14.02 Principles of Macroeconomics
Problem Set 9 Solutions

Fall 2017

1 Question 1: Technology Growth (Chapter 12)

Steady State Analysis

(a) Defining $k_t \equiv K_t / A_t N_t$, rewrite $K_{t+1} = (1-\delta)K_t + sY_t$ as

$$\begin{aligned}
K_{t+1} &= (1-\delta)K_t + sY_t \\
\Leftrightarrow \quad \frac{K_{t+1}}{A_t N_t} &= (1-\delta)\frac{K_t}{A_t N_t} + s\sqrt{\frac{K_t}{A_t N_t}} \\
\Leftrightarrow \quad \frac{K_{t+1}}{A_{t+1}N_{t+1}}\frac{A_{t+1}N_{t+1}}{A_t N_t} &= (1-\delta)k_t + s\sqrt{k_t} \\
\Leftrightarrow \quad k_{t+1}(1+g)(1+n) &= (1-\delta)k_t + s\sqrt{k_t} \\
\Leftrightarrow \quad k_{t+1} &= \frac{1}{(1+g)(1+n)}\left((1-\delta)k_t + s\sqrt{k_t}\right) \\
&\approx \frac{1}{(1+g+n)}\left((1-\delta)k_t + s\sqrt{k_t}\right)
\end{aligned}$$

The steady state satisfies

$$(g+n+\delta)k = s\sqrt{k}$$

$$\Leftrightarrow \quad k = \left(\frac{s}{g+n+\delta}\right)^2$$

Plugging in $s = 20\%, g = 4\%, n = 2\%, \delta = 10\%$, we have

(i) $\frac{K_t}{A_t N_t} \approx 1.56$ (or 1.55 without approximation).

(ii) $\frac{Y_t}{A_t N_t} = \sqrt{k} \approx 1.25$ (or 1.24 without approximation).

(iii) 0. Since k_t is constant, $\frac{Y_t}{A_t N_t} = \sqrt{k_t}$ is constant as well.

(iv) $\frac{Y_t}{N_t} = A_t\sqrt{k_t}$, and because k_t is constant, output per worker grows at rate $g = 4\%$.

(v) $Y_t = A_t N_t \sqrt{k_t}$, and because k_t is constant, total output grows at rate $(1+g)(1+n) - 1 \approx g + n = 6\%$.

(b)

(i) $\frac{K_t}{A_t N_t} \approx 1$ (or 0.99 without approximation).

(ii) $\frac{Y_t}{A_t N_t} \approx 1$ (or 0.996 without approximation).

(iii) 0.

(iv) 8%.

(v) $(1+g)(1+n) - 1 \approx g + n = 10\%$.

(c)

(i) $\frac{K_t}{A_t N_t} \approx 1$ (or 0.99 without approximation).

(ii) $\frac{Y_t}{A_t N_t} \approx 1$ (or 0.996 without approximation).

(iii) 0.

(iv) 4%.

(v) $(1+g)(1+n) - 1 \approx g + n = 10\%$.

For the same level of A_t, output per worker is given by

$$\frac{Y_t}{N_t} = A_t \sqrt{k}.$$

Consumption per worker is $(1-s)\frac{Y_t}{N_t} = (1-s)A_t\sqrt{k}$. Therefore (a) has higher consumption per worker than (c) in steady state, which implies people are better off in (a) than (c). This is because since (c) has larger population growth, they have to share with output with greater number of people and hence they each can consume less.

Transition Dynamics

(d)

$$\begin{aligned}
K_{t+1} &= (1-\delta)K_t + sY_t \\
\Leftrightarrow \quad \frac{K_{t+1}}{A_t N_t} &= (1-\delta)\frac{K_t}{A_t N_t} + s\sqrt{\frac{K_t}{A_t N_t}} \\
\Leftrightarrow \quad \frac{K_{t+1}}{A_{t+1}N_{t+1}} \frac{A_{t+1}N_{t+1}}{A_t N_t} &= (1-\delta)k_t + s\sqrt{k_t} \\
\Leftrightarrow \quad k_{t+1}(1+g)(1+n) &= (1-\delta)k_t + s\sqrt{k_t} \\
\Leftrightarrow \quad k_{t+1} &= \frac{1}{(1+g)(1+n)}\left((1-\delta)k_t + s\sqrt{k_t}\right) \\
&\approx \frac{1}{(1+g+n)}\left((1-\delta)k_t + s\sqrt{k_t}\right)
\end{aligned}$$

Therefore

$$k_{t+1} = \frac{1}{(1+g)(1+n)}\left((1-\delta)k_t + s\sqrt{k_t}\right)$$

or with approximation,

$$k_{t+1} = \frac{1}{(1+g+n)}\left((1-\delta)k_t + s\sqrt{k_t}\right)$$

(e) Figure 1 shows the solution.

(i) Output per effective worker is unchanged at time t. The economy gradually converges to the new steady state level which is lower than before. At first, the economy rapidly shrinks because of decreasing returns to scale and the economy initially has too much capital.

(ii) Same as (i), but note that total output is growing at rate $g+n$ both before and after the change in saving rate. Changes in saving rate affects the level of output, but not the growth rate.

(iv) Consumption per effective worker in the steady state is

$$\frac{(1-s)Y_t}{A_t N_t} = (1-s)\sqrt{k}$$
$$= \frac{1}{g+n+\delta} s(1-s),$$

which is single-peaked at $s = 0.5$. Therefore when s drops from 20% to 10%, steady state consumption will drop. However at time t, $(1-s)\sqrt{k_t}$ will jumps up because k_t doesn't change at time t. After time t, the consumption will follow the similar path as the output toward new steady state because consumption is proportional to the output.

(iii) Same as (iv) except that now the consumption is growing at rate g because

$$\frac{(1-s)Y_t}{N_t} = \frac{A_t}{g+n+\delta} s(1-s).$$

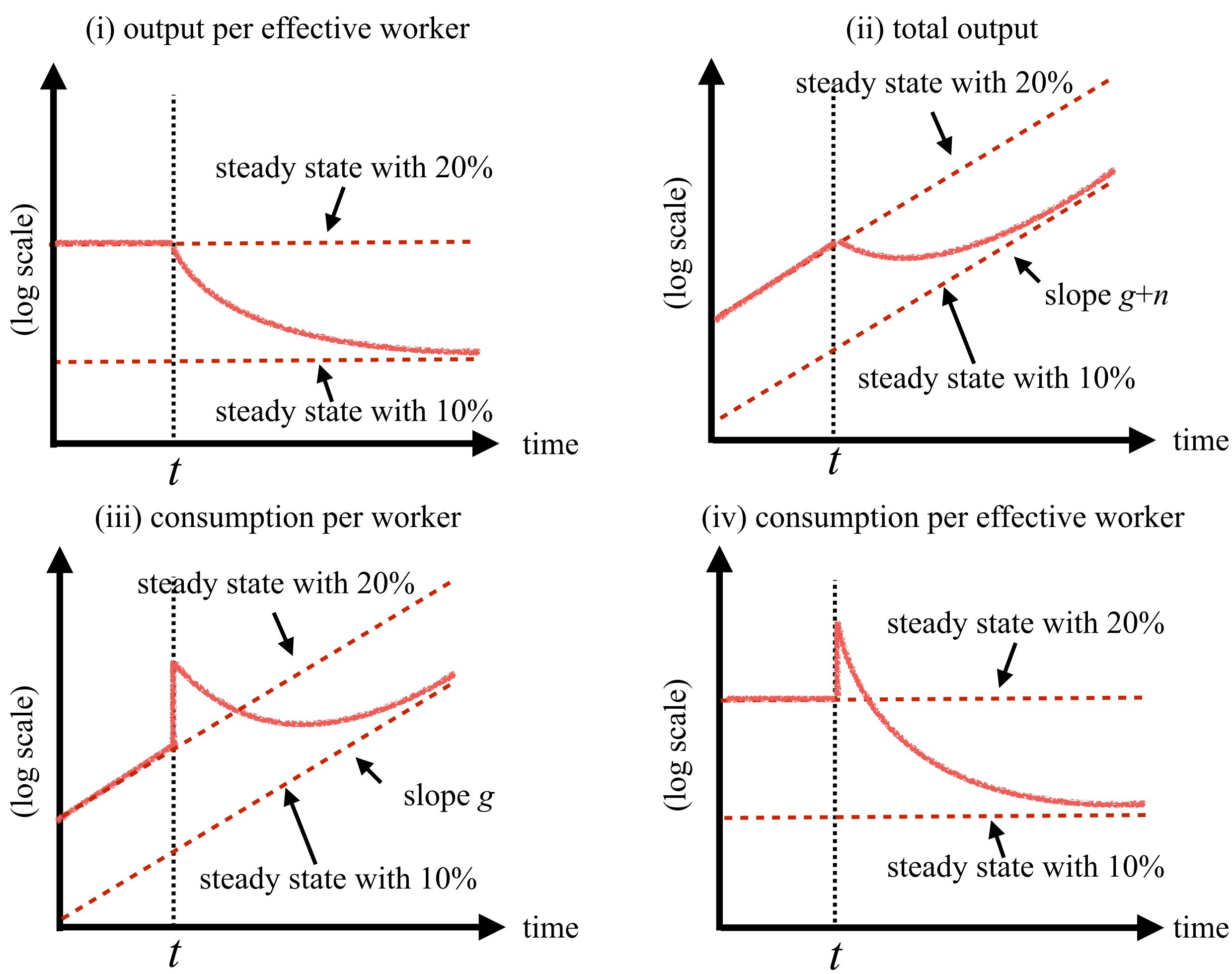

Figure 1: Solutions to (e)

(f) Figure 2 shows the solution. The only difference to (e) is that consumption at the steady state will **increase** because the original saving rate was exceeding the golden rule, and the saving rate equals golden rule level.

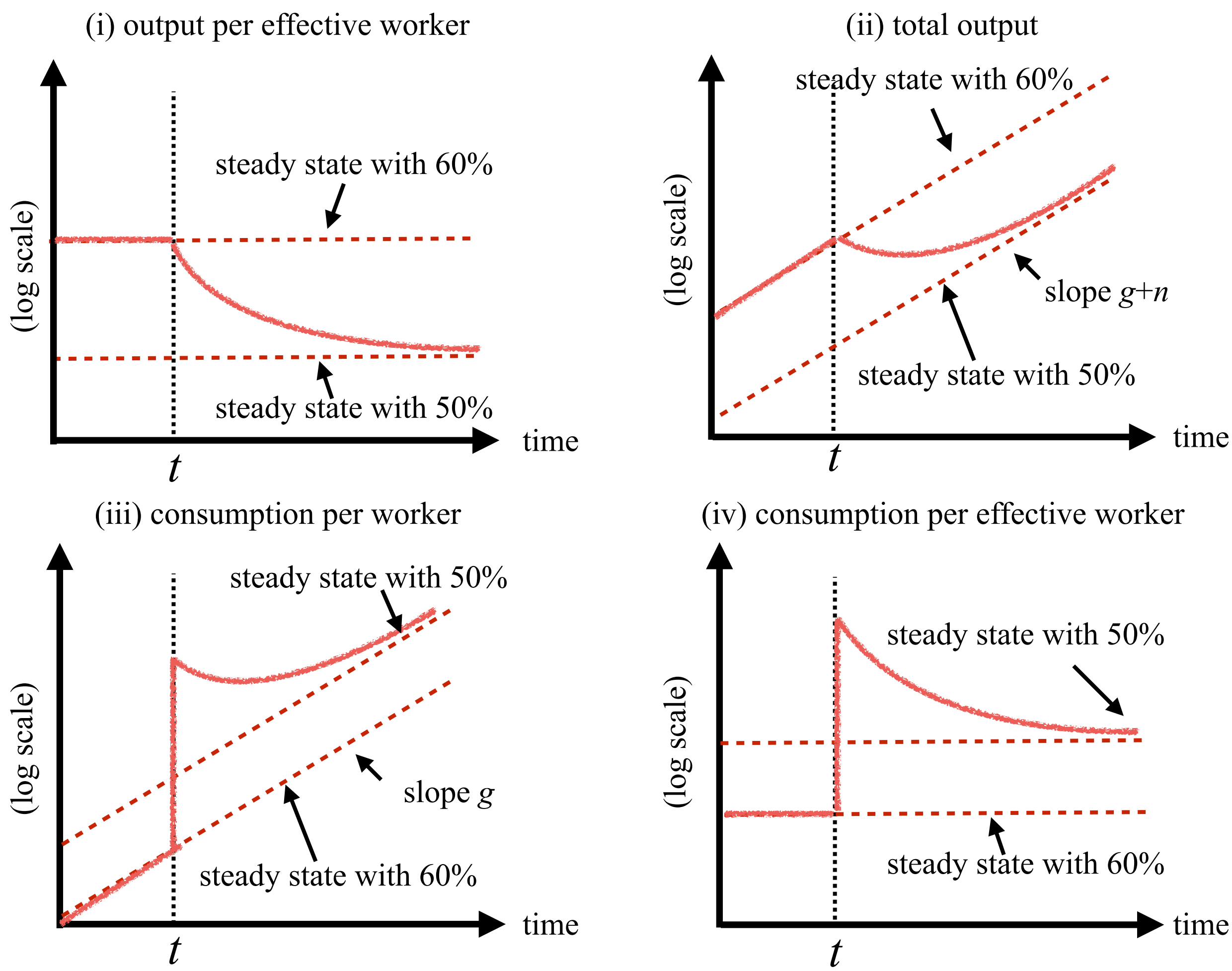

Figure 2: Solutions to (f)

2 Question 2: Productivity and the Aggregate Supply Curve (Chapter 13)

(a) Combining the given equations,

$$
\begin{aligned}
P &= (1+m)\frac{W}{A} \\
&= (1+m)\frac{1}{A}A^e P^e(1-u) \\
&= (1+m)\frac{1}{A}A^e P^e\frac{N}{L} \\
&= (1+m)\frac{A^e}{A}P^e\frac{1}{L}\frac{Y}{A}.
\end{aligned}
$$

This is a positive linear relationship between P and Y, as in Figure 3.

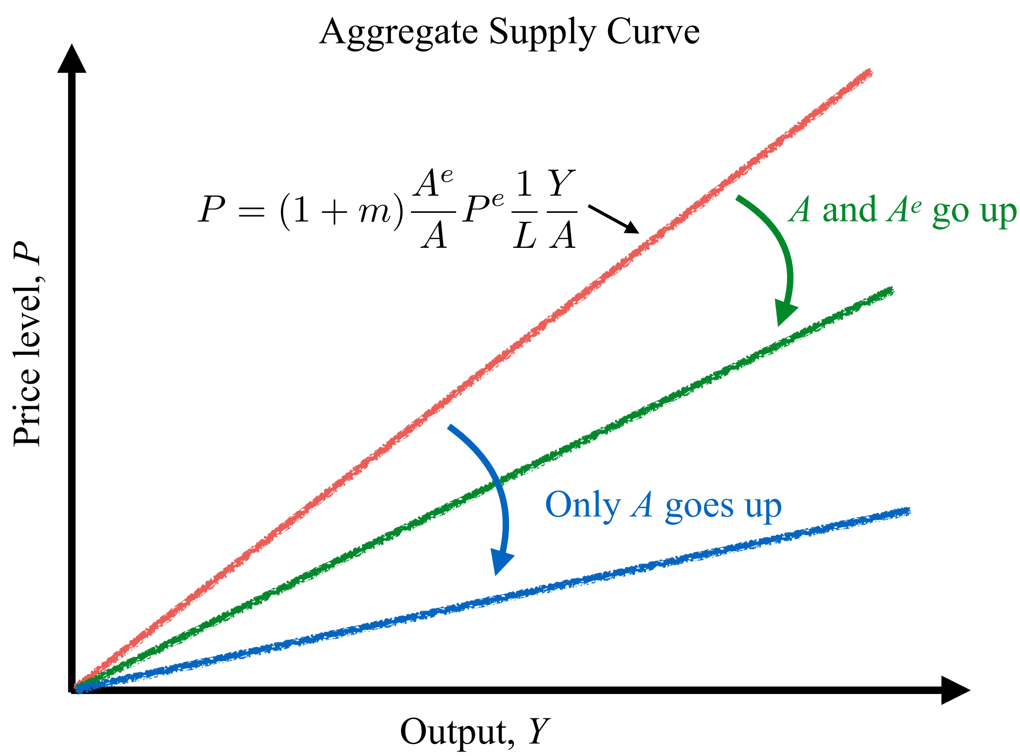

Figure 3: Aggregate Supply Curve

(b) Aggregate supply curve shifts (or tilts) down. This means that for given level of output, the price level is lower. Intuitively, as productivity increases, the economy is able to produce goods cheaper.

(c) The aggregate supply curve shifts down further when A^e is fixed. Intuitively, when workers don't adjust their expectation, they don't ask higher wages. Therefore the economy is able to produce goods even cheaper.

3 Question 3: Solow Model with Technology Growth (Chapter 12)

Consider the Solow growth model with technological change. Specifically, suppose that the total output produced at time t, Y_t, is governed by a Cobb-Douglas production function:

$$Y_t = F(K_t, A_t \cdot N_t) = K_t^{\alpha}(A_t \cdot N_t)^{1-\alpha},$$

where K_t is the amount of capital used, N_t is the amount of labor and A_t is the level of technology at time t. Suppose that the population grows at a constant rate, $n \geq 0$, the capital depreciates at rate $\delta \in (0,1)$, and the saving rate satisfies $s \in (0,1)$. Also assume that the technology grows at a constant rate at $g_A \geq 0$. That is,

$$\begin{aligned} A_{t+1} &= (1+g_A)A_t, \\ N_{t+1} &= (1+n)N_t. \end{aligned}$$

(a) Denote capital per effective worker as $k_t = \frac{K_t}{A_t N_t}$. Derive the law of motion of capital per effective worker. That is, express k_{t+1} as a function of k_t, g, n, δ, α and s.

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + sY_t \\ \Leftrightarrow \quad \frac{K_{t+1}}{A_t N_t} &= (1-\delta)\frac{K_t}{A_t N_t} + s\left(\frac{K_t}{A_t N_t}\right)^{\alpha} \\ \Leftrightarrow \quad \frac{K_{t+1}}{A_{t+1}N_{t+1}}\frac{A_{t+1}N_{t+1}}{A_t N_t} &= (1-\delta)k_t + s(k_t)^{\alpha} \\ \Leftrightarrow \quad k_{t+1}(1+g_A)(1+n) &= (1-\delta)k_t + s(k_t)^{\alpha} \\ \Leftrightarrow \quad k_{t+1} &= \frac{1}{(1+g_A)(1+n)}\left((1-\delta)k_t + s(k_t)^{\alpha}\right) \\ &\approx \frac{1}{(1+g_A+n)}\left((1-\delta)k_t + s(k_t)^{\alpha}\right) \end{aligned}$$

(b) The steady state satisfies

$$\begin{aligned} & (1+g_A+n)k = (1-\delta)k_t + s(k_t)^{\alpha} \\ \Leftrightarrow \quad & (g_A+n+\delta)k = s(k_t)^{\alpha} \\ \Leftrightarrow \quad & k = \left(\frac{s}{g_A+n+\delta}\right)^{\frac{1}{1-\alpha}}, \end{aligned}$$

or without approximation,

$$k = \left(\frac{s}{(1+g_A)(1+n)-1+\delta}\right)^{\frac{1}{1-\alpha}},$$

which does not depend on A_0. Intuitively, whatever the initial value is, the economy will reach the level of capital per effective worker that it can sustain. The steady state capital per effective negatively depends on g_A. Intuitively, higher g_A means the capital per effective worker is depreciating faster at each point in time. Therefore the economy end up having less capital per effective worker in steady state.

(c) No because for both countries, we have in steady state that

$$
\begin{aligned}
y &= k^{\alpha} \\
&= \left(\frac{s}{(1+g_A)(1+n) - 1 + \delta} \right)^{\frac{\alpha}{1-\alpha}},
\end{aligned}
$$

which does not depend on initial technology levels.

(d) Total output is given by

$$
\begin{aligned}
Y_t &= A_t N_t k^{\alpha} \\
&= A_t N_t \left(\frac{s}{(1+g_A)(1+n) - 1 + \delta} \right)^{\frac{\alpha}{1-\alpha}}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
Y_t^H / Y_t^L &= \frac{A_0^H N_0 (1+g_A)^t (1+n)^t}{A_0^L N_0 (1+g_A)^t (1+n)^t} \\
&= \frac{A_0^H}{A_0^L}.
\end{aligned}
$$

Hence the ratio Y_t^H / Y_t^L remains constant.