u}_2 \tag{59}$$

Substituting (53),(54) and (56) in (47) and (48) we get:

$$\bar{n}_1' = -\bar{n}_3\bar{u}_2 + \frac{\rho g L}{F}\cos\theta \tag{60}$$

$$\bar{n}_3' = \bar{n}_1\bar{u}_2 + \frac{\rho g L}{F}\sin\theta \tag{61}$$

Substituting (53), (55) and (56) in (49) we get,

$$\frac{K}{L^2}\bar{u}_2' = -\bar{n}_1 F \tag{62}$$

from (62) we get:

$$F = \frac{K}{L^2} \tag{63}$$

Substituting (63) in (60) , (61) and (62)

$$\bar{n}_1' = -\bar{n}_3\bar{u}_2 + \alpha\cos\theta \tag{64}$$

$$\bar{n}_3' = \bar{n}_1\bar{u}_2 + \alpha\sin\theta \tag{65}$$

$$\bar{u}_2' = -\bar{n}_1 \tag{66}$$

where $\alpha = \frac{\rho g L^3}{K}$

$\therefore$ The original 5 parameters F, g, L, ρ, K have been reduced to the 1 parameter, that is, α.
Note that different values of F, ρ, g, L, K can have the same solution provided that α remains the same.
Thus, the non-dimensional equations are:

$$\begin{aligned} x' &= \cos\theta \\ y' &= \sin\theta \\ \theta' &= u_2 \\ n_1' &= -n_3 u_2 + \alpha\cos\theta \\ n_3' &= n_1 u_2 + \alpha\sin\theta \\ u_2' &= -n_1 \end{aligned}$$

All the variables here are non-dimensionalised.

E. Boundary Conditions

We choose a coordinate system such that the filament starts at the origin. Also, at the origin, the global coordinate system and the director vectors align, thus the angle between them is 0.
Thus, at $x = 0$ we get the following boundary conditions:

$$x(0) = 0 \tag{67}$$
$$y(0) = 0 \tag{68}$$
$$\theta(0) = 0 \tag{69}$$

There is no point force acting at $x = 1$, thus, there will be no internal forces. In addition, the internal moment is 0. Thus, we get:

$$\text{From } (41)$$
$$u_2 + \frac{1}{R} = 0$$
$$\therefore u_2 = -\frac{1}{R}$$

If we take a filament of length $L = 1$ and loop it once, then:

$$R = \frac{1}{2\pi}$$

Similarly for 2 and 3 loops:

$$R = \frac{1}{2\pi \times 2} \;:\; R = \frac{1}{2\pi \times 3}$$

Thus, for n loops:

$$R = \frac{1}{2\pi n}$$

$$\therefore u_2^o = \frac{-1}{R} = -2\pi n$$

Thus, at $x = 1$ we get the following boundary conditions:

$$n_1(1) = 0 \tag{70}$$
$$n_3(1) = 0 \tag{71}$$
$$u_2(1) = -2\pi n \tag{72}$$

Therefore, the required boundary conditions are:

$$x(0) = 0$$
$$y(0) = 0$$
$$\theta(0) = 0$$
$$n_1(1) = 0$$
$$n_3(1) = 0$$
$$u_2(1) = -2\pi n$$

F. Computations

Since AUTO-07p uses pseudo-arc length continuation to compute the solutions for the above boundary value problem, we need to know the values of unknowns $x(s), y(s), n_1(s), n_3(s), \theta(s), u_2(s)$ for a certain value of α. Since for $\alpha = 0$ the equations can be solved by hand, we use this as our starting solution. Since $\alpha = 0$ corresponds to the rest shape, $u_2 = u_2^o$.

$$\begin{aligned} \because u_2 &= -2\pi n \\ \therefore u_2' &= 0 \end{aligned} \tag{73}$$

From (66) we get,

$$n_1 = 0 \tag{74}$$

From (65) we get:

$$\begin{aligned} n_3' &= 0 \\ \therefore n_3 &= \text{constant} \end{aligned}$$

Using (71) we get:

$$n_3 = 0 \tag{75}$$

$$\tag{76}$$

From (59) we get:

$$\begin{aligned} \theta' &= -2\pi n \\ \therefore \theta &= -2\pi ns + c \end{aligned}$$

Using (69) we get:

$$\theta = -2\pi ns \tag{77}$$

Substituting (77) in (57) we get:

$$\begin{aligned} x' &= \cos(-2\pi ns) \\ &= \cos(2\pi n) \\ \therefore x &= \frac{\sin(2\pi ns)}{2\pi n} + c \end{aligned}$$

Using (67) we get:

$$x = \frac{\sin(2\pi ns)}{2\pi n} \tag{78}$$

Now, Substituting (77) in (58) we get:

$$\begin{aligned} y' &= \sin(-2\pi ns) \\ &= -\sin(2\pi n) \\ \therefore y &= \frac{\cos(2\pi ns)}{2\pi n} + c \end{aligned}$$

Using (68) we get:

$$\begin{aligned} y &= \frac{\cos(2\pi ns)}{2\pi n} - \frac{1}{2\pi n} \\ \therefore y &= \frac{1}{2\pi n}(\cos(2\pi ns) - 1) \end{aligned} \tag{79}$$

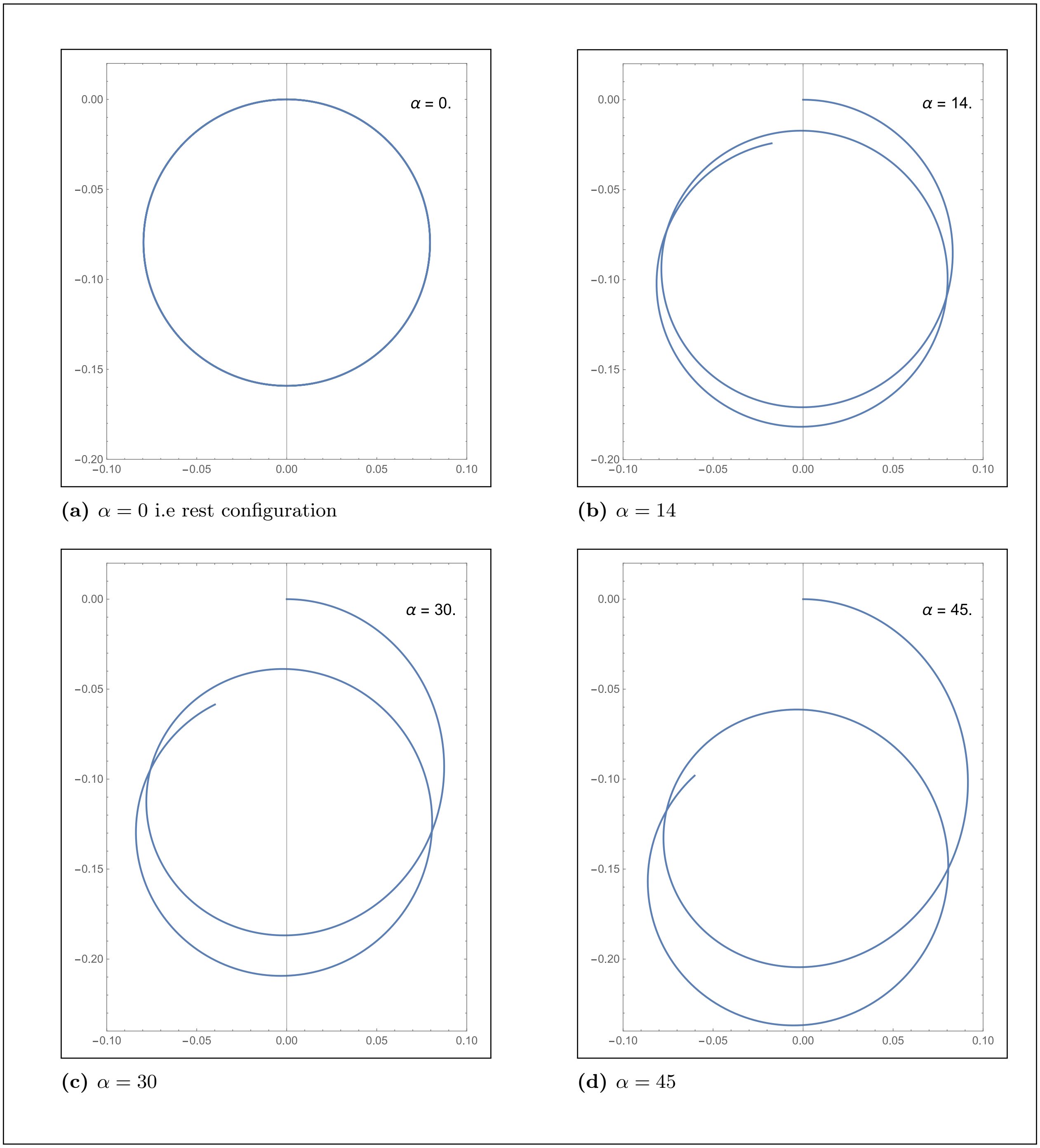
Thus, the unknown values for $\alpha = 0$ are :

$$\begin{aligned} x &= \frac{\sin(2\pi ns)}{2\pi n} \quad &;\quad y &= \frac{1}{2\pi n}(\cos(2\pi ns) - 1) \\ \theta &= -2\pi ns \quad &;\quad n_1 &= 0 \\ n_3 &= 0 \quad &;\quad u_2 &= -2\pi n \end{aligned}$$

IV. THEORETICAL RESULTS

A. Configurations

Once the differential equations and starting solution are derived, we have all the necessary inputs for the algorithm to solve the non-linear system. The configurations obtained for different values of α are:



(a) $\alpha = 0$ i.e rest configuration

(b) $\alpha = 14$

(c) $\alpha = 30$

(d) $\alpha = 45$

FIG. 2: Theoretically obtained configurations for different values of α

B. Hamiltonian

To validate the above numerics, we calculate the Hamiltonian for the same values of α.
The Hamiltonian function for an n-covered circular filament is:

$$H = n_3 + \frac{1}{2}u_2^2 - \alpha y \tag{80}$$

We use this equation without derivation.
Note that the derivative of the Hamiltonian with respect to s is 0. Thus, the graphs so obtained must be a straight line parallel to the x-axis, implying that the configurations obtained are correct.

The Hamiltonian functions obtained are as follows:

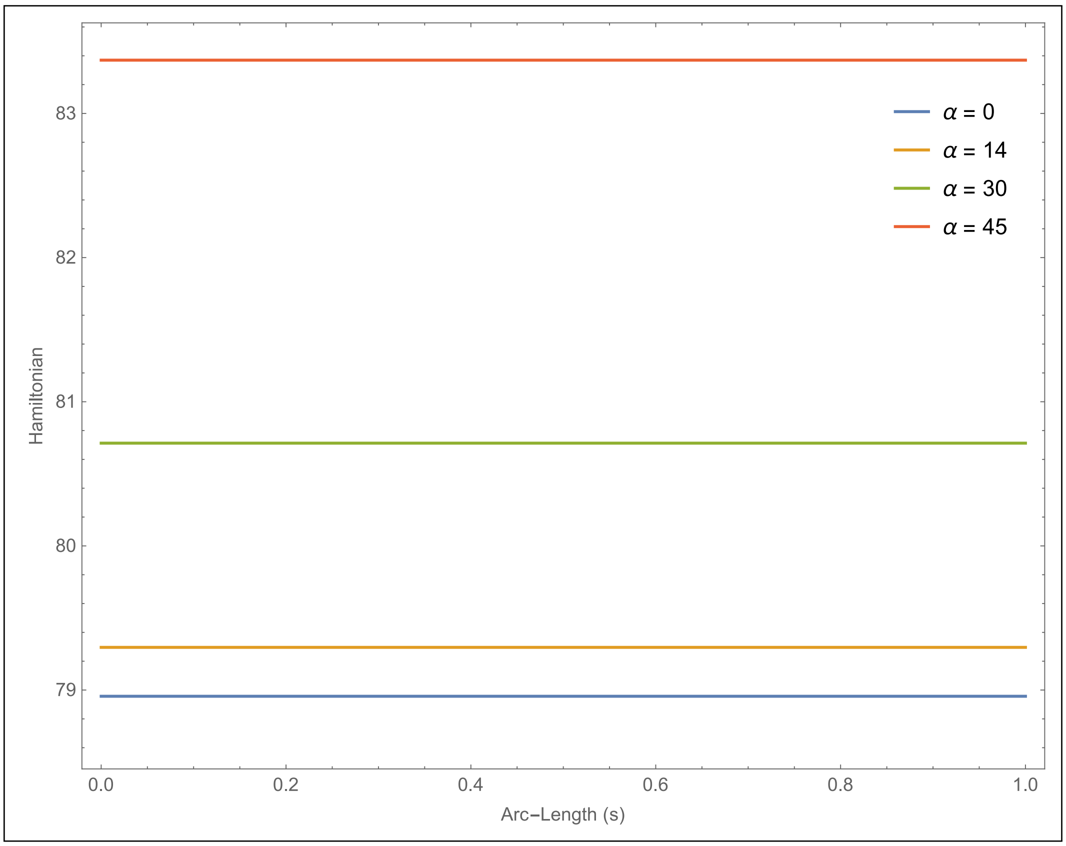
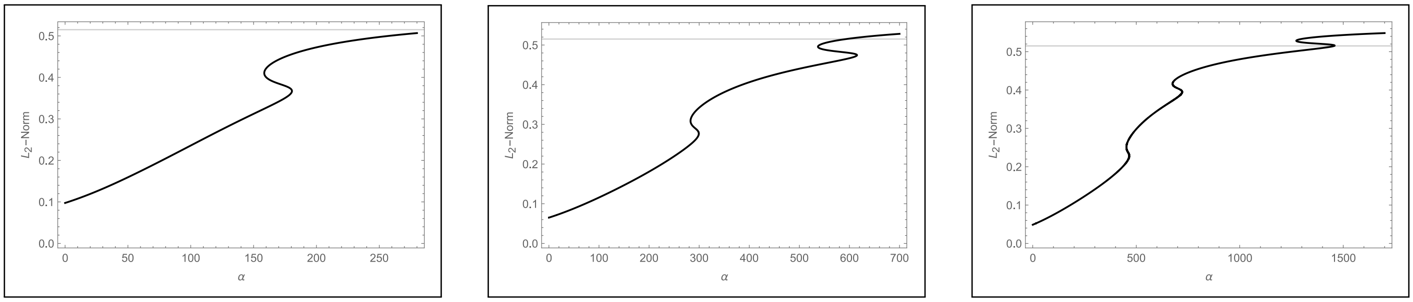


FIG. 3: Hamiltonian functions for different values of α

Since the Hamiltonian functions obtained are constants for different α,thus, we can confidently say that our numerics are correct

C. Bifurcation Diagram

In this subsection, we analyse bifurcation diagrams for different values of n. The bifurcation diagrams obtained for $n = 2, 3, 4$ are as follows



(a) $n = 2$ **(b)** $n = 3$ **(c)** $n = 4$

FIG. 4: Bifurcation diagram for n =2,3,4

The bifurcation diagram shows folds. These folds indicate that multiple configurations for the same values of α exist. For $n = 2$ we get 2 folds, for $n = 3$ there are 4 folds and for $n = 4$ there are 6 folds. This implies that for any given n, $2(n-1)$ folds exist. This is just a hypothesis: if it is true, then for $n = 5$ there should be 8 folds.

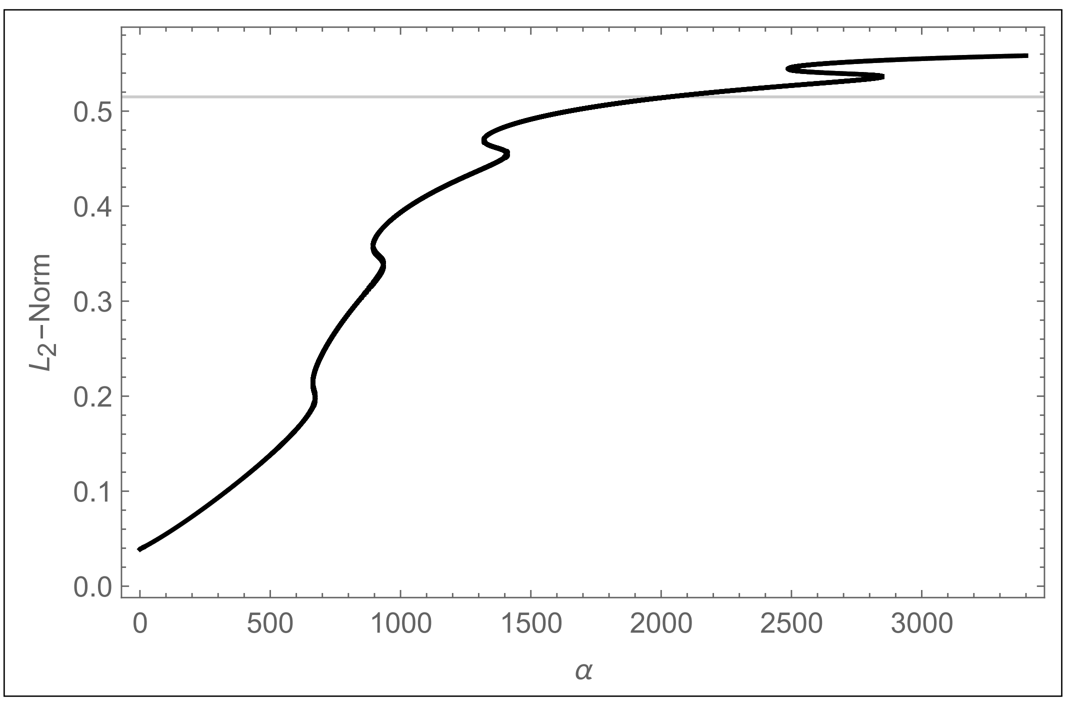


FIG. 5: Bifurcation diagram for $n = 5$

The bifurcation diagram shows 8 folds, which validates our hypothesis.

V. EXPERIMENTAL RESULTS

We experiment with a 3D printing filament with the required number of circles. The Young's modulus of this filament is unknown; thus, α becomes a fitting parameter. We can obtain the value of Young's modulus using α for which the theoretical and experimental results align after the necessary translation and rotation.

A. Experimental Setup

The experiment was performed for $n = 2$. To ensure that the angle made by the filament and the x-axis at the origin is zero, we designed a sleeve to hold the filament in place. The following image shows the design of the sleeve.

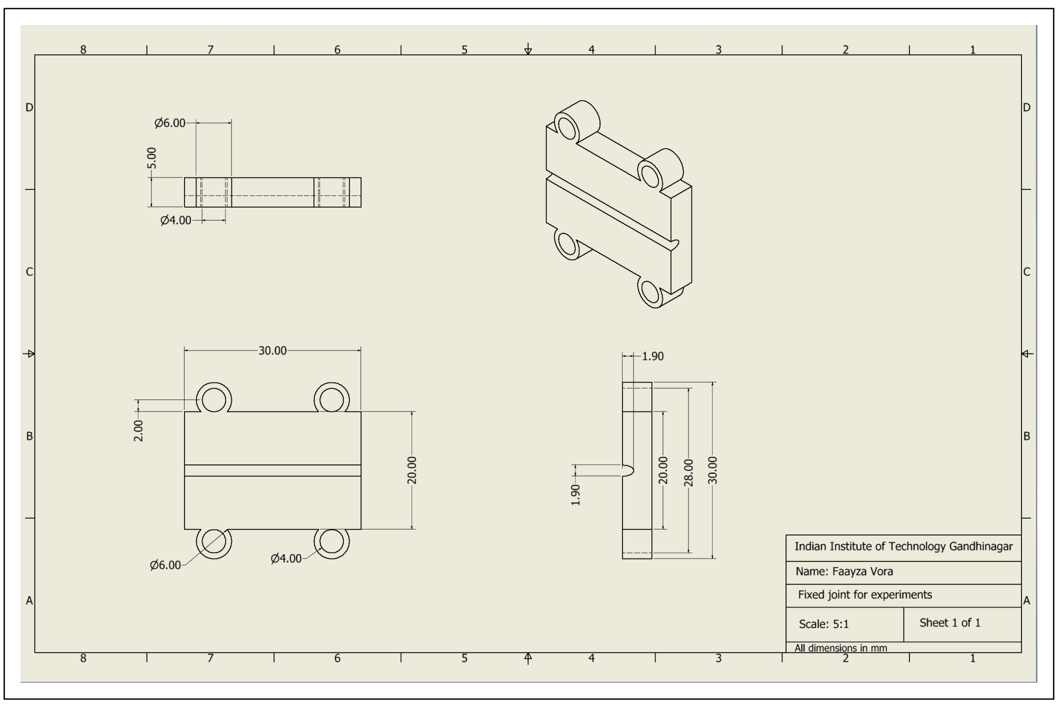


FIG. 6: Fixed joint used in experiments

Two pieces were 3D printed and stuck together, creating a hole for the filament to be inserted. The filament was then hung against a white wall, and a picture was taken with a mobile camera. During this, it was ensured that the sleeve and the mobile were levelled, and that no shadows were present.
The image used for the experiment here is:

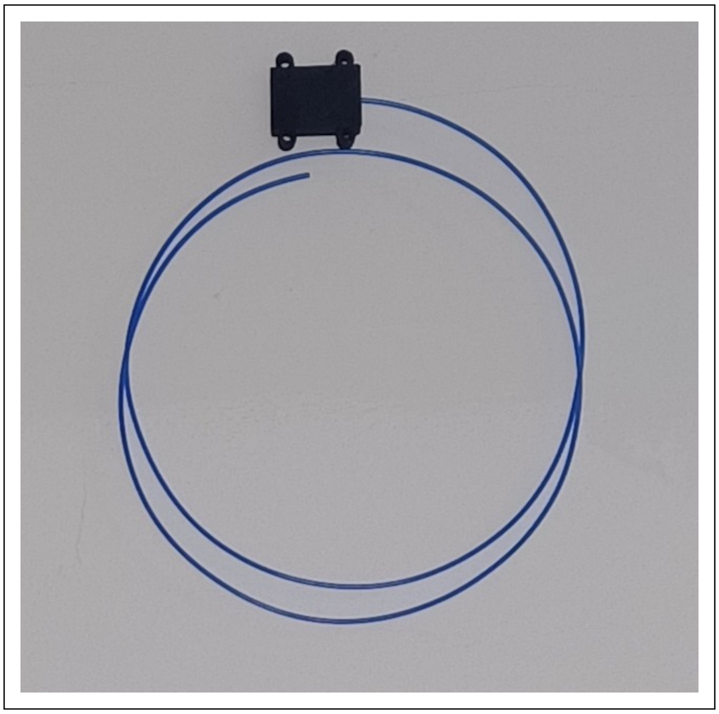


FIG. 7: Image used in experiment (cropped)

For the actual image, refer to Appendix B.

B. Experimental Data Extraction

Mathematica was used to process and extract the data from the above image. We use the EdgeDetect function of Mathematica with $r = 4$, $k = 2$ and $t = 0.0325$ to detect the edge of the filament. While doing this, the edge of the filament that crosses over itself is not detected. This makes it difficult for us to plot the centre line in these regions. Thus, we drop overlaps to compare the centre lines. This is done simply by dropping the values of the pixels that represent the overlaps.

Now the inner and the outer list are separated, and then reordered such that it follows the exact path as a filament would in a real-life scenario. The Nearest function is used to map the points on the outer list to the points on the inner list. In order to validate the correctness of this mapping, we check the diameter of the filament by using the EuclideanDistance function. The diameter obtained is in pixels. The next step would be to find the scale. We know the actual dimensions of the sleeve, i.e. $3cm \times 2cm$, which will be used to calculate the scale. The following code snippet gives us a scale of 0.34733. This means, 0.34733 pixels represent $1mm$.

```
(*Finding the scale*)
scaleStartPoint = 4872;
scaleEndPoint = 5024;
pixleDist = EuclideanDistance[configReorderedList[[scaleStartPoint]],
   configReorderedList[[scaleEndPoint]]];
scale = 30 / pixleDist // N (* actual distance in mm*)
```

FIG. 8: Code snippet for calculation of scale

The thickness found to be $1.60645mm$, the actual diameter of the filament is $1.75mm$. Thus, we can confidently say that the mapping is reliable.

Now, we divide the inner-list and Outer-List into 5 parts (Each part representing a segment of filament in between the overlaps) and use the Midpoint function to get the mid-point of a point in the inner list and its corresponding outer-list point. This gives us the centreline data, which is then exported in an Excel sheet. The inner, outer and centre lines obtained are shown in the image below:

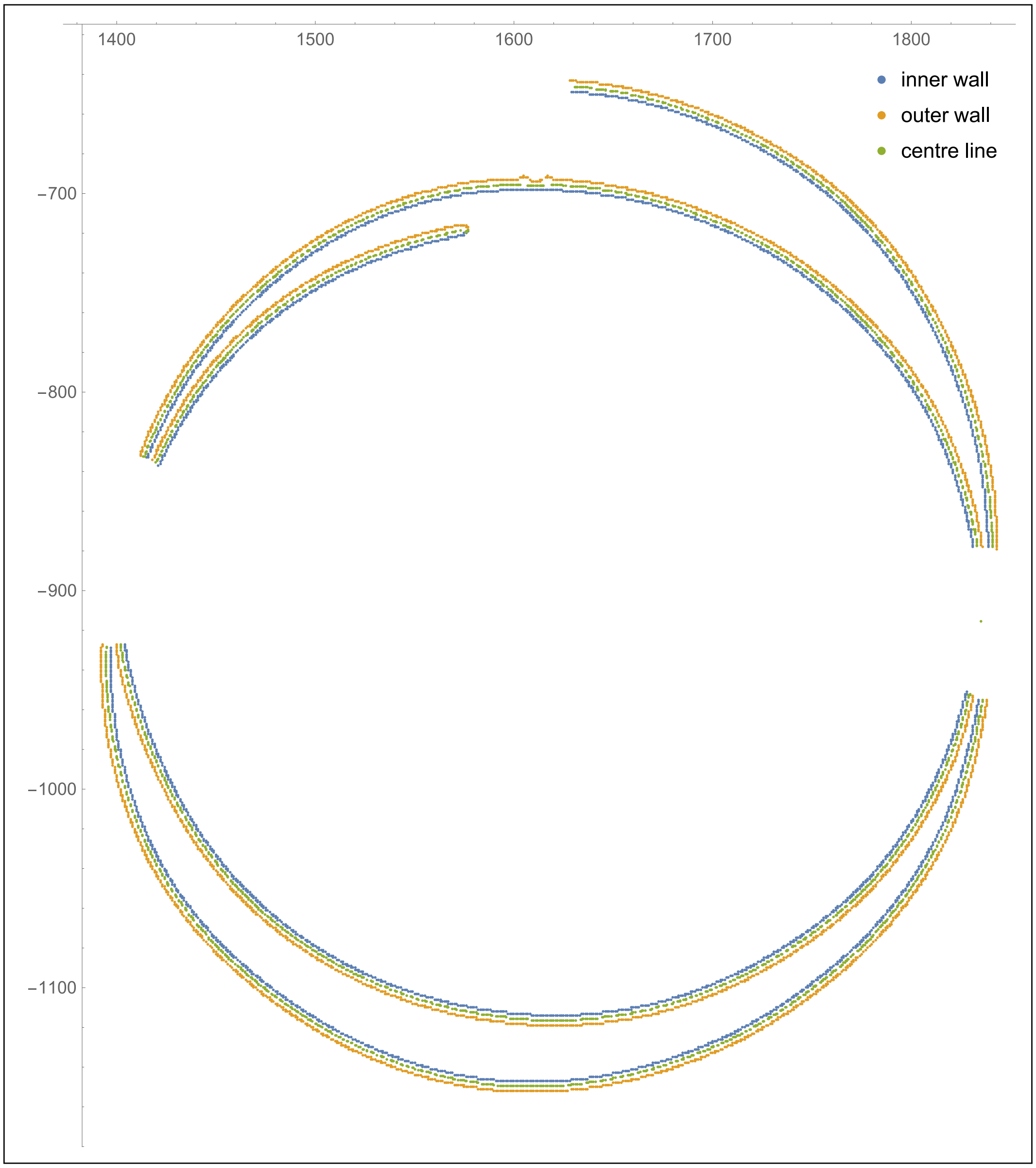
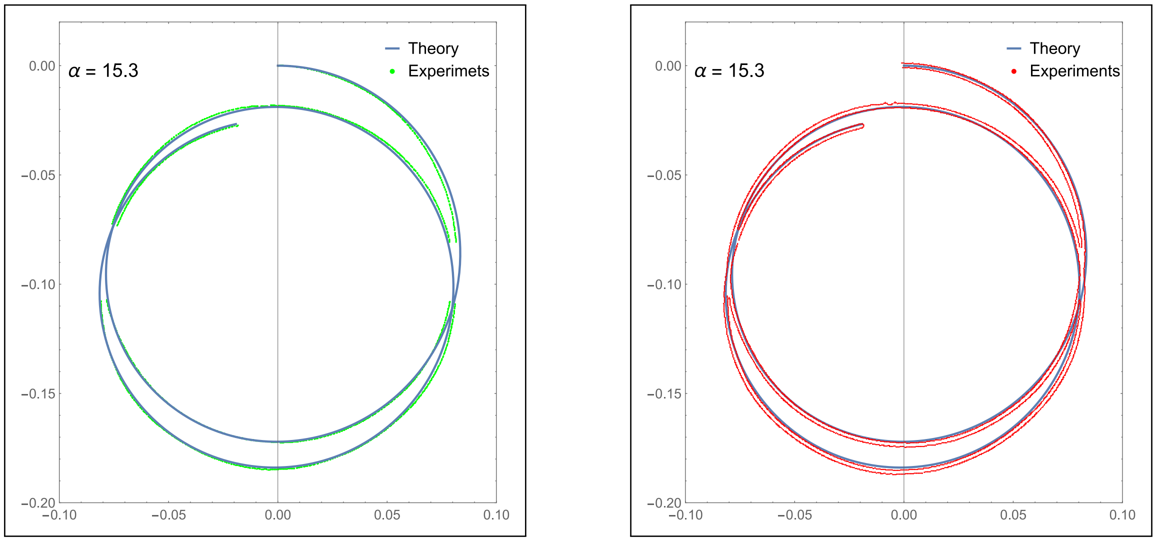


FIG. 9: Experimental data

VI. COMPARISON BETWEEN EXPERIMENT AND THEORY

The comparisons are done for the centre line and also for the inner and outer walls. The inner and outer walls are used to see how well the theory predicts the filament overlaps. Both the data are first translated by subtracting the first data point of the centre line and then rotated by an angle of 3°. To normalise the data, all the data points are supposed to be divided by the total length of the filament (since all three lists have data points for the filament in pixels, thus the length used will be the total length of the list, also in pixels). We subtract 160, 440 from the centerline's and inner and outer wall's data to minimise random error??. The images of the obtained comparisons are as follows:



(a) Experimental centreline and theory comparisons

(b) Experimental inner and outer walls and theory comparisions

FIG. 10: Comparisons

After a simple trial and error, α is found to be 15.3. Using α as 15.3, we can calculate the Young's modulus of the filament. We know that,

$$\alpha = \frac{\rho g L^3}{K} \tag{81}$$

Where $K = EI_{zz}$, Where E is Young's modulus and I_{zz} is For the filament used in this experiment, $\rho =$, $L =$, $d =$ From the above equation, Young's modulus is found to be

VII. DISCUSSIONS

Thus, we successfully computed the configurations of an n-covered circular filament modelled as a planar Kirchhoff elastic rod with a constant rest curvature. We validate the numerically obtained results by monitoring an integral called the Hamiltonian function, which is constant for all values of s and α, as expected. The experimental results are comparable to the theory, thus validating our assumption that the filament can be modelled as a Kirchhoff elastic rod.

However, there are some differences in the experimental and theoretical results. These are mainly due to random and human errors induced during the experiments. The Young's modulus is found to be *number*.This experiment can be performed for different values of n, and Young's modulus can be calculated. If the Young's modulus remains approximately the same, it implies that the experiment has been performed correctly. Moreover, this data can also be used to quantify the experimental error.

This project can be further extended by calculating the curvature of the experimental data as a function of the arc length. Since curvature is purely a geometric property of the shape of a curve and is independent of the coordinate system or scaling, the comparisons between experimental and theoretical results are less affected by factors such as measurement offset and scaling errors. Thus giving us better comparisons between theory and the experiments.

VIII. APPENDIX A: DERIVATION OF EQUILIBRIUM AND CONSTITUTIVE RELATIONS

A. Equilibrium

The force balance of the elastic rod segment $[s_1,s_2]$ is given by

$$\boldsymbol{n}(s_2) - \boldsymbol{n}(s_1) + \int_{s_1}^{s_2} \boldsymbol{p}\, ds = \boldsymbol{0} \tag{82}$$

By fundamental theorem of calculus $\boldsymbol{n}(s_2) - \boldsymbol{n}(s_1) = \int_{s_1}^{s_2} \boldsymbol{n}'(s)\, ds$. Substituting this in equation (82) we get:

$$\int_{s_1}^{s_2} (\boldsymbol{n}' + \boldsymbol{p})\, ds = \boldsymbol{0}$$

Since s_1 and s_2 are arbitrary, we get our first equilibrium condition given by:

$$\boldsymbol{n}' + \boldsymbol{p} = \boldsymbol{0} \tag{83}$$

The moment balance of the elastic rod segment $[s_1, s_2]$ is given by

$$\boldsymbol{m}(s_2) - \boldsymbol{m}(s_1) - \boldsymbol{r}(s_1) \times \boldsymbol{n}(s_1) + \boldsymbol{r}(s_2) \times \boldsymbol{n}(s_2) + \int_{s_1}^{s_2} \boldsymbol{r} \times \boldsymbol{p}\, ds + \int_{s_1}^{s_2} \boldsymbol{l}\, ds = \boldsymbol{0} \tag{84}$$

By fundamental theorem of calculus $\boldsymbol{m}(s_2) - \boldsymbol{m}(s_1) = \int_{s_1}^{s_2} \boldsymbol{m}'(s)\, ds$ and
$\boldsymbol{r}(s_2) \times \boldsymbol{n}(s_2) - \boldsymbol{r}(s_1) \times \boldsymbol{n}(s_1) = \int_{s_1}^{s_2} [\boldsymbol{r}(s) \times \boldsymbol{n}(s)]'\, ds.$
Substituting this in equation (84) we get:

$$\int_{s_1}^{s_2} (\boldsymbol{m}' + [\boldsymbol{r}(s) \times \boldsymbol{n}(s)]' + \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{l})\, ds = \boldsymbol{0}$$

Since s_1 and s_2 are arbitrary, we get

$$\boldsymbol{m}' + [\boldsymbol{r}(s) \times \boldsymbol{n}(s)]' + \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{l} = \boldsymbol{0}$$

Using the product rule, we get:

$$\begin{gathered}\boldsymbol{m}' + \boldsymbol{r}'(s) \times \boldsymbol{n}(s) + \boldsymbol{r}(s) \times \boldsymbol{n}'(s) + \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{l} = \boldsymbol{0} \\ \therefore \boldsymbol{m}' + \boldsymbol{r}'(s) \times \boldsymbol{n}(s) + \boldsymbol{r}(s) \times [\boldsymbol{n}'(s) + \boldsymbol{p}] + \boldsymbol{l} = \boldsymbol{0}\end{gathered}$$

From (83) we get $\boldsymbol{n}'(s) + \boldsymbol{p} = 0$,thus we get the second equilibrium condition given by

$$\boldsymbol{m}' + \boldsymbol{r}' \times \boldsymbol{n} + \boldsymbol{l} = \boldsymbol{0} \tag{85}$$

B. Constitutive Relations

Since the rod is assumed to be hyperelastic with straight and uniform configuration [1],there exists a scalar-valued strain energy function $W(\boldsymbol{u})$ for the rod such that:

$$\boldsymbol{m} = \frac{\partial W(\boldsymbol{u})}{\partial u} \tag{86}$$

where $W(\boldsymbol{u})$ is given by:

$$W(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{u} \cdot K\boldsymbol{u} = \sum_{i=1}^{3} \frac{1}{2} K_i u_i^2 \tag{87}$$

Substituting (87) in (86) we get:

$$\boldsymbol{m} = K\boldsymbol{u} \tag{88}$$

Since bending is about $\boldsymbol{d_2}$ only, we get:

$$\boldsymbol{m} = K u_2 \boldsymbol{d_2} \tag{89}$$

IX. APPENDIX B:ACTUAL EXPERIMENTAL IMAGE

FIG. 11: Image used for experimental data Extraction

REFERENCES

[1] H. Singh, "Planar Equilibria of an Elastic Rod Wrapped Around a Circular Capstan", *Journal of Elasticity*, vol. 151, no. 2, pp. 321–346, Oct. 2022, doi: https://doi.org/10.1007/s10659-022-09939-8.