14.02 Principles of Macroeconomics
Problem Set 8 Solutions

Fall 2017

Question 1

Suppose that a firm's production function is given by:

$$Y = F(K,N) = AK^{\alpha}N^{1-\alpha}$$

for $\alpha \in (0,1)$. The firm must rent capital from the owners of capital, and must hire workers. It pays the owners of capital a constant rental rate r for each unit of capital, and it pays workers a wage w for each unit of labor. As it is standard in economics, let's normalize the price of the good produced by the firm to $P = 1$, so that the wage and rental rates are expressed relative to this price. The firms' profits are given by:

$$\Pi(K,N) = F(K,N) - rK - wN$$

In what follows, assume that r and w are taken as given, so they are parameters.

(a)

Let

$$y = \frac{Y}{N} \qquad k = \frac{K}{N}$$

Using the production function, express y as a function of k.

Solution: Note that:

$$\begin{aligned} y &= \frac{Y}{N} \\ &= \frac{AK^{\alpha}N^{1-\alpha}}{N} \\ &= A\left(\frac{K}{N}\right)^{\alpha} \\ &= Ak^{\alpha} \end{aligned}$$

(b)

Assume the firm chooses $k \geq 0$ to maximize:

$$\frac{\Pi}{N} = y(k) - rk - w$$

What is the value $k^{\star}$ that maximizes profits per worker? Is it unique?

Solution: the objective function is strictly concave over its domain so the FOC is necessary and sufficient for a unique optimum.

$$y'(k^{\star}) - r = 0 \iff \alpha A k^{\star\alpha-1} = r \iff k^{\star} = \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}$$

(c)

Take the partial derivative of Π with respect to N and set it equal to zero. This is the necessary and sufficient condition for the firm when it chooses N optimally. Express w as a function of k and parameters.

Solution:

$$\frac{\partial \Pi}{\partial N} = (1-\alpha) AK^{\alpha} N^{-\alpha} - w = 0 \iff w = (1-\alpha) Ak^{\alpha}$$

(d)

Show that the firm's profits, when $k = k^{\star}$, are such that $\Pi = 0$.

[Hint: use the equation you derived in (b) to substitute out r, and the expression for w from (c) to substitute out w, when $k = k^{\star}$. Also, note that $\Pi = {}^{\Pi}/_{N} N$.]

Solution: plug $k^{\star}$ in the formula for profits from (b):

$$\begin{aligned}
\Pi = \frac{\Pi}{N} N &= \left[Ak^{\star\alpha} - rk^{\star} - w\right] N \\
&= \left[A \left(\frac{\alpha A}{\alpha A k^{\star\alpha-1}}\right)^{\frac{\alpha}{1-\alpha}} - \alpha A k^{\star\alpha-1} \left(\frac{\alpha A}{\alpha A k^{\star\alpha-1}}\right)^{\frac{1}{1-\alpha}} - (1-\alpha) Ak^{\star\alpha}\right] N \\
&= \left[A \left(k^{\star 1-\alpha}\right)^{\frac{\alpha}{1-\alpha}} - \alpha A k^{\star\alpha-1} \left(k^{\star 1-\alpha}\right)^{\frac{1}{1-\alpha}} - (1-\alpha) Ak^{\star\alpha}\right] N \\
&= \underbrace{\left[k^{\star\alpha} - \alpha k^{\star\alpha} - (1-\alpha) k^{\star\alpha}\right]}_{=0} AN \\
&= 0
\end{aligned}$$

This is an instance of the so called Euler's theorem for homogeneous functions. It is a general result saying that for CRS technologies, profits are going to be zero at the optimum and in equilibrium.

(e)

Now assume that the firm chooses K and N optimally, that is, to maximize Π. What condition should ${}^{K}/_{N}$ satisfy for K and N to be optimal?

Solution: Any pair of (K, N) such that:

$$\frac{K}{N} = k^{\star} = \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}$$

will be optimal, and conversely, if a pair is optimal, then it satisfies the above condition.

(f)

In subpoint (d), we showed that profits are zero. This implies that of the revenues of the firms, a certain share will accrue to capital owners, while the rest will accrue to workers. Compute these shares.

Solution: given the above discussion, we know that the capital share is:

$$\frac{rK^{\star}}{Y^{\star}} = \frac{\frac{rK^{\star}}{N^{\star}}}{\frac{Y^{\star}}{N^{\star}}} = r\frac{k^{\star}}{y(k^{\star})} = \alpha A k^{\star\alpha-1} \frac{k^{\star}}{Ak^{\star\alpha}} = \alpha$$

Obviously, the labor share is $1 - \alpha$ *because profits are zero!*

Remark: Q1(d-e-f) are highly misleading and I apologize for the confusion this has caused while solving the exercise. These subpoints will not count towards the grade of the Problem Set, while the first part of the exercise can be solved. The derivation implicitly assumes that the firm is operating at an economic equilibrium where prices are taken as given by the firm, but adjust so as to reflect the marginal products of both factors. This is why I substitute out r and w with the FOCs of the firm. This is a familiar result for those of you who have attended 14.01. In other words, prices may not arbitrary parameters if the result is to hold. To see why for general w and r none of the results in (d)-(e)-(f) hold, follow the counterexamples given by students on the Piazza thread @227.

Question 2

Suppose that the economy's production function is given by:

$$Y = AK^{\alpha}N^{1-\alpha}$$

Assume that $A = 1$, and $\alpha = 1/3$. Population N is constant.

(a)

Is this production function characterized by constant returns to scale? Explain.

Solution: Yes, it is a homogeneous function of degree 1. To see it, pick any $\lambda \neq 0$:

$$F(\lambda K, \lambda N) = A(\lambda K)^{\alpha}(\lambda N)^{1-\alpha} = \lambda AK^{\alpha}N^{1-\alpha} = \lambda F(K, N)$$

(b)

Are there decreasing returns to capital? Are there decreasing returns to labor? Justify your answers.

Solution: yes to both. Indeed, if we keep fixed either N or K, respectively, then $F(\lambda K, N) < \lambda F(K, N)$ and $F(K, \lambda N) < \lambda F(K, N)$.

(c)

Transform the production function into a relation between output per worker and capital per worker.

Solution: as in Question 1, divide both sides by N and define $y = Y/N$ and $k = K/N$:

$$y = \frac{Y}{N} = \frac{AK^{\alpha}N^{1-\alpha}}{N} = A\left(\frac{K}{N}\right)^{\alpha} = Ak^{\alpha}$$

(d)

For a given saving rate, s, and depreciation rate δ, give an expression for capital per worker in the steady state

Solution: In the steady state, it must be the case that k is constant. Remember that in the Solow model we have:

$$k_{t+1} = k_t(1-\delta) + sk_t^{\alpha}$$

Hence a fixed point of this difference equation for nonzero k_t is given by:

$$k = k(1-\delta) + sk^{\alpha}$$

divide through by k:

$$1 = 1 - \delta + sk^{\alpha-1}$$

$$k = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}$$

for $\alpha = \frac{1}{3}$ we get:

$$k = \left(\frac{s}{\delta}\right)^{\frac{3}{2}}$$

(e)

Give an expression for output per worker in the steady state.

Solution: simply plug k into the formula for output:

$$y = Ak^{\alpha} = \left(\frac{s}{\delta}\right)^{\frac{1}{2}}$$

(f)

Solve for the steady state level of output per worker when $s = 0.32$ and $\delta = 0.08$.

Solution: Plug the numbers:

$$y = \left(\frac{.32}{.08}\right)^{\frac{1}{2}} = 2$$

(g)

Suppose that the depreciation rate remains constant at $\delta = 0.08$, while the saving rate is reduced by half, to $s = 0.16$. What is the new steady-sate output per worker?

Solution:

$$y = \left(\frac{.16}{.08}\right)^{\frac{1}{2}} = \sqrt{2} \approx 1.41$$

(h)

Suppose now that both the saving rate, s, and the depreciation rate δ are equal to 0.10. What is the steady state level of capital per worker? What is the steady state level of output per worker?

Solution:

$$k = \left(\frac{.1}{.1}\right)^{\frac{3}{2}} = 1$$

$$y = \left(\frac{.1}{.1}\right)^{\frac{1}{2}} = 1$$

(i)

Suppose now that the economy is in steady state and that, at the beginning of period t, the depreciation rate increases permanently from 0.10 to 0.20. Assume here that $s = 0.10$. What will be the new steady state levels of capital per worker and output per worker?

Solution:

$$k' = \left(\frac{.1}{.2}\right)^{\frac{3}{2}} \approx 0.35$$

$$y' = \left(\frac{.1}{.2}\right)^{\frac{1}{2}} = 0.71$$

(j)

Consider the situation in (i). Compute the path of capital per worker and output per worker over the first three periods t, $t+1$, and $t+2$ after the change in the depreciation rate.

[Hint: capital does not change immediately, that is, k_t is inherited from the previous period!]

Solution: we know that $k_t = k = 1$, while k_{t+1} will be given by:

$$k_{t+1} = k_t\,(1-\delta) + sk_t^{\alpha} = (1-0.2) + 0.1 = 0.9$$

Then:

$$k_{t+2} = k_{t+1}\,(1-\delta) + sk_{t+1}^{\alpha} = 0.9\,(1-0.2) + 0.1\,(0.9)^{\frac{1}{3}} \approx 0.82$$

Output can be obtained easily by applying the production function:

$$y_t = 1$$

$$y_{t+1} = 0.9^{\frac{1}{3}} \approx 0.97$$

$$y_{t+2} \approx 0.82^{\frac{1}{3}} \approx 0.94$$

Clearly, we have convergence to the new (lower) steady state.

Question 3

Consider the following Solow growth model

$$y = Ak^{\alpha}$$

Assume that $A = 6$, $\alpha = 0.5$, and $\delta = 0.1$. We set the population growth rate, n, equal to 0.02.

(a)

If the savings rate is equal to 0.4, find the steady state levels of capital per worker, $k^{\star}$, output per worker, $y^{\star}$, and consumption per worker, $c^{\star}$.

Solution: simply recall that

$$(1+n)\,k_{t+1} = k_t\,(1-\delta) + sAk_t^{\alpha}$$

Then the steady state level of k is the fixed point for positive capital, i.e.:

$$(1+n)\,k = k\,(1-\delta) + sAk^{\alpha}$$

$$k = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\frac{1}{2}}} = \left(\frac{sA}{n+\delta}\right)^{2} = \left(\frac{6}{.02+.1}\times s\right)^{2} = (50s)^{2}$$

If $s = .4$, *then:*

$$k = 400$$

Then obviously:

$$y = 300 \times s$$

$$c = (1-s)\,y = (1-s)\,(300\times s)$$

If $s = .4$, *then:*

$$y = 120$$

$$c = 72$$

(b)

Define and find the golden rule level of (per capita) capital stock and explain why adding capital in excess of that amount will not increase per capita living standards.

Solution: The definition of golden rule s is that we require that steady-state *consumption be maximal. We simply need to choose s such that c is maximum. The objective function is strictly concave in the relevant range* $s \in (0,1)$,[1] *so the FOC is both necessary and sufficient for an interior optimum:*

$$\frac{\partial c}{\partial s} = -300s + 300\,(1-s) = 0$$

$$-s + (1-s) = 0$$

$$s = \frac{1}{2}$$

[1] *I am ignoring the points* $s = 0$ *and* $s = 1$ *because they would give rise to uninteresting economic interpretations, and also would never be optimal*

Therefore the golden rule level of per capita capital stock is

$$k = 625.$$

Note that the value we obtain is closely linked to α, but not to δ, n or A. From an economic point of view, raising the savings rate above this threshold will of course accumulate more capital, but the additional depreciation (which is linear in k) will eat away the decreasing returns from capital much faster. In other words, to sustain higher capital (via a savings rate higher than the golden rule level), we will actually need to accept lower *consumption per capita!*

REMARK FOR GRADERS: The question used misleading terminology. In particular, we meant to ask why setting a savings rate in excess of the golden rule level (so with a higher capital per worker in the steady state) is not welfare improving. However, a literal interpretation of the question could lead a student to righfully answer that adding exogenously more capital in the steady state would improve living standards. If the latter answer is presented in this way (that is, saying that we add more capital exogenously in the steady state), then it should be considered correct.

(c)

Suppose that the saving rate is at the golden rule level. What happens if the economy is initially in steady state, but then loses half of its (per capita) capital in a war and all other parameters, such as δ, n, A and α stay the same? Determine in particular how consumption and investment react, and how the growth rate of the economy changes. How does this affect the golden rule saving rate? What about the rate of return earned by capital in the short-run and long-run?

Solution: as we have seen in (b), the golden rule savings rate is constant at $s = 1/2$. If all of the other parameters are the same, it is as if we are at time $t-1$ at the steady state, but then we are thrown arbitrarily away from the steady state at t, where $k_t = \frac{1}{2}k$. Also, the capital in steady state is $k = (50 \times .5)^2 = 625$. Then we have convergence back to the steady state as usual. The per capita capital is:

$$k_t = \frac{1}{2}k = \frac{1}{2}\left(50 \times \frac{1}{2}\right)^2 = 312.5$$

Then:

$$y_t = A\left(\frac{1}{2}k\right)^{\frac{1}{2}} = 6 \times 312.5^{\frac{1}{2}} \approx 106.07$$

Therefore, we get:

$$c_t = \frac{1}{2}y_t = 53.04$$

Investment is simply $sy_t = c_t$. Then:

$$k_{t+1} = k_t\frac{1-\delta}{1+n} + \frac{s}{1+n}y_t = 312.5\left(\frac{1-0.1}{1+0.02}\right) + \frac{1}{2\,(1+0.02)}106.07 = 327.74$$

and so on and so forth we get y_{t+1}, c_{t+1} and also investment next period. In the long run, capital will have converged back to its steady state level and so will have consumption and output. The rate of return on capital per capita at t is given by:

$$\alpha A k_t^{\alpha-1} = \frac{1}{2}6\,(312.5)^{-\frac{1}{2}} \approx 16.97\%$$

in the long run:

$$\alpha A k^{\alpha-1} = \frac{1}{2}6\,(625)^{-\frac{1}{2}} = 12\%$$

which is of course lower because there are decreasing returns to capital per capita!.